4.0 Random Wave Models

4.1 Linear Random Wave Theory

In a regular wave train (one frequency) linear theory gives:

$$\eta = a\cos(kx - \omega t) \;\; ; \;\; u = \frac{a\omega\cosh k(y+d)}{\sinh(kd)}\cos(kx - \omega t);$$

where $\omega^2 = gk\tanh(kd)$

If a random sea state is defined as consisting of a large number (M > > 1) of ***linear free waves***, a linear summation gives:

$$\eta = \sum_{m=1}^{M} a_m \cos(k_m x - \omega_m t + \alpha_m) \; ; \; u = \sum_{m=1}^{M} a_m \omega_m \frac{\cosh k_m (y+d)}{\sinh(k_m d)} \cos(k_m x - \omega_m t + \alpha_m),$$

where $\omega_m^2 = gk_m \tanh(k_m d)$

If this approach is based upon a Fourier analysis of a measured wave record, it is usually referred to as a ***linear random wave theory*** (LRWT).

However, in practice many of the wave components are:

(i) Bound Waves (not free)
(ii) Non-linear

4.2 Occurrence of Bound Waves

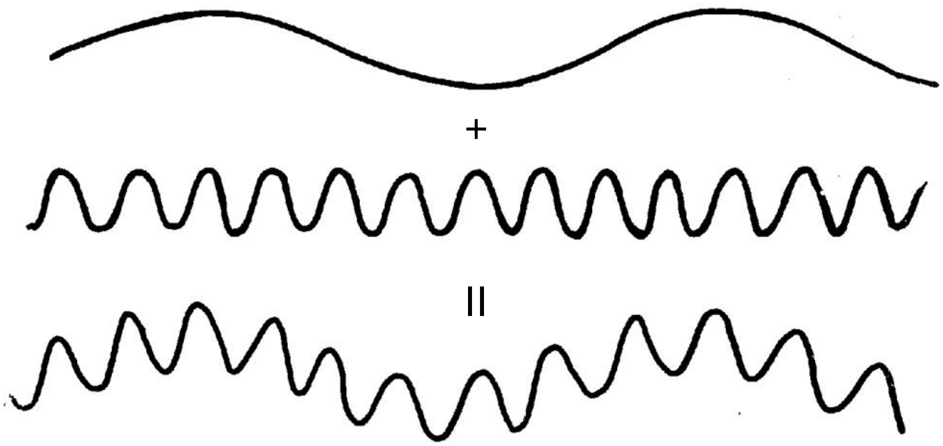
- We have already seen that in a regular wave train the development of bound harmonics is associated with the fit to the non-linear free surface boundary conditions.

- A similar effect occurs in random waves, where (in general) the bound harmonics tend to be more significant.

e.g. Consider an unsteady wave train consisting of:

- one long-wave component (low frequency)
- one short-wave component (high frequency)

Linear wave theory gives:



Long wave

Short wave

Linear solution

However, if we incorporate the nonlinearity of the problem the actual profile becomes:

Nonlinear solution

In this case the long-wave is unchanged, but the short-wave becomes modulated such that it is:

- higher and shorted on the crests of the long waves.
- Shallower and longer on the troughs of the long waves.

This change in the high-frequency free waves may be interpreted as the development of an additional bound wave necessary to provide an improved fit to:

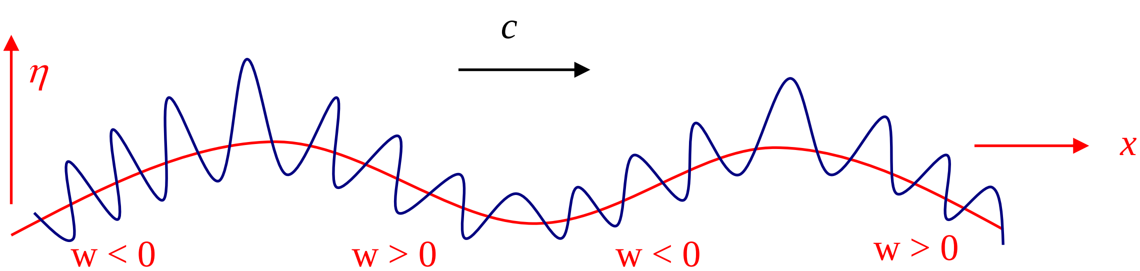
- Kinematic Free Surface boundary condition, KFSBC
- Dynamic Free Surface boundary condition, DFSBC

4.3 Physical Interpretation of Bound Waves

(i) Change in effective gravity, g'

Due to the presence of the long waves, the short waves are accelerated up and down. As a result, the effective gravity g' is:

$$g' = g + \left(\frac{\partial^2 \eta}{\partial t^2}\right)_{\text{Long wave}}$$



$$\frac{d^2\eta}{dt^2} < 0 \qquad \frac{d^2\eta}{dt^2} > 0 \qquad \frac{d^2\eta}{dt^2} < 0$$

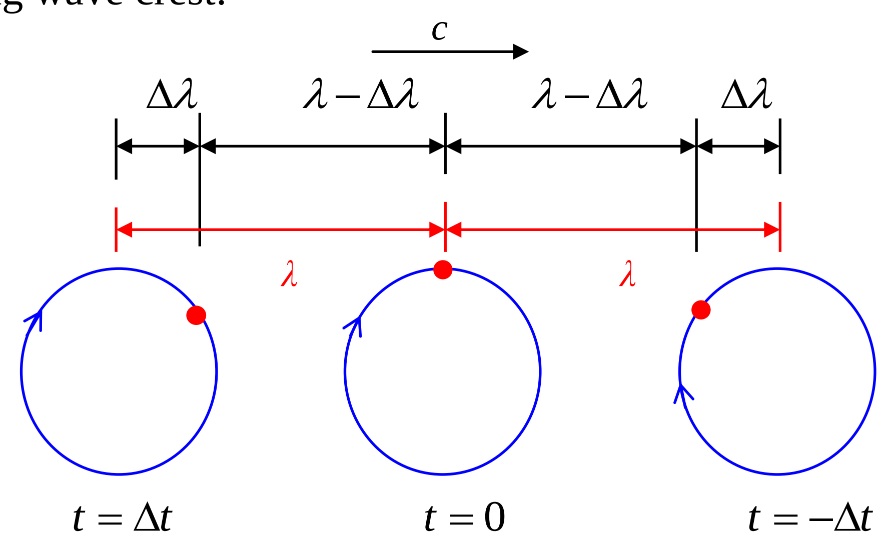
Changes in effective gravity produce amplitude modulation.

(ii) ***Change in wave length, λ***

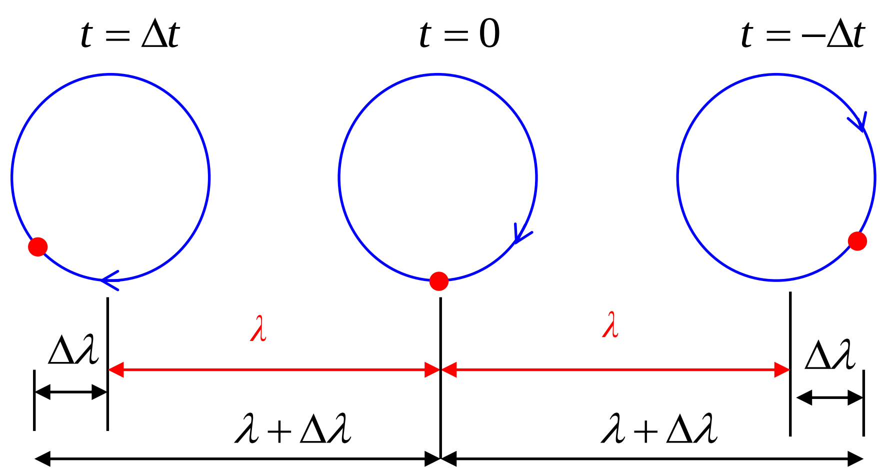
Long wave particle motion produces horizontal translation of the short waves:

e.g. In deep water

On the long wave crest:



In the long wave trough:



- Short wave length increased.

Short wave crest

Long wave orbital motion

$\Delta\lambda$ Change in λ_{SHORT}

4.4 Difficulties associated with Bound Waves

Unfortunately, calculated or measured wave spectra do not distinguish between free waves and bound waves. Hence:

$$\omega_m^2 = gk_m \tanh(k_m d)$$

is applied to all wave components. However, since the phase velocity of bound waves $c = \omega_m / k_m$ is dependent upon the associated free waves:

$\omega_m^2 \neq gk_m \tanh(k_m d)$ for bound waves.

Hence, for a given frequency the linear dispersion equation will predict an incorrect wavelength.

In practice, the identification of bound waves is very difficult. The pattern that arises is:

1st Order -	No bound harmonics
2nd Order -	Stokes' terms & 2 wave coupling
3rd Order -	Stokes' terms & 2 wave coupling & 3 wave coupling
4th Order -	Stokes' terms & 2 wave coupling & 3 wave coupling & 4 wave coupling

- One potential method is to consider a ***bi-spectral*** analysis. This will identify energy associated with frequency pairs, and can thus quantify the 2 wave coupling.
- However, high-order coupling (involving 3 or more waves) is very difficult to identify.

4.5 The effect of nonlinearity

The nonlinear terms produce their largest effect close to the water surface, where large potential errors can arise.

This is unfortunate since both u and $\partial u / \partial t$ are largest at the water surface. Hence:

- large contribution to base shear
- even larger contribution to over-turning moment (since the moment-arm is also maximised).

Furthermore, there also appears to be some confusion as to the height to which linear wave theory can be applied.

Since part of the boundary condition is applied at $y = 0$, it is often said that the theory is only valid within

$$-d \leq y \leq 0$$

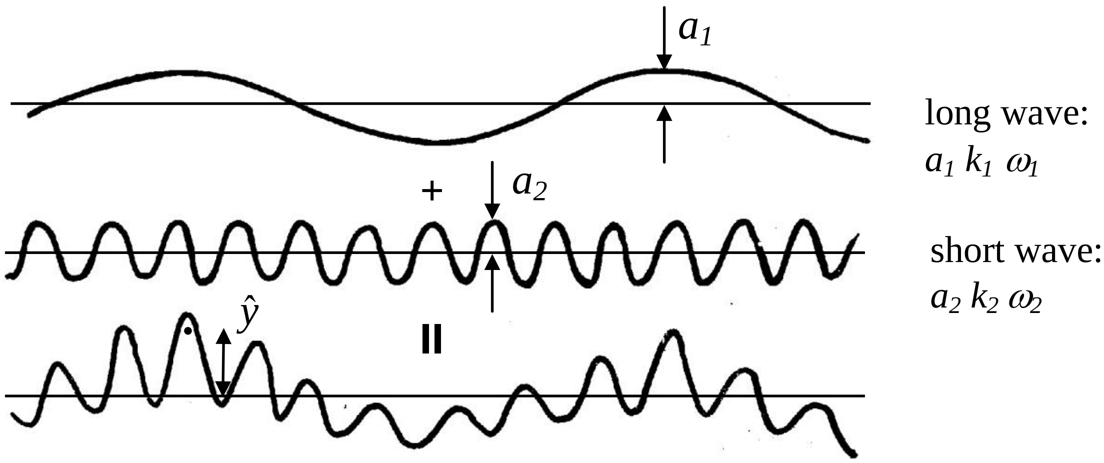
This is ***not*** true. The solution may be extended up to the water surface:

$$-d \leq y \leq a \sin(\omega t - kx),$$

but with the expectation that the 'errors' will increase as the water surface is approached.

In linear random wave theory there are problems concerning the effective water depth.

e.g. consider the long wave - short wave interaction.

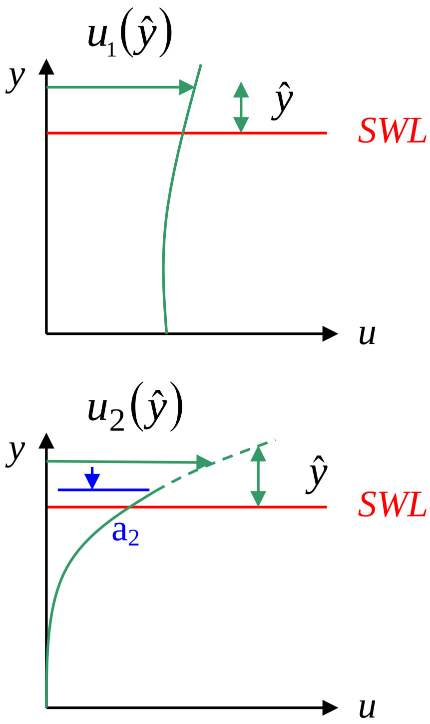


Velocity at $y = \hat{y}$ would be:

$$u = a_1\omega_1 \frac{\cosh k_1(\hat{y} + d)}{\sinh(k_1 d)} \cos(k_1 x + \omega_1 t) + a_2\omega_2 \frac{\cosh k_2(\hat{y} + d)}{\sinh(k_2 d)} \cos(k_2 x + \omega_2 t)$$

This solution will produce large errors because it does not account for the fact that the short waves are riding on the back of the long waves.

i.e. the short wave velocities are extrapolated to $y = \hat{y}$, where $\hat{y} >> a_2$ and depends on the amplitude of the long wave (a_1).



- Since $\partial u_2/\partial y$ is large for $y=0$, extrapolation of u_2 to $y=\hat{y}$ will produce large errors.

- Usually referred to as ***high-frequency contamination***.

- Results in significant over-prediction of horizontal velocities beneath the wave crest.

- Arises because the high-frequency waves propagate over the longer wave components (not correctly modelled in LRWT).

4.6 Empirical (or Stretched) Wave Solutions

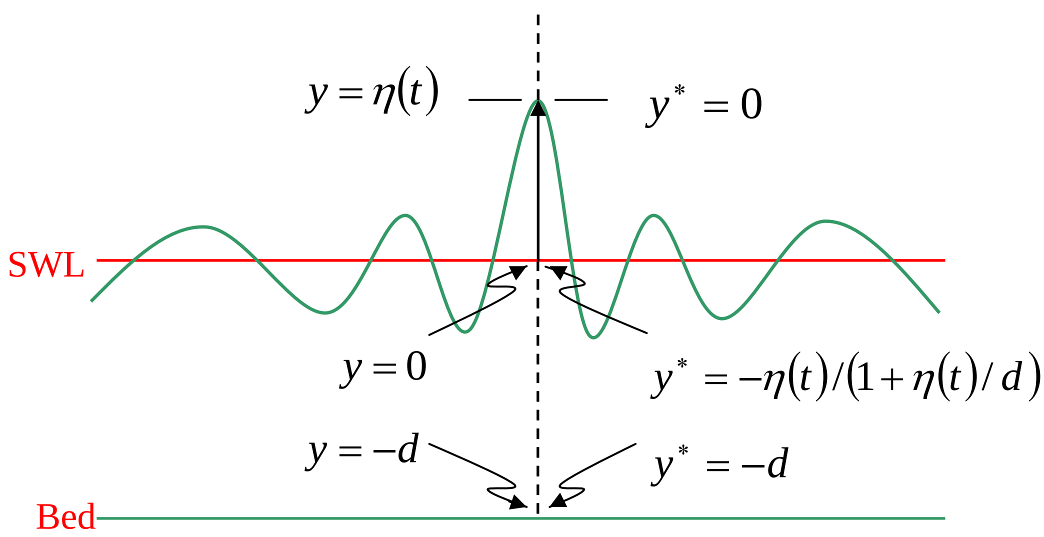
Since LRWT applied close to the water surface will typically over-predict the velocities, due to high-frequency contamination, empirical corrections have been proposed such that:

$$u(x,y,t)=\sum_{m=1}^{M} a_m\omega_m \frac{\cosh k_m\left(y^*+d\right)}{\sinh\left(k_m d\right)}\cos\left(k_m x-\omega_m t+\alpha_m\right),$$

where the only difference between this and LRWT is

$$y^*=\frac{y-\eta(t)}{\left(1+\dfrac{\eta(t)}{d}\right)}$$ where $\eta(t)$ is a measured wave profile.

Standard Co-ordinates Stretched Co-ordinates



This empirical adjustment is often referred to as ***Wheeler Stretching.***

This is one of several ***empirically corrected*** linear wave theories:

e.g.

Wheeler, J.D., (1970). *Proc. 1st Annual Offshore Technology Conf.* pp. 71-82.

Lo, J. and Dean, R.G., (1986). *Proc. 20th Int. Conf. Coastal Engng.* Vol. 1, pp. 522-36

Gudmestad, O.T. and Connor, J.J. (1986). *Applied Ocean Research*. Vol. 8, pp. 76-88.

- Wheeler's solution was the first and is perhaps the most widely used.

- All these solutions avoid the over-prediction associated with the extrapolation of the high-frequency components.

- But none of them satisfy mass continuity.

Note:

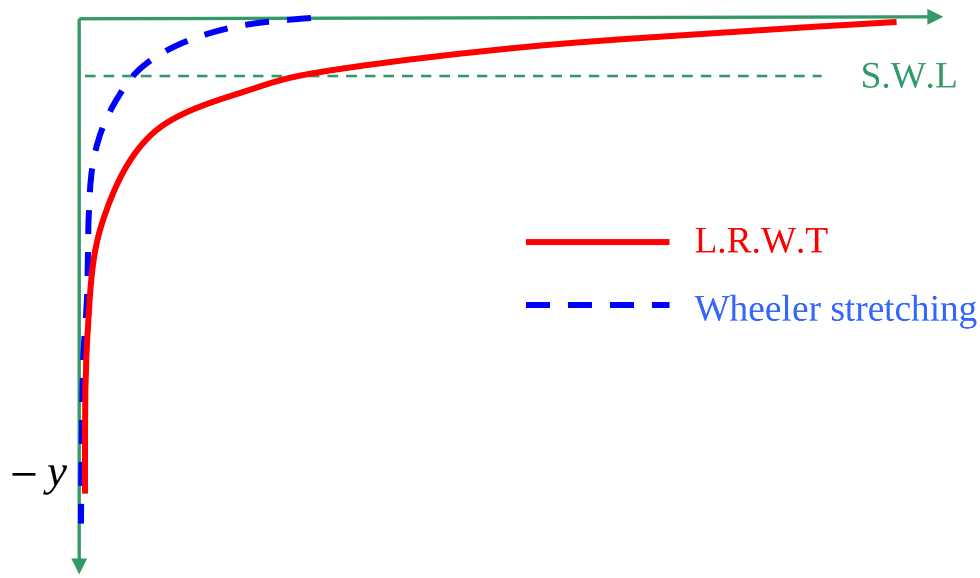
It is often said that these solutions:

(a) Satisfy mass continuity in an 'average sense'.

(b) Provide an improved fit to the free surface boundary conditions.

Although these points are true, (a) is not sufficient to model a real fluid flow, and (b) is true for the wrong reason (i.e. it simply predicts smaller velocities).

A comparison between L.R.W.T. and an 'empirical solution' will typically give:



Empirically corrected solutions are ***basically incorrect*** and ***cannot be recommended***. However, they are widely used in industry.

4.7 Second-Order Theory

Nonlinear analytical solutions are complex and difficult to achieve.

Longuet-Higgins and Stewart (1960) give the interaction between two waves as:

$$\eta = \eta_1 + \eta_2 + \frac{a_1 a_2}{2g}\{C\cos(\psi_1 - \psi_2) - D\cos(\psi_1 + \psi_2)\}$$

where $\psi_1 = (k_1 x - \omega_1 t)$; $\psi_2 = (k_2 x - \omega_2 t)$

$$\phi = \phi_1 + \phi_2 + \frac{E\cosh[(k_1 - k_2)(z+d)]\sin(\psi_1 - \psi_2)}{g(k_1 - k_2)\sinh(k_1 d - k_2 d) - (\omega_1 - \omega_2)^2 \cosh(k_1 d - k_2 d)}$$

$$+ \frac{F\cosh[(k_1 + k_2)(z+d)]\sin(\psi_1 + \psi_2)}{g(k_1 + k_2)\sinh(k_1 d + k_2 d) - (\omega_1 + \omega_2)^2 \cosh(k_1 d + k_2 d)}$$

where:

$$C = \frac{\left[2\omega_1\omega_2(\omega_1 - \omega_2)(1 + \alpha_1\alpha_2) + \omega_1^3\left(\alpha_1^2 - 1\right) - \omega_2^3\left(\alpha_2^2 - 1\right)\right](\omega_1 - \omega_2)(\alpha_1\alpha_2 - 1)}{\omega_1^2\left(\alpha_1^2 - 1\right) - 2\omega_1\omega_2(\alpha_1\alpha_2 - 1) + \omega_2^2(\alpha_2 - 1)}$$

$$+ \omega_1^2 + \omega_2^2 - \omega_1\omega_2(\alpha_1\alpha_2 + 1)$$

$$D = \frac{\left[2\omega_1\omega_2(\omega_1 + \omega_2)(1 - \alpha_1\alpha_2) - \omega_1^3\left(\alpha_1^2 - 1\right) - \omega_2^3\left(\alpha_2^2 - 1\right)\right](\omega_1 + \omega_2)(1 + \alpha_1\alpha_2)}{\omega_1^2\left(\alpha_1^2 - 1\right) - 2\omega_1\omega_2(1 + \alpha_1\alpha_2) + \omega_2^2(\alpha_2 - 1)}$$

$$+ \omega_1^2 + \omega_2^2 + \omega_1\omega_2(1 - \alpha_1\alpha_2)$$

$$E = -\frac{1}{2}a_1 a_2\left[2\omega_1\omega_2(\omega_1 - \omega_2)(1 + \alpha_1\alpha_2) + \omega_1^3\left(\alpha_1^2 - 1\right) - \omega_2^3\left(\alpha_2^2 - 1\right)\right]$$

$$F = -\frac{1}{2}a_1 a_2\left[2\omega_1\omega_2(\omega_1 + \omega_2)(1 - \alpha_1\alpha_2) - \omega_1^3\left(\alpha_1^2 - 1\right) - \omega_2^2\left(\alpha_2^2 - 1\right)\right]$$

with

$$\alpha_1 = \cosh(k_1 d) \quad ; \quad \alpha_2 = \cosh k_2 d$$

- Although this solution was originally derived for two waves, it may be applied to the interaction of many free waves within a random sea by summing up the interactions associated with each potential pair of free waves.

- Difficult to apply in practice because the free waves cannot be separated from the bound waves.

4.8 Directionality

4.8.1 Spectral representations

So far we have assumed that within a wavefield all the waves propagate in the same direction. Hence, a linear representation gives,

$$\eta = \sum_{m=1}^{M} a_m \cos(\omega_m t - k_m x + \alpha_m)$$

where x defines the direction of wave propagation.

In practice some wave components will travel at an angle θ to the mean wave direction. If directionality is introduced in design, it is usual to assume that the directional and frequency distributions are independent. A spectral representation therefore assumes,

$$S_{\eta\eta}(\omega,\theta) = D(\theta) S_{\eta\eta}(\omega)$$

where $S_{\eta\eta}(\omega)$ - frequency distribution
$D(\theta)$ - directional distribution

In other words the same directional distribution applies to all frequency components. This is not strictly true since:

Swell waves - small directional spread (effectively uni-directional)
Wind waves - may have a large directional spread.

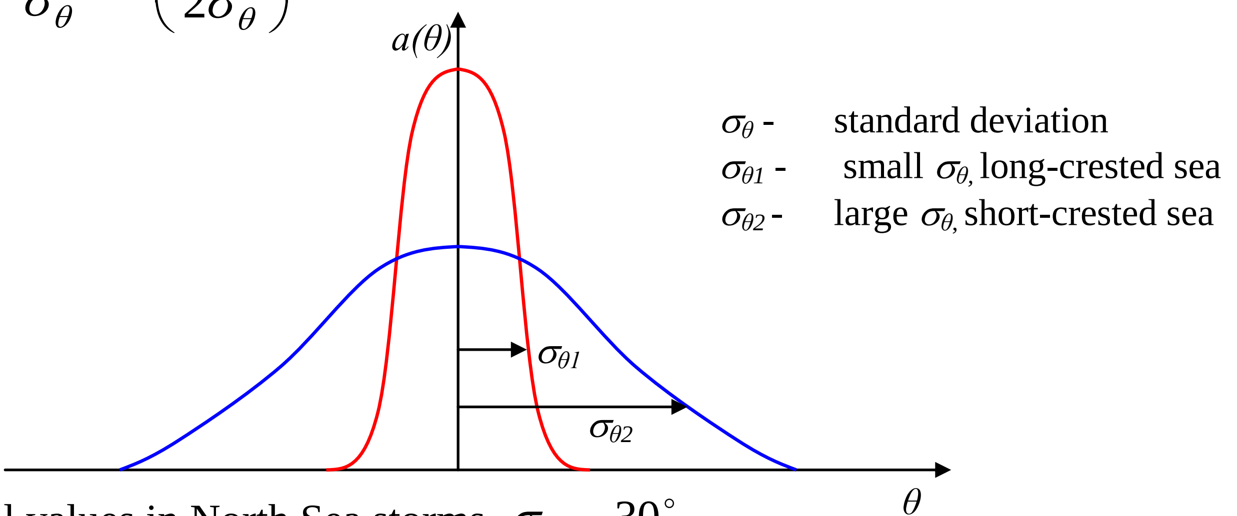
However, in the absence of a clear procedure to separate swell waves and wind waves (this is the subject of much on-going research) it is very difficult to apply anything other than a generalised directional spread.

4.8.2 Design Representations

For design purposes directionality is usually represented by either:

(i) A normal distribution:

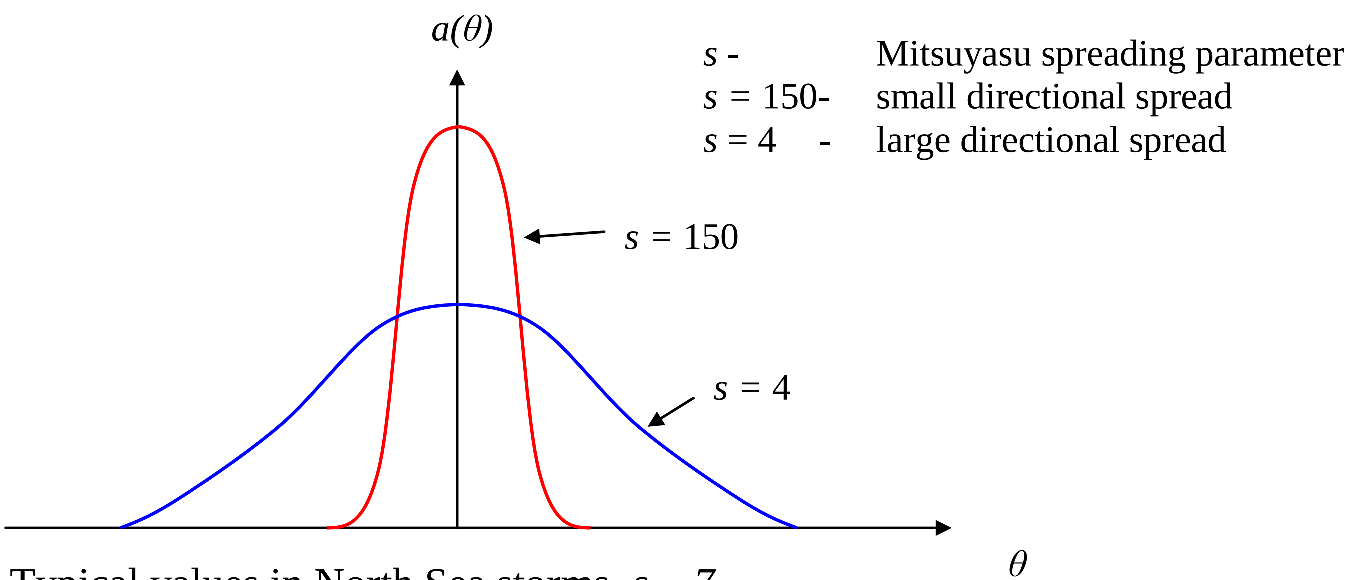
$$a(\theta) = \frac{A}{\sigma_\theta} \exp\left(\frac{-\theta^2}{2\sigma_\theta^2}\right)$$ with A = normalising coefficient



Typical values in North Sea storms, $\sigma_\theta \sim 30^\circ$

(ii) Mitsuyasu distribution

$$a(\theta) = A\cos^s\left(\frac{\theta}{2}\right)$$ with A = normalising coefficient



Typical values in North Sea storms, $s \sim 7$.

4.8.3 *Design Implications*

(i) Surface Elevation

- If one is interested in the surface elevation at a single point, then from a linear perspective directionality makes no difference to the maximum crest elevation. From a nonlinear perspective it does because of changes to the wave slope.
- If one is concerned with structures having significant spatial dimensions (large diameters or multiple legs) directionality is important.

Uni-directional waves	-	characterised by long crests of constant height.
	-	***Long-crested waves***
Multi-directional waves-		characterised by crests of limited length
	-	height varies along the length
	-	***Short-crested waves***

The greater the directionality, the more short-crested individual waves are.

- Linear solution gives

$$\eta(x,y,t) = \sum_{freq.}\sum_{dir.} a_m \cos(\omega_m t - k_{mx}x - k_{my}y)$$

where k_x - wave number in *x*-direction.
k_y - wave number in *y*-direction.

(ii) Water Particle Kinematics

- The water particle kinematics beneath a large wave are always affected by directionality.
- A linear solution gives,

$$u = \sum_{freq.}\sum_{dir.} a_m \omega_m \frac{\cosh k_m (z+d)}{\sinh(kd)} Cos(\omega_m t - k_{mx}x - k_{my}y)$$

where $k_m^2 = (k_{mx}^2 + k_{my}^2)$

- Unfortunately, we have already seen that there are many problems with applying a linear random wave theory to uni-directional waves. Similar problems arise in multi-directional waves, particularly ***high-frequency contamination***.

- To overcome this difficulty:
  - Apply a directional second-order random wave model. Extension of original work proposed by Longuet-Higgins and Steward (see section 2.3.7). Best description given by Sharma and Dean (1981). Petroleum Engng. Journal.
  - Adopt a fully nonlinear directional wave model. Bateman, Swan & Taylor (2001). This builds upon exact models (see Section 4.9).
  - Apply a ***Velocity Reduction Factor*** to account for average directional spread i.e. Calculate the uni-directional velocities and multiply by 0.7 ~ 0.8 (dependent upon σ_θ or *s*) to

account for directionality. This is perhaps the least satisfactory method, but is widely adopted.

- Design on the basis of uni-directional waves.
  - justified on the basis that the directional distributions may not be well known
  - acceptable for fixed structures (conservative for loading)
  - unacceptable for floating structures (unconservative for dynamic response i.e. roll motions)

4.9 Numerical Codes

(i) Dold and Peregrine, 1984.
Proc. 19th Int. Conf. Coastal. Engng. Vol 1, pp. 955-67.

- Exact solution for unsteady nonlinear waves.
- Based on time-marching the free surface boundary conditions:

$$\frac{d\eta}{dt} = \frac{\partial\phi}{\partial z} - \frac{\partial\eta}{\partial x}\frac{\partial\phi}{\partial x}$$

$$\frac{\partial\phi}{\partial t} = -g\eta - \tfrac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right]$$

- Initial conditions require $\eta(x)$ and $\phi(x, y = \eta)$ at given t. Seldom available, even under laboratory conditions.
- Neglects directionality.
- Not often used for design applications.

(ii) Sobey, 1992.
Applied Ocean Research. Vol. 14, pp. 93-105.

- Nonlinear numerical model.
- Based on 'local' fit to measured free-surface, $\eta(t)$.
- Computationally efficient (very quick).
- Accurate in deep water.
- Concern regarding application in shallow water.
- Later models include directionality.

(iii) Baldock and Swan, 1994.
Applied Ocean Research. Vol. 16, pp.101-112.

- Nonlinear numerical model
- Based on 'global' fit to measured free-surface, $\eta(t)$.
- Accurate solutions in any water depth.
- Computationally intensive.
- Small errors in very steep waves.

(iv) Bateman, Swan and Taylor, 2001, 2003
J. Computational Physics Vol. 174, pp. 277-305.
J. Computational Physics Vol. 186, pp. 70-92

- Exact solution for surface water waves
- Fully nonlinear
- Unsteady
- Directional
- Based on time-marching the free-surface boundary conditions
- 3-D equivalent of (i) above
- Increasingly adopted as bench-mark for design
- Limitations - non breaking/overturning waves
- flat bed

The application of complex numerical codes is increasingly common because:

- Offshore developments in more extreme wave climates.
- Growing appreciation of importance of: - nonlinearity
- directionality
- Increased computing power.
- Economic constraints.
- Technical requirements: - floating structures
- on board processing
- consequences of failure