CHAPTER 4

FOURIER SERIES AND INTEGRALS

4.1 FOURIER SERIES FOR PERIODIC FUNCTIONS

This section explains three Fourier series: **sines, cosines, and exponentials e^{ikx}**. Square waves (1 or 0 or -1) are great examples, with delta functions in the derivative. We look at a spike, a step function, and a ramp—and smoother functions too.

Start with $\sin x$. It has period 2π since $\sin(x + 2\pi) = \sin x$. It is an odd function since $\sin(-x) = -\sin x$, and it vanishes at $x = 0$ and $x = \pi$. Every function $\sin nx$ has those three properties, and Fourier looked at *infinite combinations of the sines*:

Fourier sine series $\displaystyle S(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots = \sum_{n=1}^{\infty} b_n \sin nx$ (1)

If the numbers $b_1, b_2, \ldots$ drop off quickly enough (we are foreshadowing the importance of the decay rate) then the sum $S(x)$ will inherit all three properties:

Periodic $S(x + 2\pi) = S(x)$ **Odd** $S(-x) = -S(x)$ $S(0) = S(\pi) = 0$

200 years ago, Fourier startled the mathematicians in France by suggesting that *any function $S(x)$ with those properties could be expressed as an infinite series of sines.* This idea started an enormous development of Fourier series. Our first step is to compute from $S(x)$ the number b_k that multiplies $\sin kx$.

Suppose $S(x) = \sum b_n \sin nx$. Multiply both sides by $\sin kx$. Integrate from 0 to π:

$$\int_0^{\pi} S(x) \sin kx\, dx = \int_0^{\pi} b_1 \sin x \sin kx\, dx + \cdots + \int_0^{\pi} \boldsymbol{b_k \sin kx\ \sin kx\, dx} + \cdots \quad (2)$$

On the right side, all integrals are zero except the highlighted one with $n = k$. This property of "**orthogonality**" will dominate the whole chapter. The sines make $90°$ angles in function space, when their inner products are integrals from 0 to π:

Orthogonality $\displaystyle \int_0^{\pi} \sin nx\ \sin kx\, dx = 0 \quad \text{if} \quad n \neq k\,.$ (3)

317

Zero comes quickly if we integrate $\int \cos mx \, dx = \left[\frac{\sin mx}{m}\right]_0^\pi = 0 - 0$. So we use this:

Product of sines $\sin nx \, \sin kx = \frac{1}{2} \cos(n-k)x - \frac{1}{2} \cos(n+k)x$.   (4)

Integrating $\cos mx$ with $m = n - k$ and $m = n + k$ proves orthogonality of the sines.

The exception is when $n = k$. Then we are integrating $(\sin kx)^2 = \frac{1}{2} - \frac{1}{2} \cos 2kx$:

$$\int_0^\pi \sin kx \, \sin kx \, dx = \int_0^\pi \frac{1}{2} \, dx - \int_0^\pi \frac{1}{2} \cos 2kx \, dx = \frac{\pi}{2}. \qquad (5)$$

The highlighted term in equation (2) is $b_k \pi / 2$. Multiply both sides of (2) by $2/\pi$:

Sine coefficients
$$S(-x) = -S(x) \qquad b_k = \frac{2}{\pi} \int_0^\pi S(x) \sin kx \, dx = \frac{1}{\pi} \int_{-\pi}^\pi S(x) \sin kx \, dx. \qquad (6)$$

Notice that $S(x) \sin kx$ is *even* (equal integrals from $-\pi$ to 0 and from 0 to π).

I will go immediately to the most important example of a Fourier sine series. $S(x)$ is an **odd square wave** with $SW(x) = 1$ for $0 < x < \pi$. It is drawn in Figure 4.1 as an odd function (with period 2π) that vanishes at $x = 0$ and $x = \pi$.
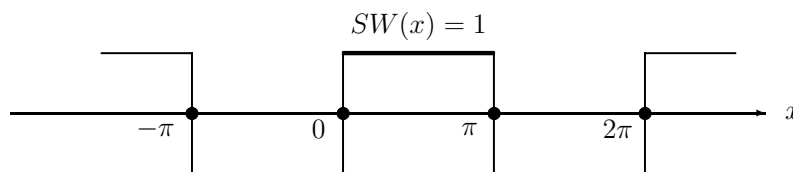

Figure 4.1: The odd square wave with $SW(x + 2\pi) = SW(x) = \{1 \text{ or } 0 \text{ or } -1\}$.

Example 1   Find the Fourier sine coefficients b_k of the square wave $SW(x)$.

Solution   For $k = 1, 2, \ldots$ use the first formula (6) with $S(x) = 1$ between 0 and π:

$$b_k = \frac{2}{\pi} \int_0^\pi \sin kx \, dx = \frac{2}{\pi} \left[\frac{-\cos kx}{k}\right]_0^\pi = \frac{2}{\pi} \left\{\frac{2}{1}, \frac{0}{2}, \frac{2}{3}, \frac{0}{4}, \frac{2}{5}, \frac{0}{6}, \ldots\right\} \qquad (7)$$

The even-numbered coefficients b_{2k} are all zero because $\cos 2k\pi = \cos 0 = 1$. The odd-numbered coefficients $b_k = 4/\pi k$ decrease at the rate $1/k$. We will see that same $1/k$ decay rate for all functions formed from *smooth pieces and jumps*.

Put those coefficients $4/\pi k$ and zero into the Fourier sine series for $SW(x)$:

Square wave $SW(x) = \frac{4}{\pi} \left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots\right]$ (8)

Figure 4.2 graphs this sum after one term, then two terms, and then five terms. You can see the all-important **Gibbs phenomenon** appearing as these "partial sums"

include more terms. Away from the jumps, we safely approach $SW(x) = 1$ or -1. At $x = \pi/2$, the series gives a beautiful alternating formula for the number π:

$$1 = \frac{4}{\pi}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right] \quad \text{so that} \quad \boldsymbol{\pi = 4\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]}. \quad (9)$$

The Gibbs phenomenon is the overshoot that moves closer and closer to the jumps. Its height approaches $1.18\ldots$ and it does not decrease with more terms of the series! Overshoot is the one greatest obstacle to calculation of all discontinuous functions (like shock waves in fluid flow). We try hard to avoid Gibbs but sometimes we can't.
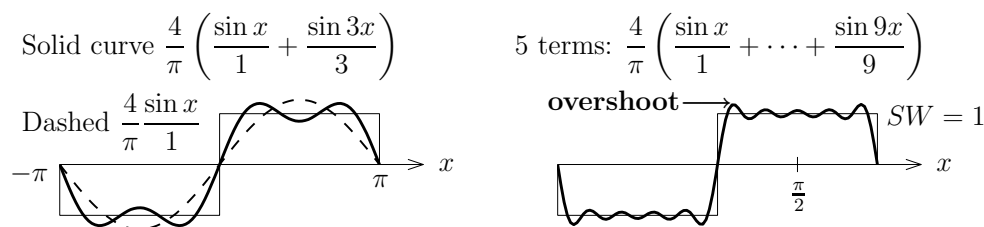
Solid curve $\dfrac{4}{\pi}\left(\dfrac{\sin x}{1} + \dfrac{\sin 3x}{3}\right)$ 5 terms: $\dfrac{4}{\pi}\left(\dfrac{\sin x}{1} + \cdots + \dfrac{\sin 9x}{9}\right)$

Dashed $\dfrac{4}{\pi}\dfrac{\sin x}{1}$

Figure 4.2: **Gibbs phenomenon**: Partial sums $\sum_1^N b_n \sin nx$ overshoot near jumps.

Fourier Coefficients are Best

Let me look again at the first term $b_1 \sin x = (4/\pi)\sin x$. This is the **closest possible approximation** to the square wave SW, by any multiple of $\sin x$ (closest in the least squares sense). To see this optimal property of the Fourier coefficients, minimize the error over all b_1:

The error is $\displaystyle\int_0^\pi (SW - b_1 \sin x)^2\, dx$ The b_1 derivative is $\displaystyle -2\int_0^\pi (SW - b_1 \sin x)\sin x\, dx$.

The integral of $\sin^2 x$ is $\pi/2$. So the derivative is zero when $b_1 = (2/\pi)\int_0^\pi S(x)\sin x\, dx$. This is exactly equation (6) for the Fourier coefficient.

Each $b_k \sin kx$ is as close as possible to $SW(x)$. We can find the coefficients b_k one at a time, *because the sines are orthogonal*. The square wave has $b_2 = 0$ because all other multiples of $\sin 2x$ increase the error. Term by term, we are "projecting the function onto each axis $\sin kx$."

Fourier Cosine Series

The cosine series applies to *even functions* with $C(-x) = C(x)$:

Cosine series $C(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots = a_0 + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx.$ (10)

Every cosine has period 2π. Figure 4.3 shows two even functions, the **repeating ramp** $RR(x)$ and the **up-down train** $UD(x)$ of delta functions. That sawtooth ramp RR is the integral of the square wave. The delta functions in UD give the derivative of the square wave. (For sines, the integral and derivative are cosines.) RR and UD will be valuable examples, one smoother than SW, one less smooth.

First we find formulas for the cosine coefficients a_0 and a_k. The constant term a_0 is the *average value* of the function $C(x)$:

$$a_0 = \textbf{Average} \qquad a_0 = \frac{1}{\pi} \int_0^\pi C(x)\,dx = \frac{1}{2\pi} \int_{-\pi}^\pi C(x)\,dx. \qquad (11)$$

I just integrated every term in the cosine series (10) from 0 to π. On the right side, the integral of a_0 is $a_0\pi$ (divide both sides by π). All other integrals are zero:

$$\int_0^\pi \cos nx\,dx = \left[\frac{\sin nx}{n}\right]_0^\pi = 0 - 0 = 0. \qquad (12)$$

In words, the constant function 1 is orthogonal to $\cos nx$ over the interval $[0, \pi]$.

The other cosine coefficients a_k come from the *orthogonality of cosines*. As with sines, we multiply both sides of (10) by $\cos kx$ and integrate from 0 to π:

$$\int_0^\pi C(x)\cos kx\,dx = \int_0^\pi a_0 \cos kx\,dx + \int_0^\pi a_1 \cos x \cos kx\,dx + \cdots + \boldsymbol{\int_0^\pi a_k (\cos kx)^2\,dx} + \cdots$$

You know what is coming. On the right side, only the highlighted term can be nonzero. Problem 4.1.1 proves this by an identity for $\cos nx \cos kx$—now (4) has a plus sign. The bold nonzero term is $\boldsymbol{a_k\pi/2}$ and we multiply both sides by $2/\pi$:

Cosine coefficients
$\boldsymbol{C(-x) = C(x)}$
$$a_k = \frac{2}{\pi} \int_0^\pi C(x)\cos kx\,dx = \frac{1}{\pi} \int_{-\pi}^\pi C(x)\cos kx\,dx. \qquad (13)$$

Again the integral over a full period from $-\pi$ to π (also 0 to 2π) is just doubled.
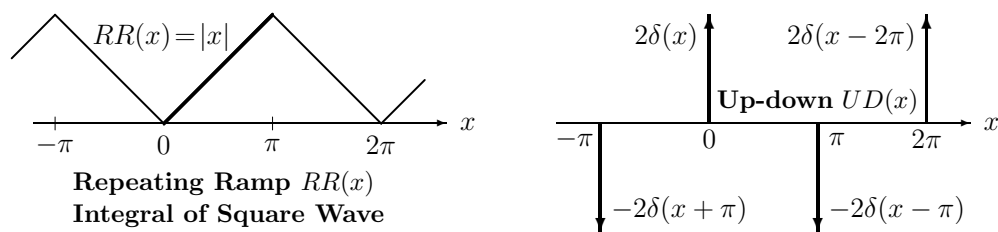

Figure 4.3: The repeating ramp RR and the up-down UD (periodic spikes) are even. The derivative of RR is the odd square wave SW. **The derivative of SW is UD.**

Example 2 Find the cosine coefficients of the ramp $RR(x)$ and the up-down $UD(x)$.

Solution The simplest way is to start with the sine series for the square wave:

$$SW(x) = \frac{4}{\pi} \left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots \right].$$

Take the derivative of every term to produce cosines in the up-down delta function:

Up-down series $\quad UD(x) = \dfrac{4}{\pi} \left[\cos x + \cos 3x + \cos 5x + \cos 7x + \cdots \right].$ (14)

Those coefficients don't decay at all. The terms in the series don't approach zero, so officially the series cannot converge. Nevertheless it is somehow correct and important. Unofficially this sum of cosines has all 1's at $x = 0$ and all -1's at $x = \pi$. Then $+\infty$ and $-\infty$ are consistent with $2\delta(x)$ and $-2\delta(x - \pi)$. The true way to recognize $\delta(x)$ is by the test $\int \delta(x) f(x)\, dx = f(0)$ and Example 3 will do this.

For the repeating ramp, we integrate the square wave series for $SW(x)$ and add the average ramp height $a_0 = \pi/2$, halfway from 0 to π:

Ramp series $\quad \boldsymbol{RR(x) = \dfrac{\pi}{2} - \dfrac{\pi}{4} \left[\dfrac{\cos x}{1^2} + \dfrac{\cos 3x}{3^2} + \dfrac{\cos 5x}{5^2} + \dfrac{\cos 7x}{7^2} + \cdots \right].}$ (15)

The constant of integration is a_0. *Those coefficients a_k drop off like $1/k^2$.* They could be computed directly from formula (13) using $\int x \cos kx\, dx$, but this requires an integration by parts (or a table of integrals or an appeal to *Mathematica* or *Maple*). It was much easier to integrate every sine separately in $SW(x)$, which makes clear the crucial point: Each "degree of smoothness" in the function is reflected in a faster decay rate of its Fourier coefficients a_k and b_k.

No decay	**Delta** functions (with spikes)
$1/k$ decay	**Step** functions (with jumps)
$1/k^2$ decay	**Ramp** functions (with corners)
$1/k^4$ decay	**Spline** functions (jumps in f''')
r^k decay with $r < 1$	**Analytic** functions like $1/(2 - \cos x)$

Each integration divides the kth coefficient by k. So the decay rate has an extra $1/k$. The "Riemann-Lebesgue lemma" says that a_k and b_k approach zero for any continuous function (in fact whenever $\int |f(x)|dx$ is finite). Analytic functions achieve a new level of smoothness—they can be differentiated forever. Their Fourier series and Taylor series in Chapter 5 converge **exponentially fast**.

The poles of $1/(2 - \cos x)$ will be complex solutions of $\cos x = 2$. Its Fourier series converges quickly because r^k decays faster than any power $1/k^p$. Analytic functions are ideal for computations—the Gibbs phenomenon will never appear.

Now we go back to $\delta(x)$ for what could be the most important example of all.

Example 3   Find the (cosine) coefficients of the *delta function* $\delta(x)$, made 2π-periodic.

Solution   The spike occurs at the start of the interval $[0, \pi]$ so safer to integrate from $-\pi$ to π. We find $a_0 = 1/2\pi$ and the other $a_k = 1/\pi$ (cosines because $\delta(x)$ is even):

$$\textbf{Average} \ \ a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(x)\, dx = \frac{1}{2\pi} \qquad \textbf{Cosines} \ \ a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \delta(x) \cos kx\, dx = \frac{1}{\pi}$$

Then the series for the delta function has all cosines in equal amounts:

Delta function $\qquad \delta(x) = \dfrac{1}{2\pi} + \dfrac{1}{\pi}\left[\cos x + \cos 2x + \cos 3x + \cdots\right].$ $\qquad$ (16)

Again this series cannot truly converge (its terms don't approach zero). But we can graph the sum after $\cos 5x$ and after $\cos 10x$. Figure 4.4 shows how these "partial sums" are doing their best to approach $\delta(x)$. They oscillate faster and faster away from $x = 0$.

Actually there is a neat formula for the partial sum $\delta_N(x)$ that stops at $\cos Nx$. Start by writing each term $2\cos\theta$ as $e^{i\theta} + e^{-i\theta}$:

$$\delta_N = \frac{1}{2\pi}\left[1 + 2\cos x + \cdots + 2\cos Nx\right] = \frac{1}{2\pi}\left[1 + e^{ix} + e^{-ix} + \cdots + e^{iNx} + e^{-iNx}\right].$$

This is a geometric progression that starts from e^{-iNx} and ends at e^{iNx}. We have powers of the same factor e^{ix}. The sum of a geometric series is known:

Partial sum
up to $\cos Nx$ $\qquad \delta_N(x) = \dfrac{1}{2\pi} \dfrac{e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x}}{e^{ix/2} - e^{-ix/2}} = \dfrac{1}{2\pi} \dfrac{\sin(N+\frac{1}{2})x}{\sin \frac{1}{2}x}.$ $\qquad$ (17)

This is the function graphed in Figure 4.4. We claim that for any N the area underneath $\delta_N(x)$ is 1. (Each cosine integrated from $-\pi$ to π gives zero. The integral of $1/2\pi$ is 1.) The central "lobe" in the graph ends when $\sin(N+\frac{1}{2})x$ comes down to zero, and that happens when $(N+\frac{1}{2})x = \pm\pi$. I think the area under that lobe (marked by bullets) approaches the same number $1.18\ldots$ that appears in the Gibbs phenomenon.

In what way does $\delta_N(x)$ approach $\delta(x)$? The terms $\cos nx$ in the series jump around at each point $x \neq 0$, not approaching zero. At $x = \pi$ we see $\frac{1}{2\pi}[1 - 2 + 2 - 2 + \cdots]$ and the sum is $1/2\pi$ or $-1/2\pi$. The bumps in the partial sums don't get smaller than $1/2\pi$. The right test for the delta function $\delta(x)$ is to multiply by a smooth $f(x) = \sum a_k \cos kx$ and integrate, because *we only know $\delta(x)$ from its integrals* $\int \delta(x)f(x)\, dx = f(0)$:

Weak convergence
of $\delta_N(x)$ to $\delta(x)$ $\qquad \displaystyle\int_{-\pi}^{\pi} \delta_N(x) f(x)\, dx = a_0 + \cdots + a_N \to f(0).$ $\qquad$ (18)

In this integrated sense (*weak sense*) the sums $\delta_N(x)$ do approach the delta function! The convergence of $a_0 + \cdots + a_N$ is the statement that at $x = 0$ the Fourier series of a smooth $f(x) = \sum a_k \cos kx$ converges to the number $f(0)$.
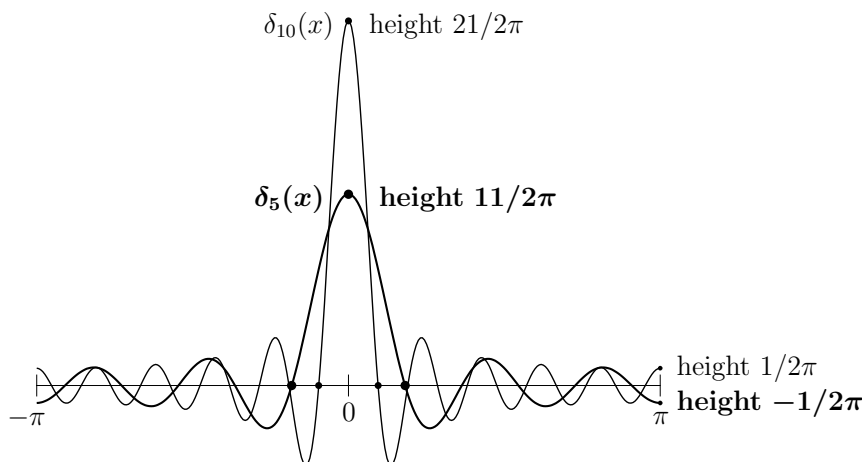
Figure 4.4: The sums $\delta_N(x) = (1 + 2\cos x + \cdots + 2\cos Nx)/2\pi$ try to approach $\delta(x)$.

Complete Series: Sines and Cosines

Over the half-period $[0, \pi]$, the sines are not orthogonal to all the cosines. In fact the integral of $\sin x$ times 1 is not zero. So for functions $F(x)$ that are not odd or even, we move to the complete series (sines plus cosines) on the full interval. Since our functions are periodic, that "full interval" can be $[-\pi, \pi]$ or $[0, 2\pi]$:

Complete Fourier series
$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx. \qquad (19)$$

On every "2π interval" all sines and cosines are mutually orthogonal. We find the Fourier coefficients a_k and b_k in the usual way: **Multiply (19) by 1 and $\cos kx$ and $\sin kx$, and integrate both sides from $-\pi$ to π:**

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x)\, dx \quad a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \cos kx\, dx \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \sin kx\, dx. \quad (20)$$

Orthogonality kills off infinitely many integrals and leaves only the one we want.

Another approach is to split $F(x) = C(x) + S(x)$ into an even part and an odd part. Then we can use the earlier cosine and sine formulas. The two parts are

$$C(x) = F_{\textbf{even}}(x) = \frac{F(x) + F(-x)}{2} \qquad S(x) = F_{\textbf{odd}}(x) = \frac{F(x) - F(-x)}{2}. \qquad (21)$$

The even part gives the a's and the odd part gives the b's. Test on a short square pulse from $x = 0$ to $x = h$—this one-sided function is not odd or even.

Example 4   Find the a's and b's if $F(x) = $ **square pulse** $= \begin{cases} \mathbf{1} & \text{for } 0 < x < h \\ \mathbf{0} & \text{for } h < x < 2\pi \end{cases}$

Solution   The integrals for a_0 and a_k and b_k stop at $x = h$ where $F(x)$ drops to zero. The coefficients decay like $1/k$ because of the jump at $x = 0$ and the drop at $x = h$:

Coefficients of square pulse $\displaystyle a_0 = \frac{1}{2\pi} \int_0^h 1 \, dx = \frac{h}{2\pi} = $ **average**

$$a_k = \frac{1}{\pi} \int_0^h \cos kx \, dx = \frac{\sin kh}{\pi k} \qquad b_k = \frac{1}{\pi} \int_0^h \sin kx \, dx = \frac{1 - \cos kh}{\pi k}. \qquad (22)$$

If we divide $F(x)$ by h, its graph is a tall thin rectangle: height $\frac{1}{h}$, base h, and area $= 1$.

When h approaches zero, $F(x)/h$ is squeezed into a very thin interval. *The tall rectangle approaches (weakly) the delta function $\delta(x)$.* The average height is area$/2\pi = 1/2\pi$. Its other coefficients a_k/h and b_k/h approach $1/\pi$ and 0, already known for $\delta(x)$:

$$\frac{F(x)}{h} \to \delta(x) \qquad \frac{a_k}{h} = \frac{1}{\pi} \frac{\sin kh}{kh} \to \frac{1}{\pi} \quad \text{and} \quad \frac{b_k}{h} = \frac{1 - \cos kh}{\pi kh} \to 0 \text{ as } h \to 0. \quad (23)$$

When the function has a jump, its Fourier series picks the halfway point. This example would converge to $F(0) = \frac{1}{2}$ and $F(h) = \frac{1}{2}$, halfway up and halfway down.

The Fourier series converges to $F(x)$ at each point where the function is smooth. This is a highly developed theory, and Carleson won the 2006 Abel Prize by proving convergence for every x except a set of measure zero. If the function has finite energy $\int |F(x)|^2 \, dx$, he showed that the Fourier series converges "almost everywhere."

Energy in Function $=$ Energy in Coefficients

There is an extremely important equation (**the energy identity**) that comes from integrating $(F(x))^2$. When we square the Fourier series of $F(x)$, and integrate from $-\pi$ to π, all the "cross terms" drop out. The only nonzero integrals come from 1^2 and $\cos^2 kx$ and $\sin^2 kx$, multiplied by a_0^2 and a_k^2 and b_k^2:

Energy in $F(x) = \int_{-\pi}^{\pi}(a_0 + \sum a_k \cos kx + \sum b_k \sin kx)^2 dx$

$$\int_{-\pi}^{\pi}(F(x))^2 dx = 2\pi a_0^2 + \pi(a_1^2 + b_1^2 + a_2^2 + b_2^2 + \cdots). \qquad (24)$$

The energy in $F(x)$ equals the energy in the coefficients. The left side is like the length squared of a vector, except *the vector is a function.* The right side comes from an infinitely long vector of a's and b's. The lengths are equal, which says that the Fourier transform from function to vector is like an orthogonal matrix. Normalized by constants $\sqrt{2\pi}$ and $\sqrt{\pi}$, we have an *orthonormal basis in function space.*

What is this function space? It is like ordinary 3-dimensional space, except the "vectors" are functions. Their length $\|f\|$ comes from integrating instead of adding: $\|f\|^2 = \int |f(x)|^2 dx$. These functions fill **Hilbert space**. The rules of geometry hold:

Length $\|f\|^2 = (f, f)$ comes from the inner product $(f, g) = \int f(x)g(x)\, dx$

Orthogonal functions $(f, g) = 0$ produce a right triangle: $\|f + g\|^2 = \|f\|^2 + \|g\|^2$

I have tried to draw Hilbert space in Figure 4.5. It has infinitely many axes. *The energy identity* (24) *is exactly the Pythagoras Law in infinite-dimensional space.*
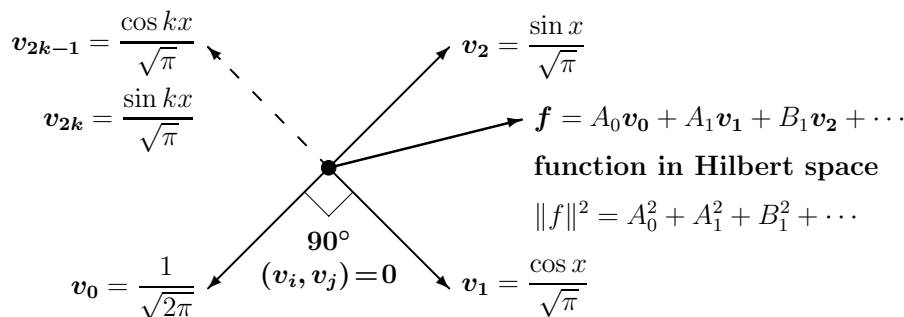

Figure 4.5: The Fourier series is a combination of orthonormal v's (sines and cosines).

Complex Exponentials $c_k e^{ikx}$

This is a small step and we have to take it. In place of separate formulas for a_0 and a_k and b_k, we will have *one formula* for all the complex coefficients c_k. And the function $F(x)$ might be complex (as in quantum mechanics). The Discrete Fourier Transform will be much simpler when we use N complex exponentials for a vector. We practice in advance with the complex infinite series for a 2π-periodic function:

Complex Fourier series $$F(x) = c_0 + c_1 e^{ix} + c_{-1} e^{-ix} + \cdots = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad (25)$$

If every $c_n = c_{-n}$, we can combine e^{inx} with e^{-inx} into $2\cos nx$. Then (25) is the cosine series for an even function. If every $c_n = -c_{-n}$, we use $e^{inx} - e^{-inx} = 2i\sin nx$. Then (25) is the sine series for an odd function and the c's are pure imaginary.

To find c_k, multiply (25) by e^{-ikx} (not e^{ikx}) and integrate from $-\pi$ to π:

$$\int_{-\pi}^{\pi} F(x)e^{-ikx} dx = \int_{-\pi}^{\pi} c_0 e^{-ikx} dx + \int_{-\pi}^{\pi} c_1 e^{ix} e^{-ikx} dx + \cdots + \int_{-\pi}^{\pi} \mathbf{c_k e^{ikx} e^{-ikx} dx} + \cdots$$

The complex exponentials are orthogonal. Every integral on the right side is zero, except for the highlighted term (when $n = k$ and $e^{ikx}e^{-ikx} = 1$). The integral of 1 is 2π. That surviving term gives the formula for c_k:

Fourier coefficients $$\int_{-\pi}^{\pi} F(x)e^{-ikx}\, dx = 2\pi c_k \quad \text{for} \quad k = 0, \pm 1, \ldots \quad (26)$$

Notice that $c_0 = a_0$ is still the average of $F(x)$, because $e^0 = 1$. The orthogonality of e^{inx} and e^{ikx} is checked by integrating, as always. But the complex inner product (F, G) takes the *complex conjugate* $\overline{G}$ *of* G. Before integrating, change e^{ikx} to e^{-ikx}:

Complex inner product   **Orthogonality of e^{inx} and e^{ikx}**

$$(F, G) = \int_{-\pi}^{\pi} F(x)\overline{G(x)}\,dx \qquad \int_{-\pi}^{\pi} e^{i(n-k)x}dx = \left[\frac{e^{i(n-k)x}}{i(n-k)}\right]_{-\pi}^{\pi} = 0\,. \qquad (27)$$

Example 5   Add the complex series for $1/(2 - e^{ix})$ and $1/(2 - e^{-ix})$. These geometric series have exponentially fast decay from $1/2^k$. The functions are analytic.

$$\left(\frac{1}{2} + \frac{e^{ix}}{4} + \frac{e^{2ix}}{8} + \cdots\right) + \left(\frac{1}{2} + \frac{e^{-ix}}{4} + \frac{e^{-2ix}}{8} + \cdots\right) = 1 + \frac{\cos x}{2} + \frac{\cos 2x}{4} + \frac{\cos 3x}{8} + \cdots$$

When we add those functions, we get a real analytic function:

$$\frac{1}{2 - e^{ix}} + \frac{1}{2 - e^{-ix}} = \frac{(2 - e^{-ix}) + (2 - e^{ix})}{(2 - e^{ix})(2 - e^{-ix})} = \frac{4 - 2\cos x}{5 - 4\cos x} \qquad (28)$$

This ratio is the infinitely smooth function whose cosine coefficients are $1/2^k$.

Example 6   Find c_k for the 2π-periodic shifted pulse $F(x) = \begin{cases} 1 & \text{for } s \leq x \leq s + h \\ 0 & \text{elsewhere in } [-\pi, \pi] \end{cases}$

Solution   The integrals (26) from $-\pi$ to π become integrals from s to $s + h$:

$$c_k = \frac{1}{2\pi} \int_{s}^{s+h} 1 \cdot e^{-ikx}\,dx = \frac{1}{2\pi}\left[\frac{e^{-ikx}}{-ik}\right]_{s}^{s+h} = e^{-iks}\left(\frac{1 - e^{-ikh}}{2\pi i k}\right)\,. \qquad (29)$$

Notice above all the simple effect of the shift by s. *It "modulates" each* c_k *by* e^{-iks}. The energy is unchanged, the integral of $|F|^2$ just shifts, and all $|e^{-iks}| = 1$:

Shift $F(x)$ to $F(x - s)$ $\longleftrightarrow$ **Multiply** c_k by e^{-iks}. $\qquad (30)$

Example 7   **Centered pulse** with shift $s = -h/2$. The square pulse becomes centered around $x = 0$. This even function equals 1 on the interval from $-h/2$ to $h/2$:

Centered by $s = -\frac{h}{2}$ $c_k = e^{ikh/2}\dfrac{1 - e^{-ikh}}{2\pi i k} = \dfrac{1}{2\pi}\dfrac{\sin(kh/2)}{k/2}\,.$

Divide by h for a tall pulse. The ratio of $\sin(kh/2)$ to $kh/2$ is the **sinc function**:

Tall pulse $\dfrac{F_{\text{centered}}}{h} = \dfrac{1}{2\pi}\displaystyle\sum_{-\infty}^{\infty} \text{sinc}\left(\dfrac{kh}{2}\right) e^{ikx} = \begin{cases} 1/h & \text{for } -h/2 \leq x \leq h/2 \\ 0 & \text{elsewhere in } [-\pi, \pi] \end{cases}$

That division by h produces area $= 1$. **Every coefficient approaches $\frac{1}{2\pi}$ as $h \to 0$.** The Fourier series for the tall thin pulse again approaches the Fourier series for $\delta(x)$.

Hilbert space can contain vectors $c = (c_0, c_1, c_{-1}, c_2, c_{-2}, \cdots)$ instead of functions $F(x)$. The length of c is $2\pi \sum |c_k|^2 = \int |F|^2 dx$. The function space is often denoted by L^2 and the vector space is ℓ^2. The energy identity is trivial (but deep). Integrating the Fourier series for $F(x)$ times $\overline{F(x)}$, orthogonality kills every $c_n \overline{c_k}$ for $n \neq k$. This leaves the $c_k \overline{c_k} = |c_k|^2$:

$$\int_{-\pi}^{\pi} |F(x)|^2 dx = \int_{-\pi}^{\pi} (\sum c_n e^{inx})(\sum \overline{c_k} e^{-ikx}) dx = 2\pi(|c_0|^2 + |c_1|^2 + |c_{-1}|^2 + \cdots). \quad (31)$$

This is Plancherel's identity: The energy in x-space equals the energy in k-space.

Finally I want to emphasize the three big rules for operating on $F(x) = \sum c_k e^{ikx}$:

1. **The derivative $\dfrac{dF}{dx}$ has Fourier coefficients** ikc_k (energy moves to high k).

2. **The integral of $F(x)$ has Fourier coefficients** $\dfrac{c_k}{ik}, k \neq 0$ (faster decay).

3. **The shift to $F(x-s)$ has Fourier coefficients** $e^{-iks} c_k$ (no change in energy).

Application: Laplace's Equation in a Circle

Our first application is to Laplace's equation. The idea is to construct $u(x,y)$ as an infinite series, choosing its coefficients to match $u_0(x,y)$ along the boundary. Everything depends on the shape of the boundary, and we take a circle of radius 1.

Begin with the simple solutions 1, $r \cos\theta$, $r \sin\theta$, $r^2 \cos 2\theta$, $r^2 \sin 2\theta$, ... to Laplace's equation. Combinations of these special solutions give all solutions in the circle:

$$u(r,\theta) = a_0 + a_1 r \cos\theta + b_1 r \sin\theta + a_2 r^2 \cos 2\theta + b_2 r^2 \sin 2\theta + \cdots \quad (32)$$

It remains to choose the constants a_k and b_k to make $u = u_0$ on the boundary. For a circle $u_0(\theta)$ is periodic, since θ and $\theta + 2\pi$ give the same point:

Set $r = 1$ $u_0(\theta) = a_0 + a_1 \cos\theta + b_1 \sin\theta + a_2 \cos 2\theta + b_2 \sin 2\theta + \cdots \quad (33)$

This is exactly the Fourier series for u_0. **The constants a_k and b_k must be the Fourier coefficients of $u_0(\theta)$.** Thus the problem is completely solved, if an infinite series (32) is acceptable as the solution.

Example 8   **Point source** $u_0 = \delta(\theta)$ at $\theta = 0$ The whole boundary is held at $u_0 = 0$, except for the source at $x = 1$, $y = 0$. Find the temperature $u(r,\theta)$ inside.

Fourier series for δ $u_0(\theta) = \dfrac{1}{2\pi} + \dfrac{1}{\pi}(\cos\theta + \cos 2\theta + \cos 3\theta + \cdots) = \dfrac{1}{2\pi} \displaystyle\sum_{-\infty}^{\infty} e^{in\theta}$

Inside the circle, each $\cos n\theta$ is multiplied by r^n:

Infinite series for u $u(r,\theta) = \dfrac{1}{2\pi} + \dfrac{1}{\pi}(r\cos\theta + r^2\cos 2\theta + r^3\cos 3\theta + \cdots)$ (34)

Poisson managed to sum this infinite series! It involves a series of powers of $re^{i\theta}$. So we know the response at every (r,θ) to the point source at $r = 1$, $\theta = 0$:

Temperature inside circle $u(r,\theta) = \dfrac{1}{2\pi}\dfrac{1 - r^2}{1 + r^2 - 2r\cos\theta}$ (35)

At the center $r = 0$, this produces the average of $u_0 = \delta(\theta)$ which is $a_0 = 1/2\pi$. On the boundary $r = 1$, this produces $u = 0$ except at the point source where $\cos 0 = 1$:

On the ray $\theta = 0$ $u(r,\theta) = \dfrac{1}{2\pi}\dfrac{1 - r^2}{1 + r^2 - 2r} = \dfrac{1}{2\pi}\dfrac{1 + r}{1 - r}.$ (36)

As r approaches 1, the solution becomes infinite as the point source requires.

Example 9   Solve for any boundary values $u_0(\theta)$ by integrating over point sources.

When the point source swings around to angle φ, the solution (35) changes from θ to $\theta - \varphi$. Integrate this "Green's function" to solve in the circle:

Poisson's formula $u(r,\theta) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} u_0(\varphi)\,\dfrac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \varphi)}\,d\varphi$ (37)

Ar $r = 0$ the fraction disappears and u is the average $\int u_0(\varphi)d\varphi/2\pi$. The steady state temperature at the center is the average temperature around the circle.

Poisson's formula illustrates a key idea. Think of any $u_0(\theta)$ as a circle of point sources. The source at angle $\varphi = \theta$ produces the solution inside the integral (37). Integrating around the circle adds up the responses to all sources and gives the response to $u_0(\theta)$.

Example 10 $u_0(\theta) = 1$ on the top half of the circle and $u_0 = -1$ on the bottom half.

Solution   The boundary values are the square wave $SW(\theta)$. Its sine series is in (8):

Square wave for $u_0(\theta)$ $SW(\theta) = \dfrac{4}{\pi}\left[\dfrac{\sin\theta}{1} + \dfrac{\sin 3\theta}{3} + \dfrac{\sin 5\theta}{5} + \cdots\right]$ (38)

Inside the circle, multiplying by r, r^2, r^3,... gives fast decay of high frequencies:

Rapid decay inside $u(r,\theta) = \dfrac{4}{\pi}\left[\dfrac{r\sin\theta}{1} + \dfrac{r^3\sin 3\theta}{3} + \dfrac{r^5\sin 5\theta}{5} + \cdots\right]$ (39)

Laplace's equation has smooth solutions, even when $u_0(\theta)$ is not smooth.

■   **WORKED EXAMPLE**   ■

A hot metal bar is moved into a freezer (zero temperature). The sides of the bar are coated so that heat only escapes at the ends. *What is the temperature $u(x,t)$ along the bar at time t?* It will approach $u = 0$ as all the heat leaves the bar.

Solution   The heat equation is $u_t = u_{xx}$. At $t = 0$ the whole bar is at a constant temperature, say $u=1$. The ends of the bar are at zero temperature for all time $t>0$. This is an **initial-boundary value problem**:

Heat equation $u_t = u_{xx}$ with $u(x,0) = 1$ and $u(0,t) = u(\pi,t) = 0.$ (40)

Those zero boundary conditions suggest a sine series. Its coefficients depend on t:

Series solution of the heat equation $u(x,t) = \sum_{1}^{\infty} b_n(t) \sin nx.$ (41)

The form of the solution shows **separation of variables**. In a comment below, we look for products $A(x)\,B(t)$ that solve the heat equation and the boundary conditions. What we reach is exactly $A(x) = \sin nx$ and the series solution (41).

Two steps remain. First, choose each $b_n(t) \sin nx$ to satisfy the heat equation:

Substitute into $u_t = u_{xx}$ $b_n'(t) \sin nx = -n^2 b_n(t) \sin nx$ $b_n(t) = e^{-n^2 t} b_n(0).$

Notice $b_n' = -n^2 b_n$. Now determine each $b_n(0)$ from the initial condition $u(x,0) = 1$ on $(0, \pi)$. Those numbers are the Fourier sine coefficients of $SW(x)$ in equation (38):

Box function/square wave $\sum_{1}^{\infty} b_n(0) \sin nx = 1$ $b_n(0) = \dfrac{4}{\pi n}$ for odd n

This completes the series solution of the initial-boundary value problem:

Bar temperature $u(x,t) = \sum_{\text{odd } n} \dfrac{4}{\pi n} e^{-n^2 t} \sin nx.$ (42)

For large n (high frequencies) the decay of $e^{-n^2 t}$ is very fast. The dominant term $(4/\pi)e^{-t} \sin x$ for large times will come from $n = 1$. This is typical of the heat equation and all diffusion, that the solution (the temperature profile) becomes very smooth as t increases.

Numerical difficulty I regret any bad news in such a beautiful solution. To compute $u(x,t)$, we would probably truncate the series in (42) to N terms. When that finite series is graphed on the website, serious bumps appear in $u_N(x,t)$. You ask if there is a physical reason but there isn't. The solution should have maximum temperature at the midpoint $x = \pi/2$, and decay smoothly to zero at the ends of the bar.

Those unphysical bumps are precisely the **Gibbs phenomenon**. The initial $u(x,0)$ is 1 on $(0, \pi)$ but its odd reflection is -1 on $(-\pi, 0)$. That jump has produced the slow $4/\pi n$ decay of the coefficients, with Gibbs oscillations near $x = 0$ and $x = \pi$. The sine series for $u(x,t)$ is not a success numerically. Would finite differences help?

Separation of variables   We found $b_n(t)$ as the coefficient of an eigenfunction $\sin nx$. Another good approach is to put $u = A(x) B(t)$ directly into $u_t = u_{xx}$:

Separation $A(x) B'(t) = A''(x) B(t)$ requires $\dfrac{A''(x)}{A(x)} = \dfrac{B'(t)}{B(t)} = \textbf{constant}$.   (43)

A''/A is constant in space, B'/B is constant in time, and they are equal:

$$\frac{A''}{A} = -\lambda \text{ gives } A = \sin \sqrt{\lambda}\, x \text{ and } \cos \sqrt{\lambda}\, x \qquad \frac{B'}{B} = -\lambda \text{ gives } B = e^{-\lambda t}$$

The products $AB = e^{-\lambda t} \sin \sqrt{\lambda}\, x$ and $e^{-\lambda t} \cos \sqrt{\lambda}\, x$ solve the heat equation for any number λ. But the boundary condition $u(0,t) = 0$ eliminates the cosines. Then $u(\pi, t) = 0$ requires $\lambda = n^2 = 1, 4, 9, \ldots$ to have $\sin \sqrt{\lambda}\, \pi = 0$. Separation of variables has recovered the functions in the series solution (42).

Finally $u(x,0) = 1$ determines the numbers $4/\pi n$ for odd n. We find zero for even n because $\sin nx$ has $n/2$ positive loops and $n/2$ negative loops. For odd n, the extra positive loop is a fraction $1/n$ of all loops, giving slow decay of the coefficients.

Heat bath (**the opposite problem**) The solution on the website is $1 - u(x,t)$, because it solves a different problem. **The bar is initially frozen at** $U(x,0) = 0$. It is placed into a heat bath at the fixed temperature $U = 1$ (or $U = T_0$). The new unknown is U and its boundary conditions are no longer zero.

The heat equation and its boundary conditions are solved first by $U_B(x,t)$. In this example $U_B \equiv 1$ is constant. Then the difference $V = U - U_B$ has zero boundary values, and its initial values are $V = -1$. Now the eigenfunction method (or separation of variables) solves for V. (The series in (42) is multiplied by -1 to account for $V(x,0) = -1$.) Adding back U_B solves the heat bath problem: $U = U_B + V = 1 - u(x,t)$.

Here $U_B \equiv 1$ is the *steady state* solution at $t = \infty$, and V is the *transient* solution. The transient starts at $V = -1$ and decays quickly to $V = 0$.

Heat bath at one end The website problem is different in another way too. The Dirichlet condition $u(\pi, t) = 1$ is replaced by the Neumann condition $u'(1,t) = 0$. Only the left end is in the heat bath. Heat flows down the metal bar and out at the far end, now located at $x = 1$. How does the solution change for fixed-free?

Again $U_B = 1$ is a steady state. The boundary conditions apply to $V = 1 - U_B$:

Fixed-free eigenfunctions $V(0) = 0$ and $V'(1) = 0$ lead to $A(x) = \sin\left(n + \dfrac{1}{2}\right)\pi x$.   (44)

Those eigenfunctions give a new form for the sum of $B_n(t)\, A_n(x)$:

Fixed-free solution $\qquad V(x,t) = \sum_{\text{odd } n} B_n(0)\, e^{-(n+\frac{1}{2})^2 \pi^2 t}\, \sin\left(n + \frac{1}{2}\right) \pi x. \qquad (45)$

All frequencies shift by $\frac{1}{2}$ and multiply by π, because $A'' = -\lambda A$ has a free end at $x = 1$. The crucial question is: **Does orthogonality still hold for** these new eigenfunctions $\sin\left(n + \frac{1}{2}\right)\pi x$ **on** $[0, 1]$? The answer is *yes* because this fixed-free "Sturm–Liouville problem" $A'' = -\lambda A$ is still symmetric.

Summary The series solutions all succeed but the truncated series all fail. We can see the overall behavior of $u(x, t)$ and $V(x, t)$. But their exact values close to the jumps are not computed well until we improve on Gibbs.

We could have solved the fixed-free problem on $[0, 1]$ with the fixed-fixed solution on $[0, 2]$. That solution will be symmetric around $x = 1$ so its slope there is zero. Then rescaling x by 2π changes $\sin(n + \frac{1}{2})\pi x$ into $\sin(2n + 1)x$. I hope you like the graphics created by Aslan Kasimov on the cse website.

Problem Set 4.1

1   Find the Fourier series on $-\pi \le x \le \pi$ for

      (a) $f(x) = \sin^3 x$, an odd function

      (b) $f(x) = |\sin x|$, an even function

      (c) $f(x) = x$

      (d) $f(x) = e^x$, using the complex form of the series.

      What are the even and odd parts of $f(x) = e^x$ and $f(x) = e^{ix}$?

2   From Parseval's formula the square wave sine coefficients satisfy

$$\pi(b_1^2 + b_2^2 + \cdots) = \int_{-\pi}^{\pi} |f(x)|^2\, dx = \int_{-\pi}^{\pi} 1\, dx = 2\pi.$$

      Derive the remarkable sum $\pi^2 = 8(1 + \frac{1}{9} + \frac{1}{25} + \cdots)$.

3   If a square pulse is centered at $x = 0$ to give

$$f(x) = 1 \quad \text{for} \quad |x| < \frac{\pi}{2}, f(x) = 0 \quad \text{for} \quad \frac{\pi}{2} < |x| < \pi,$$

      draw its graph and find its Fourier coefficients a_k and b_k.

4   Suppose f has period T instead of $2x$, so that $f(x) = f(x + T)$. Its graph from $-T/2$ to $T/2$ is repeated on each successive interval and its real and complex Fourier series are

$$f(x) = a_0 + a_1 \cos \frac{2\pi x}{T} + b_1 \sin \frac{2\pi x}{T} + \cdots = \sum_{-\infty}^{\infty} c_k\, e^{ik2\pi x/T}$$

      Multiplying by the right functions and integrating from $-T/2$ to $T/2$, find a_k, b_k, and c_k.

5   Plot the first three partial sums and the function itself:

$$x(\pi - x) = \frac{8}{\pi} \left(\frac{\sin x}{1} + \frac{\sin 3x}{27} + \frac{\sin 5x}{125} + \cdots \right), 0 < x < \pi.$$

Why is $1/k^3$ the decay rate for this function? What is the second derivative?

6   What constant function is closest in the least square sense to $f = \cos^2 x$? What multiple of $\cos x$ is closest to $f = \cos^3 x$?

7   Sketch the 2π-periodic half wave with $f(x) = \sin x$ for $0 < x < \pi$ and $f(x) = 0$ for $-\pi < x < 0$. Find its Fourier series.

8   (a) Find the lengths of the vectors $u = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$ and $v = (1, \frac{1}{3}, \frac{1}{9}, \ldots)$ in Hilbert space and test the Schwarz inequality $|u^{\mathrm{T}}v|^2 \le (u^{\mathrm{T}}u)(v^{\mathrm{T}}v)$.

   (b) For the functions $f = 1 + \frac{1}{2}e^{ix} + \frac{1}{4}e^{2ix} + \cdots$ and $g = 1 + \frac{1}{3}e^{ix} + \frac{1}{9}e^{2ix} + \cdots$ use part (a) to find the numerical value of each term in

$$\left| \int_{-\pi}^{\pi} \overline{f}(x) \, g(x) \, dx \right|^2 \le \int_{-\pi}^{\pi} |f(x)|^2 \, dx \int_{-\pi}^{\pi} |g(x)|^2 \, dx.$$

   Substitute for f and g and use orthogonality (or Parseval).

9   Find the solution to Laplace's equation with $u_0 = \theta$ on the boundary. Why is this the imaginary part of $2(z - z^2/2 + z^3/3 \cdots) = 2 \log(1 + z)$? Confirm that on the unit circle $z = e^{i\theta}$, the imaginary part of $2 \log(1 + z)$ agrees with θ.

10   If the boundary condition for Laplace's equation is $u_0 = 1$ for $0 < \theta < \pi$ and $u_0 = 0$ for $-\pi < \theta < 0$, find the Fourier series solution $u(r, \theta)$ inside the unit circle. What is u at the origin?

11   With boundary values $u_0(\theta) = 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} + \cdots$, what is the Fourier series solution to Laplace's equation in the circle? Sum the series.

12   (a) Verify that the fraction in Poisson's formula satisfies Laplace's equation.

   (b) What is the response $u(r, \theta)$ to an impulse at the point $(0, 1)$, at the angle $\varphi = \pi/2$?

   (c) If $u_0(\varphi) = 1$ in the quarter-circle $0 < \varphi < \pi/2$ and $u_0 = 0$ elsewhere, show that at points on the horizontal axis (and especially at the origin)

$$u(r, 0) = \frac{1}{2} + \frac{1}{2\pi} \tan^{-1} \left(\frac{1 - r^2}{-2r} \right) \quad \text{by using}$$

$$\int \frac{d\varphi}{b + c \cos \varphi} = \frac{1}{\sqrt{b^2 - c^2}} \tan^{-1} \left(\frac{\sqrt{b^2 - c^2} \, \sin \varphi}{c + b \cos \varphi} \right).$$

13   When the centered square pulse in Example 7 has width $h = \pi$, find

   (a) its energy $\int |F(x)|^2 \, dx$ by direct integration

   (b) its Fourier coefficients c_k as specific numbers

   (c) the sum in the energy identity (31) or (24)

   If $h = 2\pi$, why is $c_0 = 1$ the only nonzero coefficient? What is $F(x)$?

14   In Example 5, $F(x) = 1 + (\cos x)/2 + \cdots + (\cos nx)/2^n + \cdots$ is infinitely smooth:

   (a) If you take 10 derivatives, what is the Fourier series of $d^{10}F/dx^{10}$?

   (b) Does that series still converge quickly? Compare n^{10} with 2^n for n^{1024}.

15   (*A touch of complex analysis*)   The analytic function in Example 5 blows up when $4 \cos x = 5$. This cannot happen for real x, but equation (28) shows blowup if $e^{ix} = 2$ or $\frac{1}{2}$. In that case we have *poles at $x = \pm i \log 2$*. Why are there also poles at all the complex numbers $x = \pm i \log 2 + 2\pi n$?

16   (*A second touch*)   Change 2's to 3's so that equation (28) has $1/(3 - e^{ix}) + 1/(3 - e^{-ix})$. Complete that equation to find the function that gives fast decay at the rate $1/3^k$.

17   (*For complex professors only*)   Change those 2's and 3's to 1's:

$$\frac{1}{1 - e^{ix}} + \frac{1}{1 - e^{-ix}} = \frac{(1 - e^{-ix}) + (1 - e^{ix})}{(1 - e^{ix})(1 - e^{-ix})} = \frac{2 - e^{ix} - e^{-ix}}{2 - e^{ix} - e^{-ix}} = 1 \, .$$

   A constant! What happened to the pole at $e^{ix} = 1$? Where is the dangerous series $(1 + e^{ix} + \cdots) + (1 + e^{-ix} + \cdots) = 2 + 2\cos x + \cdots$ involving $\delta(x)$?

18   Following the Worked Example, solve the heat equation $u_t = u_{xx}$ from a point source $u(x, 0) = \delta(x)$ with free boundary conditions $u'(\pi, t) = u'(-\pi, t) = 0$. Use the infinite cosine series for $\delta(x)$ with time decay factors $b_n(t)$.