Sturm-Liouville Theory

Introduction

- The **Sturm-Liouville Equation** is a **homogeneous second order linear ODE**:

$$\left[p(x)y'(x)\right]' - q(x)y(x) + \lambda\rho(x)y(x) = 0$$

$$\left[py'\right]' - qy + \lambda\rho y = 0$$

  Together with the **homogeneous boundary conditions**

$$\alpha_1 y'(a) + \alpha_2 y(a) = 0$$

$$\beta_1 y'(b) + \beta_2 y(b) = 0$$

  And

  - At least one of the αs is non-zero, and likewise for the βs.
  - p, q and w are **functions** such that
    - p is **real**, **positive** and **differentiable**.
    - q is **real** and **continuous**
    - ρ is **real**, **positive** and **continuous**, and is called the **weight function**.

- The **Sturm-Liouville Equation** can also be written

$$\boxed{\mathcal{L}y = \lambda\rho y}$$

  Where $\mathcal{L}$ is the **Sturm-Liouville Operator**:

$$\boxed{\mathcal{L} = -\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) + q(x)\right]}$$

- The definition of an Hermitian operator is

$$\int_a^b f^*(x)\left[\mathcal{L}g(x)\right]\,\mathrm{d}x = \int_a^b \left[\mathcal{L}f(x)\right]^* g(x)\,\mathrm{d}x$$

  Feeding the Sturm-Liouville Operator in to the LHS:

$$-\int_a^b f^*\left[\left(pg'\right)' + qg\right]\mathrm{d}x = -\int_a^b f^*\left(pg'\right)'\,\mathrm{d}x - \int_a^b f^* qg\,\mathrm{d}x$$

  We can integrate the **first term by parts twice**:

$$\int_a^b f^* \left(pg'\right)' \,\mathrm{d}x = \left[f^*(pg')\right]_a^b - \int_a^b (f^*)' pg' \,\mathrm{d}x$$
$$= \left[f^*(pg')\right]_a^b - \left[(f^*)' pg\right]_a^b + \int_a^b [(f^*)' p]' g \,\mathrm{d}x$$

And now, assuming that the boundary terms are both 0, we have:

$$\int_a^b f^* \left(pg'\right)' \,\mathrm{d}x = \int_a^b [(f^*)' p]' g \,\mathrm{d}x$$

And therefore, the LHS above becomes:

$$-\int_a^b f^*(x)\left[\mathcal{L}g(x)\right] \,\mathrm{d}x = -\int_a^b [(f^*)' p]' g \,\mathrm{d}x - \int_a^b f^* qg \,\mathrm{d}x$$
$$= -\int_a^b [p(f^*)']' g + qf^* g \,\mathrm{d}x$$
$$= -\int_a^b \left[\mathcal{L}f(x)\right]^* g(x) \,\mathrm{d}x$$

(The last step holds because all the functions are real).

As such, the Sturm-Liouville Operator is **Hermitian *if and only if***:

$$\left[f^*(pg')\right]_a^b = 0$$
$$\left[(f^*)' pg\right]_a^b = 0$$

These are satisfied if, for **all** solutions y and g of the equation,

$$\boxed{\left[y^* pg\right]_{x=a}^{x=b} = 0}$$

NOTE: if $p = 0$ at the endpoints, then

- These points are **singularities** of the equation.
- The boundary condition above is automatically satisfied.
- We require, however, that the solution be **regular** (analytic) at the endpoints.

Formalities

- The **inner product** between two functions f and g is defined as

$$\langle f \mid g \rangle = \int_a^b f^* g \rho \,\mathrm{d}x$$

- The **adjoint**, $\mathcal{L}^\dagger$ of an operator $\mathcal{L}$ is defined by

$$\langle f \mid \mathcal{L}g \rangle = \left\langle \mathcal{L}^\dagger f \mid g \right\rangle + \text{Boundary terms}$$

If $\mathcal{L} = \mathcal{L}^\dagger$, then the operator is **self-adjoint**.

- If an operator is **self-adjoint** *and* the **boundary terms vanish**, then the operator is said to be **Hermitian**:

$$\langle f \mid \mathcal{L}g \rangle = \langle \mathcal{L}f \mid g \rangle$$

- Consider an Hermitian operator $\mathcal{L}$, with

$$\mathcal{L}y = \lambda y$$

$$\mathcal{L}z = \mu z$$

Taking inner products:

$$\langle z \mid \mathcal{L}y \rangle = \lambda \langle z \mid y \rangle$$

$$\langle y \mid \mathcal{L}z \rangle = \mu \langle y \mid z \rangle$$

Complex-conjugating the second one:

$$\langle \mathcal{L}z \mid y \rangle = \mu^* \langle z \mid y \rangle$$

Using the self-adjointness of the operator:

$$\langle z \mid \mathcal{L}y \rangle = \mu^* \langle z \mid y \rangle$$

Subtracting this from the very first equation above gives:

$$\boxed{(\lambda - \mu^*)\langle z \mid y \rangle = 0}$$

Now, let's first assume that $z = y$ and therefore $\lambda = \mu$. Then:

$$(\lambda - \lambda^*)\langle y \mid y \rangle = 0$$

Assuming that our eigenvector are non-trivial, $\langle y \mid y \rangle \neq 0$, and so

$$\boxed{\boxed{\lambda = \lambda^*}}$$

The **eigenvalues** are **real**. Now, if $z \neq y$:

$$\boxed{(\lambda - \mu)\langle z \mid y \rangle = 0}$$

And now assuming that the eigenvalues are **distinct**, we have

$$\boxed{\boxed{\langle z \mid y \rangle = 0}}$$

The **eigenvectors** are **orthogonal**.

Transforming into Sturm-Liouville Form

- Consider a general linear second order ODE:

$$y'' + g(x)y' + h(x)y + \lambda\rho(x)y = 0$$

- We can then define the **integrating factor**

$$p(x) = \exp\left(\int g \, \mathrm{d}x\right)$$

So that
  - $p' = pg$
  - p is always positive
- Multiplying the ODE through by p:

$$py'' + pgy' + phy + \lambda p\rho y = 0$$

$$(py')' + phy + \lambda p\rho y = 0$$

- This is in Sturm-Liouville form.
- **Care** must be taken to ensure that the **constraints** on p, q and ρ are **satisfied** (eg: positive, finite, etc…)

Completeness of Eigenfunctions

- The space of **eigenfunctions** is **complete** – the eigenfunctions form a **basis** for the **vector space** of **functions satisfying the boundary conditions**.
- Therefore, any function *f(x)* on the **said interval** that **satisfies** the **boundary conditions** (and even if it doesn't!) can be **expressed** as

$$f(x) = \sum_{n=0}^{\infty} a_n y_n(x)$$

- Take the **inner product** with the eigenfunction y_m

$$\langle y_m \mid f \rangle = \sum_{n=0}^{\infty} a_n \langle y_m \mid y_n \rangle = a_m \langle y_m \mid y_m \rangle$$

$$\Rightarrow \boxed{a_m = \frac{\langle y_m \mid f \rangle}{\langle y_m \mid y_m \rangle}}$$

- Assume that the y_n are **unit normalised** – we then have:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \langle y_n \mid f \rangle y_n(x) \\ &= \sum_{n=0}^{\infty} \int_a^b y_n^*(\xi) f(\xi) w(\xi) \, \mathrm{d}\xi \; y_n(x) \\ &= \int_a^b f(\xi) w(\xi) \sum_{n=0}^{\infty} y_n^*(\xi) y_n(x) \, \mathrm{d}\xi \end{aligned}$$

Which gives the **completeness relation**:

$$\boxed{w(\xi)\sum_{n=0}^{\infty} y_n^*(\xi) y_n(x) = \delta(x-\xi)}$$

Inhomogeneous Problems – Green's Functions

- Consider the **inhomogeneous problem**

$$\mathcal{L}y = f$$

- This time, though, divide $\mathcal{L}$ by ρ, so that

$$\mathcal{L}y_n = \lambda_n y_n$$

(ie: hide the weight function in $\mathcal{L}$)

- Consider the **eigenfunction expansions** of y and f:

$$f(x) = \sum_{n=0}^{\infty} a_n y_n \qquad\qquad y(x) = \sum_{n=0}^{\infty} b_n y_n$$

- Substituting into the ODE yields:

$$\sum_{n=0}^{\infty} b_n \lambda_n y_n = \sum_{n=0}^{\infty} a_n y_n$$

- Taking the inner product of both sides:

$$b_m \lambda_m \langle y_m \mid y_m \rangle = a_m \langle y_m \mid y_m \rangle$$

$$\Rightarrow b_m \lambda_m = a_m = \frac{\langle y_m \mid f \rangle}{\langle y_m \mid y_m \rangle} \qquad (*)$$

$$\Rightarrow b_m = \frac{\langle y_m \mid f \rangle}{\lambda_m \langle y_m \mid y_m \rangle}$$

- Therefore, the solution is given by:

$$y(x) = \sum_{n=0}^{\infty} \frac{\langle y_n \mid f \rangle}{\lambda_n \langle y_n \mid y_n \rangle} y_n$$

- Assuming that the eigenfunctions are **unit normalised**:

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{\lambda_n}\left(\int_a^b y_n^*(\xi) f(\xi)\rho(\xi)\,\mathrm{d}\xi\right) y_n(x)$$

$$= \int_a^b \sum_{n=0}^{\infty} \frac{1}{\lambda_n} y_n^*(\xi)\rho(\xi) y_n(x) f(\xi)\,\mathrm{d}\xi$$

$$= \int_a^b G(x,\xi) f(\xi)\,\mathrm{d}\xi$$

Where

$$\boxed{G(x,\xi) = \rho(\xi)\sum_{n=0}^{\infty} \frac{1}{\lambda_n} y_n^*(\xi) y_n(x)}$$

- As expected

$$\mathcal{L}G(x,\xi) = \rho(\xi)\sum_{n=0}^{\infty} y_n^*(\xi) y_n(x) = \delta(x-\xi)$$

(Using the completeness relation).

- Problems obviously arise if $\lambda_k = 0$. Consider two cases:
  - If $\langle y_k \mid f \rangle \neq 0$, there is no solution. This could, physically, correspond to **resonance** of the mode y_k when the forcing function f is applied.
  - If $\langle y_k \mid f \rangle = 0$, then the equation marked (*) above is **not inconsistent**, and there are solutions (though **not unique** ones) of the form

$$y = \sum_{n \neq k} \frac{\langle y_n \mid f \rangle}{\lambda_n \langle y_n \mid y_n \rangle} y_n + A y_k$$

Legendre Polynomials

- **Legendre's Equation** is

$$(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$$

Which, in **Sturm-Liouville** form is

$$\left[(1-x^2)y'\right]' + \ell(\ell+1)y = 0$$

- We restrict $-1 < x < 1$, to keep $p = 1 - x^2$ positive. At the **end-points**, $p = 0$, so the only **boundary condition** we need is **analyticity** at the **endpoints**.
- The solutions, P_n, are chosen such that

$$P_n(1) = 1$$

Which means that

$$\langle P_n \mid P_n \rangle = \frac{2}{2n+1}$$

- The **Legendre Polynomials** can be *generated* by noting that P_n is an n^{th}-order **polynomial**, that it must be **orthogonal** to **all previous polynomials** and that $P_n(1) = 1$.

