Chapter 4

Sturm-Liouville Theory and Examples

4.1 Introduction

We deal here with a class of problems that occur over and over in the solution of linear boundary value problems. The theory will be introduced followed by some examples. The primary equation is

$$-\frac{d}{dx}\left(q(x)\frac{d}{dx}\phi_n(x)\right) = \lambda_n \rho(x)\phi_n(x) \tag{4.1}$$

where $a \leq x \leq b, q, \rho \text{real}, \geq 0$. The boundary conditions are homogeneous:

$$\phi_n'(a) = c_a \phi_n(a) \tag{4.2}$$

$$\phi_n'(b) = c_b \phi_n(b) \tag{4.3}$$

and the coefficients, c_a, c_b are given constants. A useful short hand notation is

$$(f, g) \equiv \int_a^b f^*(x) g(x)\, dx \tag{4.4}$$

which is called the *inner product*. It is clear that $(f, g)^* = (g, f)$.

4.2 Reality of Eigenvalues, Orthogonality

First write the opening equation with n and then again with the index m:

$$-\frac{d}{dx}\left(q\frac{d}{dx}\phi_n\right) = \lambda_n \rho \phi_n \tag{4.5}$$

$$-\frac{d}{dx}\left(q\frac{d}{dx}\phi_m\right) = \lambda_m \rho \phi_m \tag{4.6}$$

Now multiply the first of these equations through by ϕ_m^* and integrate from a to b, then do the corresponding thing to the second equation:

$$\int_a^b -\phi_m \frac{d}{dx}\left(q\frac{d}{dx}\phi_n\right)\, dx = \lambda_n(\phi_m, \rho\phi_n) \tag{4.7}$$

$$\int_a^b -\phi_n \frac{d}{dx}\left(q\frac{d}{dx}\phi_m\right)\, dx = \lambda_m(\phi_n, \rho\phi_m) \tag{4.8}$$

Next we integrate each of the integrals by parts.

$$-\phi_m^* q\phi_n'|_a^b - \int -\phi'^*_m q\phi_n' \, dx = \lambda_n(\phi_m, \rho\phi_n) \tag{4.9}$$

$$-\phi_n^* q\phi_m'|_a^b - \int -\phi'^*_n q\phi_m' \, dx = \lambda_m(\phi_n, \rho\phi_m) \tag{4.10}$$

The leftmost terms can be simplified with use of the boundary conditions:

$$-\phi_m^*(b)c_b q(b)\phi_n(b) + \phi_n^*(a)q(a)c_a\phi_n(a) + (\phi_m', q\phi_n') = \cdots \tag{4.11}$$

and

$$-\phi_n^*(b)c_b q(b)\phi_m(b) + \phi_m^*(a)q(a)c_a\phi_m(a) + (\phi_n', q\phi_m') = \cdots \tag{4.12}$$

We want to prove two important results. The first is to show that the λ_n are real. It goes as follows. Let $n = m$ and subtract the complex conjugate of the second from the first of the pair of equations. The result is

$$(\lambda_n - \lambda_n^*)(\phi_n, \rho\phi_n) = 0 \tag{4.13}$$

The second factor is strictly positive so we must have $\lambda_n = \lambda_n^*$.

The second result we wish to prove is that the ϕ_n are orthogonal with the weight function $\rho(x)$. Just let $n \neq m$ and subtract the complex conjugate of the second of the pair from the first:

$$0 + 0 + 0 = (\lambda_n - \lambda_m)(\phi_n, q\phi_m) = 0 \tag{4.14}$$

which is the desired result. Notice that the scale or normalization of the $\phi_n(x)$ is arbitrary. We can chose it such that the functions are not just orthogonal, but orthonormal:

$$(\phi_n, \rho\phi_m) = \delta_{nm} \tag{4.15}$$

4.3 Expanding Functions into Infinite Series of the $\phi_n(x)$

Consider next a function $f(x)$ defined on the same interval. Let $f(x)$ be reasonably well behaved (we can actually tolerate discontinuities, but not divergences that are not integrable). We want to expand f into an infinite series of the ϕ_n:

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) \tag{4.16}$$

How do we compute the coefficients in the series? Simply multiply through by $\rho\phi^*(x)$ and integrate over the interval:

$$(\rho\phi_m, f) = \sum_{n=1}^{\infty} a_n(\rho\phi_n) \tag{4.17}$$

$$= \sum_n a_n \delta_{nm} = a_m \tag{4.18}$$

$$\rightarrow a_m = (\phi_m, \rho f) \tag{4.19}$$

4.3.1 Positivity of Eigenvalues in Some Cases

Return to Eq. 9 and let $n = m$. *If* the boundary conditions are such that either of the boundary terms vanishes, we have

$$\int |\phi_n'|^2 q \, dx = \lambda_n \int |\phi_n|^2 \rho \, dx \tag{4.20}$$

The *lhs* is positive and the coefficient of λ_n on the *rhs* is positive, hence we have

$$\lambda_n \geq 0 \tag{4.21}$$

There are some other cases where this can happen; for example, when $\phi_n^* q \phi_n'$ at a is equal to its value at b. This latter happens for periodic boundary conditions (e.g., on a circle).

4.3.2 Things that are so, but not proven here

It is not obvious that the functions ϕ_n are a complete set. That is they might not be sufficient to represent all functions. A good analogy is in the case of expanding a vector in terms of unit vectors. It may be that we have found a set of orthonormal basis vectors, say $\mathbf{i}$ and $\mathbf{j}$. This does not mean we can expand all vectors in terms of these two. Some vectors extend in the $\mathbf{k}$ direction. It turns out that in the case of Sturm-Louville functions, we are safe. The functions do *span* the whole space.

The eigenvalues for well behaved q and ρ are discrete if a, b are finite. They are countably infinite, running from some finite value up to infinity. If either a or b is infinite, the *spectrum* of eigenvalues might be a mixture of discrete and continuous values. If both are infinite, the spectrum is likely to be continuous. An example of the continuous spectrum is the Fourier Integral representation:

$$\phi_\omega''(t) = -\omega^2 \phi_\omega, \quad -\infty \leq t \leq \infty \tag{4.22}$$

Then the eigenfunctions are

$$\phi_\omega(t) = \frac{1}{\sqrt{2\pi}} e^{i\omega t} \tag{4.23}$$

If we want to represent a function with these eigenfunctions we must use an integral representation rather than a sum over all eigenfunctions. For example, to represent $f(t)$ we would write

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_\omega \phi_\omega(t)\, d\omega \tag{4.24}$$

and the coefficients f_ω are given by the inverse Fourier Transformation

$$f_\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \phi_\omega^*(t)\, dt \tag{4.25}$$

4.4 The Dirac Delta Function

This thing is not a function, but for many practical purposes we can consider it to be one. One suitable definition is

$$\delta(x) = \lim_{h \to 0} \frac{1}{h} \text{ for } -\frac{1}{2h} \leq x \leq \frac{1}{2h}; 0 \text{ otherwise} \tag{4.26}$$

The delta function is then a skinny box whose height is approaching a very large number, but at the same time its width is becoming vanishingly small. The area under the box is unity. The delta function is useful in representing a density of a mass which is concentrated in a point at the origin. The power of the delta function is in doing integrals over it. Consider a function that is defined and continuous at the origin, $f(x)$. It is easy to show that

$$\int_a^b f(x)\delta(x)\,dx = f(0), \quad a \leq 0 \leq b \tag{4.27}$$

If the point mass is located away from the origin at say x_0, we simply write $\delta(x - x_0)$. Besides the box representation there are many others that are equivalent, such as a gaussian whose standard deviation is approaching zero. There are a number of other useful properties such as the derivative of the delta function:

$$\int_a^b f(x)\delta'(x - x_0)\,dx = -f'(x_0) \tag{4.28}$$

This last can be proven by integrating by parts.

4.5 Examples

4.5.1 Waves on a String

We consider a string stretching from $x = 0$ to $x = L$. The vertical displacement of the string at location x and time t is given by $\psi(x, t)$. The string is clamped at each end so that the boundary conditions are $\psi(0, t) = \psi(L, t) = 0$. The equation of motion of the string (Newton's Second Law) is given by:

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2} \tag{4.29}$$

where c is the speed of the waves (c^2=horizontal tension/linear density). Apologies for not deriving the equation here, but there is no time. The identical equation governs longitudinal sound waves in a cavity with walls at $x = 0, L$. In that case $c^2 = \gamma T/R$ where γ is the ratio of specific heats, T is Kelvin temperature, and R is the gas constant for dry air.

The standard approach to this class of problems is to write the solution in factored form:

$$\psi = T(t)X(x) \tag{4.30}$$

and insert it into the governing partial differential equation:

$$T''X = c^2TX'' \tag{4.31}$$

After dividing through by TX:

$$\frac{T''}{T} = c^2\frac{X''}{X} \tag{4.32}$$

The left hand side is a function only of t, the rhs a function only of x, so each must be constant, say $-\lambda^2$:

$$T'' = -\lambda^2c^2T, \qquad X'' = -\lambda^2X \tag{4.33}$$

The solution for X is $A_1\sin\lambda x + A_2\cos\lambda x$. The only way this can satisfy $X(0) = 0$ is for $A_2 = 0$. Then we have:

$$X(x) = A_1\sin\lambda x \tag{4.34}$$

But we must have $X(L) = 0$ as well. The only way this can be accomplished is for $\lambda = n\pi/L$ where $n = 1, 2, \ldots$. Finally, then

$$X(x) = A_n\sin\frac{n\pi x}{L} \tag{4.35}$$

and we have made note of the fact that the coefficient may depend on n. We can now return to the equation for $T(t)$ and construct a solution:

$$\psi_n(x,t) = \sin\frac{n\pi x}{L}\left\{B_n\exp\left(i\frac{nc\pi t}{L}\right) + C_n\exp\left(-i\frac{nc\pi t}{L}\right)\right\} \tag{4.36}$$

We have found a solution; in fact we have found infinitely many. How are the coefficients B_n and C_n determined (note that the A_n were absorbed into the B_n and C_n). As in all Newton Second Law problems, we have to specify the initial position and velocity of the mass (distribution) in question: $\psi(x,0)$ and $\frac{\partial\psi}{\partial t}|_{t=0}$. For the string there are several interesting initial conditions to consider:

1. Plucked guitar string at $x = x_0$. In this case the initial velocity ($\psi_t(x,0)$) is zero and the initial profile is tent shaped: $\psi(x,0) = \psi_0 x/x_0$, for $x \leq x_0, \psi_0(L-x)/(L-x_0)$ for $x_0 \leq x \leq L$. ψ_0 is the displacement at the point of the pluck.

2. A piano hammer hits the string at $x = x_0$. In this case $\psi(x,0) = 0$, $\psi_t(x,0) = v_0\delta(x-x_0)$

4.5.2 The Violin (Slightly Advanced)

Another amusing case is that of the violin being excited by a bow being drawn across the string at $x = x_0$. In this case we would add a force $f(x,t) = f_0\delta(x-x_0)n(t)$. The force is applied only at $x = x_0$, and the time dependence is random, presumably white noise. Let's consider this case.

We write our forced wave equation as

$$\frac{\partial^2\psi}{\partial t^2} = c^2\frac{\partial^2\psi}{\partial x^2} + f_0\delta(x - x_0)n(t) \tag{4.37}$$

First expand the forcing into the natural modes of the unforced string:

$$f(x,t) = f_0 \sum_{m=-\infty,n=1}^{\infty} A_{n,m} \sin\frac{n\pi x}{L} \sin\frac{n\pi x_0}{L} \exp i\omega_m t \tag{4.38}$$

The ω_m are a set of discrete (angular) frequencies running from very large negative to very large positive; their connection to real frequencies is $\omega_n = \pi n/T$ where T is the length of the (long!) time interval. The lowest (magnitude of) frequency is $\omega_1 = \pi/T$. The coefficients A_n are random numbers, normally distributed, with mean zero, and variance constant (independent of n for white noise forcing and proportional to the noise amplitude squared). A_n is statistically uncorrelated with $A_m, n \neq m$.

Before getting to the violin consider the forcing to be simply

$$F(x,t) = F_{n,\omega} \sin\frac{n\pi x}{L} \exp i\omega t \tag{4.39}$$

Next take $\psi(x,t)$ to be

$$\psi(x,t) = \psi_{n,\omega}(x,t) \sin\frac{n\pi x}{L} \exp i\omega t \tag{4.40}$$

Insert this in the forced wave equation and get

$$-\left\{\omega^2 - c^2\left(\frac{n\pi}{L}\right)^2\right\}\psi_{n,\omega} = F_{n,\omega} \tag{4.41}$$

The response of this mode is

$$\psi_{n,\omega} = \frac{F_{n,\omega}}{\left\{c^2\left(\frac{n\pi}{L}\right)^2 - \omega^2\right\}} \tag{4.42}$$

This leads to a finite response unless ω, the driving frequency, is equal to one of the *natural frequencies* of the string, in this case mode n. Notice that if the forcing is only of the shape of mode n , then only mode n will be excited. This is a basic property of a linear system with time independent coefficients. If there is friction in the string, the response will be finite at $\omega = \omega_n$. Also the response curve around resonance will tend to be broader.

Returning to the bow forcing the string, we will find that all modes of the system are excited with random phase for each one. The fact that the natural frequencies of the string are multiples of the fundamental leads us to a harmonious sound. Unfortunately, this is boring.

The real violin is a coupled system with the sound cavity and the vibrating wooden structure all participating in the process. Subtle nonlinearities combined with the different frequencies of the cavity and the wooden structure in the coupled system will lead to a sound (timbre) that is not boring. It will sound like a violin!

4.5.3 Heat Conduction on a Disk

Next consider a disk of radius a. The disk has a thermal conductivity K and on its boundary ($r = a$) it is to be held at a fixed temperature, say 0° C: $\mathcal{T}(r = a, \theta, t) = 0$. The partial differential equation governing the process is

$$\frac{K}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \mathcal{T}}{\partial r}\right) + \frac{K}{r^2}\frac{\partial^2 \mathcal{T}}{\partial \theta^2} = \frac{\partial \mathcal{T}}{\partial t} \tag{4.43}$$

where we have employed standard polar coordinates. The leftmost term is the expression of $K\nabla^2\mathcal{T}$ in polar coordinates. We proceed in the now familiar way, let: $\mathcal{T} = T(t)R(r)\Theta(\theta)$, and insert it in the governing equation.

$$\frac{(rR')'}{rR} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \frac{1}{K}\frac{T'}{T} = -\lambda^2 \tag{4.44}$$

We are quickly led to

$$T(t) = T_0 \exp(-\lambda^2 K t) \tag{4.45}$$

After multiplying through by r our equation reduces to:

$$r\frac{(rR')'}{R} + \lambda^2 r^2 = -\frac{\Theta''}{\Theta} = -\nu^2 \tag{4.46}$$

where we have set the next separation constant, $-\nu^2$ on the *rhs*.

Now we can find out the constraint on $\Theta(\theta)$. We have

$$\Theta(\theta) = \Theta_0 \exp(i\nu\theta) \tag{4.47}$$

To preserve the periodicity, $\Theta(\theta + 2\pi) = \Theta(\theta)$, we must choose ν to be an integer (positive or negative). For the radial equation we obtain

$$\frac{1}{r}(rR')' + \left(\lambda^2 - \frac{\nu^2}{r^2}\right) R = 0 \tag{4.48}$$

where ν is an integer and λ is to be determined by the boundary conditions. The last equation is known as Bessel's Equation; it occurs in numerous problems involving polar or cylindrical coordinates or geometries. The solutions are of course the Bessel Functions:

$$R(r) = AJ_\nu(\lambda r) + BY_\nu(\lambda r) \tag{4.49}$$

The integer ν is the order of the Bessel Function. The $J_\nu(x)$ is the regular solution, and the $Y_\nu(x)$ is the irregular solution. The functional form of $J_\nu(x)$ is found by a power series method. We need not look into the details here. But the shapes of the functions are interesting and the first few are shown in Figs. 1 and 2. Note that the Y_ν diverge at $x = 0$. In the present problem, we can eliminate the $Y_\nu(\lambda r)$ from further consideration, since an implicit part of specifying the boundary conditions is that the solution be finite.

The boundary condition at the rim of the disk can be satisfied by forcing

$$J_\nu(\lambda_n a) = 0 \tag{4.50}$$

where $\lambda_n a =$ the $n-$th root of $J_\nu(x_n) = 0$. These roots are given in tables, and are generated in some programs such as Mathematica. You can see the first few in the figure. For J_0 the first three are 2.4048, 5.5201, 8.6537, ...

We can then state that a genreral solution to the homogeneous (i.e., no heating, just dissipation) problem is

$$\mathcal{T}(r, \theta, t) = \sum_{\nu,n} A_{\nu,n} \exp(i\nu\theta) J_\nu(\lambda_n r) \exp(-\lambda_n^2 K t) \tag{4.51}$$

And the $A_{\nu,n}$ are to be determined from the initial conditions $\mathcal{T}(r, \theta, 0)$. We can find these coefficients by fitting to this function. Note that the $J_\nu(\lambda_n r)$ are orthogonal on the interval $(0, a)$:

$$\int_0^a r J_\nu(\lambda_n r) J_\nu(\lambda_m r)\, dr = 0 \;\; \text{if } n \neq m, \;\text{otherwise a number} \tag{4.52}$$

So the Bessel Functions $J_\nu(\lambda_n r)$ form the modal shapes of the thermal decay modes. The angular part is formed by the sines and cosines of $\nu\theta$.

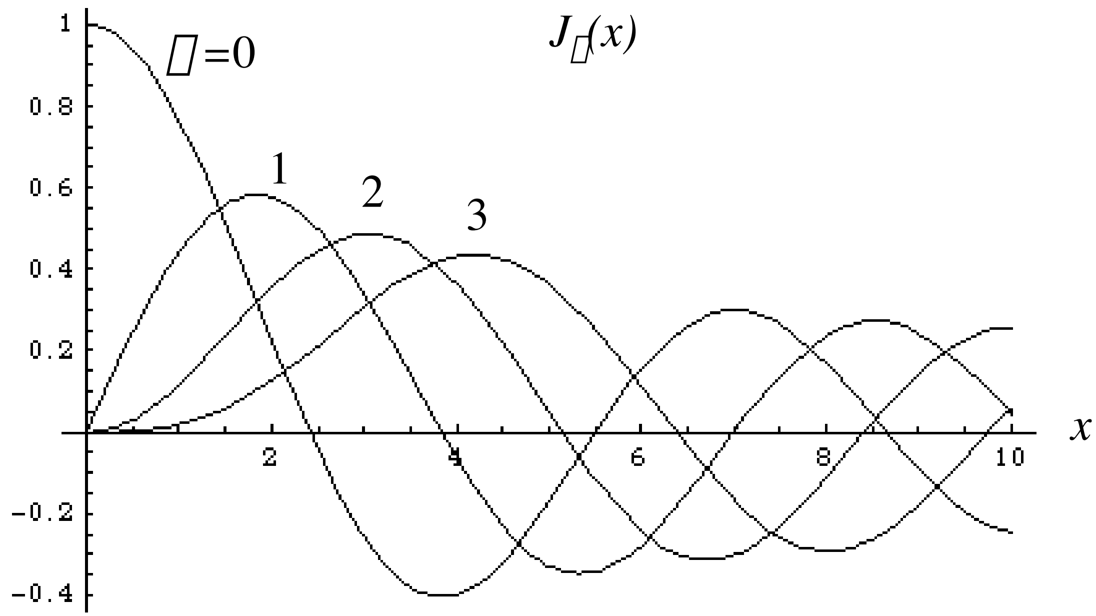

Figure 4.1: *The Bessel Functions* $J_\nu(x)$ *for* $\nu = 0, 1, 2, 3$.

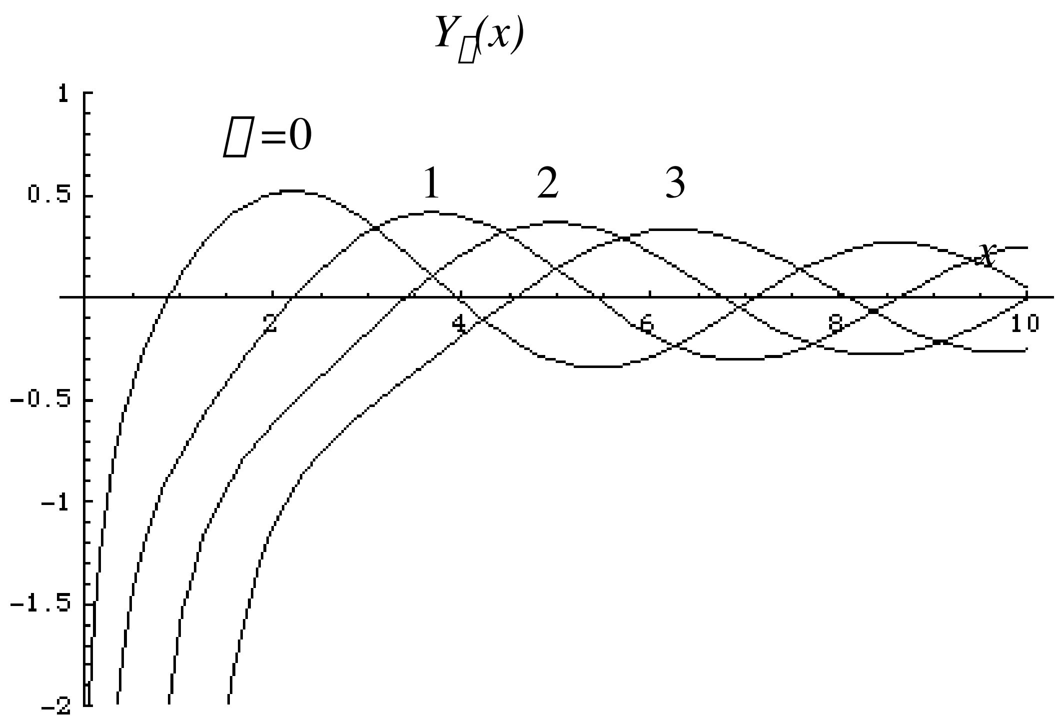

Figure 4.2: *The Bessel Functions $Y_\nu(x)$ for $\nu = 0, 1, 2, 3$.*