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Cultural–Historical Theory and Mathematics Education

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Mathematics education in the United States is currently undergoing an attempt at reform. In this chapter an alternative in the form of a Vygotskian-based approach to mathematics pedagogy is explored. While embracing teaching methods similar to those advocated within the reform movement, the Vygotskian-based curriculum, in its genetic analysis of mathematics concepts, their derivation from measurement, and representation by schematic modeling, differs substantively from both historical and current U.S. reform efforts. The teaching and curricular similarities and differences of reform practices and Vygotskian-based pedagogy reflect their respective grounding in divergent theoretical perspectives – the former in constructivism and the latter in cultural–historical theory. Here the cultural–historical approach is addressed, and some of the effects of these two pedagogical approaches on the adequacy of mathematical understanding is explored. It is necessary, however, to begin with a summary consideration of the antecedents of the current reform effort.

Mathematics education throughout the past century has come under the dominance of several learning paradigms. First was the early period of behaviorist pedagogy, succeeded by the formalism of the "new math," then the rapid reversion to "basics," and finally the emergence of constructivism, which continues to maintain its pedagogical hegemony to the present day. It is curious that throughout these periods of changing pedagogical approaches, all grounded in different philosophies of mathematics (Schmittau, 1991), a single practice persisted unchallenged. This was the practice of building children's understanding of the real number system, which Davydov (1990) asserts is the dominant subject matter of school mathematics, on the activity of counting.

The continuance of this practice is partly the result of a certain ambivalence with respect to concept development that has characterized the history of mathematics education in the United States. Behaviorism, after all, was not concerned with concept development, and the "back to basics" movement that reverted to it characteristically focused on procedural rather than conceptual competence. The "vulgar formalism" (Browder & MacLane, 1979, p. 344; cited in Hanna, 1983, p. 88) of the "new math" virtually reduced mathematics to a syntactic system, and formalist mathematics, in which the "new math" was grounded, actually generates the real numbers from the positive integers through an axiomatic system. So it is obvious why formalism not only failed to question, but actually ratified an approach to number centered on the counting numbers. The final and present period in mathematics education, unlike previous periods in which procedural competence or logical deduction was emphasized, is marked by an awareness of the importance of concepts. When clinical interviewing, a research method of choice by the mid-1980s, revealed that the direct transmission of mathematical understanding from teacher to student was not occurring despite clear explanations of mathematical content, the notion that students must "construct their own knowledge" took center stage in mathematics education. It is perhaps significant that it did so in the absence of any competing paradigm. The pendulum swing from the transmission model with its grounding in behaviorism (with some surviving formalist contaminants) was, to all appearances, extreme. Yet constructivism, as did its pedagogical predecessors, continues to ground number in counting. The fact that children typically enter school with some more or less valid knowledge of counting is doubtless a consequence of the fact that we live in a world of "stuff," most of it eminently countable. And since constructivism posits that children must construct their own concepts, what better basis could there be on which to build future mathematical understanding than children's own spontaneous counting concepts?

Unlike the mathematics teacher, the science teacher realizes that it is dangerous to assume that children's spontaneous concepts constitute an adequate basis on which to develop further understandings. When she asks these same children why a cork floats in a tub of water and a nail sinks, she may hear that it is because the nail is long and thin and the cork is more round. Now disconfirming evidence is called for, and the teacher may place a wooden matchstick and a steel ball bearing in the water, clearly challenging the children's naïve concepts by the fact that the match floats and the bearing sinks. At this point, however, the children are still very far from an understanding of density, which is a concept that cannot be grasped empirically but requires a theoretical mode of thinking for its appropriation (cf. Davydov, 1990). It is one of the concepts Vygotsky called *scientific* to distinguish them from the spontaneous concepts children form through their interactions within their everyday environment. Scientific concepts (which are not limited to the field of science) require pedagogical mediation for their appropriation. It is important to mention that only scientific concepts were considered to be true concepts by Vygotsky (Kozulin, 1990), and that virtually all mathematics concepts fit this designation (Schmittau, 1993a).

The difficulty of trying to ground children's mathematical development in their spontaneous notions of number emanating from counting, rather than reorienting them (as the science teacher must) to a more adequate theoretical development of the concept, is illustrated by Davydov (1991). He cites the fact that since number becomes identified for children with the action of counting, which only generates the positive integers, and formalist mathematics generates real numbers from these as well, a rational number (and hence a fraction) is defined as a quotient of two integers a/b such that $b \neq 0$. (This allows, for example, for 2/3, and 5/4, while properly excluding 2/0 from the realm of number.) Fractions, of course, did not evolve in this manner any more than language evolved from the rules of grammar (cf. Riegel, 1979; Schmittau, 1993b). This is a formalist definition and is in keeping with the axiomatic integer genesis of real number within that paradigm. However, since such a designation makes very little sense to children, educators divide circles into sectors and illustrate fractions from the ratios formed, thereby providing a visual interpretation of a formal definition. That this visual representation leads to less than an accurate grasp of the concept of fraction is the subject of meticulous scrutiny by Davydov, who indicts this approach on a number of counts, not the least of which is that it separates fractions from their historical origin in measurement.

Historically fractions clearly were not developed as quotients of integers. The axiomatization and formalization of mathematics that occurred in the 19th and early 20th centuries represented an attempt to reestablish mathematics on a foundation that was rigorously deductive. Hence, formalism may appropriately be viewed as a cognitive reflection – occurring very late in mathematics history – on a body of knowledge that actually developed in a very different way over a period of several thousand years. The fallacy of the "new math" was the assumption that formalist notions could be directly learned by students, who could skip the development of concepts as they had actually occurred, and instead learn mathematics by beginning at the end, so to speak, of the history of mathematical development. The primary reason for the failure of the "new math" was that ordinary students could not learn mathematics in this way. Rigorous deduction and formal logic were not the paths of conceptual genesis.

Further, it is significant that the formalist reestablishment of the category of real number as an emanation of the positive integers (or counting numbers) has the character of a generative metonymy. In his provocative book *Women, Fire, and Dangerous Things: What Categories Reveal About the Mind,* Lakoff (1987) discusses the manner in which the real numbers constitute a generative category, that is, one characterized by its generation from a member or subgroup of members according to rules. Lakoff observes that the set of single-digit numbers generates all the counting numbers through the rules of positionality in our base 10 numeration system. The rational numbers are then defined as quotients of these, and the irrationals as infinite nonrepeating decimals composed of the digits 0 through 9. Lakoff further notes that generative categories tend toward metonymy, as the generative subcategory becomes representative of the category as a whole.

Our research (Schmittau, 1994) indicates that this development of the real numbers as a generative category is not confined to formalism, but occurs whenever the counting numbers are taken as primary, that is, when the concept of number is allowed to develop from the action of counting. Consequently the entire category of real numbers may be interpreted by students in terms of the counting numbers, and the smaller the representatives, the better. There are, moreover, other far-reaching consequences of the acceptance of the counting numbers as a basis for the development of the concept of number. Since fractions and irrational numbers cannot be generated through counting, not only do many students – and even adults – fail to see fractions and irrationals as numbers (Skemp, 1987; Schmittau, 1994), but they may inadequately conceptualize the so-called fundamental operations (i.e., addition, subtraction, multiplication, and division) on these numbers as well. By way of illustration, we shall focus on one of these, the operation (or more properly the *action*) of multiplication.

Conventional pedagogical practice in the United States (by which we shall mean common textbook approaches that in practice become the basis for curriculum) define multiplication as repeated addition. Hence, 5×4 means $5 + 5 + 5 + 5$. This is, of course, an extension of the generative metonymic, since one can repeat an action such as adding 5 to itself only an integral (but not a fractional or irrational) number of times. Textbooks sometimes present other "models" of multiplication, such as arrays in which circles, squares, or other symbols appear in equal groups. It is generally unclear whether these constitute the same notion – that is, one is just repeatedly adding the same number of objects in each group – or whether they represent disparate concepts (in which case one might well wonder why they are both called *multiplication*). Increasingly, rectangular models are finding their way into textbooks as well and often prove helpful in providing meaning to the operation, but again absent the requisite conceptual connections, it is unclear whether in and of themselves they will be sufficient to transform the learning of multiplication from instrumental (a collection of rules) to relational (an integrated system of knowledge) (Skemp, 1978).

A Vygotskian Learning Paradigm for Number and Multiplication

However, in the curriculum developed and researched by V. V. Davydov and his colleagues in Russia for more than 40 years and grounded in

Vygotskian cultural–historical psychology, a very different approach to the genesis of both number and fundamental actions such as multiplication is taken. Number is developed out of the action of measurement rather than counting.

Generation of Number from Measurement

Preparatory activities for the development of measurement in Davydov's curriculum reflect the essence of mathematics as the science of quantity and relation. The first-grade course (Davydov, Gorbov, Mikulina, & Saveleva, 1999) begins with the comparison of two quantities (length, area, volume, or weight), which differ sufficiently to permit a visual determination of their equality or inequality without placing them in spatial proximity. In the case of weights, merely hefting them in the hands is sufficient to determine which is greater. Next children are presented with quantities that do not differ so significantly and therefore require alignment to effect a determination as to which is greater. They may be asked to compare the length of a pencil and a pen, for example, or the area of a textbook and a notebook, or the volume of liquid in two identically shaped containers. Two weights may be so close that a balance is necessary to make a determination about which of them is greater. No sooner have students mastered these requisite alignments than they are confronted with a task requiring them to compare quantities that cannot be aligned. They might be asked to compare the height of a bookcase and the length of the teacher's desk, the area of the classroom door and that of the overhead projector screen, or the volume of liquid in two containers having very different shapes. Now the children must find an intermediary, such as a piece of rope to compare the lengths, or a third container into which to pour the original liquids to determine which of them has greater volume.

Once children have become comfortable with these methods, they will then be confronted with the task of comparing two long line segments with only an intermediary unit such as a short strip of paper to use for this purpose. They must now lay off the strip on each of the segments as many times as required: That is, they must *measure* each one. The measure is then expressed as a ratio of the length of the original segment to the length of the unit. For example, if the length of the original segment is designated A and the length of the strip of paper is designated U, then A/U is the required measure. This measure may be a whole number or a fraction, or even an irrational number. Measurements resulting in fractions (or irrationals) are not encountered in the first grade, of course, but occur later in the child's education and significantly do not require a reconceptualization of number when they do occur. In curricula where number develops from the action of counting, however, successive reconceptualizations of both the concept of number and the various operations performed on numbers are required each time a new type of number is introduced. Thus the genesis of number from measure gives greater coherence to the category of real number and spares the student such successive conceptual upheavals, which as Skemp (1987) attests and our own research (Schmittau, 1994) shows, are rarely accomplished.

Progressive Task Difficulty

The first-grade curriculum of Davydov not only is grounded in cultural–historical theory, following the anthropological and historical development of mathematics and framing significant moments in this development in ways psychologically accessible to children, but accomplishes this through a stream of progressively more difficult problems, without demarcation into chapters or sections. The teaching methods employed greatly resemble those advocated by constructivism, but with very different theoretical foundations. Vygotsky and Luria (1993) carried out an extensive investigation of the development of primates, traditional peoples, and children and concluded that cognitive development occurs only when members of these groups are confronted with a problem for which previous solution methods are inadequate. Hence, the progressively more difficult problems of comparison of quantity in the first-grade curriculum described above reflect this view. No sooner do children master one solution method than they are confronted with a problem for which this method is no longer adequate.

The following classroom episode described by Lee (2002) is illustrative. The first graders have just learned that if $A > B$, they can conclude that $B < A$ without reverting to concrete objects. The teacher cuts a paper plate into three parts labeled A, B, and C (with areas $A > B > C$) and places them into an envelope out of sight of the students. She then presents the task: If $A > B$, then B __ C. All children write $B < C$ and cite their previous conclusion from $A > B$ (viz., $B < A$) as the reason. They have drawn a false conclusion based on syntactic similarity. The teacher points out that C does not appear in the initial inequality, but the children are unmoved. They see their error when presented with the plate pieces, but the teacher's attempts to elicit a correct conclusion without such concrete aids are unsuccessful. So the teacher tries another approach.

She asks the children to compare the height of classmates Mike (T) and Sue (C), eliciting $T > C$. She then inquires as to how T compares with the height of an unknown first grader, Ellen (E). Mike promptly writes $T > E$, explaining that this must be true since he is the tallest first grader! Having made an obviously ineffective choice of students, the teacher then asks the children to compare Mike's height with the height B of another child, Bobby, whom they do not know. A flurry of questions about Bobby's grade, age, and so on, ensues, to which the teacher responds that she either does not know or cannot tell. The children finally agree that the correct answer is $T\ ?\ B$, since they do not know Bobby and cannot conclude anything about the relationship between the heights of the two boys. And the fact that $T > C$ was of no importance to their argument.

Clearly Davydov's curriculum is anything but didactic. At this writing, we have completed the implementation of the first 3 years of his program in a school setting in the Northeast (to our knowledge a first in the United States), and we have found the problem solving–inquiry focus challenging for both students and teacher. It has typically taken our American children a year to develop the intense focus and sustained concentration required consistently and productively to engage with the problems, which appear to continuously expand their zones of proximal development (Vygotsky, 1934/1986). The problems themselves are very interesting to the children, but the challenge is unrelenting, and there is never a day when they can simply "kick back" and do "fun stuff" or drill on "facts." After Vygotsky, for whom learning leads development, Davydov's program, in both curriculum and teaching methodology, has as its intended goal not only a deep understanding of mathematics but cognitive development itself.

Genetic Analysis of Concepts

In his *Types of Generalization in Instruction: Logical and Psychological Problems in the Structuring of School Curricula*, Davydov (1990) explains this orientation toward cognitive development. He cites a study of Krutetskii in which students unfamiliar with the square of a sum were presented with the basic example $(a + b)^2$ and taught its meaning. They were then presented with another square of a sum, $(C + D + E)(E + C + D)$, whose surface features were very different from those of the original example. Many students, whom Krutetskii identified as average, had to be given intermediate examples such as $(3x - 6y)^2$ and 51^2 before they were able to discern the conceptual structure of $(C + D + E)(E + C + D)$ as the square of a sum (i.e., $[(C + D) + E][(C + D) + E]$, which, if $C + D = K$, is $(K + E)^2$). A few students immediately grasped the *theoretical essence* of the first example $(a + b)^2$ and easily discerned it in $(C + D + E)(E + C + D)$, which was judged to be the most syntactically different example in the series (there were eight examples in all). Rather than labeling these students "gifted," Davydov noted that their mental activity was qualitatively different from that of the less capable students.

> Confronting a specific problem they primarily tried to discover its "essence," to distinguish the main lines by abstracting themselves from its particular features – from its concrete form . . . striving to delineate the internal connections among its conditions (this is peculiar to theoretical generalization). (Davydov, 1990, p. 133)

Davydov observed that theoretical generalization is necessary for the appropriation of Vygotskian scientific concepts and set about the task of attempting to develop in ordinary students this ability, which is generally evidenced by only the most capable. Hence, his curriculum is a rich synergy of content and method designed not only to enable students to grasp mathematics at a deep conceptual level, but to develop their ability to think theoretically.

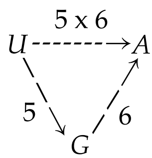
Before such a curriculum can be created, however, there must be an epistemological analysis of the concepts in question that encompasses both historical and conceptual analyses. This often entails a lengthy and arduous process, but a necessary one, since symbolic forms of thought (typical of mathematics) "absorb" the genesis of a concept, making it "necessary to trace all of the *historically* available methods of solving the same problems in order to see the initial forms behind the abbreviated curtailed thought processes [represented symbolically], to find the laws and rules for this curtailment and then to detail the complete structure of the thought processes being analyzed" (Davydov, 1990, p. 322). This genetic analysis is reflected in the development of number from measure in Davydov's curriculum, since historically it became necessary to admit the results of measure, such as irrational numbers, into the system of real numbers (otherwise such common quantities as the diagonal of a unit square or the circumference of a circle could not be designated numerically). This was not accomplished without upheaval, since the Greeks had relegated irrationals to the category of "magnitudes" while admitting only integers as numbers. By developing the real numbers through measurement, this historically Herculean cognitive restructuring by students is avoided.

The approach to multiplication in Davydov's curriculum also reflects the understanding gained from a genetic analysis of the concept. The first-grade curriculum actually lays the groundwork for multiplication by presenting children with many tasks that require them both to build and to measure quantities. And they use a schematic form to designate these actions. For example, the designation

$$U \xrightarrow{|||} A$$

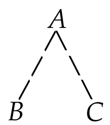
indicates that three units have been used to build or measure quantity A. The symbol $U \xrightarrow{||||} ?$ indicates that the student must *build* a quantity using four units. The unit is specified and may be one or more line segments, squares, or other shapes, which then must be combined to build the quantity. Alternately, the symbol $U \xrightarrow{?} A$ indicates that the student must *measure* quantity A using unit U, and thereby determine the value of the ?. The students do many varieties of such problems. Then they are confronted in the second grade with a situation in which they must do a measurement of a very large quantity with a very small unit, and the process is thus a deliberately tedious one (Davydov, 1992).

For example, following Davydov (1992), children may be told to pretend that they are working for the local animal shelter and must give each kitten a very small paper cup of water poured from a large pitcher. They need to know how many kittens will receive water. The process is tedious, and there are other larger glasses on the table, but no mention is made of them. Eventually a child will suggest that we find out how many little paper cups of water one of the larger glasses will hold and then determine how many of the larger glasses we can fill from the pitcher. For example, a glass may hold five of the paper cups, and the pitcher may hold six glasses. Now the situation must be schematized a bit differently. Since we found it too tedious to do a straightforward measure of the volume of the pitcher with the unit paper cup, we cannot represent our measure as we did previously, by designating the number of units *U* in quantity *A*. Now our schematic must represent the *change in unit* from a smaller unit *U* (here the little paper cup) to a larger unit *G* (the glass) with which we then measured the volume of water. The children therefore indicate this action as follows:



Multiplication is now defined as a method for taking an indirect measurement by means of a change in unit (from a smaller to a larger unit) (Davydov, 1992). This reflects Lebesgue's (1960; cited in Davydov, 1992) stress on multiplication as a change in the system of units. One can see how the need for such a process as multiplication arose historically as the numerosity of quantities increased with cultural complexity. Here multiplication is not reduced to addition, which is a different action (of composition rather than of measurement).

It is important to note the use made of mathematical models or schematics, such as the building, measurement, and multiplication models, in Davydov's curriculum, which preserve in representational form the mathematical action that constitutes the essence of the concept in question. In my research in Russia with Davydov and his colleagues, I saw the power of such models in classrooms where I observed Davydov's program being taught and have now observed it even more extensively in our implementation of the program here in the United States. A particularly powerful (albeit deceptively simple) schematic is the part–whole model from which first graders write three equations derived from their actions with quantities before numbers are introduced. This model suggests putting together or taking apart a set of objects or quantities.



$$A = B + C$$
$$A - B = C$$
$$A - C = B$$

Since this schematic represents the essence of actions of composing and decomposing quantities, adding and subtracting are not perceived as formally separated operations, but as complementary actions. The whole (A) must be found from composing the parts (B and C); a part must be the difference between the whole and the remaining part(s). Children have no difficulty with missing addend problems as a result. Children in the United States, however, typically find missing addend problems such as the following difficult: "John has 14 baseball cards. Eric gave him 6 cards. How many cards did John have originally?" The sentence representing this problem appears to indicate addition: $? + 6 = 14$. However, it is necessary to *subtract* 6 from 14 to obtain the solution. No such confusion arises if the preceding schematic is employed to analyze and represent such a problem, as 14 is the whole, 6 is one part, and the other part is found by subtraction.

Now that we have completed the implementation of the first 3 years (these years constitute the 3 years of Russian elementary school) of Davydov's curriculum in a U.S. setting, our research has confirmed the effects of these models firsthand. The power to analyze situations such models afford children cannot be overstated. Neither can their ability to connect the conceptual content of mathematics at very deep and important levels. The function of a model is, after all, either to render hidden features visible or to render particular (or essential) features salient. Hence, appropriately constructed models might be expected to give students the ability to grasp conceptual structure at its most abstract level, thereby enabling them to ascend from the abstract to the concrete, as Hegel, whose influence on Vygotsky was considerable, advocated. In addition, these schematics allow conceptual *connections* (the sine qua non of learning) between mathematical actions previously viewed as separate operations. Finally, they provide students with the *tools of analysis* required for problem solving.

Although with the publication of the National Council of Teachers of Mathematics (NCTM, 1989, 2000) standards, the U.S. curriculum has shifted in recent years from procedural and algorithmic dominance to more work with concrete materials, it lacks the critical intermediate work with schematic models, the genetic analysis, and the emphasis on conceptual *essence* that are so central to Davydov's curriculum.

A CROSS-CULTURAL STUDY OF THE CONCEPTUAL STRUCTURE OF MULTIPLICATION

How does the understanding of students who experience a curriculum designed in such a way as to foster the development of a generative metonymic structure for the categories of real number and multiplication differ from that of students instructed in Davydov's curriculum, which develops the concepts of number and multiplication very differently?

A comparative study conducted with 40 secondary and university students in the United States and 24 elementary and secondary students in Russia addressed this question (Schmittau, 1994). The U.S. university students represented a diversity of course majors and varying backgrounds in mathematics (high school geometry through calculus, statistics, and linear algebra). The secondary student component consisted almost entirely of high school students, 90% of these rated "very good" or high-achieving in mathematics by teachers and mathematics grades. The Russian students consisted of fourth and fifth graders and a cohort of ninth- and tenth-grade students, all of whom had experienced Davydov's curriculum during their elementary years, the first 3 years of Russian schooling. After these 3 years, the older students had experienced a variety of mostly traditional approaches to the teaching of mathematics. The Russian elementary students were rated either good or average by their teachers, and all Russian secondary students were rated average.

Our investigation of conceptual structure took into account the fact that commonly held assumptions in psychology predicating the structure of conceptual categories on genus and differentia have given way in recent years to massive evidence of family resemblance and comparison-to-exemplar structures (Lakoff, 1987). Rosch (1973) was the first to establish evidence of such category organization. She found that when subjects were asked to rate instances of fruit on a scale of 1 to 7 for degree of membership in the category, a prototypic instance emerged to which all other instances were compared. An apple, for example, might receive a rating of 1, designating it as an exemplary member of the category "fruit," and an olive might receive a 7, indicating that the subject did not regard it as a good example of a fruit or perhaps did not consider it to be a fruit at all. The rating for "fig" might fall somewhere between these two instances. Rosch determined that the characteristics of the apple, especially that it was juicy and sweet, were believed by subjects to be essential to fruit. Hence, they judged all other instances of the category on this basis, and the apple functioned as a prototype for the category. Her work has been widely replicated, and evidence of prototypicality has been confirmed even in such highly structured domains as science and mathematics. Armstrong, Gleitman, and Gleitman (1983), for example, extended Rosch's work to the categories of odd and even numbers and found prototype effects for both.

Subjects in our study were assigned the task of rating instances of multiplication (on a scale of 1 to 7) for degree of membership in the category. The instances to be rated included integers, fractions, irrationals, monomial and binomial products, and a product of length and width yielding rectangular area. Upon completion of the rating task, subjects were asked the question "What is multiplication?" This question emanates from the Vygotskian *method of concept definition* (Luria, 1981, p. 56), in which subjects are asked, "What is – ?" with respect to the concept of interest. After this, subjects were asked with respect to each instance of multiplication appearing on the rating task, "In what sense do you consider this (particular instance of integer, irrational, or binomial multiplication, for example) to be multiplication?" A flexible clinical interview format was employed in probing subjects' responses. This third measure was a variant of the Vygotskian *comparison and differentiation* method (Luria, 1981, p. 58), in which the designated instance and the subject's own meaning for multiplication are juxtaposed.

Results on the rating task indicated that for the American students multiplication possessed a prototypic structure. Every U.S. student assigned the positive integer instance 4×3 a rating of 1 but rated other instances as considerably less representative of multiplication, thereby indicating the exemplariness of the cardinal instance. Triangulation of the data yielded confirmation from the second measure. In response to the question "What is multiplication?" all the U.S. subjects stated that it was repeated addition. Finally, on the third measure, in more than 90% of the cases in which students gave evidence that an instance of multiplication had any meaning for them, this meaning was linked to the exemplar or prototypic instance. For example, after the cardinal instance 4×3, the monomial product ab received the most favorable ratings. Twenty-three of the U.S. students found it meaningful, and all substituted small positive integers for a and b, thereby establishing linkage to the positive integer prototype for multiplication. Only one student noted that a and b could represent any real numbers, and that the substitution of positive integers did not resolve whatever conceptual difficulties existed for the multiplication of other types of real numbers (fractions, for example). The results were similar for the instance of binomial multiplication $(2x + y)(x + 3y)$; only 12 of the U.S. students reported that binomial multiplication had any meaning for them at all. Of those for whom it did, all illustrated its meaning by substituting small whole numbers for x and y. The most popular choices among the university students were 1 and 2, which yielded a product of 4×7 and effected a reduction to the counting number prototype. In effect, these subjects deformed the generalized algebraic product into their limited understanding of binomial multiplication predicated on cardinality.

Another disturbing finding was that half of the U.S. university students and two-thirds of the secondary students indicated that they did not see the area of a rectangle as multiplication. These subjects were unable to draw a grid in a rectangle that would illustrate how its area is a product of length and width. They could not go beyond the simple substitution of small whole numbers for b and h in the formula $A = bh$ (area = base × height), whereby they again effected a reduction to the cardinal number prototype. Moreover, they accomplished this merely by substitution of counting numbers into the formula, which they were able to do in order to produce a value for A without perceiving any apparent connection to a rectangle at all. They also gave evidence of considerable confusion between area and perimeter.

By way of contrast, the Russian students did not give evidence of prototypicality on the ratings task. The younger students actually rated the rectangular area instance $A = bh$ as more exemplary of the category than 4×3, and many commented that this counting number instance was too easy and, therefore, uninteresting to them. Nor did they characterize the meaning of multiplication as repeated addition; rather the essential change in the system of units was reflected in their conceptualization of area. None of the Russian students confused area with perimeter, and even the youngest students were very explicit about the conceptual transitions necessary to establish rectangular area as multiplication. All were explicit about the change of unit, from a small square to a row of such squares, which then must be repeated to form the rectangle. This is the essence of rectangular area, and it emanates directly from the conceptual essence of multiplication. None of the U.S. students had this understanding. The protocols of virtually all of the Russian students, however, even the youngest, consistently identified first the change in quantity from the base b (or height h) of the rectangle to the area of a rectangular strip having dimensions $b \times 1$ (or $h \times 1$). They also explicitly noted the change in unit from a single unit square within the rectangle to a rectangular strip of such squares (Fig. 11.1).

Similarly, *every* Russian student, including beginning fourth graders who had never been introduced to binomial multiplication, was able to obtain the product of two binomials and explain in what sense it represented multiplication. Unlike the U.S. students, they did not reduce either the monomial or the binomial factors to small whole numbers in order to understand the action to be performed as multiplication. Instead, they expressed this understanding at a higher level of generalization, that of algebraic abstraction. Only later, when requested to do so, did they substitute specific numbers to obtain a product. This typifies the ascent from the abstract to the concrete advocated by Hegel. Unlike the U.S. university students who substituted the smallest whole numbers they could think of for x and y, the Russian children, when asked to illustrate their abstract understandings with a concrete solution, chose numbers such as 64, 206, and 103.9 as factors. These children evidenced a confidence not found in the

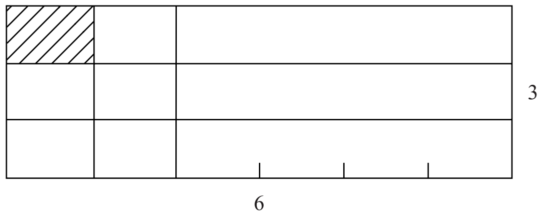


FIGURE 11.1a Model of area by a Russian student illustrating change in unit from a single square to a rectangular strip of such squares.

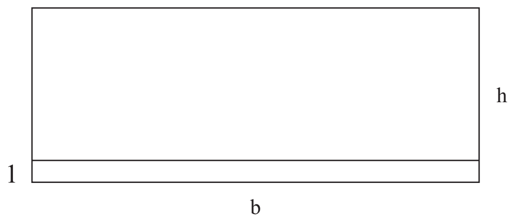


FIGURE 11.1b Model of area by a Russian student illustrating transition from linear dimension b to a rectangular strip of dimensions $b \times 1$.

American subjects, whose age and subject matter background advantages might have been expected to result in the generalized understandings actually shown by the Russian children who were uninstructed in binomial multiplication. Some of the Russian students explained binomial products by drawing a rectangular model with dimensions $2x + y$ and $x + 3y$, then showing a strip of dimensions $2x + y$ by 1 repeating $x + 3y$ times. (Fig. 11.2).

The U.S. students who converted fractions to decimals reported that they mentally removed the decimal points (thereby effecting a reduction to the positive integer prototype), multiplied the resulting integers, and then invoked the "rule" to reposition the decimal point in the product. None knew how or why the "rule" worked. A fifth-grade Russian student made a similar transition from fractions to decimals, writing:

$$\frac{2}{3} = \frac{20}{30} = .6 \quad \text{and} \quad \frac{4}{5} = \frac{40}{50} = .8 \quad \text{Then} \quad 0.6 \times 0.8 = .48$$

In contrast to his U.S. counterparts, this child, when questioned about how he saw this as multiplication, explained without hesitation, ".08 repeats 6 times."

The product of irrationals (Π and $\sqrt{2}$) had meaning for only one secondary and two U.S. university students, who explained it correctly by successive approximation of two nonrepeating decimals. For many students, however, the multiplicative difficulties were compounded by the added failure to understand the irrational numbers themselves. Some regarded

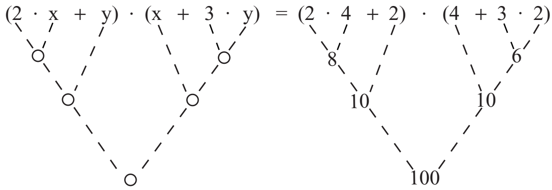


FIGURE 11.2a Model of binomial multiplication by a Russian fourth-grade student.

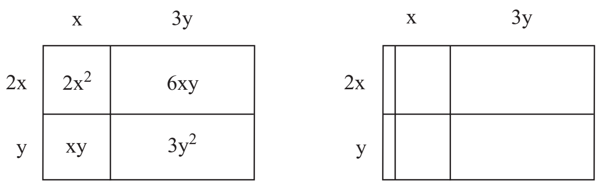


FIGURE 11.2b Model of binomial multiplication by a Russian student showing repetition of a rectangular strip of dimensions $2x + y$ by 1.

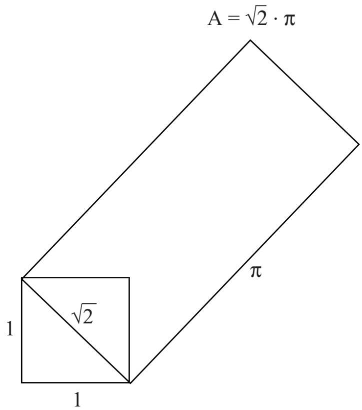


FIGURE 11.3 Russian ninth-grade student's model of $\sqrt{2} \cdot \Pi$ as the area of a rectangle.

Π and $\sqrt{2}$ as "mere symbols" to be consigned to a calculator for solution; others insisted that 2 does not have a square root. The older Russian students used successive decimal approximation as well as area models for this problem. One sketched $\sqrt{2} \cdot \Pi$ as the area of a circle having radius $\sqrt[4]{2}$; another marked off $\sqrt{2}$ as the diagonal of a unit square, then drew a rectangle using this as one side and Π as the other. The area she identified as $\sqrt{2} \cdot \Pi$ (Fig. 11.3). Those who used successive approximation were challenged to explain how 1.4 (an approximation for $\sqrt{2}$) could repeat 3.14 (an approximation for Π) times. Their immediate explanation was that first 314 was multiplied by 14 (or repeated 14 times), and then the required divisions by 10 and 100 were performed, resulting in the relocation of the decimal point. The Russian students never mentioned "rules"; they spoke of "actions" instead, and the meaning of such actions was consistent throughout a variety of algorithmic reformulations (cf. Schmittau, 1993b, for a more extensive discussion of grounding mathematical meaning in action).

Davydov's curriculum maintains students' mathematical actions at Leontiev's (1983) level of goal-directed action, whereas the "rules" U.S. students referred to occur at the operational level where actions have become routinized. The algorithm for multiplication of decimals is one example. Fortunately, constructivist influences are focusing more attention on goal-directed action in U.S. classrooms, but difficulty in linking conceptualization to the algorithm often occurs, with computation consigned to a calculator. Dependency on a calculator for the simplest computations has fueled the current "back to basics" movement in the United States. Ironically, while constructivism rails appropriately against mindless drill on algorithms, it promotes calculator usage, which is the ultimate mechanization of human action, "transmitting to the machine those elements that begin to be formalized in human activity itself" (Tikhomirov, 1981, p. 275). From a Vygotskian perspective, the algorithm is an important cultural–historical product, and great pains are taken in Davydov's curriculum to trace its historical and conceptual links to fundamental mathematical actions, of which the algorithm is a symbolic trace. As a result, our children who have completed 3 years of Davydov's curriculum here in the northeastern United States not only have a deep conceptual understanding of the mathematics involved, but are accurately multiplying three-digit numbers and dividing three-digit numbers into six- and seven-digit numbers. The conceptual versus procedural debate in the United States reflects a false dichotomy; an algorithm is a *symbolic trace of the meaningful mathematical actions* required to solve a problem. We move to manipulation of the symbols (such as numerals) when cultural factors bring about an increase in complexity whereby action on objects becomes tedious and consequently prone to error.

The dysfunctional manner in which American students reduced conceptually complex structures to cardinal instances reflected the fact that this category was for them structured around the counting number prototype. We originally anticipated that this category, developed pedagogically in the form of a generative metonymy, might have a formalistic structure, but we found no evidence that any student had succeeded in apprehending it as a generative metonymy with formalist connections among the instances. (None, for example, defined a fraction as a quotient of two integers a/b, such that $b \neq 0$.) Perhaps this, together with the difficulties encountered by students during the "new math" era, reflects the human need to traverse individually a cognitive path similar to that taken by the culture as a whole in the original development of these concepts (Vygotsky & Luria, 1993). Clearly, the cultural-historical development followed by Davydov's curriculum resulted in far greater conceptual coherence for the category of multiplication for real numbers.

MULTIPLICATION AS A VYGOTSKIAN SCIENTIFIC CONCEPT

There is, however, one final and extremely important consideration. Davydov (1990) extended Vygotsky's research into spontaneous and scientific concepts, finding a primary distinction in their manner of formation. The process of empirical abstraction, of identifying similarities and differences at the level of appearances, is sufficient only for the formation of spontaneous concepts. What can be empirically abstracted concerning a phenomenon such as the diurnal cycle, for example, is the "fact" of the Sun's revolution about the Earth. The rotation of the Earth on its axis, the real cause of the Sun's "rising" in the east and "setting" in the west, cannot be apprehended at the phenomenological level (Lektorsky, 1984; Kozulin, 1990), but requires the development of a theoretical mode of thought (Davydov, 1990). This is the case for mathematical concepts as well, but Davydov observes that because pedagogy has for the most part advanced no further than the level of Lockean empiricism, such empirical methods as comparison and contrast are reinforced throughout schooling.

What our combination of Rosch's and Vygotsky's research methods detected in the U.S. subjects were the results of attempts at formation of a scientific concept through the cognitively dysfunctional means of empirical abstraction. Prototypic organization, a common occurrence in generative metonymic categories (Lakoff, 1987), develops empirically on the basis of representativeness of features and is extended through a comparison-to-exemplar process. We may consider the construction of the category "fruit" investigated by Rosch (1973). One who has appropriated the scientific concept as "that which contains the seeds" has apprehended a theoretical essence that is not apparent among a variety of surface features. Such an individual might be expected to approach pertinent new botanical knowledge in a fundamentally different way than those to whom a fruit is quintessentially an apple.

In the case of mathematics the consequences of empirical abstraction are more devastating, however. Once a premature cognitive commitment (Langer, 1989) has been made to a cardinal structure, one cannot determine empirically by a process of comparison of their differential features what multiplication might mean for various types of numbers, such as fractions, irrationals, and their algebraic formulations (Schmittau, 1993b). The result is not a true scientific concept but a pseudoconceptual generalization, the Vygotskian designation for many of the so-called alternate conceptualizations or misconceptions found in the data of U.S. subjects, but conspicuously absent in the protocols of the Russian students. We saw no evidence, for example, of such common misconceptions as "multiplication makes bigger," the apparent result of conceptualizing multiplication within the framework of cardinality. Because the Russian children apprehended the theoretical essence of multiplication, the concept retained its constancy of meaning across contexts and, hence, could confidently be extended into new ones.

The pedagogical experiences of the Russian students, however, were the result of an extensive historical, conceptual, and psychological analysis on the part of Davydov and his colleagues. The generation of the real numbers through actions of measuring (rather than their derivative formation as "quotients," for example, of numbers that arise through actions of counting) avoids the scholastic repetition of the historical development of the concept of real number, in which 2,000 years were required to unite the products of counting and the products of measuring into one conceptual system. It is here that considerations of Davydov's work and its theoretical basis have the potential to open up new perspectives in our own reform process. In addition to providing a prototype of pedagogy informed by Vygotskian psychology, they have much to contribute to considerations of epistemological and psychological foundations for curriculum and instruction.

THE EXTENSION OF MULTIPLICATION TO EXPONENTIATION: ANOTHER GENERATIVE METONYMY

It is significant that the generative metonomy is not confined to multiplication in American mathematics pedagogy. When multiplication is extended to exponentiation, for example, the basis of this extension is again the counting numbers. Typically the textbook and classroom treatment of this subject begins with the definition of an exponent as repeated multiplication. That is, x^3 is defined as $\mathrm{x} \cdot \mathrm{x} \cdot \mathrm{x}$, or the repeated multiplication of x by itself. Consequently $5^4 = 5 \times 5 \times 5 \times 5$, which is analogous to the definition of multiplication as repeated addition. Hence, as multiplication was defined as a simple extension of addition, rather than a separate mathematical action or operation, we now have exponentiation as a simple extension of multiplication, and another category that is developed as a generative metonymy. However, just as with multiplication, students must encounter and be able to understand exponents that are fractional or irrational, and the generative metonymic approach is not sufficient to account for these since it is predicated on counting numbers.

We researched the understanding of university students with respect to this category and found so little understanding of this concept among students who were not mathematics majors that often they told us that an exponent was a little number in the upper-right-hand corner next to another number or letter, but they did not know what this little number meant. We presented a "fantasy" problem of plant growth, which was not designed to mimic botanical reality, but to explore the concept of exponentiation from a cultural–historical perspective rather than as the generative metonymic category it has become. The plant is first noticed (on day 1) and found to be 3 cm in height. It is measured at the same time on successive days and found to have heights of 9, 27, and 81 cm, respectively. Students are asked to assume this pattern is representative and to give the heights on several days previous to the first day on which the plant was observed. This yields heights of 1, 1/3, 1/9, and so on, and generates the nonpositive integer exponents for powers of 3 (3^0, 3^{-1}, 3^{-2}, etc.). Then students are asked the plant's height 12 hours before it was first measured. Even students who have nearly completed master's degrees in mathematics find this surprisingly difficult. They want to say that the height is $3^{1/2}$, which they "know" (i.e., have been told and accepted) is $\sqrt{3}$, but find this difficult to establish.

This problem follows the cultural–historical development of exponents and logarithms, which involved mathematicians in the juxtaposition of arithmetic and geometric sequences similar to those that constitute the domain and range of the plant growth function. In solving the problem, which approaches the development of exponents through the analysis of an exponential *function*, a student is constantly working back and forth across these two sequences, the arithmetic representing time and the geometric, growth. Such a development is consistent with cultural–historical theory, provides greater conceptual coherence for the category, and prevents its development as a generative metonymy emanating from the positive integers.

CONCLUSION

I have noted several differences between constructivism and cultural–historical theory, especially as these pertain to mathematics pedagogy. There is another important difference. From a Vygotskian perspective, the scientific concept has been constructed historically by the culture, a product of "universal generic thought" (Davydov, 1990, p. 311). In order to allow its appropriation by the individual, such a concept must be subjected to genetic and psychological analyses and pedagogically mediated. A student has very little chance of "constructing" the scientific concept of multiplication independently. Further, "within the theoretical learning approach, 'the child as an independent learner is considered to be a result, rather than a premise of the learning process'" (Kozulin, 1995, p. 121; cited in Karpov & Haywood, 1998, p. 33). This explains the underlying difference beneath the surface similarities in classroom teaching from a constructivist and a cultural–historical perspective. Because the problem solving done within the curricular structure in Davydov's program is designed to develop the cognitive abilities of theoretical generalization, the approach to the subject matter is fundamentally different, although in both cases the teacher may function as a facilitator and the instruction is in neither case didactic.

It is scarcely possible to close this discussion without commenting on currently popular attempts within mathematics education to frame Vygotsky as a "social constructivist." In light of all that has been said here, it would appear that such attempts not only are ill conceived, but, in fact, miss the mark by a wide margin. At the very least, they obscure the deep theoretical and pedagogical differences between constructivism and cultural–historical theory that are reflected both in the construction of curricula and in the actual processes of teaching and learning mathematics.

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