Undergraduate Notes in Mathematics

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A First Course of Partial Differential Equations in Physical Sciences and Engineering

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Preface

Partial differential equations are often used to construct models of the most basic theories underlying physics and engineering. The goal of this book is to develop the most basic ideas from the theory of partial differential equations, and apply them to the simplest models arising from the above mentioned fields.
It is not easy to master the theory of partial differential equations. Unlike the theory of ordinary differential equations, which relies on the fundamental existence and uniqueness theorem, there is no single theorem which is central to the subject. Instead, there are separate theories used for each of the major types of partial differential equations that commonly arise.
It is worth pointing out that the preponderance of differential equations arising in applications, in science, in engineering, and within mathematics itself, are of either first or second order, with the latter being by far the most prevalent. We will mainly cover these two classes of PDEs.
This book is intended for a first course in partial differential equations at the advanced undergraduate level for students in engineering and physical sciences. It is assumed that the student has had the standard three semester calculus sequence, and a course in ordinary differential equations.

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Contents

Preliminaries

In this chapter we include some results from calculus which we will use often in the study of partial differential equations. Details and proof of these results can be found in most calculus books.

1 Some Results of Calculus

The first result provides a mean of showing when a function is zero on an interval.

Theorem 1.1
(a) Suppose that f is continuous on an interval $I \subset \mathbb{R}$ such that $\int_a^b f(x)dx = 0$ for all subintervals $[a, b] \subset I$. Then $f(x) = 0$ for all $x \in I$.
(b) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and non-negative. If $\int_a^b f(x)dx = 0$ then $f(x) = 0$ on $[a, b]$.
(c) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous such that $\int_a^b f(x)g(x)dx = 0$ for all continuous functions g on $[a, b]$. Then $f(x) = 0$ on $[a, b]$.

Proof.
(a) Fix $a \in I$. Let $x \in I$. By the Fundamental Theorem of Calculus we have

$$0 = \frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Since x was arbitrary, we have $f(x) = 0$ for all $x \in I$.
(b) Suppose the contrary. That is, suppose that $x_0 \in [a, b]$ such that $f(x_0) > 0$. By the continuity of $f(x)$ at x_0, there is a $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \frac{f(x_0)}{2}$. That is, $|x - x_0| < \delta$ implies $f(x) > \frac{f(x_0)}{2} > 0$.

In words, there exists an open interval $I \subset [a,b]$ centered at x_0 such that $f(x) > 0$ for all $x \in I$. Hence, because $f(x) \geq 0$ we must have

$$\int_a^b f(x)dx \geq \int_I f(x)dx > 0$$

which contradicts our assumption that the integral is zero. We conclude that $f(x) = 0$ on $[a,b]$.
(c) This follows from (b) by taking $g(x) = f(x)$ ■

Remark 1.1
The above theorem remains valid for functions in two variables. For example, if $f(x,y)$ is defined for x in an interval I and y in an interval J such that

$$\int_a^b \int_c^d f(x,y)dxdy = 0$$

for all $[a,b] \subset J$ and $[c,d] \subset I$ then $f(x,y) = 0$ over the rectangle $I \times J$.

Example 1.1
Let $f, g : [a,b] \to \mathbb{R}$ be continuous and such that $f(x) \leq g(x)$ for all x in $[a,b]$. Show that if $\int_a^b (g(x) - f(x))dx = 0$ then $f(x) \equiv g(x)$ on $[a,b]$.

Solution.
Apply part (b) of previous theorem to the function $h(x) = g(x) - f(x)$ ■

Partial Derivatives
For multivariable functions, there are two common notations for partial derivatives, and we shall employ them interchangeably. The first is the Leibnitz notation that employs the symbol ∂ to denote partial derivative. The second, a more compact notation, is to use subscripts to indicate partial derivatives. For example, u_t represents $\frac{\partial u}{\partial t}$, while u_{xx} represents $\frac{\partial^2 u}{\partial x^2}$, and u_{xxt} becomes $\frac{\partial^3 u}{\partial^2 x \partial t}$.

An important formula of differentiation is the so-called **chain rule.** If $u = u(x,y)$ where $x = x(s,t)$ and $y = y(s,t)$ then

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}.$$

Likewise,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}.$$

Example 1.2
Compute the partial derivatives indicated:
(a) $\frac{\partial}{\partial y}(y^2 \sin xy)$
(b) $\frac{\partial^2}{\partial x^2}[e^{x+y}]^2$

Solution.
(a) We have $\frac{\partial}{\partial y}(y^2 \sin xy) = \sin xy \frac{\partial}{\partial y}(y^2) + y^2 \frac{\partial}{\partial y}(\sin xy) = 2y \sin xy + xy^2 \cos xy$.
(b) We have $\frac{\partial}{\partial x}[e^{x+y}]^2 = \frac{\partial}{\partial x}e^{2(x+y)} = 2e^{2(x+y)}$. Thus, $\frac{\partial^2}{\partial x^2}[e^{x+y}]^2 = \frac{\partial}{\partial x}2e^{2(x+y)} = 4e^{2(x+y)}$ ■

Example 1.3
Suppose $u(x, y) = \sin(x^2 + y^2)$, where $x = te^s$ and $y = s + t$. Find u_s and u_t.

Solution.
We have

$$\begin{aligned} u_s =& u_x x_s + u_y y_s = 2x \cos(x^2 + y^2) te^s + 2y \cos(x^2 + y^2) \\ =& [2te^s + 2(s + t)] \cos[t^2 e^{2s} + (s + t)^2] \end{aligned}$$

Likewise,

$$\begin{aligned} u_t =& u_x x_t + u_y y_t = 2x \cos(x^2 + y^2) e^s + 2y \cos(x^2 + y^2) \\ =& [2te^s + 2(s + t)] \cos[t^2 e^{2s} + (s + t)^2] \ ■ \end{aligned}$$

Often we must differentiate an integral with respect to a parameter which may appear in the limits of integration, or in the integrand.
Let $f(x, t)$ be a continuous function in the rectangle $\{a \le x \le b\} \times \{c \le t \le d\}$. Assume that $\frac{\partial f}{\partial t}$ is continuous on this rectangle. Define the function

$$J(t) = \int_{a(t)}^{b(t)} f(x, t)dx$$

where $a(t)$ and $b(t)$ are continuously differentiable functions of t such that $a \le a(t) \le b(t) \le b$. Recall that a function $f(x)$ is said to be **continnuously differentiable** if the derivative $f'(x)$ exists, and is itself a continuous function.

Theorem 1.2

$$\begin{aligned}\frac{dJ}{dt} =& \frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx\\ =& f(b(t),t)b'(t) - f(a(t),t)a'(t) + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x,t)dx\end{aligned}$$

Example 1.4
Consider the heat problem

$$u_t = ku_{xx} - \alpha u, \quad \alpha > 0,\ k > 0, \quad 0 < x < L,\ t > 0$$

with boundary conditions $u_x(0,t) = 0 = u_x(L,t)$ and initial condition $u(x,0) = f(x)$. Let $E(t) = \frac{1}{2}\int_0^L u^2 dx$.
(a) Show that $E'(t) \le 0$.
(b) Show that $E(t) \le \int_0^L \frac{1}{2}|f(x)|^2 dx$.

Solution.
(a) We have

$$\begin{aligned}\frac{dE}{dt} =& \frac{1}{2}\int_0^L \frac{\partial}{\partial t}u^2(x,t)dx\\ =& \int_0^L u(x,t)u_t(x,t)dx = k\int_0^L u(x,t)u_{xx}(x,t)dx - \alpha\int_0^L u^2(x,t)dx\\ =& ku(x,t)u_x(x,t)|_0^L - k\int_0^L u_x^2(x,t)dx - \alpha\int_0^L u^2(x,t)dx\\ =& -k\int_0^L u_x^2(x,t)dx - \alpha\int_0^L u^2(x,t)dx \le 0.\end{aligned}$$

(b) From (a) we conclude that $E(t)$ is a decreasing function of $t > 0$. Thus,

$$E(t) \le E(0) = \frac{1}{2}\int_0^L u^2(x,0)dx = \int_0^L \frac{1}{2}|f(x)|^2 dx \blacksquare$$

The Least Upper Bound
A function $f : D \to \mathbb{R}$ is said to be **bounded from above** in D if there is a constant M such that $f(x) \le M$ for all $x \in D$. We call M an upper bound

of f. Note that the numbers, $M+1, M+2, \cdots$ are also upper bounds of f. The smallest upper bound of f is called the **least upper bound** or the **supremum.** If M is the supremum of f in D we write

$$M = \sup\{f(x) : x \in D\}.$$

Note that if N is any upper bound of f in D then $M \leq N$.

Example 1.5
Find the supremum of $f(x) = \sin x$.

Solution.
The graph of f is bounded between -1 and 1. Thus, $\sup\{f(x) : x \in \mathbb{R}\} = 1$ ■

Example 1.6
Find

$$\sup\left\{\left|\epsilon^2 \sin\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right)\right| : x \in \mathbb{R},\ t > 0\right\}$$

Solution.
The answer is

$$\sup\left\{\left|\epsilon^2 \sin\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right)\right| : x \in \mathbb{R}, t > 0\right\} = \epsilon^2 \blacksquare$$

Practice Problems

Exercise 1.1
Compute the partial derivatives indicated:
(a) $\frac{\partial}{\partial x}(y^2 \sin xy)$
(b) $\frac{\partial^2}{\partial x^2}(e^{x^2 y})$
(c) $\frac{\partial^4}{\partial x \partial y^2 \partial z}\left(z \ln\left(\frac{x^2}{y}\right)\right)$.

Exercise 1.2
Find all the first partial derivatives of the functions:
(a) $f(x,y) = x^4 + 6\sqrt{y}$
(b) $f(x,y,z) = x^2 y - 10y^2 z^3 + 43x - 7\tan(4y)$
(c) $f(s,t) = t^7 \ln(s^2) + \frac{9}{t^3} - \sqrt[7]{s^4}$
(d) $f(x,y) = \cos\left(\frac{4}{x}\right)e^{x^2 y - 5y^3}$
(e) $f(u,v) = \frac{9u}{u^2+5v}$
(f) $f(x,y,z) = \frac{x \sin y}{z^2}$
(g) $f(x,y) = \sqrt{x^2 + \ln(5x - 3y^2)}$

Exercise 1.3
Let $f(x,y) = e^{3x}\cos y$. Compute $f_x(0, 2\pi)$.

Exercise 1.4
If $z = e^x \sin y,\ x = st^2$, and $y = s^2 t$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Exercise 1.5
In the equation

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = x - 2y$$

identify the independent variable(s) and the dependent variable.

Exercise 1.6
Let f be an odd function, that is, $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Show that for all $a \in \mathbb{R}$ we have

$$\int_{-a}^{a} f(x)dx = 0.$$

Exercise 1.7
Let f be an even function, that is, $f(-x) = f(x)$ for all $x \in \mathbb{R}$. Show that for all $a \in \mathbb{R}$ we have

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

Exercise 1.8
Use the product rule of derivatives to derive the formula of integration by parts

$$\int uv'dx = uv - \int u'vdx.$$

Exercise 1.9
Let $u_\epsilon(x,t) = \epsilon^2 \sin\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right)$. Find u_{tt} and u_{xx}.

Exercise 1.10
Let $u_\epsilon(x,t) = \epsilon^2 \sin\left(\frac{x}{\epsilon}\right) \sinh\left(\frac{t}{\epsilon}\right)$, where

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Find u_{tt} and u_{xx}.

Exercise 1.11
Find

$$\sup\left\{\left|\epsilon^2 \sinh\left(\frac{t}{\epsilon}\right) \sin\left(\frac{x}{\epsilon}\right)\right| : x \in \mathbb{R}\right\}.$$

Exercise 1.12
Let $u_n(x,t) = 1 + \frac{e^{n^2 t}}{n} \sin nx$.
(a) Find $\sup\{|u_n(x,0) - 1| : x \in \mathbb{R}\}$.
(b) Find $\sup\{|u_n(x,t) - 1| : x \in \mathbb{R}\}$.

2 Sequences of Functions: Pointwise and Uniform Convergence

Later in this book we will be constructing solutions to PDEs involving infinite sums of sines and cosines. These infinite sums or series are called **Fourier series.** Fourier series are examples of series of functions. Convergence of series of functions is defined in terms of convergence of a sequence of functions. In this section we study the two types of convergence of sequences of functions.

Recall that a sequence of numbers $\{a_n\}_{n=1}^{\infty}$ is said to **converge** to a number L if and only if for every given $\epsilon > 0$ there is a positive integer $N = N(\epsilon)$ such that for all $n \geq N$ we have$|a_n - L| < \epsilon$.

What is the analogue concept of convergence when the terms of the sequence are variables? Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}$ consider a function $f_n : D \to \mathbb{R}$. Thus, we obtain a sequence of functions $\{f_n\}_{n=1}^{\infty}$. For such a sequence, there are two types of convergenve that we consider in this section: pointwise convergence and uniform convergence.

We say that $\{f_n\}_{n=1}^{\infty}$ **converges pointwise** on D to a function $f : D \to \mathbb{R}$ if and only if for a given $a \in D$ and $\epsilon > 0$ there is a positive integer $N = N(a, \epsilon)$ such that if $n \geq N$ then $|f_n(a) - f(a)| < \epsilon$. In symbol, we write

$$\lim_{n\to\infty} f_n(a) = f(a).$$

It is important to note that N is a function of both a and ϵ.

Example 2.1
Define $f_n : [0, \infty) \to \mathbb{R}$ by $f_n(x) = \frac{nx}{1+n^2x^2}$. Show that the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to the function $f(x) = 0$ for all $x \geq 0$.

Solution.
For all $x \geq 0$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = 0 \blacksquare$$

Example 2.2
For each positive integer n let $f_n : (0, \infty) \to \mathbb{R}$ be given by $f_n(x) = nx$. Show that $\{f_n\}_{n=1}^{\infty}$ does not converge pointwise on $D = (0, \infty)$.

Solution.
This follows from the fact that $\lim_{n\to\infty} nx = \infty$ for all $x \in D$ ■

As pointed out above, for pointwise convergence, the positive integer N depends on both the given x and ϵ. A stronger convergence concept can be defined where N depends only on ϵ.
Let D be a subset of $\mathbb{R}$ and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on D. We say that $\{f_n\}_{n=1}^{\infty}$ **converges uniformly** on D to a function $f : D \to \mathbb{R}$ if and only if for all $\epsilon > 0$ there is a positive integer $N = N(\epsilon)$ such that if $n \geq N$ then $|f_n(x) - f(x)| < \epsilon$ for all $x \in D$.
This definition says that the integer N depends only on the given ϵ so that for $n \geq N$, the graph of $f_n(x)$ is bounded above by the graph of $f(x) + \epsilon$ and below by the graph of $f(x) - \epsilon$.

Example 2.3
For each positive integer n let $f_n : [0,1] \to \mathbb{R}$ be given by $f_n(x) = \frac{x}{n}$. Show that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the zero function.

Solution.
Let $\epsilon > 0$ be given. Let N be a positive integer such that $N > \frac{1}{\epsilon}$. Then for $n \geq N$ we have
$$|f_n(x) - f(x)| = \frac{|x|}{n} \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon$$
for all $x \in [0,1]$ ■

Clearly, uniform convergence implies pointwise convergence to the same limit function. However, the converse is not true in general.

Example 2.4
Define $f_n : [0,\infty) \to \mathbb{R}$ by $f_n(x) = \frac{nx}{1+n^2x^2}$. By Example 2.1, this sequence converges pointwise to $f(x) = 0$. Let $\epsilon = \frac{1}{3}$. Show that there is no positive integer N with the property $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \geq 0$. Hence, the given sequence does not converge uniformly to $f(x)$.

Solution.
For any positive integer N and for $n \geq N$ we have
$$\left| f_n\left(\frac{1}{n}\right) - f\left(\frac{1}{n}\right)\right| = \frac{1}{2} > \epsilon \ ■$$

Exercise 2.1 below shows a sequence of continuous functions converging pointwise to a discontinuous function. That is, pointwise convergence does not preserve the property of continuity. One of the interesting features of uniform convergence is that it preserves continuity as shown in the next example.

Example 2.5
Suppose that for each $n \geq 1$ the function $f_n : D \to \mathbb{R}$ is continuous in D. Suppose that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f. Let $a \in D$.
(a) Let $\epsilon > 0$ be given. Show that there is a positive integer N such that if $n \geq N$ then $|f_n(x) - f(x)| < \frac{\epsilon}{3}$ for all $x \in D$.
(b) Show that there is a $\delta > 0$ such that for all $|x - a| < \delta$ we have $|f_N(x) - f_N(a)| < \frac{\epsilon}{3}$.
(c) Using (a) and (b) show that for $|x - a| < \delta$ we have $|f(x) - f(a)| < \epsilon$. Hence, f is continuous in D since a was arbitrary. Symbolically we write

$$\lim_{x \to a} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to a} f_n(x).$$

Solution.
(a) This follows from the definition of uniform convergence.
(b) This follows from the fact that f_N is continuous at $a \in D$.
(c) For $|x - a| < \delta$ we have $|f(x) - f(a)| = |f(a) - f_N(a) + f_N(a) - f_N(x) + f_N(x) - f(x)| \leq |f_N(a) - f(a)| + |f_N(a) - f_N(x)| + |f_N(x) - f(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$ ■

Does pointwise convergenvce preserve integration? In real analysis, it is proven that pointwise convergence does not preserve integrability. That is, the pointwise limit of a sequence of integrable functions need not be integrable. Even when a sequence of functions converges pointwise, the process of interchanging limits and integration is not true in general.
Contrary to pointwise convergence, uniform convergence preserves integration. Moreover, limits and integration can be interchanged. That is, if $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on a closed interval $[a, b]$ then

$$\lim_{n \to \infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n \to \infty} f_n(x)dx.$$

Now, what about differentiablility? Again, pointwise convergence fails in general to conserve the differentiability property. See Exercise 2.1. Does uniform convergence preserve differentiability? The answer is still no as shown in the next example.

Example 2.6
Consider the family of functions $f_n : [-1, 1]$ given by $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$.
(a) Show that f_n is differentiable for each $n \geq 1$.
(b) Show that for all $x \in [-1, 1]$ we have

$$|f_n(x) - f(x)| \leq \frac{1}{\sqrt{n}}$$

where $f(x) = |x|$. Hint: Note that $\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2} \geq \frac{1}{\sqrt{n}}$.
(c) Let $\epsilon > 0$ be given. Show that there is a positive integer N such that for $n \geq N$ we have

$$|f_n(x) - f(x)| < \epsilon \text{ for all } x \in [-1, 1].$$

Thus, $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the non-differentiable function $f(x) = |x|$.

Solution.
(a) f_n is the composition of two differentiable functions so it is differentiable with derivative

$$f_n'(x) = x\left[x^2 + \frac{1}{n}\right]^{-\frac{1}{2}}.$$

(b) We have

$$\begin{aligned}
|f_n(x) - f(x)| = &\left|\sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2}\right| = \left|\frac{(\sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2})(\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2})}{\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2}}\right| \\
= &\frac{\frac{1}{n}}{\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2}} \\
\leq &\frac{\frac{1}{n}}{\frac{1}{\sqrt{n}}} = \frac{1}{\sqrt{n}}
\end{aligned}$$

(c) Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ we can find a positive integer N such that for all $n \geq N$ we have $\frac{1}{\sqrt{n}} < \epsilon$. Now the answer to the question follows from this and part (b) ■

Even when uniform convergence occurs, the process of interchanging limits and differentiation may fail as shown in the next example.

Example 2.7
Consider the functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \frac{\sin nx}{n}$.
(a) Show that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the function $f(x) = 0$.
(b) Note that $\{f_n\}_{n=1}^{\infty}$ and f are differentiable functions. Show that

$$\lim_{n\to\infty} f_n'(x) \neq f'(x) = \left[\lim_{n\to\infty} f_n(x)\right]'.$$

That is, one cannot, in general, interchange limits and derivatives.

Solution.
(a) Let $\epsilon > 0$ be given. Let N be a positive integer such that $N > \frac{1}{\epsilon}$. Then for $n \geq N$ we have

$$|f_n(x) - f(x)| = \left|\frac{\sin nx}{n}\right| \leq \frac{1}{n} < \epsilon$$

and this is true for all $x \in \mathbb{R}$. Hence, $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the function $f(x) = 0$.
(b) We have $\lim_{n\to\infty} f_n'(\pi) = \lim_{n\to\infty} \cos n\pi = \lim_{n\to\infty}(-1)^n$ which does not converge. However, $f'(\pi) = 0$ ■

Pointwise convergence was not enough to preserve differentiability, and neither was uniform convergence by itself. Even with uniform convergence the process of interchanging limits with derivatives is not true in general. However, if we combine pointwise convergence with uniform convergence we can indeed preserve differentiability and also switch the limit process with the process of differentiation.

Theorem 2.3
Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of differentiable functions on $[a, b]$ that converges pointwise to some function f defined on $[a, b]$. If $\{f_n'\}_{n=1}^{\infty}$ converges uniformly on $[a, b]$ to a function g, then the function f is differentiable with derivative equals to g. Thus,

$$\lim_{n\to\infty} f_n'(x) = g(x) = f'(x) = \left[\lim_{n\to\infty} f_n(x)\right]'.$$

Finally, we conclude this section with the following important result that is useful when a given sequence is bounded.

Theorem 2.4
Consider a sequence $f_n : D \to \mathbb{R}$. Then this sequence converges uniformly to $f : D \to \mathbb{R}$ if and only if

$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in D\} = 0.$$

Example 2.8
Show that the sequence defined by $f_n(x) = \frac{\cos x}{n}$ converges uniformly to the zero function.

Solution.
We have

$$0 \leq \sup\{|\frac{\cos x}{n}| : x \in \mathbb{R}\} \leq \frac{1}{n}.$$

Now apply the squeeze rule for sequences we find that

$$\lim_{n\to\infty} \sup\{|\frac{\cos x}{n}| : x \in \mathbb{R}\} = 0$$

which implies that the given sequence converges uniformly to the zero function on $\mathbb{R}$ ■

Practice Problems

Exercise 2.1
Define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = x^n$. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

(a) Show that the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f.
(b) Show that the sequence $\{f_n\}_{n=1}^{\infty}$ does not converge uniformly to f. Hint: Suppose otherwise. Let $\epsilon = 0.5$ and get a contradiction by using a point $(0.5)^{\frac{1}{N}} < x < 1$.

Exercise 2.2
Consider the sequence of functions

$$f_n(x) = \frac{nx + x^2}{n^2}$$

defined for all x in $\mathbb{R}$. Show that this sequence converges pointwise to a function f to be determined.

Exercise 2.3
Consider the sequence of functions

$$f_n(x) = \frac{\sin(nx+3)}{\sqrt{n+1}}$$

defined for all x in $\mathbb{R}$. Show that this sequence converges pointwise to a function f to be determined.

Exercise 2.4
Consider the sequence of functions defined by $f_n(x) = n^2x^n$ for all $0 \le x \le 1$. Show that this sequence does not converge pointwise to any function.

Exercise 2.5
Consider the sequence of functions defined by $f_n(x) = (\cos x)^n$ for all $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Show that this sequence converges pointwise to a noncontinuous function to be determined.

Exercise 2.6
Consider the sequence of functions $f_n(x) = x - \frac{x^n}{n}$ defined on $[0,1)$.
(a) Does $\{f_n\}_{n=1}^{\infty}$ converge to some limit function? If so, find the limit function and show whether the convergence is pointwise or uniform.
(b) Does $\{f_n'\}_{n=1}^{\infty}$ converge to some limit function? If so, find the limit function and show whether the convergence is pointwise or uniform.

Exercise 2.7
Let $f_n(x) = \frac{x^n}{1+x^n}$ for $x \in [0,2]$.
(a) Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ on $[0,2]$.
(b) Does $f_n \to f$ uniformly on $[0,2]$?

Exercise 2.8
For each $n \in \mathbb{N}$ define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \frac{n+\cos x}{2n+\sin^2 x}$.
(a) Show that $f_n \to \frac{1}{2}$ uniformly.
(b) Find $\lim_{n\to\infty} \int_2^7 f_n(x)dx$.

Exercise 2.9
Show that the sequence defined by $f_n(x) = (\cos x)^n$ does not converge uniformly on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Exercise 2.10
Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions such that

$$\sup\{|f_n(x)| : 2 \le x \le 5\} \le \frac{2^n}{1+4^n}.$$

(a) Show that this sequence converges uniformly to a function f to be found.
(b) What is the value of the limit $\lim_{n\to\infty} \int_2^5 f_n(x)dx$?

Review of Some ODEs Results

Later on in this book, we will encounter problems where a given partial differential is reduced to an ordinary differential function by means of a given change of variables. Then techniques from the theory of ODE are required in solving the transformed ODE. In this chapter, we include some of the results from ODE theory that will be needed in our future discussions.

3 The Method of Integrating Factor

In this section, we discuss a technique for solving the first order linear non-homogeneous equation

$$y' + p(t)y = g(t) \tag{3.1}$$

where $p(t)$ and $g(t)$ are continuous on the open interval $a < t < b$.
Since $p(t)$ is continuous, it has an antiderivative namely $\int p(t)dt$. Let $\mu(t) = e^{\int p(t)dt}$. Multiply Equation (3.1) by $\mu(t)$ and notice that the left hand side of the resulting equation is the derivative of a product. Indeed,

$$\frac{d}{dt}(\mu(t)y) = \mu(t)g(t).$$

Integrate both sides of the last equation with respect to t to obtain

$$\mu(t)y = \int \mu(t)g(t)dt + C$$

Hence,

$$y(t) = \frac{1}{\mu(t)}\int \mu(t)g(t)dt + \frac{C}{\mu(t)}$$

or

$$y(t) = e^{-\int p(t)dt}\int e^{\int p(t)dt}g(t)dt + Ce^{-\int p(t)dt}$$

Notice that the second term of the previous expression is just the general solution for the homogeneous equation

$$y' + p(t)y = 0$$

whereas the first term is a solution to the nonhomogeneous equation. That is, the general solution to Equation (3.1) is the sum of a particular solution of the nonhomogeneous equation and the general solution of the homogeneous equation.

Example 3.1
Solve the initial value problem

$$y' - \frac{y}{t} = 4t, \quad y(1) = 5.$$

Solution.
We have $p(t) = -\frac{1}{t}$ so that $\mu(t) = \frac{1}{t}$. Multiplying the given equation by the integrating factor and using the product rule we notice that

$$\left(\frac{1}{t}y\right)' = 4.$$

Integrating with respect to t and then solving for y we find that the general solution is given by

$$y(t) = t\int 4dt + Ct = 4t^2 + Ct.$$

Since $y(1) = 5$, we find $C = 1$ and hence the unique solution to the IVP is $y(t) = 4t^2 + t, \ 0 < t < \infty$ ■

Example 3.2
Find the general solution to the equation

$$y' + \frac{2}{t}y = \ln t, \ t > 0.$$

Solution.
The integrating factor is $\mu(t) = e^{\int \frac{2}{t}dt} = t^2$. Multiplying the given equation by t^2 to obtain

$$(t^2y)' = t^2 \ln t.$$

Integrating with respect to t we find

$$t^2 y = \int t^2 \ln t dt + C.$$

The integral on the right-hand side is evaluated using integration by parts with $u = \ln t, dv = t^2 dt, du = \frac{dt}{t}, v = \frac{t^3}{3}$ obtaining

$$t^2 y = \frac{t^3}{3} \ln t - \frac{t^3}{9} + C$$

Thus,

$$y = \frac{t}{3} \ln t - \frac{t}{9} + \frac{C}{t^2} \blacksquare$$

Practice Problems

Exercise 3.1
Solve the IVP: $y' + 2ty = t, \quad y(0) = 0.$

Exercise 3.2
Find the general solution: $y' + 3y = t + e^{-2t}$.

Exercise 3.3
Find the general solution: $y' + \frac{1}{t}y = 3\cos t, \; t > 0.$

Exercise 3.4
Find the general solution: $y' + 2y = \cos(3t)$.

Exercise 3.5
Find the general solution: $y' + (\cos t)y = -3\cos t$.

Exercise 3.6
Given that the solution to the IVP $ty' + 4y = \alpha t^2, \; y(1) = -\frac{1}{3}$ exists on the interval $-\infty < t < \infty$. What is the value of the constant α?

Exercise 3.7
Suppose that $y(t) = Ce^{-2t} + t + 1$ is the general solution to the equation $y' + p(t)y = g(t)$. Determine the functions $p(t)$ and $g(t)$.

Exercise 3.8
Suppose that $y(t) = -2e^{-t} + e^t + \sin t$ is the unique solution to the IVP $y' + y = g(t), \; y(0) = y_0$. Determine the constant y_0 and the function $g(t)$.

Exercise 3.9
Find the value (if any) of the unique solution to the IVP $y' + (1 + \cos t)y = 1 + \cos t, \; y(0) = 3$ in the long run?

Exercise 3.10
Solve

$$au_x + bu_y + cu = 0$$

by using the change of variables $s = ax + by$ and $t = bx - ay$.

Sample Exam Questions

Exercise 3.11
Solve the initial value problem $ty' = y + t, \quad y(1) = 7$.

Exercise 3.12
Show that if a and λ are positive constants, and b is any real number, then every solution of the equation

$$y' + ay = be^{-\lambda t}$$

has the property that $y \to 0$ as $t \to \infty$. Hint: Consider the cases $a = \lambda$ and $a \neq \lambda$ separately.

Exercise 3.13
Solve the initial-value problem $y' + y = e^t y^2, y(0) = 1$ using the substitution $u(t) = \frac{1}{y(t)}$

Exercise 3.14
Solve the initial-value problem $ty' + 2y = t^2 - t + 1, \quad y(1) = \frac{1}{2}$

Exercise 3.15
Solve $y' - \frac{1}{t}y = \sin t, \quad y(1) = 3$. Express your answer in terms of the **sine integral**, $Si(t) = \int_0^t \frac{\sin s}{s} ds$.

4 The Method of Separation of Variables for ODEs

The method of separation of variables that you have seen in the theory of ordinary differential equations has an analogue in the theory of partial differential equations (Section 17). In this section, we review the method for ordinary differentiable equations.

A first order differential equation is **separable** if it can be written with one variable only on the left and the other variable only on the right:

$$f(y)y' = g(t)$$

To solve this equation, we proceed as follows. Let $F(t)$ be an antiderivative of $f(t)$ and $G(t)$ be an antiderivative of $g(t)$. Then by the Chain Rule

$$\frac{d}{dt}F(y) = \frac{dF}{dy}\frac{dy}{dt} = f(y)y'$$

Thus,

$$f(y)y' - g(t) = \frac{d}{dt}F(y) - \frac{d}{dt}G(t) = \frac{d}{dt}[F(y) - G(t)] = 0$$

It follows that

$$F(y) - G(t) = C$$

which is equivalent to

$$\int f(y)y'dt = \int g(t)dt + C$$

As you can see, the result is generally an implicit equation involving a function of y and a function of t. It may or may not be possible to solve this to get y explicitly as a function of t. For an initial value problem, substitute the values of t and y by t_0 and y_0 to get the value of C.

Remark 4.2

If F is a differentiable function of y and y is a differentiable function of t and both F and y are given then the chain rule allows us to find $\frac{dF}{dt}$ given by

$$\frac{dF}{dt} = \frac{dF}{dy} \cdot \frac{dy}{dt}$$

For separable equations, we are given $f(y)y' = \frac{dF}{dt}$ and we are asked to find $F(y)$. This process is referred to as "reversing the chain rule."

Example 4.1
Solve the initial value problem $y' = 6ty^2, \quad y(1) = \frac{1}{25}$.

Solution.
Separating the variables and integrating both sides we obtain

$$\int \frac{y'}{y^2} dt = \int 6tdt$$

or

$$-\int \frac{d}{dt}\left(\frac{1}{y}\right) dt = \int 6tdt$$

Thus,

$$-\frac{1}{y(t)} = 3t^2 + C$$

Since $y(1) = \frac{1}{25}$, we find $C = -28$. The unique solution to the IVP is then given explicitly by

$$y(t) = \frac{1}{28 - 3t^2}$$

Example 4.2
Solve the IVP $yy' = 4\sin(2t), \quad y(0) = 1$.

Solution.
This is a separable differential equation. Integrating both sides we find

$$\int \frac{d}{dt}\left(\frac{y^2}{2}\right) dt = 4\int \sin(2t)dt$$

Thus,

$$y^2 = -4\cos(2t) + C$$

Since $y(0) = 1$, we find $C = 5$. Now, solving explicitly for $y(t)$ we find

$$y(t) = \pm\sqrt{-4\cos t + 5}$$

Since $y(0) = 1$, we have $y(t) = \sqrt{-4\cos t + 5}$. The interval of existence of the solution is the interval $-\infty < t < \infty$ ■

Practice Problems

Exercise 4.1
Solve the (separable) differential equation

$$y' = te^{t^2 - \ln y^2}.$$

Exercise 4.2
Solve the (separable) differential equation

$$y' = \frac{t^2 y - 4y}{t + 2}.$$

Exercise 4.3
Solve the (separable) differential equation

$$ty' = 2(y - 4).$$

Exercise 4.4
Solve the (separable) differential equation

$$y' = 2y(2 - y).$$

Exercise 4.5
Solve the IVP

$$y' = \frac{4\sin(2t)}{y}, \quad y(0) = 1.$$

Exercise 4.6
Solve the IVP:

$$yy' = \sin t, \quad y(\frac{\pi}{2}) = -2.$$

Exercise 4.7
Solve the IVP:

$$y' + y + 1 = 0, \quad y(1) = 0.$$

Exercise 4.8
Solve the IVP:

$$y' - ty^3 = 0, \quad y(0) = 2.$$

Exercise 4.9
Solve the IVP:
$$y' = 1 + y^2, \quad y(\frac{\pi}{4}) = -1.$$

Exercise 4.10
Solve the IVP:
$$y' = t - ty^2, \quad y(0) = \frac{1}{2}.$$

Sample Exam Questions

Exercise 4.11
For what values of the constants α, y_0, and integer n is the function $y(t) = (4+t)^{-\frac{1}{2}}$ a solution of the initial value problem?

$$y' + \alpha y^n = 0, \quad y(0) = y_0.$$

Exercise 4.12
Solve the equation $3u_y + u_{xy} = 0$ by using the substitution $v = u_y$.

Exercise 4.13
Solve the IVP

$$(2y - \sin y)y' = \sin t - t, \quad y(0) = 0.$$

Exercise 4.14
State an initial value problem, with initial condition imposed at $t_0 = 2$, having implicit solution $y^3 + t^2 + \sin y = 4$.

Exercise 4.15
Can the differential equation

$$\frac{dy}{dx} = x^2 - xy$$

be solved by the method of separation of variables? Explain.

5 Second Order Linear ODEs

When solving second order partial differential equations such as the heat, wave, and Laplace's equations using the method of separation of variables for PDEs one ends up confronting second order linear ODEs. Thus, it is deemed necessary to review some of the techniques used in solving second order linear ordinary differential equations which we do in this section.

We start first by considering the second order linear ODE with constant coefficients given by

$$ay'' + by' + cy = 0 \tag{5.1}$$

where a, b and c are constants with $a \neq 0$.

Notice first that for $b = 0$ and $c \neq 0$ the function y'' is a constant multiple of y. So it makes sense to look for a function with such property. One such function is $y(t) = e^{rt}$. Substituting this function into (5.1) leads to

$$ay'' + by' + cy = ar^2e^{rt} + bre^{rt} + ce^{rt} = (ar^2 + br + c)e^{rt} = 0$$

Since $e^{rt} > 0$ for all t, the previous equation leads to

$$ar^2 + br + c = 0 \tag{5.2}$$

Thus, a function $y(t) = e^{rt}$ is a solution to (5.1) when r satisfies equation (5.2). We call (5.2) the **characteristic equation** for (5.1) and the polynomial $C(r) = ar^2 + br + c$ is called the **characteristic polynomial**.

The characteristic equation is a quadratic equation. Thus, this equation can have two distinct real solutions, two equal solutions, or two conjugate complex solutions depending on the sign of the expression $b^2 - 4ac$. Hence, we consider the following three cases:

Case 1: $b^2 - 4ac > 0$.

In this case, equation (5.2) have two distinct real roots $r_1 = \frac{-b-\sqrt{b^2-4ac}}{4a}$ and $r_2 = \frac{-b+\sqrt{b^2-4ac}}{4a}$. The general solution to (5.1) is given by

$$y(t) = c_1e^{r_1t} + c_2e^{r_2t}$$

where c_1 and c_2 are arbitrary constants.

Example 5.1

Solve the initial value problem

$$y'' - y' - 6y = 0,\ y(0) = 1,\ y'(0) = 2.$$

Describe the behavior of the solution $y(t)$ as $t \to -\infty$ and $t \to \infty$.

Solution.
The characteristic polynomial is $C(r) = r^2 - r - 6 = (r-3)(r+2)$ so that the characteristic equation $r^2 - r - 6 = 0$ has the solutions $r_1 = 3$ and $r_2 = -2$. The general solution is then given by

$$y(t) = c_1 e^{3t} + c_2 e^{-2t}.$$

Taking the derivative to obtain

$$y'(t) = 3c_1 e^{3t} - 2c_2 e^{-2t}.$$

The conditions $y(0) = 1$ and $y'(0) = 2$ lead to the system

$$\begin{array}{rcl} c_1 + c_2 & = & 1 \\ 3c_1 - 2c_2 & = & 2. \end{array}$$

Solving this system by the method of elimination we find $c_1 = \frac{4}{5}$ and $c_2 = \frac{1}{5}$. Hence, the unique solution to the initial value problem is

$$y(t) = \frac{1}{5}(4e^{3t} + e^{-2t}).$$

As $t \to -\infty$, $e^{3t} \to 0$ and $e^{-2t} \to \infty$. Thus, $y(t) \to \infty$. Similarly, $y(t) \to \infty$ as $t \to \infty$ ■

Case 2: $b^2 - 4ac = 0$.
In this case, the characteristic equation has the single root $r = -\frac{b}{2a}$. The general solution to (5.1) is given by

$$y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 t e^{-\frac{b}{2a}t}$$

where c_1 and c_2 are arbitrary constants.

Example 5.2
Solve the initial value problem: $y'' + 2y' + y = 0,\ y(0) = 1,\ y_1'(0) = -1.$

Solution.
The characteristic equation $r^2 + 2r + 1 = 0$ has a repeated root: $r_1 = r_2 = -1$. Thus, the general solution is given by

$$y(t) = c_1 e^{-t} + c_2 t e^{-t}.$$

The two conditions $y(0) = 1$ and $y'(0) = -1$ lead to $c_1 = 1$ and $c_2 = 0$. Hence, the unique solution is $y(t) = e^{-t}$ ■

Case 3: $b^2 - 4ac < 0$.
In this case, the complex roots of equation (5.1) are given by

$$r_{1,2} = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}$$

where $i = \sqrt{-1}$. The general solution is given by

$$y(t) = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$$

where $\alpha = -\frac{b}{2a}$, $\beta = \frac{\sqrt{4ac-b^2}}{2a}$, and c_1 and c_2 are real numbers.

Example 5.3
Solve the initial value problem

$$y'' - 10y' + 29y = 0,\ y(0) = 1,\ y'(0) = 3.$$

Solution.
The characteristic equation $r^2 - 10r + 29 = 0$ has the complex roots $r_{1,2} = 5 \pm 2i$. Thus, the general solution is given by the expression

$$y(t) = e^{5t}(c_1 \cos 2t + c_2 \sin 2t).$$

Finding y' we obtain

$$y'(t) = e^{5t}[(5c_1 + 2c_2) \cos 2t + (5c_2 - 2c_1) \sin 2t].$$

The initial conditions yield $c_1 = 1$ and $c_2 = -1$. Thus, the unique solution to the initial value problem is

$$y(t) = e^{5t}(\cos 2t - \sin 2t) \ ■$$

An Eigenvalue Problem
Consider the question of finding a nontrivial twice differentiable function u satisfying the ordinary differential equation

$$\frac{d^2u}{dx^2} = \lambda u,\ 0 < x < 1.$$

subject to the boundary conditions $u(0) = u(1) = 0$. This problem is referred to as the eigenvalue problem for the following reason: Define the function $L \equiv \frac{d^2}{dx^2}$. Then the given equation can be written as $Lu = \lambda u$. In linear algebra, λ is called an **eigenvalue** of L with corresponding **eigenvector** u. Different solutions to the eigenvalue problem are obtained depending on the sign of λ. Suppose first that $\lambda = 0$. Then $u(x) = C_1x + C_2$ for arbitrary constants C_1 and C_2. Using the boundary conditions we find $C_1 = C_2 = 0$. Hence, $u \equiv 0$.
Suppose that $\lambda > 0$. Then $u(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$. Again, the boundary conditions imply that $u \equiv 0$.
Now, suppose that $\lambda < 0$. Then $u(x) = A\cos\sqrt{-\lambda}x + B\sin\sqrt{-\lambda}x$. Using the condition $u(0) = 0$ to obtain $A = 0$. Using the condition $u(1) = 0$ and assuming we are looking for non-trivial solution u we expect to have $\sin\sqrt{-\lambda} = 0$. This happens when $\lambda = \lambda_n = -(n\pi)^2$ where $n \in \mathbb{N}$. We call λ_n an eigenvalue with corresponding **eigenfunction** $u_n(x) = \sin n\pi x$.
Finally, using the principle of superposition we find that the general solution to the eigenvalue problem is given by

$$u(x) = \sum_{n=1}^{\infty} A_n \sin n\pi x$$

where the convergence is pointwise convergence (See Section 2).

Euler Equations
A second order linear differential equations of the form

$$ax^2y'' + bxy' + cy = 0$$

where a, b, c are constants is called an **Euler equation.**
To solve Euler equation, one starts with solutions of the form $y = x^r$ (with $x > 0$) where r is to be determined. Plugging this into the differential equation to get

$$\begin{aligned} ax^2r(r-1)x^{r-2} + bxrx^{r-1} + cx^r =& 0 \\ (ar^2 - ar + br + c)x^r =& 0 \\ ar^2 - (a-b)r + c =& 0 \end{aligned}$$

This last equation is a quadratic equation in r and so we will have three cases to look at : Real distinct roots, double roots, and complex conjugate roots.

If the quadratic equation has two distinct real roots r_1 and r_2 then the general solution is given by

$$y(x) = Ax^{r_1} + Bx^{r_2}.$$

If the quadratic equation has two equal roots $r_1 = r_2 = r$ then the general solution is given by

$$y(x) = x^r(A + B\ln x).$$

If the quadratic equation has two complex conjugate solutions $r_{1,2} = \alpha \pm i\beta$ then the general solution is given by

$$y(x) = x^\alpha(A\cos(\beta\ln x) + B\sin(\beta\ln x)).$$

Example 5.4
Solve the initial value problem

$$2x^2y'' + 3xy' - 15y = 0$$

$$y(1) = 0,\ y'(1) = 1.$$

Solution.
Letting $y = x^r$ we obtain the quadratic equation $2r^2 + r - 15 = 0$ whose roots are $r_1 = \frac{5}{2}$ and $r_2 = -3$. Hence, the general solution is given by

$$y(x) = Ax^{\frac{5}{2}} + Bx^{-3}.$$

The condition $y(1) = 0$ implies $A + B = 0$. The condition $y'(1) = 1$ implies $\frac{5}{2}A - 3B = 1$. Solving this system of two unknowns we find $A = \frac{2}{11}$ and $B = -\frac{2}{11}$. Hence, the unique solution is given by

$$y = \frac{2}{11}x^{\frac{5}{2}} - \frac{2}{11}x^{-3} \blacksquare$$

Second Order Linear nonhomogeneous ODE: The Method of Undetermined Coefficients
We consider the nonhomogeneous second order

$$ay'' + by' + cy = g(t),\ a < t < b.$$

We know that the general solution has the structure

$$y(t) = c_1y_1(t) + c_2y_2(t) + y_p(t)$$

where $y_p(t)$ is a particular solution to the nonhomogeneous equation. We will write $y(t) = y_h(t) + y_p(t)$ where $y_h(t) = c_1y_1(t) + c_2y_2(t)$.
One way to finding y_p is by using the method of undetermined coefficients. The idea behind the method of undetermined coefficients is to look for $y_p(t)$ which is of a form like that of $g(t)$. This is possible only for special functions $g(t)$, but these special cases arise quite frequently in applications.
We will assume that $g(t)$ being simple means it is some combination of terms like $e^{rt}, \cos(kt), \sin(kt)$, and polynomials $a_nt^n + a_{n-1}t^{n-1} + \cdots a_1t + a_0$. Based on those terms we will put together a candidate y_p that has some constants in it we need to solve for: Those are the undetermined coefficients this method is named for.
In the following table we list examples of $g(t)$ along with the corresponding form of the particular solution.

Form of $g(t)$	Form of $y_p(t)$
$a_nt^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0$	$t^r[A_nt^n + A_{n-1}t^{n-1} + \cdots + A_1t + A_0$
$[a_nt^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0]e^{\alpha t}$	$t^r[A_nt^n + A_{n-1}t^{n-1} + \cdots + A_1t + A_0]e^{\alpha t}$
$[a_nt^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0]\cos\alpha t$	$t^r[(A_nt^n + A_{n-1}t^{n-1} + \cdots + A_1t + A_0)\cos\alpha t$
or	$+(B_nt^n + B_{n-1}t^{n-1} + \cdots + B_1t + B_0)\sin\alpha t]$
$[a_nt^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0]\sin\alpha t$	
$e^{\alpha t}[a_nt^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0]\sin\beta t$	$t^r[(A_nt^n + A_{n-1}t^{n-1} + \cdots + A_1t + A_0)e^{\alpha t}\cos\beta t$
or	$+(B_nt^n + B_{n-1}t^{n-1} + \cdots + B_1t + B_0)e^{\alpha t}\sin\beta t]$
$e^{\alpha t}[a_nt^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0]\cos\beta t$	

The number r is chosen to be the smallest nonnegative integer such that no term in the assumed form is a solution of the homogeneous equation $ay'' + by' + cy = 0$. The value of r will be 0, 1, or 2.

Example 5.5
List an appropriate form for a particular solution of
(a) $y'' + 4y = t^2e^{3t}$.
(b) $y'' + 4y = te^{2t}\cos t$.
(c) $y'' + 4y = 2t^2 + 5\sin 2t + e^{3t}$.
(d) $y'' + 4y = t^2\cos 2t$.

Solution.
The general solution to the homogeneous equation is $y_h(t) = c_1\cos 2t + c_2\sin 2t$.
(a) For $g(t) = t^2e^{3t}$, an appropriate particular solution has the form $y_p(t) =$

$t^r(A_2t^2 + A_1t + A_0)e^{3t}$. We take $r = 0$ since no term in the assumed form for y_p is present in the expression of $y_h(t)$. Thus

$$y_p(t) = (A_2t^2 + A_1t + A_0)e^{3t}$$

(b) An appropriate form is

$$y_p(t) = t^r[(A_1t + A_0)e^{2t}\cos t + (B_1t + B_0)e^{2t}\sin t]$$

We take $r = 0$ since no term in the assumed form for y_p is present in the expression of $y_h(t)$. Thus

$$y_p(t) = (A_1t + A_0)e^{2t}\cos t + (B_1t + B_0)e^{2t}\sin t$$

(c)

$$y_p(t) = A_2t^2 + A_1t + A_0 + B_0t\cos 2t + C_0t\sin 2t + D_0e^{3t}$$

(d)

$$y_p(t) = t(A_2t^2 + A_1t + A_0)\cos 2t + t(B_2t^2 + B_1t + B_0)\sin 2t \blacksquare$$

Example 5.6
Find the general solution of

$$y'' - 2y' - 3y = 4t - 5 + 6te^{2t}$$

Solution.
The characteristic equation of the homogeneous equation is $r^2 - 2r - 3 = 0$ with roots $r_1 = -1$ and $r_2 = 3$. Thus,

$$y_h(t) = c_1e^{-t} + c_2e^{3t}$$

A guess for the particular solution is $y_p(t) = At + B + Cte^{2t} + De^{2t}$. Inserting this into the differential equation leads to

$$-3At - 2A - 3B - 3Cte^{2t} + (2C - 3D)e^{2t} = 4t - 5 + 6te^{2t}$$

From this identity we obtain $-3A = 4$ so that $A = -\frac{4}{3}$. Also, $-2A - 3B = -5$ so that $B = \frac{23}{9}$. Since $-3C = 6$ we find $C = -2$. From $2C - 3D = 0$ we find $D = -\frac{4}{3}$. It follows that

$$y(t) = c_1e^{-t} + c_2e^{3t} - \frac{4}{3}t + \frac{23}{9} - \left(2t + \frac{4}{3}\right)e^{2t} \blacksquare$$

Practice Problems

Exercise 5.1
Solve the initial value problem

$$y'' - 4y' + 3y = 0,\ y(0) = -1,\ y'(0) = 1$$

Describe the behavior of the solution $y(t)$ as $t \to -\infty$ and $t \to \infty$.

Exercise 5.2
Solve the initial value problem

$$y'' + 4y' + 2y = 0,\ y(0) = 0,\ y'(0) = 4$$

Describe the behavior of the solution $y(t)$ as $t \to -\infty$ and $t \to \infty$.

Exercise 5.3
Solve the initial value problem

$$2y'' - y = 0,\ y(0) = -2,\ y'(0) = \sqrt{2}$$

Describe the behavior of the solution $y(t)$ as $t \to -\infty$ and $t \to \infty$.

Exercise 5.4
Find a homogeneous second-order linear ordinary differential equation whose general solution is $y(t) = c_1e^{2t} + c_2e^{-t}$.

Exercise 5.5
Solve the IVP

$$9y'' - 6y' + y = 0,\ y(3) = -2,\ y'(3) = -\frac{5}{3}$$

Exercise 5.6
Solve the IVP

$$25y'' + 20y' + 4y = 0,\ y(5) = 4e^{-2},\ y'(5) = -\frac{3}{5}e^{-2}$$

Exercise 5.7
The graph of a solution $y(t)$ of the differential equation $4y'' + 4y' + y = 0$ passes through the points $(1, e^{-\frac{1}{2}})$ and $(2, 0)$. Determine $y(0)$ and $y'(0)$.

Exercise 5.8
Find the general solution of $y'' - 6y' + 9y = 0$.

Exercise 5.9
Solve the IVP

$$y'' + 2y' + 2y = 0,\ y(0) = 3,\ y'(0) = -1$$

Exercise 5.10
Solve the IVP

$$2y'' - 2y' + y = 0,\ y(-\pi) = 1,\ y'(-\pi) = -1$$

Exercise 5.11
Find the general solution of

$$y'' - y' + y = 2\sin 3t$$

Exercise 5.12
Find the general solution of

$$y'' + 4y' - 2y = 2t^2 - 3t + 6$$

Sample Exam Questions

Exercise 5.13
Find the general solution to the following differential equation.

$$x^2y'' - 7xy' + 16y = 0.$$

Exercise 5.14
Find the general solution to the following differential equation.

$$x^2y'' + 3xy' + 4y = 0.$$

Exercise 5.15
Consider the differential equation

$$\frac{d^2y}{dx^2} + \lambda y = 0.$$

Determine the eigenvalues λ and the corresponding eigenfunctions if y satisfies the following boundary conditions:
(a) $y(0) = y(\pi) = 0$
(b) $y(0) = y'(L) = 0$
(c) $y'(0) = y(1) = 0.$

Exercise 5.16
Show by direct computation that the eigenvalue problems

$$(ky'(x))' + \lambda y(x) = 0,\ k > 0$$

with the following boundary conditions have no negative eigenvalues λ :
(a) $y(0) = y(L) = 0$
(b) $y'(0) = y'(L) = 0$
(c) $y(L) = y(-L), \quad y'(L) = y'(-L).$

Exercise 5.17
Solve the initial-value problem: $2y'' + 5y' - 3y = 0,\ y(0) = 2,\ y'(0) = 1.$

Exercise 5.18
Find the general solution of

$$y'' - y' = 5e^t - \sin 2t$$

Exercise 5.19
Solve using undetermined coefficients:

$$y'' + y' - 2y = t + \sin 2t, y(0) = 1, y'(0) = 0$$

Introduction to PDEs

Many fields in engineering and the physical sciences require the study of ODE and PDE. Examples of those fields are acoustics, aerodynamics, elasticity, electrodynamics, fluid dynamics, geophysics (seismic wave propagation), heat transfer, meteorology, oceanography, optics, petroleum engineering, plasma physics (ionized liquids and gases), quantum mechanics.
So the study of partial differential equation is of great importance to the above mentioned fields. The purpose of this chapter is to introduce the reader to the basic terms of partial differential equations.

6 The Basic Concepts

The goal of this section is to introduce the reader to the basic concepts and notations that will be used in the remainder of this book.
A **differential equation** is an equation that involves an unknown scalar function (the dependent variable) and one or more of its derivatives. For example,

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 3y = -3 \tag{6.1}$$

or

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + u = 0. \tag{6.2}$$

If the unknown function is a function in one single variable then the differential equation is called an **ordinary differential equation.** An example of an ordinary differential equation is Equation (6.1). In contrast, when the unknown function is a function of two or more independent variables then the differential equation is called a **partial differential equation**, in short PDE. Equation (6.2) is an example of a partial differential equation. In this book we will be focusing on partial differential equations.

Example 6.1
Identify which variables are dependent variable or independent variable(s) for the following differential equations.
(a) $\frac{d^4y}{dx^4} - x^2 + y = 0$
(b) $u_{tt} + xu_{tx} = 0.$
(c) $x\frac{dx}{dt} = 4.$
(d) $\frac{\partial y}{\partial u} - 4\frac{\partial y}{\partial v} = u + 3y.$

Solution.
(a) Independent variable is x and the dependent variable is y.
(b) Independent variables are x and t and the dependent variable is u.
(c) Independent variable is t and the dependent variable is x.
(d) Independent variables are u and v and the dependent variable is y ■

Example 6.2
Classify the following as either ODE or PDE.
(a) $u_t = c^2u_{xx}$.
(b) $y'' - 4y' + 5y = 0.$
(c) $u_t + cu_x = 5.$

Solution.
(a) PDE (b) ODE (c) PDE ■

The **order** of a partial differential equation is the highest order derivative occurring in the equation. Thus, (6.2) is a second order partial differential equation.

Example 6.3
Find the order of each of the following partial differential equations:
(a) $xu_x + yu_y = x^2 + y^2$
(b) $uu_x + u_y = 2$
(c) $u_{tt} - c^2u_{xx} = f(x,t)$
(d) $u_t + uu_x + u_{xxx} = 0$
(e) $u_{tt} + u_{xxxx} = 0.$

Solution.
(a) First order (b) First order (c) Second order (d) Third order (e) Fourth order ■

A partial differential equation is called **linear** if it is linear in the unknown function and all its derivatives with coefficients depend only on the independent variables. For example, a first order linear partial differential equation has the form

$$A(x,y)u_x + B(x,y)u_y + C(x,y)u = D(x,y)$$

whereas a second order linear partial differential equation has the form

$$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y + F(x,y)u = G(x,y).$$

A partial differential equation is called **quasi-linear** if the highest-order derivatives which appear in the equation are of degree 1(regardless of the manner in which lower-order derivatives and unknown functions occur in the equation). For example, a first order quasi-linear partial differential equation has the form

$$A(x,y,u)u_x + B(x,y,u)u_y = C(x,y,u)$$

whereas a second order quasi-linear partial differential equation has the form

$$A(x,y,u,u_x,u_y)u_{xx} + B(x,y,u,u_x,u_y)u_{xy} + C(x,y,u,u_x,u_y)u_{yy} = D(x,y,u,u_x,u_y).$$

A partial differential equation is **semi-linear** if it is quasi-linear and the coefficients of the highest-order derivatives are functions of independent variables only. For example, a first order semi-linear partial differential equation has the form

$$A(x,y)u_x + B(x,y)u_y = C(x,y,u)$$

whereas a second order semi-linear partial differential equation has the form

$$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} = D(x,y,u,u_x,u_y).$$

Note that linear and semi-linear partial differential equations are special cases of quasi-linear equations.
A partial differential equation that is not linear is called **nonlinear**. For example, $u_x^2 + 2u_{xy} = 0$.
As for ODEs, linear PDEs are usually simpler to analyze/solve than nonlinear PDEs.

Example 6.4
Determine whether the given PDE is linear, quasilinear, semilinear, or nonlinear:
(a) $xu_x + yu_y = x^2 + y^2$
(b) $uu_x + u_y = 2$
(c) $u_{tt} - c^2u_{xx} = f(x,t)$
(d) $u_t + uu_x + u_{xxx} = 0$
(e) $u_{tt}^2 + u_{xxxx} = 0$.

Solution.
(a) Linear, quasilinear, semilinear.
(b) Quasilinear, nonlinear.
(c) Linear, quasilinear, semilinear.
(d) Quasilinear, semilinear, nonlinear.
(e) Quasilinear, semilinear, nonlinear ■

A more precise definition of a linear differential equation begins with the concept of a **linear differential operator** L. The operator L is assembled by summing the basic partial derivative operators, with coefficients depending on the independent variables. The operator acts on sufficiently smooth functions depending on the relevant independent variables. **Linearity** imposes two key requirements:

$$L[u+v] = L[u] + L[v] \text{ and } L[\alpha u] = \alpha L[u],$$

for any two (sufficiently smooth) functions u, v and any constant α.

Example 6.5
Define a linear differential operator for the PDE

$$u_t = c^2 u_{xx}.$$

Solution.
Let $L[u] = u_t - c^2u_{xx}$. Then one can easily check that $L[u+v] = L[u] + L[v]$ and $L[\alpha u] = \alpha L[u]$ ■

A linear partial differential equation is called **homogeneous** if every term of the equation involves the unknown function or its partial derivatives. A linear partial differential equation that is not homogeneous is called **nonhomogeneous.** In this case, there is a term in the equation that involves only

the independent variables.
A homogeneous linear partial differential equation has the form

$$L[u] = 0$$

where L is a linear differential operator.

Example 6.6
Determine whether the equation is homogeneous or nonhomogeneous:
(a) $xu_x + yu_y = x^2 + y^2$.
(b) $u_{tt} = c^2 u_{xx}$.
(c) $u_{xx} + u_{yy} = 0$.

Solution.
(a) Nonhomogeneous because of $x^2 + y^2$.
(b) Homogeneous.
(c) Homogeneous ■

Finally, we shall be employing a few basic notational conventions regarding the variables that appear in our differential equations. We always use t to denote time, while x, y, z will represent (Cartesian) space coordinates. Polar coordinates r, θ will also be used when needed, and our notational conventions appear at the appropriate places in the exposition.
An **equilibrium equation** models an unchanging physical system, and so only involves the space variables. The time variable t appears when modeling *dynamical* , meaning time-varying, processes. Both time and space coordinates are independent variables.

Practice Problems

Exercise 6.1
Classify the following equations as either ODE or PDE.

(a) $(y''')^4 + \frac{t^2}{(y')^2+4} = 0$

(b) $\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{y-x}{y+x}$

(c) $y'' - 4y = 0$

Exercise 6.2
Write the equation

$$u_{xx} + 2u_{xy} + u_{yy} = 0$$

in the coordinates $s = x,\ t = x - y.$

Exercise 6.3
Write the equation

$$u_{xx} - 2u_{xy} + 5u_{yy} = 0$$

in the coordinates $s = x + y,\ t = 2x.$

Exercise 6.4
For each of the following PDEs, state its order and whether it is linear or nonlinear. If it is linear, also state whether it is homogeneous or nonhomogeneous:
(a) $uu_x + x^2u_{yyy} + \sin x = 0$
(b) $u_x + e^{x^2}u_y = 0$
(c) $u_{tt} + (\sin y)u_{yy} - e^t \cos y = 0.$

Exercise 6.5
For each of the following PDEs, determine its order and whether it is linear or not. For linear PDEs, state also whether the equation is homogeneous or not. For nonlinear PDEs, circle all term(s) that are not linear.
(a) $x^2u_{xx} + e^xu = xu_{xyy}$
(b) $e^yu_{xxx} + e^xu = -\sin y + 10xu_y$
(c) $y^2u_{xx} + e^xuu_x = 2xu_y + u$
(d) $u_xu_{xxy} + e^xuu_y = 5x^2u_x$
(e) $u_t = k^2(u_{xx} + u_{yy}) + f(x, y, t).$

Exercise 6.6
Which of the following PDEs are linear?
(a) **Laplace's equation:** $u_{xx} + u_{yy} = 0$.
(b) **Convection (transport) equation:** $u_t + cu_x = 0$.
(c) **Minimal surface equation:** $(1+Z_y^2)Z_{xx} - 2Z_xZ_yZ_{xy} + (1+Z_x^2)Z_{yy} = 0$.
(d) **Korteweg-Vries equation:** $u_t + 6uu_x = u_{xxx}$.

Exercise 6.7
Classify the following differential equations as ODEs or PDEs, linear or nonlinear, and determine their order. For the linear equations, determine whether or not they are homogeneous.
(a) The **diffusion equation** for $u(x,t)$:

$$u_t = ku_{xx}.$$

(b) The **wave equation** for $w(x,t)$:

$$w_{tt} = c^2 w_{xx}.$$

(c) The **thin film equation** for $h(x,t)$:

$$h_t = -(hh_{xxx})_x.$$

(d) The **forced harmonic oscillator** for $y(t)$:

$$y_{tt} + \omega^2 y = F\cos(\omega t).$$

(e) The **Poisson Equation** for the electric potential $\Phi(x,y,z)$:

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 4\pi\rho(x,y,z).$$

where $\rho(x,y,z)$ is a known charge density.
(f) **Burger's equation** for $h(x,t)$:

$$h_t + hh_x = \nu h_{xx}.$$

Exercise 6.8
Write down the general form of a linear second order differential equation of a function in three variables.

Exercise 6.9
Give the orders of the following PDEs, and classify them as linear or nonlinear. If the PDE is linear, specify whether it is homogeneous or nonhomogeneous.
(a) $x^2u_{xxy} + y^2u_{yy} - \log(1 + y^2)u = 0$
(b) $u_x + u^3 = 1$
(c) $u_{xxyy} + e^xu_x = y$
(d) $uu_{xx} + u_{yy} - u = 0$
(e) $u_{xx} + u_t = 3u.$

Exercise 6.10
Consider the second-order PDE

$$u_{xx} + 4u_{xy} + 4u_{yy} = 0.$$

Use the change of variables $v(x, y) = y - 2x$ and $w(x, y) = x$ to show that $u_{ww} = 0$.

Sample Exam Questions

Exercise 6.11
Write the one dimensional wave equation $u_{tt} = c^2 u_{xx}$ in the coordinates $v = x + ct$ and $w = x - ct$.

Exercise 6.12
Write the PDE

$$u_{xx} + 2u_{xy} - 3u_{yy} = 0$$

in the coordinates $v(x, y) = y - 3x$ and $w(x, y) = x + y$.

Exercise 6.13
Write the PDE

$$au_x + bu_y = 0$$

in the coordinates $s(x, y) = ax + by$ and $t(x, y) = bx - ay$. Assume $a^2 + b^2 > 0$.

Exercise 6.14
Write the PDE

$$u_x + u_y = 1$$

in the coordinates $s = x + y$ and $t = x - y$.

Exercise 6.15
Write the PDE

$$au_t + bu_x = u, \quad a, b \neq 0$$

in the coordinates $v = ax - bt$ and $w = \frac{1}{a}t$.

7 Solutions and Related Topics

By a **classical solution** or **strong solution** to a partial differential equation we mean a function that satisfies the equation. To **solve** a PDE is to find all its classical solutions. In the case of only two independent variables x and y, a solution $u(x, y)$ is visualized geometrically as a surface, called a **solution surface** or an **integral surface** in the (x, y, u) space.
A formula that expresses all the solutions of a PDE is called the **general solution** of the equation.

Example 7.1
Show that $u(x, t) = e^{-\lambda^2\alpha^2 t}(\cos\lambda x - \sin\lambda x)$ is a solution to the equation $u_t - \alpha^2 u_{xx} = 0$.

Solution.
Since

$$\begin{aligned} u_t - \alpha^2 u_{xx} =& - \lambda^2\alpha^2 e^{-\lambda^2\alpha^2 t}(\cos\lambda x \\ & - \sin\lambda x) - \alpha^2 e^{-\lambda^2\alpha^2 t}(-\lambda^2\cos\lambda x + \lambda^2\sin\lambda x) = 0 \end{aligned}$$

the given function is a solution to the given equation ■

Example 7.2
Find the general solution of $u_{xy} = 0$.

Solution.
Integrating first we respect to y we find $u_x(x, y) = f(x)$, where f is an arbitrary differentiable function. Integrating u_x with respect to x we find $u(x, y) = \int f(x)dx + g(y)$, where g is an arbitrary differentiable function ■

Note that the general solution in the previous example involves two arbitrary functions. In general, the general solution of a partial differential equation is an expression that involves arbitrary functions. This is in contrast to the general solution of an ordinary differential equation which involves arbitrary constants.
Usually, a classical solution enjoys properties such as **smootheness** (i.e. a function that has continuous derivatives up to some desired order over some domain.) and continuity. However, in the theory of nonlinear pdes, there are solutions that do not require the smoothness property. Such solutions are

called **weak solutions** or **generalized solutions.** We illustrate this concept using equations rather than pdes. Consider the equation $x^2 - y^2 = 0$. The function $y = x$ is a classical solution of this equation. This solution is infinitely differentiable function. On the other hand, the function $y = |x|$ is also a solution to the given equation. However, this solution is not differentiable at 0. We call such a solution a weak solution. In this book, the word solution will refer to a classical solution.

Example 7.3
Show that $u(x,t) = t + \frac{1}{2}x^2$ is a classical solution to the PDE

$$u_t = u_{xx}. \tag{7.1}$$

Solution.
Assume that the domain of definition of u is $D \subset \mathbb{R}^2$. Since $u, u_t, u_x, u_{tx}, u_{xx}$ exist and are continuous in D(i.e., u is smooth in D) and u satisfies equation (7.1), we conclude that u is a classical solution to the given PDE ■

Now, consider the linear differential operator L as defined in the previous section. The defining properties of linearity immediately imply the key facts concerning homogeneous linear (differential) equations.

Theorem 7.1
The sum of two solutions to a homogeneous linear differential equation is again a solution, as is the product of a solution by any constant.

Proof.
Let u_1, u_2 be solutions, meaning that $L[u_1] = 0$ and $L[u_2] = 0$. Then, thanks to linearity,

$$L[u_1 + u_2] = L[u_1] + L[u_2] = 0,$$

and hence their sum $u_1 + u_2$ is a solution. Similarly, if α is any constant, and u any solution, then

$$L[\alpha u] = \alpha L[u] = \alpha 0 = 0,$$

and so the scalar multiple αu is also a solution ■

The following result is known as the **superposition principle** for homogeneous linear equations.

Theorem 7.2
If $u_1, \cdots, u_n$ are solutions to a common homogeneous linear partial differential equation $L[u] = 0$, then the linear combination $u = c_1u_1 + \cdots + c_nu_n$ is a solution for any choice of constants $c_1, \cdots, c_n$.

Proof.
The key fact is that, thanks to the linearity of L, for any sufficiently smooth functions $u_1, \cdots, u_n$ and any constants $c_1, \cdots, c_n$,

$$\begin{aligned} L[u] =& L[c_1u_1 + \cdots + c_nu_n] = L[c_1u_1 + \cdots + c_{n-1}u_{n-1}] + L[c_nu_n] \\ =& \cdots = L[c_1u_1] + \cdots + L[c_nu_n] = c_1L[u_1] + \cdots + c_nL[u_n]. \end{aligned}$$

In particular, if the functions are solutions, so $L[u_1] = 0, \cdots, L[u_n] = 0$, then the right hand side of the above equation vanishes, proving that u is also a solution to the homogeneous equation $L[u] = 0$ ■

In physical applications, homogeneous linear equations model unforced systems that are subject to their own internal constraints. External forcing is represented by an additional term that does not involve the dependent variable. This results in the nonhomogeneous equation

$$L[u] = f$$

where L is a linear partial differential operator, u is the dependent variable, and f is a given non-zero function of the independent variables alone.
You already learned the basic philosophy for solving of nonhomogeneous linear equations in your study of elementary ordinary differential equations. Step one is to determine the general solution to the homogeneous equation. Step two is to find a particular solution to the nonhomogeneous version. The general solution to the nonhomogeneous equation is then obtained by adding the two together. Here is the general version of this procedure:

Theorem 7.3
Let u_i be a particular solution to the nonhomogeneous linear equation $L[u] = f$. Then the general solution to $L[u] = f$ is given by $u = u_i + u_h$, where u_h is the general solution to the corresponding homogeneous equation $L[u] = 0$.

Proof.
Let us first show that $u = u_i + u_h$ is also a solution to $L[u] = f$. By linearity,

$$L[u] = L[u_i + u_h] = L[u_i] + L[u_h] = f + 0 = f.$$

To show that every solution to the nonhomogeneous equation can be expressed in this manner, suppose u satisfies $L[u] = f$. Set $u_h = u - u_i$. Then, by linearity,

$$L[u_h] = L[u - u_i] = L[u] - L[u_i] = 0,$$

and hence u_h is a solution to the homogeneous differential equation. Thus, $u = u_i + u_h$ has the required form ■

In physical applications, one can interpret the particular solution u_i as a response of the system to the external forcing function, while the solution u_h to the homogeneous equation represents the system's internal, unforced motion. The general solution to a linear nonhomogeneous equation is thus a combination of the external and internal responses.

As you have noticed by now, one solution of a linear PDE leads to the creation of lots of solutions. In contrast, nonlinear equations are much tougher to deal with, for example, knowledge of several solutions does not necessarily help in constructing others. Indeed, even finding one solution to a nonlinear partial differential equation can be quite a challenge.

In this introductory course, we will primarily − but not exclusively − concentrate on analyzing the most basic linear partial differential equations. But we will have occasion to briefly foray into the nonlinear realm, to appreciate some recent developments in this fascinating area of contemporary research and applications.

As observed above, a general solution of a partial differential equation has infinitely many solutions. In almost all cases, this general solution is of little use since it has to satisfy other supplementary conditions, usually called initial or boundary conditions. These conditions determine the unique solution of interest.

A **boundary value problem** is a partial differential equation where either the unknown function or its derivatives have values assigned on the physical boundary of the domain in which the problem is specified. These conditions are called **boundary conditions**. For example,

$$\begin{aligned} u_{xx} + u_{yy} =& 0 & \text{if } 0 < x, y < 1 \\ u(x,0) = u(x,1) =& 0 & \text{if } 0 < x < 1 \\ u_x(0,y) = u_x(1,y) =& 0 & \text{if } 0 < y < 1. \end{aligned}$$

There are three types of boundary conditions which arise frequently in formulating physical problems:
1. **Dirichlet Boundary Conditions:** In this case, the dependent function u is prescribed on the boundary of the bounded domain. For example, if the bounded domain is the rectangular plate $0 < x < L_1$ and $0 < y < L_2$, the boundary conditions $u(0,y), u(L_1,y), u(x,0)$, and $u(x,L_2)$ are prescribed.
The boundary conditions are called **homogeneous** if the dependent variable is zero at any point on the boundary, otherwise the boundary conditions are called **nonhomogeneous.**
2. **Neumann Boundary Conditions:** In this case, first partial derivatives are prescribed on the boundary of the bounded domain. For example, the Neuman boundary conditions for a rod of length L, where $0 < x < L$, are of the form $u_x(0,t) = \alpha$ and $u_x(L,t) = \beta$, where α and β are constants.
3. **Robin or mixed Boundary Conditions:** This occurs when the dependent variable and its first partial derivatives are prescribed on the boundary of the bounded domain.

An **initial valur problem** (or **Cauchy problem**) is a partial differential equation together with a set of additional conditions on the solution or its derivatives at either a given point or a given curve in the domain of the solution. These conditions are called **initial value conditions.** For example, the **transport equation**

$$\begin{aligned} u_t(x,t) + cu_x(x,t) =& 0 \\ u(x,0) =& f(x) \end{aligned}$$

is a Cauchy problem.
It can be shown that initial conditions for a PDE are necessary and sufficient for the existence of a unique solution.
We say that an initial and/or boundary value problem associated with a PDE is **well-posed** if it has a solution which is *unique* and depends continuously on the data given in the problem. The last condition, namely the continuous dependence is important in physical problems. This condition means that the solution changes by a small amount when the conditions change a little. Such solutions are said to be **stable**.

Example 7.4

For $x \in \mathbb{R}$ and $t > 0$ we consider the initial value problem

$$\begin{aligned} u_{tt} - u_{xx} =& 0 \\ u(x,0) = u_t(x,0) =& 0 \end{aligned}$$

Clearly, $u(x,t) = 0$ is a solution to this problem.
(a) Let $0 < \epsilon << 1$ be a very small number. Show that the function $u_\epsilon(x,t) = \epsilon^2 \sin\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right)$ is a solution to the problem

$$\begin{aligned} u_{tt} - u_{xx} =& 0 \\ u(x,0) =& 0 \\ u_t(x,0) =& \epsilon \sin\left(\frac{x}{\epsilon}\right) \end{aligned}$$

(b) Show that $\sup\{|u_\epsilon(x,t) - u(x,t)| : x \in \mathbb{R}, t > 0\} = \epsilon^2$. Thus, a small change in the initial data leads to a small change in the solution. Hence, the initial value problem is well-posed.

Solution.
(a) We have

$$\begin{aligned} \frac{\partial u_\epsilon}{\partial t} =& \epsilon \sin\left(\frac{x}{\epsilon}\right) \cos\left(\frac{t}{\epsilon}\right) \\ \frac{\partial^2 u_\epsilon}{\partial t^2} =& - \sin\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right) \\ \frac{\partial u_\epsilon}{\partial x} =& \epsilon \cos\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right) \\ \frac{\partial^2 u_\epsilon}{\partial x^2} =& - \sin\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right) \end{aligned}$$

Thus, $\frac{\partial^2 u_\epsilon}{\partial t^2} - \frac{\partial^2 u_\epsilon}{\partial x^2} = 0$. Moreover, $u_\epsilon(x,0) = 0$ and $\frac{\partial}{\partial t} u_\epsilon(x,0) = \epsilon \sin\left(\frac{x}{\epsilon}\right)$.
(b) We have

$$\begin{aligned} \sup\{|u_\epsilon(x,t) - u(x,t)| : x \in \mathbb{R}, t > 0\} =& \epsilon^2 \sup\left\{\left|\sin\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right)\right| : x \in \mathbb{R}, t > 0\right\} \\ =& \epsilon^2 \blacksquare \end{aligned}$$

A problem that is not well-posed is referred to as an **ill-posed** problem. We illustrate this concept in the next example.

Example 7.5
For $x \in \mathbb{R}$ and $t > 0$ we consider the initial value problem

$$\begin{aligned} u_{tt} + u_{xx} =& 0 \\ u(x,0) = u_t(x,0) =& 0 \end{aligned}$$

Clearly, $u(x,t) = 0$ is a solution to this problem.
(a) Let $0 < \epsilon << 1$ be a very small number. Show that the function $u_\epsilon(x,t) = \epsilon^2 \sin\left(\frac{x}{\epsilon}\right) \sinh\left(\frac{t}{\epsilon}\right)$, where

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

is a solution to the problem

$$\begin{aligned} u_{tt} + u_{xx} =& 0 \\ u(x,0) =& 0 \\ u_t(x,0) =& \epsilon \sin\left(\frac{x}{\epsilon}\right) \end{aligned}$$

(b) Show that $\sup\{|\frac{\partial}{\partial t} u_\epsilon(x,0) - u_t(x,0)| : x \in \mathbb{R}\} = \epsilon$ and $\sup\{|u_\epsilon(x,t) - u(x,t)| : x \in \mathbb{R}\} = \epsilon^2 \left|\sinh\left(\frac{t}{\epsilon}\right)\right|$.
(c) Find $\lim_{t\to\infty} \sup\{|u_\epsilon(x,t) - u(x,t)| : x \in \mathbb{R}\}$.

Solution.
(a) We have

$$\begin{aligned} \frac{\partial u_\epsilon}{\partial t} =& \epsilon \sin\left(\frac{x}{\epsilon}\right) \cosh\left(\frac{t}{\epsilon}\right) \\ \frac{\partial^2 u_\epsilon}{\partial t^2} =& \sin\left(\frac{x}{\epsilon}\right) \sinh\left(\frac{t}{\epsilon}\right) \\ \frac{\partial u_\epsilon}{\partial x} =& \epsilon \cos\left(\frac{x}{\epsilon}\right) \sinh\left(\frac{t}{\epsilon}\right) \\ \frac{\partial^2 u_\epsilon}{\partial x^2} =& -\sin\left(\frac{x}{\epsilon}\right) \sinh\left(\frac{t}{\epsilon}\right) \end{aligned}$$

Thus, $\frac{\partial^2 u_\epsilon}{\partial t^2} + \frac{\partial^2 u_\epsilon}{\partial x^2} = 0$. Moreover, $u_\epsilon(x,0) = 0$ and $\frac{\partial}{\partial t} u_\epsilon(x,0) = \epsilon \sin\left(\frac{x}{\epsilon}\right)$.
(b) We have

$$\begin{aligned} \sup\{|\frac{\partial}{\partial t} u_\epsilon(x,0) - u_t(x,0)| : x \in \mathbb{R}\} =& \sup\{\left|\epsilon \sin\left(\frac{x}{\epsilon}\right)\right| : x \in \mathbb{R}\} \\ =& \epsilon \sup\{\left|\sin\left(\frac{x}{\epsilon}\right)\right| : x \in \mathbb{R}\} = \epsilon \end{aligned}$$

and

$$\begin{aligned}\sup\{|u_\epsilon(x,t)-u(x,t)| : x\in\mathbb{R}\} =& \epsilon^2\sup\{\left|\sinh\left(\frac{t}{\epsilon}\right)\sin\left(\frac{x}{\epsilon}\right)\right| : x\in\mathbb{R}\}\\ =& \epsilon^2\left|\sinh\left(\frac{t}{\epsilon}\right)\right|.\end{aligned}$$

(c) We have

$$\lim_{t\to\infty}\sup\{|u_\epsilon(x,t)-u(x,t)| : x\in\mathbb{R}\} = \lim_{t\to\infty}\epsilon^2\left|\sinh\left(\frac{t}{\epsilon}\right)\right| = \infty.$$

Thus, a small change in the initial data leads to a catastrophically change in the solution. Hence, the given problem is ill-posed ■

Practice Problems

Exercise 7.1
Determine a and b so that $u(x,y)=e^{ax+by}$ is a solution to the equation

$$u_{xxxx}+u_{yyyy}+2u_{xxyy}=0.$$

Exercise 7.2
Consider the following differential equation

$$tu_{xx}-u_t=0.$$

Suppose $u(t,x)=X(x)T(t)$. Show that there is a constant λ such that $X''=\lambda X$ and $T''=\lambda tT$.

Exercise 7.3
Consider the initial value problem

$$xu_x+(x+1)yu_y=0,\quad x,y>1$$

$$u(1,1)=e.$$

Show that $u(x,y)=\frac{xe^x}{y}$ is the solution to this problem.

Exercise 7.4
Show that $u(x,y)=e^{-2y}\sin{(x-y)}$ is the solution to the initial value problem

$$\begin{cases} u_x+u_y+2u=0 & \text{for } x,y>0\\ u(x,0)=\sin x \end{cases}$$

Exercise 7.5
Solve each of the following differential equations:
(a) $\frac{du}{dx}=0$ where $u=u(x)$.
(b) $\frac{\partial u}{\partial x}=0$ where $u=u(x,y)$.

Exercise 7.6
Solve each of the following differential equations:
(a) $\frac{d^2u}{dx^2}=0$ where $u=u(x)$.
(b) $\frac{\partial^2 u}{\partial x\partial y}=0$ where $u=u(x,y)$.

Exercise 7.7
Show that $u(x,y) = f(y+2x) + xg(y+2x)$, where f and g are two arbitrary twice differentiable functions, satisfy the equation

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0.$$

Exercise 7.8
Find the differential equation whose general solution is given by $u(x,t) = f(x-ct)+g(x+ct)$, where f and g are arbitrary twice differentiable functions in one variable.

Exercise 7.9
Let $p : \mathbb{R} \to \mathbb{R}$ be a differentiable function in one variable. Prove that

$$u_t = p(u)u_x$$

has a solution satisfying $u(x,t) = f(x + p(u)t)$, where f is an arbitrary differentiable function. Then find the general solution to $u_t = (\sin u)u_x$.

Exercise 7.10
Find the general solution to the pde

$$u_{xx} + 2u_{xy} + u_{yy} = 0.$$

Hint: See Exercise 6.2.

Sample Exam Questions

Exercise 7.11
Let $u(x,t)$ be a function such that u_{xx} exists and $u(0,t)=u(L,t)=0$ for all $t\in\mathbb{R}$. Prove that
$$\int_0^L u_{xx}(x,t)u(x,t)dx \le 0.$$

Exercise 7.12
Consider the initial value problem
$$u_t+u_{xx}=0,\ x\in\mathbb{R},\ t>0$$
$$u(x,0)=1.$$
(a) Show that $u(x,t)\equiv 1$ is a solution to this problem.
(b) Show that $u_n(x,t) = 1+\frac{e^{n^2t}}{n}\sin nx$ is a solution to the initial value problem
$$u_t+u_{xx}=0,\ x\in\mathbb{R},\ t>0$$
$$u(x,0)=1+\frac{\sin nx}{n}.$$
(c) Find $\sup\{|u_n(x,0)-1| : x\in\mathbb{R}\}$.
(d) Find $\sup\{|u_n(x,t)-1| : x\in\mathbb{R}\}$.
(e) Show that the problem is ill-posed.

Exercise 7.13
Find the general solution of each of the following PDEs by means of direct integration.
(a) $u_x=3x^2+y^2,\ u=u(x,y)$.
(b) $u_{xy}=x^2y,\ u=u(x,y)$.
(c) $u_{xyz}=0,\ u=u(x,y,z)$.
(d) $u_{xtt}=e^{2x+3t},\ u=u(x,t)$.

Exercise 7.14
Consider the second-order PDE
$$u_{xx}+4u_{xy}+4u_{yy}=0.$$
(a) Use the change of variables $v(x,y)=y-2x$ and $w(x,y)=x$ to show that $u_{ww}=0$.
(b) Find the general solution to the given PDE.

Exercise 7.15
Derive the general solution to the PDE

$$u_{tt} = c^2 u_{xx}$$

by using the change of variables $v = x + ct$ and $w = x - ct$.

First Order Partial Differential Equations

Many problems in the mathematical, physical, and engineering sciences deal with the formulation and the solution of first order partial differential equations. Our first task is to understand simple first order equations. In applications, first order partial differential equations are most commonly used to describe dynamical processes, and so time, t, is one of the independent variables. Most of our discussion will focus on dynamical models in a single space dimension, bearing in mind that most of the methods can be readily extended to higher dimensional situations. First order partial differential equations and systems model a wide variety of wave phenomena, including transport of solvents in fluids, flood waves, acoustics, gas dynamics, glacier motion, traffic flow, and also a variety of biological and ecological systems. From a mathematical point of view, first order partial differential equations have the advantage of providing conceptual basis that can be utilized in the study of higher order partial differential equations.
In this chapter we introduce the basic definitions of first order partial differential equations. We then derive the one dimensional spatial transport eqution and discuss some methods of solutions. One general method of solvability for quasilinear first order partial differential equation, known as the method of characteristics, is analyzed.

8 Classification of First Order PDEs

In this section, we present the basic definitions pertained to first order PDE. By a **first order differential equation** in two variables x and y we mean any equation of the form

$$F(x, y, u, u_x, u_y) = 0. \tag{8.1}$$

In what follows the functions a, b, and c are assumed to be continuously differentiable functions. If Equation (8.1) can be written in the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{8.2}$$

then we say that the equation is **quasilinear**. The following are examples of quasilinear equations:

$$uu_x + u_y + cu^2 = 0$$

$$x(y^2 + u)u_x - y(x^2 + u)u_y = (x^2 - y^2)u.$$

If Equation (8.1) can be written in the form

$$a(x, y)u_x + b(x, y)u_y = c(x, y, u) \tag{8.3}$$

then we say that the equation is **semilinear**. The following are examples of semilinear equations:

$$xu_x + yu_y = u^2 + x^2$$

$$(x + 1)^2 u_x + (y - 1)^2 u_y = (x + y)u^2.$$

If Equation (8.1) can be written in the form

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y) \tag{8.4}$$

then we say that the equation is **linear**. Examples of linear equations are:

$$xu_x + yu_y = cu$$

$$(y - z)y_x + (z - x)u_y + (x - y)u_z = 0.$$

A first order pde that is not linear is said to be **nonlinear.** Examples of nonlinear equations are:

$$u_x + cu_y^2 = xy$$

$$u_x^2 + u_y^2 = c.$$

First order partial differential equations are classified as either linear or nonlinear. Clearly, linear equations are a special kind of quasilinear equation

(8.2) if a and b are functions of x and y only and c is a linear function of u. Likewise, semilinear equations are quasilinear equations if a and b are functions of x and y only. Also, semilinear equations (8.4) reduces to a linear equation if c is linear in u.
A linear equation is called **homogeneous** if $d(x,y) \equiv 0$ and **nonhomogeneous** if $d(x,y) \neq 0$. Examples of linear homogeneous equations are:

$$xu_x + yu_y = cu$$

$$(y-z)u_x + (z-x)u_y + (x-y)u_z = 0.$$

Examples of nonhomogeneous equations are:

$$u_x + (x+y)u_y - u = e^x$$

$$yu_x + xu_y = xy.$$

Recall that for an ordinary linear differential equation, the general solution depends mainly on arbitrary constants. Unlike ODEs, in linear partial differential equations, the general solution depends on arbitrary functions.

Example 8.1
Solve the equation $u_t(x,t) = 0$.

Solution.
The general solution is given by $u(x,t) = f(x)$ where f is an arbitrary differentiable function of x ■

Example 8.2
Consider the transport equation

$$au_t(x,t) + bu_x(x,t) = 0$$

where a and b are constants. Show that $u(x,t) = f(bt - ax)$ is a solution to the given equation, where f is an arbitrary differentiable function in one variable.

Solution.
Let $v(x,t) = bt - ax$. Using the chain rule we see that $u_t(x,t) = bf_v(v)$ and $u_x(x,t) = -af_v(v)$. Hence, $au_t(x,t) + bu_x(x,t) = abf_v(v) - abf_v(v) = 0$ ■

Practice Problems

Exercise 8.1
Classify each of the following PDE as linear, quasilinear, semi-linear, or non-linear.
(a) $xu_x + yu_y = \sin(xy)$.
(b) $u_t + uu_x = 0$
(c) $u_x^2 + u^3u_y^4 = 0$.
(d) $(x+3)u_x + xy^2u_y = u^3$.

Exercise 8.2
Show that $u(x,y) = e^x f(2x-y)$, where f is a differentiable function of one variable, is a solution to the equation

$$u_x + 2u_y - u = 0.$$

Exercise 8.3
Show that $u(x,y) = x\sqrt{xy}$ satisfies the equation

$$xu_x - yu_y = u$$

subject to

$$u(y,y) = y^2,\ y \geq 0.$$

Exercise 8.4
Show that $u(x,y) = \cos(x^2+y^2)$ satisfies the equation

$$-yu_x + xu_y = 0$$

subject to

$$u(0,y) = \cos y^2.$$

Exercise 8.5
Show that $u(x,y) = y - \frac{1}{2}(x^2-y^2)$ satisfies the equation

$$\frac{1}{x}u_x + \frac{1}{y}u_y = \frac{1}{y}$$

subject to $u(x,1) = \frac{1}{2}(3-x^2)$.

Exercise 8.6
Find a relationship between a and b if $u(x, y) = f(ax + by)$ is a solution to the equation $3u_x - 7u_y = 0$ for any differentiable function f.

Exercise 8.7
Suppose L is a linear operator, that is, $L(\alpha u+\beta v) = \alpha L(u)+\beta L(v)$. Consider the homogeneous and nonhomogeneous linear equations

$$Lu = 0$$

$$Lu = f$$

where f is some function. Suppose v is a solution to the homogeneous equation, and w is a solution to the nonhomogeneous equation. Show $u = av + w$ is a solution to the nonhomogeneous equation for any constant a.

Exercise 8.8
Reduce the partial differential equation

$$au_x + bu_y + cu = 0$$

to a first order ODE by introducing the change of variables $s = ax + by$ and $t = bx - ay$.

Exercise 8.9
Solve the partial differential equation

$$u_x + u_y = 1$$

by introducing the change of variables $s = x + y$ and $t = x - y$.

Sample Exam Questions

Exercise 8.10
Show that $u(x,y) = e^{-4x}f(2x-3y)$ is a solution to the first-order PDE

$$3u_x + 2u_y + 12u = 0.$$

Exercise 8.11
Derive the general solution of the PDE

$$au_t + bu_x = u, \quad a, b \neq 0$$

by using the change of variables $v = ax - bt$ and $w = \frac{1}{a}t$.

Exercise 8.12
Derive the general solution of the PDE

$$au_x + bu_y = 0, \quad a, b \neq 0$$

by using the change of variables $s(x,y) = ax + by$ and $t(x,y) = bx - ay$. Assume $a^2 + b^2 > 0$.

Exercise 8.13
Write the equation

$$u_t + cu_x + \lambda u = f(x,y)$$

in the coordinates $v = x - ct,\ w = t$.

Exercise 8.14
Suppose that $u(x,t) = w(x - ct)$ is a solution to the PDE

$$xu_x + tu_t = Au$$

where A and c are constants. Let $v = x - ct$. Write the differential equation with unknown function $w(v)$.

9 The One Dimensional Spatial Transport Equations

Modeling is the process of writing a differential equation to describe a physical situation. In this section we discuss the one-dimensional transport equation and discuss an analytical method for solving it.

Linear Transport Equation for Fluid Flows
We shall describe the transport of a dissolved chemical by water that is traveling with uniform velocity c through a long thin tube G with uniform cross section A. (The very same discussion applies to the description of the transport of gas by air moving through a pipe.) We identify G with the open interval (a, b), and the velocity $c > 0$ is in the (rightward) positive direction of the $x-$axis. We will assume that the concentration of the chemical is constant across the cross section A at each point x so that the chemical changes in the $x-$direction and thus the term one-dimensional spatial equation. See Figure 9.1

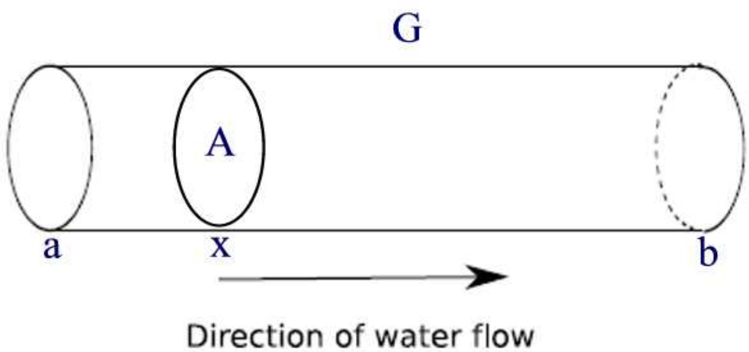


Figure 9.1

Let $u(x, t)$ be a continuously differentiable function denoting the concentration of the chemical (i.e. amount of chemical/area) at position x at time t. Then at time t, the amount of chemical stored in a section of the tube between positions a and x is given by the definite integral

$$\int_a^x Au(s,t)ds.$$

Since the water is flowing at a speed c, so at time $h+t$ the same quantity of chemical will be

$$\int_a^x Au(s,t)ds = \int_{a+ch}^{x+ch} Au(s,t+h)ds.$$

Taking the derivative of both sides with respect to x we find

$$u(x,t) = u(x+ch, t+h).$$

Now taking the derivative of this last equation with respect to h we find

$$0 = u_t(x+ch, t+h) + cu_x(x+ch, t+h).$$

Taking the limit of this last equation as h approaches 0 we find

$$u_t(x,t) + cu_x(x,t) = 0 \tag{9.1}$$

for all (x,t). This equation is called the **transport equation** in one-dimensional space. It is a linear, homogeneous first order partial differential equation.

Example 9.1
Show that $u(x,t) = f(x-ct)$ is a solution to (9.1), where f is an arbitrary differentiable function in one variable.

Solution.
Using the chain rule we find

$$u_t = -cf'(x-ct) \text{ and } u_x = f'(x-ct).$$

Hence, by substituting these results into the equation we find

$$u_t + cu_x = -cf'(x-ct) + cf'(x-ct) = 0.$$

The solution $u(x,t) = f(x-ct)$ is called the **right traveling wave,** since the graph of the function $f(x-ct)$ at a given time t is the graph of $f(x)$ shifted to the right by the value ct. Thus, with growing time, the function $f(x)$ is moving without changes to the right at the speed c ■

An initial value condition determines a unique solution to the transport equation as stated in the next theorem.

Theorem 9.4
Let g be a continuously differentiable function. Then there is a unique continuously differentiable solution $u(x,t)$ to the IVP

$$au_x(x,t) + bu_t(x,t) = 0$$
$$u(x,0) = g(x).$$

Indeed, u is given explicitly by the formula

$$u(x,t) = f(bx - at), \ g(x) = f(bx).$$

Method of Solutions: The Coordinate Method
We will solve (9.1) by solving the more general equation

$$au_x + bu_y = 0 \tag{9.2}$$

where $a^2 + b^2 > 0$.
We introduce a new rectangular system by the substitution

$$s = ax + by, \quad t = bx - ay$$

According to the chain rule for the derivative of a composite function, we have

$$\begin{aligned} u_x =& u_s s_x + u_t t_x = au_s + bu_t \\ u_y =& u_s s_y + u_t t_y = bu_s - au_t \end{aligned}$$

Substituting these into (9.2) to obtain

$$a^2 u_s + abu_t + b^2 u_s - abu_t = 0$$

or

$$(a^2 + b^2)u_s = 0$$

and since $a^2 + b^2 > 0$ we obtain

$$u_s = 0.$$

Solving this equation, we find

$$u(s,t) = f(t)$$

where f is an arbitrary differentiable function of one variable. Now, in terms of x and y we find

$$u(x,y) = f(bx - ay).$$

Example 9.2
Use the coordinate method to find the solution to $u_t - 3u_x = 0,\ u(x,0) = e^{-x^2}$.

Solution.
Let $v = -3x + t$ and $w = x + 3t$. Then $u_x = -3u_v + u_w$ and $u_t = u_v + 3u_w$. Substituting these into the given equation we find $10u_v = 0$ or $u_v = 0$. Hence, $u(v,w) = f(w)$ or $u(x,t) = f(x+3t)$ where f is a differentiable function in one variable. Since $u(x,0) = e^{-x^2}$, we find $e^{-x^2} = f(x)$. Hence, $u(x,t) = e^{-(x+3t)^2}$ ■

Transport Equation with Decay: The Method of Characteristic Coordinates
A **transport equation with decay** is an equation given by

$$u_t + cu_x + \lambda u = f(x,t)$$

where λ and c are constants and f is a given function representing external resources. Note that the decay is characterized by the term λu.
To solve this equation, we introduce the **characteristic coordinates** given by

$$v = x - ct, \quad w = t.$$

Using the chain rule, we find

$$\begin{aligned} u_t =& u_v v_t + u_w w_t = -cu_v + u_w \\ u_x =& u_v v_x + u_w w_x = u_v. \end{aligned}$$

Substituting these into the original equation we obtain the equation

$$u_w + \lambda u = f(v + cw, w)$$

which can be solved by the method of integrating factor. We illustrate this approach in the next example.

Example 9.3
Find the general solution of the transport equation

$$u_t + u_x - u = t.$$

Solution.
The characteristic coordinates are

$$v = x - t, \quad w = t.$$

These transform the original equation to the first order ODE

$$u_w - u = w.$$

Using the method of integrating factor, we find

$$\frac{d}{dw}(e^{-w}u) = we^{-w}$$

and solving this equation we find

$$u(v, w) = -(1 + w) + e^w f(v)$$

and in terms of x and t we find

$$u(x, t) = f(x - t)e^t - (1 + t) \blacksquare$$

A more general method for solving quasilinear first order partial differential equations, known as the method of characteristics, will be discussed in the next section.

Practice Problems

Exercise 9.1
Use the coordinate method to find the solution to $u_t + 3u_x = 0,\ u(x,0) = \sin x$.

Exercise 9.2
Use the coordinate method, solve the equation $au_x + bu_y + cu = 0$.

Exercise 9.3
Use the coordinate method, solve the equation $u_x + 2u_y = \cos(y - 2x)$ with the initial condition $u(0,y) = f(y)$, where $f : \mathbb{R} \to \mathbb{R}$ is a given function.

Exercise 9.4
Show that the initial value problem $u_t + u_x = x,\ u(x,x) = 1$ has no solution.

Exercise 9.5
Solve the transport equation $u_t + 2u_x = -3u$ with initial condition $u(x,0) = \frac{1}{1+x^2}$.

Exercise 9.6
Solve $u_t + u_x - 3u = t$ with initial condition $u(x,0) = x^2$.

Exercise 9.7
Show that the decay term λu in the transport equation with decay

$$u_t + cu_x + \lambda u = 0$$

can be eliminated by the substitution $w = ue^{\lambda t}$.

Exercise 9.8
Use the coordinate method to solve

$$u_x + u_y = u^2$$

$$u(x,0) = h(x)$$

Exercise 9.9 *(Well-Posed)*
Let u be the unique solution to the IVP

$$u_t + cu_x = 0$$

$$u(x,0) = f(x)$$

and v be the unique solution to the IVP

$$u_t + cu_x = 0$$

$$u(x,0) = g(x)$$

where f and g are continuously differentiable functions.
(a) Show that $w(x,t) = u(x,t) - v(x,t)$ is the unique solution to the IVP

$$u_t + cu_x = 0$$

$$u(x,0) = f(x) - g(x)$$

(b) Write an explicit formula for w in terms of f and g.
(c) Use (b) to conclude that the transport problem is well-posed. That is, a small change in the initial data leads to a small change in the solution.

Exercise 9.10
Solve the initial boundary value problem

$$u_t + cu_x = -\lambda u, \ x > 0, \ t > 0$$

$$u(x,0) = 0, \ u(0,t) = g(t), \ t > 0.$$

Sample Exam Questions

Exercise 9.11
Solve the first-order equation $2u_t+3u_x = 0$ with the initial condition $u(x, 0) = \sin x$.

Exercise 9.12
Solve the PDE

$$u_x + u_y = 1$$

using the coordinate method.

Exercise 9.13
Consider the first order linear homogeneous PDE

$$Au_x + Bu_y + Cu = 0$$

where A, B, and C are constants with $A \neq 0$.
(a) Determine a, b, c, d in terms of A, B, C such that $ad - bc \neq 0$ and so that the change of variables $v = ax + by$ and $w = cx + dy$ will reduce the given PDE to a first order PDE of the form $\alpha u_v + \beta u = 0$.
(b) Use (a) to find the general solution of the given PDE.

Exercise 9.14
Use the result of the previous problem to solve the PDE

$$u_x + u_y + u = 0.$$

10 The Method of Characteristics

In this section we develop a method for finding the general solution of a quasilinear first order partial differential equation. This method is called the **method of characteristics** or **Lagrange's method.** This method of solution can be described by the following result.

Theorem 10.1
The general solution of the quasilinear first order PDE

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \tag{10.1}$$

is given by

$$f(v,w) = 0 \tag{10.2}$$

where f is an arbitrary differentiable function of $v(x,y,u)$ and $w(x,y,u)$ and v =constant= c_1, w =constant= c_2 are solutions to the ODE system

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}. \tag{10.3}$$

Equations (10.3) are called the **characteristic equations** in non-parametric forms. The corresponding parametric forms are given by the system of ODEs

$$\begin{aligned}\frac{dx}{ds} =& a\\ \frac{dy}{ds} =& b\\ \frac{du}{ds} =& c\end{aligned}$$

Remark 10.1
Sometimes (10.2) is written explicitly as $v = g(w)$ or $w = g(v)$ where g is an arbitrary differentiable function.

Example 10.1
Find the general solution of the PDE $x^2u_x + y^2u_y = (x+y)u$.

Solution.
The characteristic equations for this PDE are $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{du}{(x+y)u}$. Using the

first two fractions, we have $\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$ and this implies $\frac{x-y}{xy} = c_1$. Also, we can solve for x obtaining $x = \frac{1}{\frac{1}{y}-c_1}$ and $x + y = \frac{y}{1-c_1y} + y$. Using the last two fractions we find $\frac{dy}{y^2} = \frac{du}{u(x+y)} \Longrightarrow \frac{1}{y^2}\left(\frac{y}{1-c_1y} + y\right)dy = \frac{du}{u} \Longrightarrow \left(\frac{1}{y-c_1y^2} + \frac{1}{y}\right)dy = \frac{du}{u} \Longrightarrow \left(\frac{1-c_1y}{y-c_1y^2} + \frac{c_1}{1-c_1y} + \frac{1}{y}\right)dy = \frac{du}{u} \Longrightarrow \int\left(\frac{2}{y} + \frac{c_1}{1-c_1y}\right)dy = \int \frac{du}{u} \Longrightarrow 2\ln|y| - \ln|1-c_1y| = \ln|u| + c_2 \Longrightarrow \frac{y^2}{1-c_1y} = cu \Longrightarrow y \cdot \frac{y}{1-c_1y} = cu \Longrightarrow y\frac{1}{\frac{1}{y}-c_1} = cu \Longrightarrow xy = cu$. Hence, the general solution is

$$f\left(\frac{x-y}{xy}, \frac{xy}{u}\right) = 0$$

where f is an arbitrary differentiable function ■

Example 10.2
Find the general solution of the PDE $yuu_x + xuu_y = xy$.

Solution.
The characteristic equations are $\frac{dx}{yu} = \frac{dy}{xu} = \frac{du}{xy}$. Using the first two fractions we find $x^2 - y^2 = c_1$. Using the last two fractions we find $u^2 - y^2 = c_2$. Hence, the general solution is $f(x^2 - y^2, u^2 - y^2) = 0$ or $u^2 = y^2 + g(x^2 - y^2)$, where f and g are arbitrary differentiable functions ■

Example 10.3
Find the general solution of the PDE $x(y^2 - u^2)u_x - y(u^2 + x^2)y_y = (x^2 + y^2)u$.

Solution.
The characteristic equations are $\frac{dx}{x(y^2-u^2)} = \frac{dy}{-y(u^2+x^2)} = \frac{du}{(x^2+y^2)u}$. Using a property of proportions we can write

$$\frac{xdx + ydy + udu}{x^2(y^2 - u^2) - y^2(u^2 + x^2) + u^2(x^2 + y^2)} = \frac{du}{(x^2 + y^2)u}.$$

That is

$$\frac{xdx + ydy + udu}{0} = \frac{du}{(x^2 + y^2)u}$$

or

$$xdx + ydy + udu = 0.$$

Hence, we find $x^2 + y^2 + u^2 = c_1$. Also,

$$\frac{\frac{dx}{x} - \frac{dy}{y}}{y^2 - u^2 + u^2 + x^2} = \frac{du}{(x^2 + y^2)u}$$

or

$$\frac{dx}{x} - \frac{dy}{y} = \frac{du}{u}.$$

Hence, we find $\ln\left|\frac{yu}{x}\right|$ =constant or $\frac{yu}{x} = c_2$. The general solution is given by

$$f\left(x^2 + y^2 + u^2, \frac{yu}{x}\right) = 0$$

or

$$u = \frac{x}{y}g(x^2 + y^2 + u^2)$$

where f and g are arbitrary differentiable functions ■

Example 10.4
Solve the transport equation using the method of characteristics

$$u_t + cu_x = 0.$$

Solution.
The characteristic equations are given by

$$\frac{dt}{1} = \frac{dx}{c} = \frac{du}{0}.$$

Solving the first two fractions we find $x - ct = k$. The last fraction implies $u = k'$. The general solution is given by $f(x - ct, u) = 0$ or $u = g(x - ct)$ ■

Solution curves to the ODE

$$\frac{dy}{dx} = \frac{b}{a}$$

are called **characteristic curves** or simply **characteristics.** These are curves in the $xy-$plane.

Example 10.5
Find the characteristics of $\cos y u_x + u_y + xu = 0$.

Solution.
Solving the equation $\frac{dy}{dx} = \frac{1}{\cos y}$ by the separation of variable method we find $\sin y - x = k$ ■

Example 10.6
Find the characteristics of $u_x + 2u_y - u = 0$.

Solution.
We have $a = 1$ and $b = 2$. Thus, $\frac{dy}{dx} = 2$ so that the characteristics are given by $2x - y = k$ ■

Practice Problem

Exercise 10.1
Find the characteristics of the PDE

$$xu_x - yu_y = u.$$

Exercise 10.2
Find the characteristics of the PDE

$$-yu_x + xu_y = 0.$$

Exercise 10.3
Find the characteristics of the PDE

$$(x+y)(u_x + u_y) = u - 1.$$

Exercise 10.4
Find the general solution of the PDE $xu_x + yu_y = 1 + u^2$.

Exercise 10.5
Find the general solution of the PDE $\ln(y+u)u_x + u_y = -1$.

Exercise 10.6
Find the general solution of the PDE $xu_x + yu_y = u$.

Exercise 10.7
Find the general solution of the PDE $xu_x + yu_y = nu$.

Exercise 10.8
Find the general solution of the PDE $x(y-u)u_x + y(u-x)u_y = u(x-y)$.

Exercise 10.9
Find the general solution of the PDE $u(u^2+xy)(xu_x - yu_y) = x^4$.

Exercise 10.10
Find the general solution of the PDE $(y+xu)u_x - (x+yu)u_y = x^2 - y^2$.

Exercise 10.11
Find the general solution of the PDE $(y^2+u^2)u_x - xyu_y + xu = 0$.

Exercise 10.12
Find the general form of solutions to

$$u_x + 2u_y = u$$

and sketch some of the characteristics. Hint: define a new variable $v = e^{-x}u$. What equation does v satisfy?

Exercise 10.13
Find the general form of solutions to

$$(1 + x^2)u_x + u_y = 0$$

and sketch some of the characteristics.

Sample Exam Questions

Exercise 10.14
Find the general solution of the equation

$$u_x + yu_y = u.$$

Exercise 10.15
Find the characteristics associated with the PDE

$$u_x + xu_y + 3u = 2.$$

Exercise 10.16
Find the general solution of the first order PDE

$$u_x + yu_y + xu = 0.$$

Exercise 10.17
Find the characteristics of the PDE

$$\frac{1}{x}u_x + \frac{1}{y}u_y = 0.$$

Exercise 10.18
Find the characteristics of the PDE

$$\frac{1}{x}u_x + \frac{1}{y}u_y = \frac{1}{y}.$$

11 The Cauchy Problem for First Order Quasilinear Equations

When solving a partial differential equation, it is seldom the case that one tries to study the properties of the general solution of such equations. In general, one deals with those partial differential equations whose solutions satisfy certain supplementary conditions. In the case of a first order partial differential equation, we determine the particular solution by formulating an initial value porblem also known as a **Cauchy problem.**
In this section, we discuss the Cauchy problem for the first order quasilinear partial differential equation

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u). \tag{11.1}$$

Recall that the initial value problem of a first order ordinary differential equation asks for a solution of the equation which has a given value at a given point in $\mathbb{R}$. The Cauchy problem for the PDE (11.1) asks for a solution of (11.1) which has given values on a given curve in $\mathbb{R}^2$. A precise statement of the problem is given next.

Initial Value Problem or Cauchy Problem
Let C be a given curve in $\mathbb{R}^2$ defined parametrically by the equations

$$x = x_0(t), \quad y = y_0(t)$$

where x_0, y_0 are continuously differentiable functions on some interval I. Let $u_0(t)$ be a given continuously differentiable function on I. The Cauchy problem for (11.1) asks for a continuously differentiable function $u = u(x, y)$ defined in a domain $\Omega \subset \mathbb{R}^2$ containing the curve C and such that:
(1) $u = u(x, y)$ is a solution of (11.1) in Ω.
(2) On the curve C, u equals the given function $u_0(t)$, i.e.

$$u(x_0(t), y_0(t)) = u_0(t), \ t \in I. \tag{11.2}$$

We call C the **initial curve** of the problem, $u_0(t)$ the **initial data**, and (11.2) the **initial condition** of the problem. See Figure 11.1.

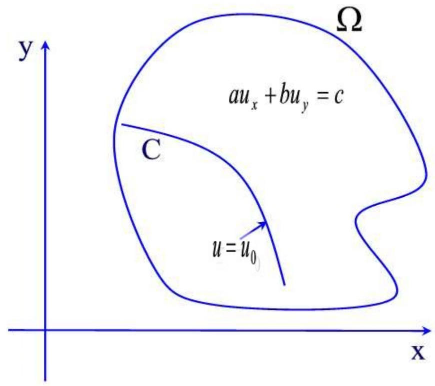


Figure 11.1

If we view a solution $u = u(x, y)$ of (11.1) as an integral surface of (11.1), we can give a simple geometrical statement of the problem: Find a solution surface of (11.1) containing the curve Γ described parametrically by the equations

$$\Gamma : x = x_0(t),\ y = y_0(t),\ u = u_0(t), \qquad t \in I.$$

Note that the projection of this curve in the $xy-$plane is just the curve C. The following theorem asserts that under certain conditions the Cauchy problem (11.1) - (11.2) has a unique solution.

Theorem 11.1
Suppose that $x_0(t), y_0(t)$, and $u_0(t)$ are continuously differentiable functions of t in an interval I, and that a, b, and c are functions of x, y, and u with continuous first order partial derivatives with respect to their argument in some domain D of $(x, y, u)-$space containing the initial curve

$$\Gamma : x = x_0(t),\ y = y_0(t),\ u = u_0(t)$$

where $t \in I$. Then for each point $(x_0(t), y_0(t), u_0(t))$ on Γ that satisfies the condition

$$a(x_0(t), y_0(t), u_0(t))\frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t))\frac{dx_0}{dt}(t) \neq 0. \qquad (11.3)$$

there exists a unique solution $u = u(x, y)$ of (11.1) in a neighborhood U of $(x_0(t), y_0(t))$ such that the initial condition (11.2) is satisfied for every point on C contained in U. See Figure 11.2.

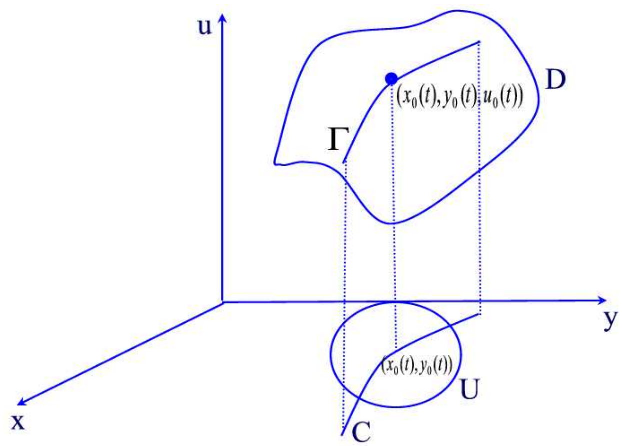


Figure 11.2

Note that condition (11.3) implies that

$$\frac{dy_0(t)}{dx_0(t)} \neq \frac{b(x_0, y_0, u_0)}{a(x_0, y_0, u_0)}$$

which means that the vector $(a(x_0, y_0, u_0), b(x_0, y_0, u_0), c(x_0, y_0, u_0))$ is not tangent to Γ. (Recall that the normal vector to C has components $\left(\frac{dy_0(t)}{dt}, -\frac{dx_0(t)}{t}\right)$ so that a vector (a, b) is tangent to C if $a\frac{dy_0(t)}{dt} - b\frac{dx_0(t)}{dt} = 0$.) It follows that the Cauchy problem has a unique solution if C is nowhere characteristic.
We construct the desired solution using the method of characteristics as follows: Pick a point $(x_0(t), y_0(t), u_0(t)) \in \Gamma$. Using this as the initial value we solve the system of ODEs consisting of the characteristic equations in parametric form

$$\begin{aligned}
\frac{dx}{ds} =& a(x(s), y(s), u(s)) \\
\frac{dy}{ds} =& b(x(s), y(s), u(s)) \\
\frac{du}{ds} =& c(x(s), y(s), u(s))
\end{aligned}$$

satisfying the initial condition

$$(x(0), y(0), u(0)) = (x_0(t), y_0(t), u_0(t)).$$

The solution depends on the parameter s so it consists of a triples of functions

$$x = x(s,t), \quad y = y(s,t), \quad u = u(s,t). \tag{11.4}$$

This system represents the parametric representation of the integral surface of the problem in which the curve Γ corresponds to $s = 0$. The solution u is recovered by solving the first two equations in (11.4) for

$$t = t(x,y), \quad s = s(x,y)$$

and substituting these into the third equation to obtain $u(x,y) = u(s(x,y), t(x,y))$.

Example 11.1
Solve the Cauchy problem

$$\begin{aligned} u_x + u_y =& 1 \\ u(x,0) =& f(x). \end{aligned}$$

Solution.
The initial curve in $\mathbb{R}^3$ can be given parametrically as

$$\Gamma : x_0(t) = t, \; y_0(t) = 0, \; u_0(t) = f(t).$$

We have

$$a(x_0(t), y_0(t), u_0(t))\frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t))\frac{dx_0}{dt}(t) = -1 \neq 0$$

so by the above theorem the given Cauchy problem has a unique solution. To find this solution, we solve the system of ODEs

$$\begin{aligned} \frac{dx}{ds} =& 1 \\ \frac{dy}{ds} =& 1 \\ \frac{du}{ds} =& 1. \end{aligned}$$

Solving this system we find

$$x(s,t) = s + \alpha(t), \quad y(s,t) = s + \beta(t), \quad u(s) = s + \gamma(t).$$

But $x(0,t) = t$ so that $\alpha(t) = t$. Similarly, $y(0,t) = 0$ so that $\beta(t) = 0$ and $u(0,t) = f(t)$ implies $\gamma(t) = f(t)$. Hence, the unique solution is given parametrically by the equations

$$x(s,t) = t + s, \quad y(s,t) = s, \quad u(s,t) = s + f(t).$$

Solving the first two equations for s and t we find

$$s = y, \quad t = x - y$$

and substituting these into the third equation we find

$$u(x,y) = y + f(x-y).$$

Alternative Computation
We can apply the results of the previous section to find the unique solution. If we solve the characteristic equations in non-parametric form

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{1}$$

we find $x - y = c_1$ and $u - x = c_2$. Thus, the general solution of the PDE is given by $u = x + F(x-y)$. Using the Cauchy data $u(x,0) = f(x)$ we find $f(x) = x + F(x)$ which implies that $F(x) = f(x) - x$. Hence, the unique solution is given by

$$u(x,y) = x + f(x-y) - (x-y) = y + f(x-y) \blacksquare$$

If condition (11.3) is not satisfied than C is a characteristic curve. If the curve Γ satisfies the characteristic equations than the problem has infinitely many solutions. To see this, pick an arbitrary point $P_0 = (x_0, y_0, u_0)$ on Γ. Pick a new initial curve Γ' passing through P_0 which is not tangent to Γ at P_0. In this case, condition (11.3) is satisfied and the new Cauchy problem has a unique solution. Since there are infinitely many ways of selecting Γ', we obtain infinitely many solutions. We illustrate this case in the next example.

Example 11.2
Solve the Cauchy problem

$$\begin{aligned} u_x + u_y &= 1 \\ u(x,x) &= x. \end{aligned}$$

Solution.
The initial curve in $\mathbb{R}^3$ can be given parametrically as

$$\Gamma : x_0(t) = t,\ \ y_0(t) = t,\ \ u_0(t) = t.$$

We have

$$a(x_0(t), y_0(t), u_0(t))\frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t))\frac{dx_0}{dt}(t) = 0.$$

As in Example 11.1, the general solution of the PDE is $u(x, y) = y + f(x - y)$ where f is an arbitrary differentiable function. Using the Cauchy data $u(x, x) = x$ we find $f(0) = 0$. Thus, the solution is given by

$$u(x, y) = y + f(x - y)$$

where f is an arbitrary function such that $f(0) = 0$. There are infinitely many choices for f. Hence, the problem has infinitely many solutions. Note that Γ satisfies the characteristic equations ■

If condition (11.3) is not satisfied and if Γ does not satisfy the characteristic equations then it can be shown that the Cauchy problem has no solutions. We illustrate this case next.

Example 11.3
Solve the Cauchy problem

$$\begin{aligned} u_x + u_y =& 1 \\ u(x, x) =& 1. \end{aligned}$$

Solution.
The initial curve in $\mathbb{R}^3$ can be given parametrically as

$$\Gamma : x_0(t) = t,\ \ y_0(t) = t,\ \ u_0(t) = 1.$$

We have

$$a(x_0(t), y_0(t), u_0(t))\frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t))\frac{dx_0}{dt}(t) = 0.$$

Solving the characteristic equations in parametric form we find

$$x(s, t) = s + \alpha(t),\quad y(s, t) = s + \beta(t),\quad u(s, t) = s + \gamma(t).$$

Clearly, Γ does not satisfy the characteristic equations. Now, the general solution to the PDE is given by $u = y + f(x - y)$. Using the Cauchy data $u(x, x) = 1$ we find $f(0) = 1 - x$, which is not possible since the LHS is a fixed number whereas the RHS is a variable expression. Hence, the problem has no solutions ■

Example 11.4
Solve the Cauchy problem

$$\begin{aligned} u_x - u_y =& 1 \\ u(x, 0) =& x^2. \end{aligned} \tag{11.5}$$

Solution.
The initial curve is given parametrically by

$$\Gamma : \; x_0(t) = t, \quad y_0(t) = 0, \quad u_0(t) = t^2.$$

We have

$$a(x_0(t), y_0(t), u_0(t))\frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t))\frac{dx_0}{dt}(t) = 1 \neq 0$$

so the Cauchy problem has a unique solution.
The characteristic equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{1}.$$

Using the first two fractions we find $x + y = c_1$. Using the first and the third fractions we find $u - x = c_2$. Thus, the general solution can be represented by

$$u = x + f(x + y)$$

where f is an arbitrary differentiable function. Using the Cauchy data $u(x, 0) = x^2$ we find $x^2 - x = f(x)$. Hence, the unique solution is given by

$$u = x + (x + y)^2 - (x + y) = (x + y)^2 - y \; ■$$

Example 11.5
Solve the initial value problem

$$u_t + uu_x = x, \quad u(x, 0) = 1.$$

Solution.
The initial curve is given parametrically by

$$\Gamma: \ x_0(t) = t, \quad y_0(t) = 0, \quad u_0(t) = 1.$$

We have

$$a(x_0(t), y_0(t), u_0(t))\frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t))\frac{dx_0}{dt}(t) = -1 \neq 0$$

so the Cauchy problem has a unique solution.
The characteristic equations are

$$\frac{dt}{1} = \frac{dx}{u} = \frac{du}{x}.$$

Since

$$\frac{dt}{1} = \frac{d(x+u)}{x+u}$$

we find that $(x+u)e^{-t} = c_1$. Now, using the last two fractions we find $u^2 - x^2 = c_2$. Hence, the general solution is given by

$$f((x+u)e^{-t}, u^2 - x^2) = 0$$

where f is an arbitrary differentiable function. Using the Cauchy data we find $c_1 = 1 + x$ and $c_2 = 1 - x^2 = 2(1+x) - (1+x)^2 = 2c_1 - c_1^2$. Thus,

$$u^2 - x^2 = 2(x+u)e^{-t} - (x+u)^2 e^{-2t}$$

or

$$u - x = 2e^{-t} - (x+u)e^{-2t}.$$

This can be reduced further as follows: $u + ue^{-2t} = x + 2e^{-t} - xe^{-2t} = 2e^{-t} + x(1 - e^{-2t}) \Longrightarrow u = \frac{2e^{-t}}{1+e^{-2t}} + x\frac{1-e^{-2t}}{1+e^{-2t}} = \text{sech}(t) + x\tanh(t)$ ■

Example 11.6
Solve the initial value problem

$$uu_x + u_y = 1$$

with the initial curve

$$\Gamma: x_0(t) = 2t^2, \quad y_0(t) = 2t, \quad u_0(t) = 0, \quad t > 0.$$

Solution.
We have

$$a(x_0(t), y_0(t), u_0(t))\frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t))\frac{dx_0}{dt}(t) = -4t \neq 0,\ t > 0$$

so the Cauchy problem has a unique solution.
The characteristic equations in parametric form are given by the system of ODEs

$$\begin{aligned}\frac{dx}{ds} =& u\\ \frac{dy}{ds} =& 1\\ \frac{du}{ds} =& 1.\end{aligned}$$

Thus, the solution of this system depends on two parameters s and t. Solving the last two equations we find

$$y(s,t) = s + \beta(t), \quad u(s,t) = s + \gamma(t).$$

Solving the first equation with u being replaced by $s + \gamma(t)$ we find

$$x(s,t) = \frac{1}{2}s^2 + \gamma(t)s + \alpha(t).$$

Using the initial conditions

$$x(0,t) = 2t^2,\ y(0,t) = 2t,\ u(0,t) = 0$$

we find

$$x(s,t) = \frac{1}{2}s^2 + 2t^2, \quad y(s,t) = s + 2t, \quad u(s,t) = s.$$

Eliminating s and t we find

$$(u-y)^2 + u^2 = 2x.$$

Solving this quadratic equation in u to find

$$2u = y \pm (4x - y^2)^{\frac{1}{2}}.$$

The solution surface satisfying $u = 0$ on $y^2 = 2x$ is given by

$$2u = y - (4x - y^2)^{\frac{1}{2}}.$$

This represents a solution surface only when $y^2 < 4x$. The solution does not exist for $y^2 > 4x$ ■

Practice Problems

Exercise 11.1
Solve
$$(y-u)u_x + (u-x)u_y = x-y$$
with the condition $u(x, \frac{1}{x}) = 0$.

Exercise 11.2
Solve the linear equation
$$yu_x + xu_y = u$$
with the Cauchy data $u(x,0) = x^3$.

Exercise 11.3
Solve
$$x(y^2+u)u_x - y(x^2+u)u_y = (x^2-y^2)u$$
with the Cauchy data $u(x,-x) = 1$.

Exercise 11.4
Solve
$$xu_x + yu_y = xe^{-u}$$
with the Cauchy data $u(x,x^2) = 0$.

Exercise 11.5
Solve the initial value problem
$$xu_x + u_y = 0, \quad u(x,0) = f(x)$$
using the characteristic equations in parametric form.

Exercise 11.6
Solve the initial value problem
$$u_t + au_x = 0, \quad u(x,0) = f(x).$$

Exercise 11.7
Solve the initial value problem
$$au_x + u_y = u^2, \quad u(x,0) = \cos x$$

Exercise 11.8
Solve the initial value problem

$$u_x + xu_y = u, \quad u(1, y) = h(y).$$

Exercise 11.9
Solve the initial value problem

$$uu_x + u_y = 0, \quad u(x, 0) = f(x).$$

Exercise 11.10
Solve the initial value problem

$$\sqrt{1 - x^2}u_x + u_y = 0, \quad u(0, y) = y.$$

Sample Exam Questions

Exercise 11.11
Consider

$$xu_x + 2yu_y = 0.$$

(i) Find and sketch the characteristics.
(ii) Find the solution with $u(1, y) = e^y$.
(iii) What happens if you try to find the solution satisfying either $u(0, y) = g(y)$ or $u(x, 0) = h(x)$ for given functions g and h?
(iv) Explain, using your picture of the characteristics, what goes wrong at $(x, y) = (0, 0)$.

Exercise 11.12
Solve the equation $u_x + u_y = u$ subject to the condition $u(x, 0) = \cos x$.

Exercise 11.13
(a) Find the general solution of the equation

$$u_x + yu_y = u.$$

(b) Find the solution satisfying the Cauchy data $u(x, 3e^x) = 2$.
(c) Find the solution satisfying the Cauchy data $u(x, e^x) = e^x$.

Exercise 11.14
Solve the Cauchy problem

$$u_x + 4u_y = x(u + 1)$$

$$u(x, 5x) = 1.$$

Exercise 11.15
Solve the Cauchy problem

$$u_x - u_y = u$$

$$u(x, -x) = \sin x.$$

Exercise 11.16
(a) Find the characteristics of the equation

$$yu_x + xu_y = 0.$$

(b) Sketch some of the characteristics.
(c) Find the solution satisfying the boundary condition $u(0,y) = e^{-y^2}$.
(d) In which region of the plane is the solution uniquely determined?

Exercise 11.17
Consider the equation $u_x + yu_y = 0$. Is there a solution satisfying the extra condition
(a) $u(x,0) = 1$
(b) $u(x,0) = x$?
If yes, give a formula; if no, explain why.

Second Order Linear Partial Differential Equations

In this chapter we consider the three fundamental second order linear partial differential equations of parabolic, hyperbolic, and elliptic type. These types arise in many applications such as the wave equation, the heat equation and the Laplace's equation. We will study the solvability of each of these equations.

12 Second Order PDEs in Two Variables

In this section we will briefly review second order partial differential equations.

A **second order partial differential equation in the variables** x and y is an equation of the form

$$F(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}) = 0. \tag{12.1}$$

If Equation (12.1) can be written in the form

$$A(x, y, u, u_x, u_y)u_{xx}+B(x, y, u, u_x, u_y)u_{xy}+C(x, y, u, u_x, u_y)u_{yy} = D(x, y, u, u_x, u_y) \tag{12.2}$$

then we say that the equation is **quasilinear**.

If Equation (12.1) can be written in the form

$$A(x, y)u_{xx} + B(x, y)u_{xy} + C(x, y)u_{yy} = D(x, y, u, u_x, u_y) \tag{12.3}$$

then we say that the equation is **semilinear**.

If Equation (12.1) can be written in the form

$$A(x, y)u_{xx}+B(x, y)u_{xy}+C(x, y)u_{yy}+D(x, y)u_x+E(x, y)u_y+F(x, y)u = G(x, y) \tag{12.4}$$

then we say that the equation is **linear**.
A linear equation is said to be **homogeneous** when $G(x, y) \equiv 0$ and **non-homogeneous** otherwise.
Equation (12.4) resembles the general equation of a conic section

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

which is classified as either parabolic, hyperbolic, or elliptic based on the sign of the **discriminant** $B^2 - 4AC$. We do the same for a second order linear partial differential equation:
• **Hyperbolic:** This occurs if $B^2 - 4AC > 0$ at a given point in the domain of u.
• **Parabolic:** This occurs if $B^2 - 4AC = 0$ at a given point in the domain of u.
• **Elliptic:** This occurs if $B^2 - 4AC < 0$ at a given point in the domain of u.

Example 12.1
Determine whether the equation $u_{xx} + xu_{yy} = 0$ is hyperbolic, parabolic or elliptic.

Solution.
Here we are given $A = 1, B = 0$, and $C = x$. Since $B^2 - 4AC = -4x$, the given equation is hyperbolic if $x < 0$, parabolic if $x = 0$ and elliptic if $x > 0$ ■

Second order partial differential equations arise in many areas of scientific applications. In what follows we list some of the well-known models that are of great interest:
1. The *heat equation* in one-dimensional space is given by

$$u_t = ku_{xx}$$

where k is a constant.
2. The *wave equation* in one-dimensional space is given by

$$u_{tt} = c^2 u_{xx}$$

where c is a constant.
3. The *Laplace equation* is given by

$$\Delta u = u_{xx} + u_{yy} = 0.$$

Practice Problems

Exercise 12.1
Classify each of the following equation as hyperbolic, parabolic, or elliptic:
(a) Wave propagation: $u_{tt} = c^2 u_{xx},\ c > 0$.
(b) Heat conduction: $u_t = cu_{xx},\ c > 0$.
(c) Laplace's equation: $\Delta u = u_{xx} + u_{yy} = 0$.

Exercise 12.2
Classify the following linear scalar PDE with constant coefficents as hyperbolic, parabolic or elliptic.
(a) $u_{xx} + 4u_{xy} + 5u_{yy} + u_x + 2u_y = 0$.
(b) $u_{xx} - 4u_{xy} + 4u_{yy} + 3u_x + 4u = 0$.
(c) $u_{xx} + 2u_{xy} - 3u_{yy} + 2u_x + 6u_y = 0$.

Exercise 12.3
Find the region(s) in the $xy-$plane where the equation

$$(1+x)u_{xx} + 2xyu_{xy} - y^2 u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch these regions.

Exercise 12.4
Show that $u(x,t) = \cos x \sin t$ is a solution to the problem

$$\begin{aligned} u_{tt} &= u_{xx} \\ u(x,0) &= 0 \\ u_t(x,0) &= \cos x \\ u_x(0,t) &= 0 \end{aligned}$$

for all $x, t > 0$.

Exercise 12.5
Classify each of the following PDE as linear, quasilinear, semi-linear, or nonlinear.
(a) $u_t + uu_x = uu_{xx}$
(b) $xu_{tt} + tu_{xx} + u^3 u_x^2 = t + 1$
(c) $u_{tt} = c^2 u_{xx}$
(d) $u_{tt}^2 + u_x = 0$.

Exercise 12.6
Show that, for all $(x, y) \neq (0, 0)$, $u(x, y) = \ln(x^2 + y^2)$ is a solution of

$$u_{xx} + u_{yy} = 0,$$

and that, for all $(x, y, z) \neq (0, 0, 0)$, $u(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ is a solution of

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

Exercise 12.7
Consider the eigenvalue problem

$$u_{xx} = \lambda u, \quad 0 < x < L$$

$$u_x(0) = k_0 u(0)$$
$$u_x(L) = -k_L u(L)$$

with Robin boundary conditions, where k_0 and k_L are given positive numbers and $u = u(x)$. Can this system have a nontrivial solution $u \not\equiv 0$ for $\lambda > 0$? Hint: Multiply the first equation by u and integrate over $x \in [0, L]$.

Exercise 12.8
Show that $u(x, y) = f(x)g(y)$, where f and g are arbitrary differentiable functions, is a solution to the PDE

$$uu_{xy} = u_x u_y.$$

Exercise 12.9
Show that for any $n \in \mathbb{N}$, the function $u_n(x, y) = \sin nx \sinh ny$ is a solution to the Laplace equation

$$\Delta u = u_{xx} + u_{yy} = 0.$$

Exercise 12.10
Solve

$$u_{xy} = xy.$$

Sample Exam Questions

Exercise 12.11
Classify each of the following second-oder PDEs according to whether they are hyperbolic, parabolic, or elliptic:
(a) $2u_{xx} - 4u_{xy} + 7u_{yy} - u = 0.$
(b) $u_{xx} - 2\cos x u_{xy} - \sin^2 x u_{yy} = 0.$
(c) $yu_{xx} + 2(x-1)u_{xy} - (y+2)u_{yy} = 0.$

Exercise 12.12
Let $c > 0$. By computing u_x, u_{xx}, u_t, and u_{tt} show that

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

is a solution to the PDE

$$u_{tt} = c^2 u_{xx}$$

where f is twice differentiable function and g is a differentiable function. Then compute and simplify $u(x,0)$ and $u_t(x,0)$.

Exercise 12.13
Consider the second-order PDE

$$yu_{xx} + u_{xy} - x^2 u_{yy} - u_x - u = 0.$$

Determine the region D in $\mathbb{R}^2$, if such a region exists, that makes this PDE: (a) hyperbolic, (b) parabolic, (c) elliptic.

Exercise 12.14
Consider the second-order hyperbolic PDE

$$u_{xx} + 2u_{xy} - 3u_{yy} = 0.$$

Use the change of variables $v(x,y) = y - 3x$ and $w(x,y) = x + y$ to solve the given equation.

Exercise 12.15
Solve the Cauchy problem

$$u_{xx} + 2u_{xy} - 3u_{yy} = 0.$$

$$u(x,2x) = 1, \; u_x(x,2x) = x.$$

13 Hyperbolic Type: The Wave equation

The wave equation has many physical applications from sound waves in air to magnetic waves in the Sun's atmosphere. However, the simplest systems to visualize and describe are waves on a stretched elastic string.
Initially the string is horizontal with two fixed ends say a left end L and a right end R. Then from end L we shake the string and we notice a wave propogate through the string. The aim is to try and determine the vertical displacement from the $x-$axis of the string, $u(x,t)$, as a function of position x and time t. A displacement of a tiny piece of the string between points P and Q is shown in Figure 13.1.

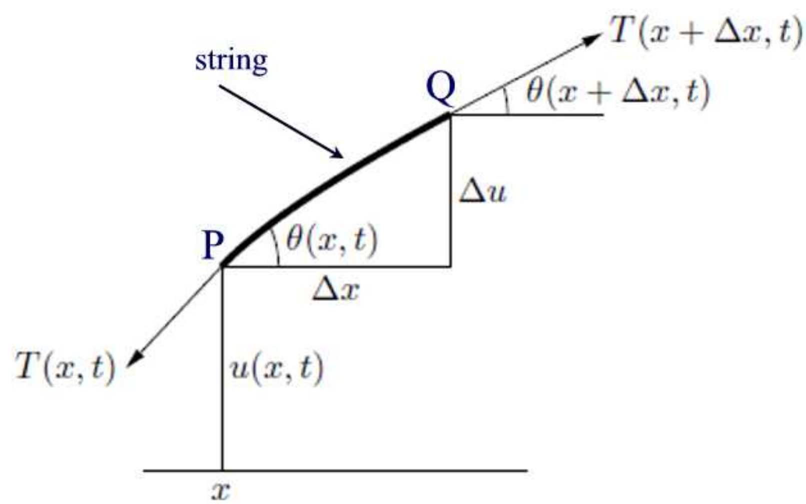


Figure 13.1

where
• $\theta(x,t)$ is the angle between the string and a horizontal line at position x and time t;
• $T(x,t)$ is the tension in the string at position x and time t;
• $\rho(x)$ is the mass density of the string at position x.
To derive the wave equation we need to make some simplifying assumptions:
(1) The density of the string, ρ, is constant so that the mass of the string between P and Q is simply ρ times the length of the string between P and

Q, where the length of the string is Δs given by

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta u)^2} = \Delta x\sqrt{1 + \left(\frac{\Delta u}{\Delta x}\right)^2} \approx \Delta x\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}$$

(2) The displacement, $u(x,t)$, and its derivatives are assumed small so that

$$\Delta s \approx \Delta x$$

and the mass of the portion of the string is

$$\rho\Delta x.$$

(3) The only forces acting on this portion of the string are the tensions $T(x,t)$ at P and $T(x+\Delta x,t)$ at Q. (In physics, tension is the magnitude of the pulling force exerted by a string). The gravitational force is neglected.
(4) Our tiny string element moves only vertically. Then the net horizontal force on it must be zero.
Next, we consider the forces acting on the typical string portion shown in Figure 13.1. These forces are:
(i) tension pulling to the right, which has magnitude $T(x+\Delta x,t)$, and acts at an angle $\theta(x+\Delta x,t)$ above the horizontal.
(ii) tension pulling to the left, which has magnitude $T(x,t)$, and acts at an angle $\theta(x,t)$ above the horizontal.
Now we resolve the forces into their horizontal and vertical components.
• **Horizontal:** The net horizontal force of the tiny string is $T(x+\Delta x,t)\cos\theta(x+\Delta x,t) - T(x,t)\cos\theta(x,t)$. Since there is no horizontal motion, we must have

$$T(x,t)\cos\theta(x,t) = T(x+\Delta x,t)\cos\theta(x+\Delta x,t) = T. \tag{13.1}$$

• **Vertical:** At P the tension force is $-T(x,t)\sin\theta(x,t)$ whereas at Q the force is $T(x+\Delta x,t)\sin\theta(x+\Delta x,t)$. Then Newton's Law of motion

$$\text{mass} \times \text{acceleration} = \text{Applied Forces}$$

gives

$$\rho\Delta x\frac{\partial^2 u}{\partial t^2} = T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t).$$

Dividing by T and using (13.1) we obtain

$$\begin{aligned}\frac{\rho}{T}\Delta x\frac{\partial^2 u}{\partial t^2} =&\frac{T(x+\Delta x,t)\sin\theta(x+\Delta x,t)}{T(x+\Delta x,t)\cos\theta(x+\Delta x,t)} - \frac{T(x,t)\sin\theta(x,t)}{T(x,t)\cos\theta(x,t)}\\ =&\tan\theta(x+\Delta x,t)-\tan\theta(x,t).\end{aligned}$$

But

$$\tan\theta(x,t)=\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}=u_x(x,t).$$

Likewise,

$$\tan\theta(x+\Delta x,t)=u_x(x+\Delta x,t).$$

Hence, we get

$$\frac{\rho}{T}\Delta x u_{tt}(x,t)=u_x(x+\Delta x,t)-u_x(x,t).$$

Dividing by Δx and letting $\Delta x\to 0$ we obtain

$$\frac{\rho}{T}u_{tt}(x,t)=u_{xx}(x,t)$$

or

$$u_{tt}(x,t)=c^2u_{xx}(x,t) \tag{13.2}$$

where $c^2=\frac{T}{\rho}$. We call c the **wave speed.**

D'Alembert Solution of (13.2)
Let $v=x+ct$ and $w=x-ct$. Then by application of the chain rule we find

$$\begin{aligned}u_t =&c(u_v-u_w)\\ u_x =&u_v+u_w\\ u_{tt} =&c^2(u_{vv}-2u_{vw}+u_{ww})\\ u_{xx} =&u_{vv}+2u_{vw}+u_{ww}.\end{aligned}$$

Substituting into (13.2) we obtain

$$c^2(u_{vv}+2u_{vw}+u_{ww})=c^2(u_{vv}-2u_{vw}+u_{ww})$$

and this simplifies to

$$4c^2u_{vw}=0 \text{ or } u_{vw}=0.$$

It follows that

$$u(v,w) = f(v) + g(w)$$

where f and g are arbitrary differentiable functions. Now, writing u in terms of x and y we find the general solution

$$u(x,y) = f(x+ct) + g(x-ct).$$

D'Alembert's solution involves two arbitrary functions that are determined (normally) by two initial conditions.

Example 13.1
Find the solution to the Cauchy problem

$$\begin{aligned} u_{tt} =& c^2 u_{xx} \\ u(x,0) =& v(x) \\ u_t(x,0) =& w(x). \end{aligned}$$

Solution.
We have

$$u(x,0) = f(x) + g(x) = v(x)$$

and

$$u_t(x,0) = cf'(x) - cg'(x) = w(x)$$

which implies that

$$f(x) - g(x) = \frac{1}{c}W(x) = \frac{1}{c}\int w(x)dx.$$

Therefore,

$$g(x) = \frac{1}{2}(v(x) - \frac{1}{c}W(x)).$$

Hence,

$$f(x) = \frac{1}{2}(v(x) + \frac{1}{c}W(x)).$$

Finally,

$$\begin{aligned} u(x,t) =& \frac{1}{2}[v(x-ct) + v(x+ct) + \frac{1}{c}(W(x+ct) - W(x-ct))] \\ =& \frac{1}{2}[v(x-ct) + v(x+ct) + \frac{1}{c}\int_{x-ct}^{x+ct} w(s)ds] \blacksquare \end{aligned}$$

Practice Problems

Exercise 13.1
Show that if $v(x,t)$ and $w(x,t)$ satisfy equation (13.2) then $\alpha v + \beta w$ is also a solution to (13.2), where α and β are constants.

Exercise 13.2
Show that any linear time independent function $u(x,t) = ax + b$ is a solution to equation (13.2).

Exercise 13.3
Find a solution to (13.2) that satisfies the homogeneous conditions $u(x,0) = u(0,t) = u(L,t) = 0$.

Exercise 13.4
Solve the initial value problem

$$\begin{aligned} u_{tt} =& 9u_{xx} \\ u(x,0) =& \cos x \\ u_t(x,0) =& 0. \end{aligned}$$

Exercise 13.5
Solve the initial value problem

$$\begin{aligned} u_{tt} =& u_{xx} \\ u(x,0) =& \frac{1}{1+x^2} \\ u_t(x,0) =& 0. \end{aligned}$$

Exercise 13.6
Solve the initial value problem

$$\begin{aligned} u_{tt} =& 4u_{xx} \\ u(x,0) =& 1 \\ u_t(x,0) =& \cos(2\pi x). \end{aligned}$$

Exercise 13.7
Solve the initial value problem

$$\begin{aligned} u_{tt} =& 25u_{xx} \\ u(x,0) =& v(x) \\ u_t(x,0) =& 0 \end{aligned}$$

where

$$v(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0. \end{cases}$$

Exercise 13.8
Solve the initial value problem

$$\begin{aligned} u_{tt} =& c^2 u_{xx} \\ u(x,0) =& e^{-x^2} \\ u_t(x,0) =& \cos^2 x. \end{aligned}$$

Exercise 13.9
Prove that the wave equation, $u_{tt} = c^2 u_{xx}$ satisfies the following properties, which are known as invariance properties. If $u(x,t)$ is a solution, then
(i) Any translate, $u(x-y,t)$ where y is a fixed constant, is also a solution.
(ii) Any derivative, say $u_x(x,t)$, is also a solution.
(iii) Any dilation, $u(ax,at)$, is a solution, for any fixed constant a.

Exercise 13.10
Find $v(r)$ if $u(r,t) = \frac{v(r)}{r}\cos nt$ is a solution to the PDE

$$u_{rr} + \frac{2}{r}u_r = u_{tt}.$$

Sample Exam Questions

Exercise 13.11
Find the solution of the wave equation on the real line $(-\infty < x < +\infty)$ with the initial conditions

$$u(x,0) = e^x, \quad u_t(x,0) = \sin x.$$

Exercise 13.12
The total energy of the string (the sum of the kinetic and potential energies) is defined as

$$E(t) = \frac{1}{2}\int_0^L (u_t^2 + c^2 u_x^2)dx.$$

(a) Using the wave equation derive the equation of conservation of energy

$$\frac{dE(t)}{dt} = c^2(u_t(L,t)u_x(L,t) - u_t(0,t)u_x(0,t)).$$

(b) Assuming fixed ends boundary conditions, that is the ends of the string are fixed so that $u(0,t) = u(L,t) = 0$, for all $t > 0$, show that the energy is constant.
(c) Assuming free ends boundary conditions for both $x = 0$ and $x = L$, that is both $u(0,t)$ and $u(L,t)$ vary with t, show that the energy is constant.

Exercise 13.13
For a wave equation with damping

$$u_{tt} - c^2 u_{xx} + du_t = 0, \quad d > 0, \ 0 < x < L$$

with the fixed ends boundary conditions show that the total energy decreases.

Exercise 13.14
(a) Verify that for any twice differentiable $R(x)$ the function

$$u(x,t) = R(x - ct)$$

is a solution of the wave equation $u_{tt} = c^2 u_{xx}$. Such solutions are called traveling waves.
(b) Show that the potential and kinetic energies (see Exercise 13.12) are equal for the traveling wave solution in (a).

Exercise 13.15
Find the solution of the Cauchy wave equation

$$u_{tt} = 4u_{xx}$$

$$u(x,0) = x^2, \quad u_t(x,0) = \sin 2x.$$

Simplify your answer as much as possible.

14 Parabolic Type: The Heat Equation in One-Dimensional Space

In this section, We will look at a model for describing the distribution of temperature in a solid material as a function of time and space.

Before we begin our discussion of the mathematics of the heat equation, we must first determine what is meant by the term **heat**? Heat is type of energy known as **thermal energy**. Heat travels in waves like other forms of energy, and can change the matter it touches. It can heat it up and cause chemical reactions like burning to occur.

Heat can be released through a chemical reaction (such as the nuclear reactions that make the Sun "burn") or can be trapped for a limited time by insulators. It is often released along with other kinds of energy such as light waves or sound waves. For example, a burning candle releases light and heat waves. On the other hand, an explosion releases light, heat, and sound waves. The most common units of heat are BTU (British Thermal Unit), Calorie and Joule.

Consider now a rod made of homogeneous heat conducting material (i.e. it is composed of the exact same material and no foreign bodies are in it) of uniform density ρ and constant cross section A, placed along the $x-$axis from $x = 0$ to $x = L$ as shown in Figure 14.1.

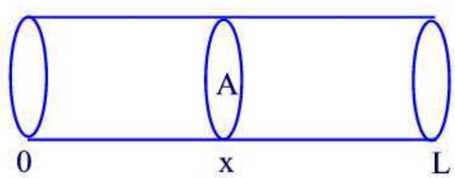


Figure 14.1

Assume the heat flows only in the $x-$direction, with the lateral sides well insulated, and the only way heat can enter or leave the rod is at either end. Also we assume that the temperature of the rod is constant at any point of the cross section. In other words, temperature will only vary in x and we can hence consider the rod to be a one spatial dimensional rod. We will also assume that heat energy in any piece of the rod is conserved.

Let $u(x, t)$ be the temperature of the cross section at the point x and the time t. Consider a portion U of the rod from x to $x + \Delta x$ of length Δx as shown in Figure 14.2.

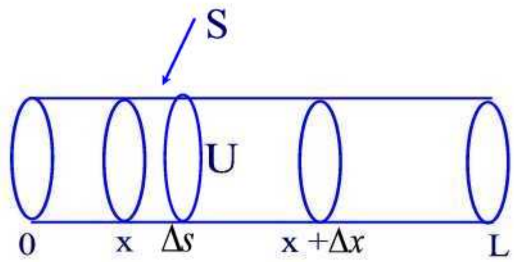


Figure 14.2

Consider the portion S of U of height Δs. From the theory of heat conduction, the quantity of heat ΔQ from x to $x+\Delta s$ at time t is given by

$$\Delta Q = c\rho u(x,t)\Delta V$$

where ΔV is the volume of S and c is the **specific heat**, that is, the amount of heat energy that it takes to raise one unit of mass of the material by one unit of temperature.
But S is a cylinder of height Δs and area of base A so that $\Delta V = A\Delta s$. Hence,

$$\Delta Q = c\rho A u(x,t)\Delta s.$$

The quantity of heat in the portion U is given by

$$Q(t) = \int_x^{x+\Delta x} c\rho A u(s,t)ds.$$

By differentiating we take the partial of u to find the change in heat with respect to time.

$$\frac{dQ}{dt} = \int_x^{x+\Delta x} c\rho A u_t(s,t)ds.$$

Assuming that u is continuously differentiable, we can apply the mean value theorem for integrals and find $x \le \xi \le x+\Delta x$ such that

$$\int_x^{x+\Delta x} u_t(s,t)ds = \Delta x u_t(\xi,t).$$

Thus, the rate of change of heat in U is given by

$$\frac{dQ}{dt} = c\rho A\Delta x u_t(\xi,t).$$

On the other hand, by Fourier (or Fick's) law of heat conduction, the rate of heat flow through any cross section is proportional to the area A and the negative gradient of the temperature normal to the cross section, and heat flows in the direction of decreasing temperature. Thus, the rate of heat flowing in U through the cross section at x is $-KAu_x(x,t)$ and the rate of heat flowing out of U through the cross section at $x+\Delta x$ is $-KAu_x(x+\Delta x, t)$, where K is the **thermal conductivity** of the rod.
Now, the conservation of energy law states

rate of change of heat in U = rate of heat flowing in $-$ rate of heat flowing out

or mathematically written as,

$$c\rho A\Delta x u_t(\xi, t) = -KAu_x(x,t) + KAu_x(x+\Delta x, t)$$

or

$$c\rho A\Delta x u_t(\xi, t) = KA[u_x(x+\Delta x, t) - u_x(x,t)].$$

Dividing this last equation by $cA\rho\Delta x$ and letting $\Delta x \to 0$ we obtain

$$u_t(x,t) = ku_{xx}(x,t) \tag{14.1}$$

where $k = \frac{K}{c\rho}$ is called the **diffusivity constant.**
Equation (14.1) is the one dimensional heat equation which is second order, linear, homogeneous, and of parabolic type.
The non-homogeneous heat equation

$$u_t = ku_{xx} + f(x)$$

is known as the heat equation with an external **heat source** $f(x)$. An example of an exterenal heat source is the heat generated from a candle placed under the bar.
The function

$$E(t) = \int_0^L c\rho u(x,t)dx$$

is called the **total thermal energy** at time t of the entire rod.

Example 14.1
The two ends of a uniform rod of length L are insulated. There is a constant source of thermal energy $q_0 \neq 0$ and the temperature is initially

$u(x,0) = f(x)$.
(a) Write the equation and the boundary conditions for this model.
(b) Calculate the total thermal energy of the entire rod.

Solution.
(a) The model is given by the PDE

$$c\rho u_t(x,t) = Ku_{xx} + q_0$$

with boundary conditions

$$u_x(0,t) = u_x(L,t) = 0.$$

(b) First note that

$$\begin{aligned}\frac{d}{dt}\int_0^L c\rho u(x,t)dx =& \int_0^L c\rho u_t(x,t)dx = \int_0^L Ku_{xx}dx + \int_0^L q_0 dx\\ =& Ku_x\big|_0^L + q_0L = q_0L\end{aligned}$$

since $u_x(0,t) = u_x(L,t) = 0$. Integrating in time from 0 to t we find

$$E(t) = q_0Lt + C.$$

But $C = E(0) = \int_0^L c\rho u(x,0)dx = \int_0^L c\rho f(x)dx$. Hence, the total thermal energy is given by

$$E(t) = \int_0^L c\rho f(x)dx + q_0Lt \blacksquare$$

Initial Boundary Value Problems
In order to solve the heat equation we must give the problem some initial conditions. If you recall from the theory of ODE, the number of conditions required for solving initial value problems always matched the highest order of the derivative in the equation.
In partial differential equations the same idea holds except now we have to pay attention to the variable we are differentiating with respect to as well. So, for the heat equation we have got a first order time derivative and so we will need one initial condition and a second order spatial derivative and so we will need two boundary conditions.

For the initial condition, we define the temperature of every point along the rod at time $t = 0$ by

$$u(x, 0) = f(x)$$

where f is a given (prescribed) function of x. This function is known as the **initial temperature distribution.**
The boundary conditions will tell us something about what the temperature is doing at the ends of the bar. The conditions are given by

$$u(0, t) = T_0 \text{ and } u(L, t) = T_L.$$

and they are called as the **Dirichlet conditions.** In this case, the general form of the heat equation initial boundary value problem is to find $u(x, t)$ satisfying

$$\begin{aligned} u_t(x, t) =& k u_{xx}(x, t), \quad 0 \leq x \leq L, \ t > 0 \\ u(x, 0) =& f(x), \quad 0 \leq x \leq L \\ u(0, t) =& T_0, u(L, t) = T_L, \quad t > 0. \end{aligned}$$

In the case of insulated endpoints, i.e. there is no heat flow out of them, we use the boundary conditions

$$u_x(0, t) = u_x(L, t) = 0.$$

These conditions are examples of what is known as **Neumann boundary** conditions. In this case, the general form of the heat equation initial boundary value problem is to find $u(x, t)$ satisfying

$$\begin{aligned} u_t(x, t) =& k u_{xx}(x, t), \quad 0 \leq x \leq L, \ t > 0 \\ u(x, 0) =& f(x), \quad 0 \leq x \leq L \\ u_x(0, t) =& u_x(L, t) = 0, \quad t > 0. \end{aligned}$$

Practice Problems

Exercise 14.1
Show that if $u(x,t)$ and $v(x,t)$ satisfy equation (14.1) then $\alpha u + \beta v$ is also a solution to (14.1), where α and β are constants.

Exercise 14.2
Show that any linear time independent function $u(x,t) = ax + b$ is a solution to equation (14.1).

Exercise 14.3
Find a linear time independent solution u to (14.1) that satisfies $u(0,t) = T_0$ and $u(L,T) = T_L$.

Exercise 14.4
Show that to solve (14.1) with the boundary conditions $u(0,t) = T_0$ and $u(L,t) = T_L$ it suffices to solve (14.1) with the **homogeneous boundary conditions** $u(0,t) = u(L,t) = 0$.

Exercise 14.5
Find a solution to (14.1) that satisfies the conditions $u(x,0) = u(0,t) = u(L,t) = 0$.

Exercise 14.6
Let (I) denote equation (14.1) together with intial condition $u(x,0) = f(x)$, where f is not the zero function, and the homogeneous boundary conditions $u(0,t) = u(L,t) = 0$. Suppose a nontrivial solution to (I) can be written in the form $u(x,t) = X(x)T(t)$. Show that X and T satisfy the ODE

$$X'' - \frac{\lambda}{k}X = 0 \text{ and } T' - \lambda T = 0$$

for some constant λ.

Exercise 14.7
Consider again the solution $u(x,t) = X(x)T(t)$. Clearly, $T(t) = T(0)e^{\lambda t}$. Suppose that $\lambda > 0$.
(a) Show that $X(x) = Ae^{x\sqrt{\alpha}} + Be^{-x\sqrt{\alpha}}$, where $\alpha = \frac{\lambda}{k}$ and A and B are arbitrary constants.
(b) Show that A and B satisfy the two equations $A + B = 0$ and $A(e^{L\sqrt{\alpha}} - e^{-L\sqrt{\alpha}}) = 0$.
(c) Show that $A = 0$ leads to a contradiction.
(d) Using (b) and (c) show that $e^{L\sqrt{\alpha}} = e^{-L\sqrt{\alpha}}$. Show that this equality leads to a contradiction. We conclude that $\lambda < 0$.

Exercise 14.8
Consider the results of the previous exercise.
(a) Show that $X(x) = c_1 \cos \beta x + c_2 \sin \beta x$ where $\beta = \sqrt{\frac{-\lambda}{k}}$.
(b) Show that $\lambda = \lambda_n = -\frac{kn^2\pi^2}{L^2}$, where n is an integer.

Exercise 14.9
Show that $u(x,t) = \sum_{k=1}^{n} u_k(x,t)$, where $u_n(x,t) = c_n e^{-\frac{kn^2\pi^2}{L^2}t} \sin\left(\frac{n\pi}{L}\right) x$ satisfies (14.1) and the homogeneous boundary conditions.

Exercise 14.10
Suppose that a wire is stretched between 0 and a. Describe the boundary conditions for the temperature $u(x,t)$ when
(i) the left end is kept at 0 degrees and the right end is kept at 100 degrees; and
(ii) when both ends are insulated.

Exercise 14.11
Let $u_t = u_{xx}$ for $0 < x < \pi$ and $t > 0$ with boundary conditions $u(0,t) = 0 = u(\pi,t)$ and initial condition $u(x,0) = f(x)$. Let $E(t) = \int_0^\pi (u_t^2 + u_x^2)dx$. Show that $E'(t) < 0$.

Exercise 14.12
Suppose

$$u_t = u_{xx} + 4, \ u_x(0,t) = 5, \ u_x(L,t) = 6, \ u(x,0) = f(x).$$

Calculate the total thermal energy of the one-dimensional rod (as a function of time).

Sample Exam Questions

Exercise 14.13
Consider the heat equation

$$u_t = ku_{xx}$$

for $x \in (0,1)$ and $t > 0$, with boundary conditions $u(0,t) = 2$ and $u(1,t) = 3$ for $t > 0$ and initial conditions $u(x,0) = x$ for $x \in (0,1)$. A function $v(x)$ that satisfies the equation $v''(x) = 0$, with conditions $v(0) = 2$ and $v(1) = 3$ is called a **steady-state solution.** That is, the steady-state solutions of the heat equation are those solutions that don't depend on time. Find $v(x)$.

Exercise 14.14
Consider the equation for the one-dimensional rod of length L with given heat energy source:

$$u_t = u_{xx} + q(x).$$

Assume that the initial temperature distribution is given by $u(x,0) = f(x)$. Find the equilibrium (steady state) temperature distribution in the following cases.
(a) $q(x) = 0, u(0) = 0, u(L) = T$.
(b) $q(x) = 0, u_x(0) = 0, u(L) = T$.
(c) $q(x) = 0, u(0) = T, u_x(L) = \alpha$.

Exercise 14.15
Consider the equation for the one-dimensional rod of length L with insulated ends:

$$c\rho u_t = Ku_{xx}, \quad u_x(0,t) = u_x(L,t) = 0.$$

(a) Give the expression for the total thermal energy of the rod.
(b) Show using the equation and the boundary conditions that the total thermal energy is constant.

Exercise 14.16
Suppose

$$u_t = u_{xx} + x, \quad u(x,0) = f(x), \; u_x(0,t) = \beta, \; u_x(L,t) = 7.$$

(a) Calculate the total thermal energy of the one-dimensional rod (as a function of time).
(b) From part (a) find the value of β for which a steady-state solution exist.
(c) For the above value of β find the steady state solution.

15 An Introduction to Fourier Series

In this and the next section we will have a brief look to the subject of Fourier series. The point here is to do just enough to allow us to do some basic solutions to partial differential equations later in the book.
Motivation: In Calculus we have seen that certain functions may be represented as power series by means of the Taylor expansions. These functions must have infinitely many derivatives, and the series provide a good approximation only in some (often small) vicinity of a reference point.
Fourier series constructed of trigonometric rather than power functions, and can be used for functions not only not differentiable, but even discontinuous at some points. The main limitation of Fourier series is that the underlying function should be periodic.
Recall from calculus that a **function series** is a series where the summands are functions. Examples of function series include power series, Laurent series, Fourier series, etc.
Unlike series of numbers, there exist many types of convergence of series of functions, namely, pointwise, uniform, etc. We say that a series of functions $\sum_{n=1}^{\infty} f_n(x)$ **converges pointwise** to a function f if and only if the sequence of partial sums

$$S_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$$

converges pointwise to f. We write

$$\sum_{n=1}^{\infty} f_n(x) = \lim_{n\to\infty} S_n(x) = f(x).$$

Likewise, we say that a series of functions $\sum_{n=1}^{\infty} f_n(x)$ **converges uniformly** to a function f if and only if the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ converges uniformly to f.
In this section we introduce a type of series of functions known as **Fourier series**. They are given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right], \quad -L \le x \le L \qquad (15.1)$$

where a_n and b_n are called the **Fourier coefficients.** The expression on the right is called a **trigonometric series.** Note that we begin the series with $\frac{a_0}{2}$ as opposed to simply a_0 to simplify the coefficient formula for a_n that we

will derive later in this section.
The main questions we want to consider next are the questions of determining which functions can be represented by Fourier series and if so how to compute the coefficients a_n and b_n.
Before answering these questions, we look at some of the properties of Fourier series.

Periodicity Property
Recall that a function f is said to be **periodic** with period $T > 0$ if $f(x+T) = f(x)$ for all $x, x+T$ in the domain of f. The smallest value of T for which f is periodic is called the **fundamental period.** A graph of a $T-$periodic function is shown in Figure 15.1.

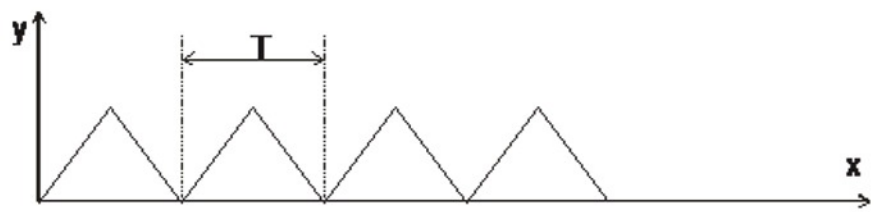


Figure 15.1

For a $T-$periodic function we have

$$f(x) = f(x+T) = f(x+2T) = \cdots .$$

Note that the definite integral of a $T-$periodic function is the same over any interval of length T. By Exercise 15.1 below, if f and g are two periodic functions with common period T, then the product fg and an arbitrary linear combination $c_1 f + c_2 g$ are also periodic with period T. It is an easy exercise to show that the Fourier series (15.1) is periodic with fundamental period $2L$.

Orthogonality Property
Recall from Calculus that for each pair of vectors $\vec{u}$ and $\vec{v}$ we associate a scalar quantity $\vec{u} \cdot \vec{v}$ called the **dot product** of $\vec{u}$ and $\vec{v}$. We say that $\vec{u}$ and $\vec{v}$ are **orthogonal** if and only if $\vec{u} \cdot \vec{v} = 0$. We want to define a similar concept for functions.
Let f and g be two functions with domain the closed interval $[a, b]$. We define

a function that takes a pair of functions to a scalar. Symbolically, we write

$$< f, g >= \int_a^b f(x)g(x)dx.$$

We call $< f, g >$ the **inner product** of f and g. We say that f and g are **orthogonal** if and only if $< f, g >= 0$. A set of functions is said to be **mutually orthogonal** if each distinct pair of functions in the set is orthogonal.

Example 15.1
Show that the set $\left\{1, \cos\left(\frac{n\pi}{L}x\right), \sin\left(\frac{n\pi}{L}x\right) : n \in \mathbb{N}\right\}$ is mutually orthogonal in $[-L, L]$.

Solution.
We have

$$\int_{-L}^{L} 1 \cdot \cos\left(\frac{n\pi}{L}x\right)dx = \frac{L}{n\pi}\left[\sin\left(\frac{n\pi}{L}x\right)\right]_{-L}^{L} = 0$$

and

$$\int_{-L}^{L} 1 \cdot \sin\left(\frac{n\pi}{L}x\right)dx = -\frac{L}{n\pi}\left[\cos\left(\frac{n\pi}{L}x\right)\right]_{-L}^{L} = 0.$$

Now, for $n \neq m$ we have

$$\begin{aligned}\int_{-L}^{L} \cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx =& \frac{1}{2}\int_{-L}^{L}\left[\cos\left(\frac{(m+n)\pi}{L}x\right) + \cos\left(\frac{(m-n)\pi}{L}x\right)\right]dx\\ =& \frac{1}{2}\left[\frac{L}{(m+n)\pi}\sin\left(\frac{(m+n)\pi}{L}x\right)\right.\\ &\left.+ \frac{L}{(m-n)\pi}\sin\left(\frac{(m-n)\pi}{L}x\right)\right]_{-L}^{L} = 0\end{aligned}$$

where we used the trigonometric identity

$$\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)].$$

In the exercises below, we show that

$$\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = 0$$

and

$$\int_{-L}^{L} \cos\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = 0 \blacksquare$$

The reason we care about these functions being orthogonal is because we will exploit this fact to develop a formula for the coefficients in our Fourier series.

Now, in order to answer the first question mentioned earlier, that is, which functions can be expressed as a Fourier series expansion, we need to introduce some mathematical concepts.
A function $f(x)$ is said to be **piecewise continuous** on $[a, b]$ if it is continuous in $[a, b]$ execept possibly at finitely many points of discontinuity within the interval $[a, b]$, and at each point of discontinuity, the right- and left-handed limits of f exist. An example of a piecewise continuous function is the function

$$f(x) = \begin{cases} x & 0 \leq x < 1 \\ x^2 - x & 1 \leq x \leq 2. \end{cases}$$

We will say that f is **piecewise smooth** in $[a, b]$ if and only if $f(x)$ as well as its derivatives are piecewise continuous.
The following theorem, proven in more advanced books, ensures that a Fourier decomposition can be found for any function which is piecewise smooth.

Theorem 15.1
Let f be a $2L$-periodic function. If f is a piecewise smooth on $[-L, L]$ then for all points of discontinuity $x \in (-L, L)$ we have

$$\frac{f(x^-) + f(x^+)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right].$$

where as for points of continuity $x \in (-L, L)$ we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right].$$

Remark 15.1
(1) Almost all functions occurring in practice are piecewise smooth functions.
(2) Given a non-periodic function f on $[-L, L]$. The above theorem applies to the periodic extension F of f where $F(x + 2nL) = f(x)$ $(n \in \mathbb{Z})$ and $F(x) = f(x)$ on $[-L, L]$.

Convergence Results of Fourier Series

We list few of the results regarding the convergence of Fourier series:
(1) The type of convergence in the above theorem is pointwise convergence.
(2) The convergence is uniform for a continuous function f on $[-L, L]$ such that $f(-L) = f(L)$.
(3) The convergence is uniform whenever $\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2)$ is convergent.
(4) If $f(x)$ is periodic, continuous, and has a piecewise continuous derivative, then the Fourier Series corresponding to f converges uniformly to $f(x)$ for the entire real line.
(5) The convergence is uniform on any closed interval that does not contain a point of discontinuity.

Euler-Fourier Formulas

Next, we will answer the second question mentioned earlier, that is, the question of finding formulas for the coefficients a_n and b_n. These formulas for a_n and b_n are called Euler-Fourier formulas which we derive next. We will assume that the RHS in (15.1) converges uniformly to $f(x)$ on the interval $[-L, L]$. Integrating both sides of (15.1) we obtain

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{L} \frac{a_0}{2}dx + \int_{-L}^{L} \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right] dx.$$

Since the trigonometric series is assumed to be uniformly convergent, from Section 2, we can interchange the order of integration and summation to obtain

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{L} \frac{a_0}{2}dx + \sum_{n=1}^{\infty} \int_{-L}^{L} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right] dx.$$

But

$$\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)dx = \frac{L}{n\pi} \sin\left(\frac{n\pi}{L}x\right)\Big]_{-L}^{L} = 0$$

and likewise

$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)dx = -\frac{L}{n\pi} \cos\left(\frac{n\pi}{L}x\right)\Big]_{-L}^{L} = 0.$$

Thus,

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx.$$

To find the other Fourier coefficients, we recall the results of Exercises 15.2 - 15.3 below.

$$\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx = \begin{cases} L & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = \begin{cases} L & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx = 0, \ \forall m, n.$$

Now, to find the formula for the Fourier coefficients a_m for $m > 0$, we multiply both sides of (15.1) by $\cos\left(\frac{m\pi}{L}x\right)$ and integrate from $-L$ to L to otbain

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi}{L}x\right) = \int_{-L}^{L} \frac{a_0}{2}\cos\left(\frac{m\pi}{L}x\right)dx + \sum_{n=1}^{\infty}\left[a_n\int_{-L}^{L}\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx\right.$$
$$\left. + b_n\int_{-L}^{L}\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\right]dx.$$

Hence,

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi}{L}x\right)dx = a_m L$$

and therefore

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi}{L}x\right)dx.$$

Likewise, we can show that

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi}{L}x\right)dx.$$

Example 15.2
Find the Fourier series expansion of

$$f(x) = \begin{cases} 0, & x \leq 0 \\ x, & x > 0 \end{cases}$$

on the interval $[-\pi, \pi]$.

Solution.
We have

$$\begin{aligned}
a_0 =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} xdx = \frac{\pi}{2}\\
a_n =& \frac{1}{\pi}\int_0^{\pi} x\cos nxdx = \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} = \frac{(-1)^n - 1}{\pi n^2}\\
b_n =& \frac{1}{\pi}\int_0^{\pi} x\sin nxdx = \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} = \frac{(-1)^{n+1}}{n}
\end{aligned}$$

Hence,

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos(nx) + \frac{(-1)^{n+1}}{n}\sin(nx)\right] \blacksquare$$

Example 15.3
Apply Theorem 15.1 to the function in Example 15.2.

Solution.
Let F be a periodic extension of f of period 2π. Thus, $f(x) = F(x)$ on the interval $[-\pi, \pi]$. Clearly, F is a piecewise smooth function so that by the previous thereom we can write

$$\frac{\pi}{4}+\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos(nx) + \frac{(-1)^{n+1}}{n}\sin(nx)\right] = \begin{cases} \frac{\pi}{2}, & \text{if } x = -\pi\\ f(x), & \text{if } -\pi < x < \pi\\ \frac{\pi}{2}, & \text{if } x = \pi\end{cases}$$

Taking $x = \pi$ we have the identity

$$\frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{\pi n^2}(-1)^n = \frac{\pi}{2}$$

which can be simplified to

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

This provides a method for computing an approximate value of π $\blacksquare$

Remark 15.2
An example of a function that does not have a Fourier series representation is the function $f(x) = \frac{1}{x^2}$ on $[-L, L]$. For example, the coefficient a_0 for this function does not exist. Thus, not every function can be written as a Fourier series expansion.

The final topic of discussion here is the topic of differentiation and integration of Fourier series. In particular we want to know if we can differentiate a Fourier series term by term and have the result be the Fourier series of the derivative of the function. Likewise we want to know if we can integrate a Fourier series term by term and arrive at the Fourier series of the integral of the function. Answers to these questions are provided next.

Theorem 15.2
A Fourier series of a *piecewise smooth* function f can always be integrated term by term and the result is a convergent infinite series that always converges to $\int_{-L}^{L} f(x)dx$ even if the original series has jumps.

Theorem 15.3
A Fourier series of a continuous function $f(x)$ can be differentiated term by term if $f'(x)$ is *piecewise smooth.* The result of the differentiation is the Fourier series of $f'(x)$.

Practice Problems

Exercise 15.1
Let f and g be two functions with common domain D and common period T. Show that
(a) fg is periodic of period T.
(b) $c_1f + c_2g$ is periodic of period T, where c_1 and c_2 are real numbers.

Exercise 15.2
Show that for $m \neq n$ we have
(a) $\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = 0$ and
(b) $\int_{-L}^{L} \cos\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = 0$.

Exercise 15.3
Compute the following integrals:
(a) $\int_{-L}^{L} \cos^2\left(\frac{n\pi}{L}x\right)dx$.
(b) $\int_{-L}^{L} \sin^2\left(\frac{n\pi}{L}x\right)dx$.
(c) $\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx$.

Exercise 15.4
Find the Fourier coefficients of

$$f(x) = \begin{cases} -\pi, & -\pi \leq x < 0 \\ \pi, & 0 < x < \pi \\ 0, & x = 0, \pi \end{cases}$$

on the interval $[-\pi, \pi]$.

Exercise 15.5
Find the Fourier series of $f(x) = x^2 - \frac{1}{2}$ on the interval $[-1, 1]$.

Exercise 15.6
Find the Fourier series of the function

$$f(x) = \begin{cases} -1, & -2\pi < x < -\pi \\ 0, & -\pi < x < \pi \\ 1, & \pi < x < 2\pi. \end{cases}$$

Exercise 15.7
Find the Fourier series of the function

$$f(x) = \begin{cases} 1 + x, & -2 \leq x \leq 0 \\ 1 - x, & 0 < x \leq 2. \end{cases}$$

Exercise 15.8
Show that $f(x) = \frac{1}{x}$ is not piecewise continuous on $[-1, 1]$.

Exercise 15.9
Assume that $f(x)$ is continuous and has period $2L$. Prove that

$$\int_{-L}^{L} f(x)dx = \int_{-L+a}^{L+a} f(x)dx$$

is independent of $a \in \mathbb{R}$. In particular, it does not matter over which interval the Fourier coefficients are computed as long as the interval length is $2L$. [Remark: This result is also true for piecewise continuous functions].

Exercise 15.10
Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 2 & 1 \leq x < 3 \end{cases}$$

and extended periodically with period 3 to $\mathbb{R}$ so that $f(x + 3) = f(x)$ for all x.
(i) Find the Fourier series of $f(x)$.
(ii) Discuss its limit: In particular, does the Fourier series converge pointwise or uniformly to its limit, and what is this limit?
(iii) Plot the graph of $f(x)$ and the limit of the Fourier series.

Sample Exam Questions

Exercise 15.11
For the following functions $f(x)$ on the interval $-L < x < L$, determine the coefficients $a_n,\ n = 0, 1, 2, \cdots$ and $b_n, n \in \mathbb{N}$ of the Fourier series expansion.
(a) $f(x) = 1$.
(b) $f(x) = 2 + \sin\left(\frac{\pi x}{L}\right)$.
(c) $f(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x > 0. \end{cases}$
(d) $f(x) = x$.

Exercise 15.12
Let $f(t)$ be the function with period 2π defined as

$$f(t) = \begin{cases} 2 & \text{if } 0 \leq x \leq \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} < x \leq 2\pi \end{cases}$$

$f(t)$ has a Fourier series and that series is equal to

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt).$$

Find $\frac{a_3}{b_3}$.

Exercise 15.13
Let $f(x) = x^3$ on $[-\pi, \pi]$, extended periodically to all of $\mathbb{R}$. Find the Fourier coefficients $a_n,\ n = 1, 2, 3, \cdots$.

Exercise 15.14
Let $f(x)$ be the square wave function

$$f(x) = \begin{cases} -\pi & -\pi \leq x < 0 \\ \pi & 0 \leq x \leq \pi \end{cases}$$

extended periodically to all of $\mathbb{R}$. To what value does the Fourier series of $f(x)$ converge when $x = 0$?

Exercise 15.15
(a) Find the Fourier series of

$$f(x) = \begin{cases} 1 & -\pi \leq x < 0 \\ 2 & 0 \leq x \leq \pi \end{cases}$$

extended periodically to all of $\mathbb{R}$. Simplify your coefficients as much as possible.
(b) Use (a) to evaluate the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)}$. Hint: Evaluate the Fourier series at $x = \frac{\pi}{2}$.

16 Fourier Sines Series and Fourier Cosines Series

In this section we discuss some important properties of Fourier series when the underlying function f is either even or odd.
A function f is **odd** if it satisfies $f(-x) = -f(x)$ for all x in the domain of f whereas f is **even** if it satisfies $f(-x) = f(x)$ for all x in the domain of f. Now, we recall from Exercises (1.6)-(1.7) the following facts about even and odd functions. If $f(x)$ is even then

$$\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx.$$

If f is odd then

$$\int_{-L}^{L} f(x)dx = 0.$$

Using just these basic facts we can figure out some important properties of the Fourier series we get for odd or even functions.

Example 16.1
Show the following
(a) If f and g are either both even or both odd then fg is even.
(b) If f is odd and g is even then fg is odd.

Solution.
(a) Suppose that both f and g are even. Then $(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x)$. That is, fg is even. Now, suppose that both f and g are odd. Then $(fg)(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = (fg)(x)$. That is, fg is even.
(b) f is odd and g is even. Then $(fg)(-x) = f(-x)g(-x) = -f(x)g(x) = -(fg)(x)$. That is, fg is odd ■

Example 16.2
(a) Find the value of the integral $\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$ when f is even.
(b) Find the value of the integral $\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$ when f is odd.

Solution.
(a) Since the function $\sin\left(\frac{n\pi}{L}x\right)$ is odd and f is even, we have that $f(x)\sin\left(\frac{n\pi}{L}x\right)$

is odd so that

$$\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx = 0$$

(b) Since the function $\cos\left(\frac{n\pi}{L}x\right)$ is even and f is odd, we have that $f(x)\cos\left(\frac{n\pi}{L}x\right)$ is odd so that

$$\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx = 0 \blacksquare$$

Even and Odd Extensions

Let $f : [0, L] \to \mathbb{R}$ be a piecewise smooth function. We define the **odd extension** of this function on the interval $-L \le x \le L$ by

$$f_{odd}(x) = \begin{cases} f(x) & 0 < x \le L \\ -f(-x) & -L \le x < 0 \\ 0 & x = 0. \end{cases}$$

This function will be odd on the interval $[-L, L]$, and will be equal to $f(x)$ on the interval $(0, L]$. We can then further extend this function to the entire real line by defining it to be $2L$ periodic. Let $\overline{f}_{odd}$ denote this extension. We note that $\overline{f}_{odd}$ is an odd function and piecewise smooth so that by Theorem 15.1 it possesses a Fourier series expansion, and from the fact that it is odd all of the a_n's are zero. Moreover, in the interval $[0, L]$ we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right). \tag{16.1}$$

We call (16.1) the **Fourier sine series** of f.
The coefficients b_n are given by the formula

$$\begin{aligned} b_n =& \frac{1}{L}\int_{-L}^{L} \overline{f}_{odd} \sin\left(\frac{n\pi}{L}x\right)dx = \frac{2}{L}\int_{0}^{L} \overline{f}_{odd} \sin\left(\frac{n\pi}{L}x\right)dx \\ =& \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx \end{aligned}$$

since $\overline{f}_{odd}\sin\left(\frac{n\pi}{L}x\right)$ is an even function.
Likewise, we can define the **even extension** of f on the interval $-L \le x \le L$ by

$$f_{even}(x) = \begin{cases} f(x) & 0 \le x \le L \\ f(-x) & -L \le x < 0. \end{cases}$$

We can then further extend this function to the entire real line by defining it to be $2L$ periodic. Let $\overline{f}_{even}$ denote this extension. Again, we note that $\overline{f}_{even}$ is equal to the original function $f(x)$ on the interval upon which $f(x)$ is defined. Since $\overline{f}_{even}$ is piecewise smooth, by Theorem 15.1 it possesses a Fourier series expansion, and from the fact that it is even all of the b_n's are zero. Moreover, in the interval $[0, L]$ we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right). \tag{16.2}$$

We call (16.2) the **Fourier cosine series** of f. The coefficients a_n are given by

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx,\ n = 0, 1, 2, \cdots.$$

Example 16.3
Graph the odd and even extensions of the function $f(x) = x,\ 0 \le x \le 1.$

Solution.
We have $f_{odd}(x) = x$ for $-1 \le x \le 1$. The odd extension of f is shown in Figure 16.1(a). Likewise,

$$f_{even}(x) = \begin{cases} x & 0 \le x \le 1 \\ -x & -1 \le x < 0. \end{cases}$$

The even extension is shown in Figure 16.1(b) ■

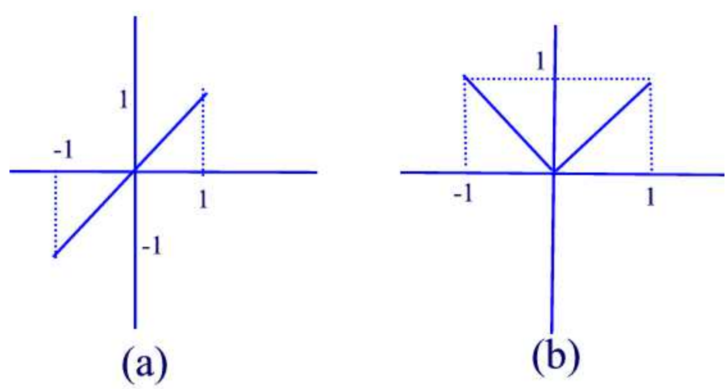


Figure 16.1

Example 16.4
Find the Fourier sine series of the function

$$f(x) = \begin{cases} x, & 0 \leq x \leq \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \leq x \leq \pi. \end{cases}$$

Solution.
We have

$$b_n = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x \sin nx dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x)\sin nx dx\right].$$

Using integration by parts we find

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x \sin nx dx &= \left[-\frac{x}{n}\cos nx\right]_0^{\frac{\pi}{2}} + \frac{1}{n}\int_0^{\frac{\pi}{2}} \cos nx dx \\ &= -\frac{\pi \cos(n\pi/2)}{2n} + \frac{1}{n^2}[\sin nx]_0^{\frac{\pi}{2}} \\ &= -\frac{\pi \cos(n\pi/2)}{2n} + \frac{\sin(n\pi/2)}{n^2}\end{aligned}$$

while

$$\begin{aligned}\int_{\frac{\pi}{2}}^{\pi} (\pi - x)\sin nx dx &= \left[-\frac{(\pi - x)}{n}\cos nx\right]_{\frac{\pi}{2}}^{\pi} - \frac{1}{n}\int_{\frac{\pi}{2}}^{\pi} \cos nx dx \\ &= \frac{\pi \cos(n\pi/2)}{2n} - \frac{1}{n^2}[\sin nx]_{\frac{\pi}{2}}^{\pi} \\ &= \frac{\pi \cos(n\pi/2)}{2n} + \frac{\sin(n\pi/2)}{n^2}\end{aligned}$$

Thus,

$$b_n = \frac{4\sin(n\pi/2)}{\pi n^2},$$

and the Fourier sine series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} \frac{4\sin(n\pi/2)}{\pi n^2}\sin nx = \sum_{n=1}^{\infty} \frac{4(-1)^{2n-1}}{\pi(2n-1)^2}\sin(2n-1)x \blacksquare$$

Practice Problems

Exercise 16.1
Give an example of a function that is both even and odd.

Exercise 16.2
Graph the odd and even extensions of the function $f(x) = 1,\ 0 \le x \le 1$.

Exercise 16.3
Graph the odd and even extensions of the function $f(x) = L - x$ for $0 \le x \le L$.

Exercise 16.4
Graph the odd and even extensions of the function $f(x) = 1 + x^2$ for $0 \le x \le L$.

Exercise 16.5
Find the Fourier cosine series of the function

$$f(x) = \begin{cases} x, & 0 \le x \le \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \le x \le \pi \end{cases}$$

Exercise 16.6
Find the Fourier cosine series of $f(x) = x$ on the interval $[0, \pi]$.

Exercise 16.7
Find the Fourier sine series of $f(x) = 1$ on the interval $[0, \pi]$.

Exercise 16.8
Find the Fourier sine series of $f(x) = \cos x$ on the interval $[0, \pi]$.

Exercise 16.9
Find the Fourier cosine series of $f(x) = e^{2x}$ on the interval $[0, 1]$.

Sample Exam Questions

Exercise 16.10
For the following functions on the interval $[0, L]$, find the coefficients b_n of the Fourier sine expansion.
(a) $f(x) = \sin\left(\frac{2\pi}{L}x\right)$.
(b) $f(x) = 1$
(c) $f(x) = \cos\left(\frac{\pi}{L}x\right)$.

Exercise 16.11
For the following functions on the interval $[0, L]$, find the coefficients a_n of the Fourier cosine expansion.
(a) $f(x) = 5 + \cos\left(\frac{\pi}{L}x\right)$.
(b) $f(x) = x$
(c)

$$f(x) = \begin{cases} 1 & 0 < x \leq \frac{L}{2} \\ 0 & \frac{L}{2} < x \leq L \end{cases}$$

Exercise 16.12
Consider a function $f(x)$, defined on $0 \leq x \leq L$, which is even (symmetric) around $x = \frac{L}{2}$. Show that the even coefficients (n even) of the Fourier sine series are zero.

Exercise 16.13
Consider a function $f(x)$, defined on $0 \leq x \leq L$, which is odd around $x = \frac{L}{2}$. Show that the even coefficients (n even) of the Fourier cosine series are zero.

Exercise 16.14
The Fourier sine series of $f(x) = \cos\left(\frac{\pi x}{L}\right)$ for $0 \leq x \leq L$ is given by

$$\cos\left(\frac{\pi x}{L}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad n \in \mathbb{N}$$

where

$$b_1 = 0, \quad b_n = \frac{2n}{(n^2-1)\pi}[1 + (-1)^n].$$

Using term-by-term integration, find the Fourier cosine series of $\sin\left(\frac{n\pi x}{L}\right)$.

Exercise 16.15
Consider the function

$$f(x) = \begin{cases} 1 & 0 \le x < 1 \\ 2 & 1 \le x < 2 \end{cases}$$

(a) Sketch the even extension of f.
(b) Find a_0 in the Fourier series for the even extension of f.
(c) Find a_n $(n = 1, 2, \cdots)$ in the Fourier series for the even extension of f.
(d) Find b_n in the Fourier series for the even extension of f.
(e) Write the Fourier series for the even extension of f.

17 Separation of Variables for PDEs

Finding analytic solutions to PDEs is essentially impossible. Most of the PDE techniques involve a mixture of analytic, qualitative and numeric approaches. Of course, there are some easy PDEs too. If you are lucky your PDE has a solution with separable variables. In this section we discuss the application of the method of separation of variables in the solution of PDEs. In developing a solution to a partial differential equation by separation of variables, one assumes that it is possible to separate the contributions of the independent variables into separate functions that each involve only one independent variable. Thus, the method consists of the following steps
1. Factorize the (unknown) dependent variable of the PDE into a product of functions, each of the factors being a function of one independent variable. That is,

$$u(x,y) = X(x)Y(y).$$

2. Substitute into the PDE, and divide the resulting equation by $X(x)Y(y)$.
3. Then the problem turns into a set of separated ODEs (one for $X(x)$ and one for $Y(y)$.)
4. The general solution of the ODEs is found, and boundary initial conditions are imposed.
5. $u(x,y)$ is formed by multiplying together $X(x)$ and $Y(y)$.

We illustrate these steps in the next two examples.

Example 17.1
Find all the solutions of the form $u(x,t) = X(x)T(t)$ of the equation

$$u_{xx} - u_x = u_t$$

Solution.
It is very easy to find the derivatives of a separable function:

$$u_x = X'(x)T(t), u_t = X(x)T'(t) \text{ and } u_{xx} = X''(x)T(t)$$

this is basically a consequence of the fact that differentiation with respect to x sees t as a constant, and vice versa. Now the equation $u_{xx} - u_x = u_t$ becomes

$$X''(x)T(t) - X'(x)T(t) = X(x)T'(t).$$

We can separate variables further. Division by $X(x)T(t)$ gives

$$\frac{X''(x) - X'(x)}{X(x)} = \frac{T'(t)}{T(t)}.$$

The expression on the LHS is a function of x whereas the one on the RHS is a function of t only. They both have to be constant. That is,

$$\frac{X''(x) - X'(x)}{X(x)} = \frac{T'(t)}{T(t)} = \lambda.$$

Thus, we have the following ODEs:

$$X'' - X' - \lambda X = 0 \text{ and } T' = \lambda T.$$

The second equation is easy to solve: $T(t) = Ce^{\lambda t}$. The first equation is solved via the characteristic equation $\omega^2 - \omega - \lambda = 0$, whose solutions are

$$\omega = \frac{1 \pm \sqrt{1+4\lambda}}{2}.$$

If $\lambda > -\frac{1}{4}$ then

$$X(x) = Ae^{\frac{1+\sqrt{1+4\lambda}}{2}x} + Be^{\frac{1-\sqrt{1+4\lambda}}{2}x}.$$

In this case,

$$u(x,t) = De^{\frac{1+\sqrt{1+4\lambda}}{2}x}e^{\lambda t} + Ee^{\frac{1-\sqrt{1+4\lambda}}{2}x}e^{\lambda t}.$$

If $\lambda = -\frac{1}{4}$ then

$$X(x) = Ae^{\frac{x}{2}} + Bxe^{\frac{x}{2}}$$

and in this case

$$u(x,t) = (D + Ex)e^{\frac{x}{2} - \frac{t}{4}}.$$

If $\lambda < -\frac{1}{4}$ then

$$X(x) = Ae^{\frac{x}{2}} \cos\left(\frac{\sqrt{-(1+4\lambda)}}{2}x\right) + Be^{\frac{x}{2}} \sin\left(\frac{\sqrt{-(1+4\lambda)}}{2}x\right).$$

In this case,

$$u(x,t) = D'e^{\frac{x}{2}+\lambda t} \cos\left(\frac{\sqrt{-(1+4\lambda)}}{2}x\right) + B'e^{\frac{x}{2}+\lambda t} \sin\left(\frac{\sqrt{-(1+4\lambda)}}{2}x\right) \blacksquare$$

Example 17.2
Solve Laplace's equation using the separation of variables method

$$\Delta u = u_{xx} + u_{yy} = 0.$$

Solution.
We look for a solution of the form $u(x,y) = X(x)Y(y)$. Substituting in the Laplace's equation, we obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Assuming $X(x)Y(y)$ is nonzero, dividing for $X(x)Y(y)$ and subtracting $\frac{Y''(y)}{Y(y)}$ from both sides, we find:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

The left hand side is a function of x while the right hand side is a function of y. This says that they must equal to a constant. That is,

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

where λ is a constant. This results in the following two ODEs

$$X'' - \lambda X = 0 \text{ and } Y'' + \lambda Y = 0.$$

The solutions of these equations depend on the sign of λ.
• If $\lambda > 0$ then the solutions are given

$$\begin{aligned} X(x) =& Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x} \\ Y(y) =& C\cos(\sqrt{\lambda}y) + D\sin(\sqrt{\lambda}y) \end{aligned}$$

where A, B, C, and D are constants. In this case,

$$\begin{aligned} u(x,t) =& k_1 e^{\sqrt{\lambda}x}\cos(\sqrt{\lambda}y) + k_2 e^{\sqrt{\lambda}x}\sin(\sqrt{\lambda}y) \\ &+ k_3 e^{-\sqrt{\lambda}x}\cos(\sqrt{\lambda}y) + k_4 e^{-\sqrt{\lambda}x}\sin(\sqrt{\lambda}y). \end{aligned}$$

• If $\lambda = 0$ then

$$\begin{aligned} X(x) =& Ax + B \\ Y(y) =& Cy + D \end{aligned}$$

where A, B, and C are arbitrary constants. In this case,

$$u(x,y) = k_1xy + k_2x + k_3y + k_4.$$

• If $\lambda < 0$ then

$$\begin{aligned}X(x) =& A\cos(\sqrt{-\lambda}x) + B\sin(\sqrt{-\lambda}x)\\ Y(y) =& Ce^{\sqrt{-\lambda}y} + De^{-\sqrt{-\lambda}y}\end{aligned}$$

where A, B, C, and D are arbitrary constants. In this case,

$$\begin{aligned}u(x,y) =& k_1\cos(\sqrt{-\lambda}x)e^{\sqrt{-\lambda}y} + k_2\cos(\sqrt{-\lambda}x)e^{-\sqrt{-\lambda}y}\\ &+ k_3\sin(\sqrt{-\lambda}x)e^{\sqrt{-\lambda}y} + k_4\sin(\sqrt{-\lambda}x)e^{-\sqrt{-\lambda}y} \blacksquare\end{aligned}$$

Example 17.3
Solve using the separation of variables method.

$$yu_x - xu_y = 0.$$

Solution.
Substitute $u(x,y) = X(x)Y(y)$ into the given equation we find

$$yX'Y - xXY' = 0.$$

This can be separated into

$$\frac{X'}{xX} = \frac{Y'}{yY}.$$

The left hand side is a function of x while the right hand side is a function of y. This says that they must equal to a constant. That is,

$$\frac{X'}{xX} = \frac{Y'}{yY} = \lambda$$

where λ is a constant. This results in the following two ODEs

$$X' - \lambda xX = 0 \text{ and } Y' - \lambda yY = 0.$$

Solving these equations using the method of separation of variable for ODEs we find $X(x) = Ae^{\frac{\lambda x^2}{2}}$ and $Y(y) = Be^{\frac{\lambda y^2}{2}}$. Thus, the general solution is given by

$$u(x,y) = Ce^{\frac{\lambda(x^2+y^2)}{2}} \blacksquare$$

Practice Problems

Exercise 17.1
Solve using the separation of variables method

$$\Delta u + \lambda u = 0.$$

Exercise 17.2
Solve using the separation of variables method

$$u_t = ku_{xx}.$$

Exercise 17.3
Derive the system of ordinary differential equations for $R(r)$ and $\Theta(\theta)$ that is satisfied by solutions to

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

Exercise 17.4
Derive the system of ordinary differential equations and boundary conditions for $X(x)$ and $T(t)$ that is satisfied by solutions to

$$u_{tt} = u_{xx} - 2u, \quad 0 < x < 1, \; t > 0$$

$$u(0,t) = 0 = u(1,t) \quad t > 0$$

of the form $u(x,t) = X(x)T(t)$. (Note: you do not need to solve for X and T.)

Exercise 17.5
Derive the system of ordinary differential equations and boundary conditions for $X(x)$ and $T(t)$ that is satisfied by solutions to

$$u_t = ku_{xx}, \quad 0 < x < L, \; t > 0$$

$$u(x,0) = f(x), \; u_x(0,t) = 0 = u_x(L,t) \quad t > 0$$

of the form $u(x,t) = X(x)T(t)$. (Note: you do not need to solve for X and T.)

Exercise 17.6
Find all product solutions of the PDE $u_x + u_t = 0$.

Exercise 17.7
Derive the system of ordinary differential equations for $X(x)$ and $Y(y)$ that is satisfied by solutions to

$$3u_{yy} - 5u_{xxxy} + 7u_{xxy} = 0.$$

of the form $u(x,y) = X(x)Y(y)$.

Exercise 17.8
Find the general solution by the method of separation of variables.

$$u_{xy} + u = 0.$$

Exercise 17.9
Find the general solution by the method of separation of variables.

$$u_x - yu_y = 0.$$

Sample Exam Questions

Exercise 17.10
Find the general solution by the method of separation of variables.

$$u_{tt} - u_{xx} = 0.$$

Exercise 17.11
For the following PDEs find the ODEs implied by the method of separation of variables.
(a) $u_t = kr(ru_r)_r$
(b) $u_t = ku_{xx} - \alpha u$
(c) $u_t = ku_{xx} - au_x$
(d) $u_{xx} + u_{yy} = 0$
(e) $u_t = ku_{xxxx}$.

Exercise 17.12
Find all solutions to the following partial differential equation that can be obtained via the separation of variables.

$$u_x - u_y = 0.$$

Exercise 17.13
Separate the PDE $u_{xx} - u_y + u_{yy} = u$ into two ODEs with a parameter. You do not need to solve the ODEs.

18 Solutions of the Heat Equation by the Separation of Variables Method

In this section we apply the method of separation of variables in solving the one spatial dimension of the heat equation.

The Heat Equation with Dirichlet Boundary Conditions
Consider the problem of finding all nontrivial solutions to the heat equation $u_t = ku_{xx}$ that satisfies the initial time condition $u(x,0) = f(x)$ and the Dirichlet boundary conditions $u(0,t) = T_0$ and $u(L,t) = T_L$.
From Exercise 14.4, it suffices to solve the problem with the Dirichlet boundary conditions being replaced by the homogeneous boundary conditions $u(0,t) = u(L,t) = 0$ (that is, the endpoints are assumed to be at zero temperature) with u not the trivial solution. Let's assume that the solution can be written in the form $u(x,t) = X(x)T(t)$. Substituting into the heat equation we obtain

$$\frac{X''}{X} = \frac{T'}{kT}.$$

Since the LHS only depends on x and the RHS only depends on t, there must be a constant λ such that

$$\tfrac{X''}{X} = \lambda \text{ and } \tfrac{T'}{kT} = \lambda.$$

This gives the two ordinary differential equations

$$X'' - \lambda X = 0 \text{ and } T' - k\lambda T = 0.$$

As far as the boundary conditions, we have

$$u(0,t) = 0 = X(0)T(t) \Longrightarrow X(0) = 0$$

and

$$u(L,t) = 0 = X(L)T(t) \Longrightarrow X(L) = 0.$$

Note that T is not the zero function for otherwise $u \equiv 0$ and this contradicts our assumption that u is the non-trivial solution.
Next, we consider the three cases of the sign of λ.

Case 1: $\lambda = 0$
In this case, $X'' = 0$. Solving this equation we find $X(x) = ax + b$. Since $X(0) = 0$ we find $b = 0$. Since $X(L) = 0$ we find $a = 0$. Hence, $X \equiv 0$ and $u(x,t) \equiv 0$. That is, u is the trivial solution.

Case 2: $\lambda > 0$
In this case, $X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$. Again, the conditions $X(0) = X(L) = 0$ imply $A = B = 0$ and hence the solution is the trivial solution.

Case 3: $\lambda < 0$
In this case, $X(x) = A\cos\sqrt{-\lambda}x + B\sin\sqrt{-\lambda}x$. The condition $X(0) = 0$ implies $A = 0$. The condition $X(L) = 0$ implies $B\sin\sqrt{-\lambda}L = 0$. We must have $B \neq 0$ otherwise $X(x) = 0$ and this leads to the trivial solution. Since $B \neq 0$, we obtain $\sin\sqrt{-\lambda}L = 0$ or $\sqrt{-\lambda}L = n\pi$ where $n \in \mathbb{N}$. Solving for λ we find $\lambda = -\frac{n^2\pi^2}{L^2}$. Thus, we obtain infinitely many solutions given by

$$X_n(x) = A_n \sin\frac{n\pi}{L}x, \quad n \in \mathbb{N}.$$

Now, solving the equation

$$T' - \lambda kT = 0$$

by the method of separation of variables we obtain

$$T_n(t) = B_n e^{-\frac{n^2\pi^2}{L^2}kt}, \ n \in \mathbb{N}.$$

Hence, the functions

$$u_n(x,t) = C_n \sin\left(\frac{n\pi}{L}x\right)e^{-\frac{n^2\pi^2}{L^2}kt}, \ n \in \mathbb{N}$$

satisfy $u_t = ku_{xx}$ and the boundary conditions $u(0,t) = u(L,t) = 0$.
Now, in order for these solutions to satisfy the initial value condition $u(x,0) = f(x)$, we invoke the superposition principle of linear PDE to write

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right)e^{-\frac{n^2\pi^2}{L^2}kt}. \tag{18.1}$$

To determine the unknown constants C_n we use the initial condition $u(x,0) = f(x)$ in (18.1) to obtain

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right).$$

Since the right-hand side is the Fourier sine series of f on the interval $[0, L]$, the coefficients C_n are given by

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx. \tag{18.2}$$

Thus, the solution to the heat equation is given by (18.1) with the C_n's calculated from (18.2).

Remark 18.1
According to Exercise 14.4, the solution to the heat equation with non-homogeneous condition $u(0,t) = T_0$ and $u(L,t) = T_L$ is given by

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right)e^{-\frac{n^2\pi^2}{L^2}kt} + T_0 + \frac{T_L - T_0}{L}x.$$

The Heat Equation with Neumann Boundary Conditions
When both ends of the bar are insulated, that is, there is no heat flow out of them, we use the boundary conditions

$$u_x(0,t) = u_x(L,t) = 0.$$

In this case, the general form of the heat equation initial boundary value problem is to find $u(x,t)$ satisfying

$$\begin{aligned} u_t(x,t) =& k u_{xx}(x,t), \quad 0 \le x \le L, \ t > 0 \\ u(x,0) =& f(x), \quad 0 \le x \le L \\ u_x(0,t) =& u_x(L,t) = 0, \quad t > 0. \end{aligned}$$

Since $0 = u_x(0,t) = X'(0)T(t)$ we obtain $X'(0) = 0$. Likewise, $0 = u_x(L,t) = X'(L)T(t)$ implies $X'(L) = 0$. Now, differentiating $X(x) = A\cos\sqrt{-\lambda}x + B\sin\sqrt{-\lambda}x$ with respect to x we find

$$X'(x) = -\sqrt{-\lambda}A\sin\sqrt{-\lambda}x + \sqrt{-\lambda}B\cos\sqrt{-\lambda}x.$$

The conditions $X'(0) = X'(L) = 0$ imply $\sqrt{-\lambda}B = 0$ and $\sqrt{-\lambda}A\sin\sqrt{-\lambda}L = 0$. Hence, $B = 0$ and $\lambda = -\frac{n^2\pi^2}{L^2}$ and

$$X_n(x) = A_n \cos\left(\frac{n\pi}{L}x\right), \ n = 0, 1, 2, \cdots$$

and

$$u_n(x,t) = C_n \cos\left(\frac{n\pi}{L}x\right)e^{-\frac{n^2\pi^2}{L^2}kt}.$$

By the superposition principle, the required solution to the heat equation with Neumann boundary conditions is given by

$$u(x,t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi}{L}x\right)e^{-\frac{n^2\pi^2}{L^2}kt}$$

where

$$C_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx, \ n \in \mathbb{N}.$$

Practice Problems

Exercise 18.1
Find the temperature in a bar of length 2 whose ends are kept at zero and lateral surface insulated if the initial temperature is $f(x) = \sin\left(\frac{\pi}{2}x\right) + 3\sin\left(\frac{5\pi}{2}x\right)$.

Exercise 18.2
Find the temperature in a homogeneous bar of heat conducting material of length L with its end points kept at zero and initial temperature distribution given by $f(x) = \frac{dx}{L^2}(L-x),\ 0 \le x \le L$.

Exercise 18.3
Find the temperature in a thin metal rod of length L, with both ends insulated (so that there is no passage of heat through the ends) and with initial temperature in the rod $f(x) = \sin\left(\frac{\pi}{L}x\right)$.

Exercise 18.4
Solve the following heat equation with Dirichlet boundary conditions

$$u_t = ku_{xx}$$
$$u(0,t) = u(L,t) = 0$$
$$u(x,0) = \begin{cases} 1 & 0 \le x < \frac{L}{2} \\ 2 & \frac{L}{2} \le x \le L. \end{cases}$$

Exercise 18.5
Solve

$$u_t = ku_{xx}$$
$$u(0,t) = u(L,t) = 0$$
$$u(x,0) = 6\sin\left(\frac{9\pi}{L}x\right).$$

Exercise 18.6
Solve

$$u_t = ku_{xx}$$

subject to

$$u_x(0,t) = u_x(L,t) = 0$$
$$u(x,0) = \begin{cases} 0 & 0 \le x < \frac{L}{2} \\ 1 & \frac{L}{2} \le x \le L. \end{cases}$$

Exercise 18.7
Solve

$$u_t = ku_{xx}$$

subject to

$$u_x(0,t) = u_x(L,t) = 0$$
$$u(x,0) = 6 + 4\cos\left(\frac{3\pi}{L}x\right).$$

Exercise 18.8
Solve

$$u_t = ku_{xx}$$

subject to

$$u_x(0,t) = u_x(L,t) = 0$$
$$u(x,0) = -3\cos\left(\frac{8\pi}{L}x\right).$$

Exercise 18.9
Find the general solution $u(x,t)$ of

$$u_t = u_{xx} - u, \quad 0 < x < L, \ t > 0$$

$$u_x(0,t) = 0 = u_x(L,t), \quad t > 0.$$

Briefly describe its behavior as $t \to \infty$.

Sample Exam Questions

Exercise 18.10 *(Energy method)*
Let u_1 and u_2 be two solutions to the Robin boundary value problem

$$u_t = u_{xx} - u, \quad 0 < x < 1, \; t > 0$$

$$u_x(0,t) = u_x(1,t) = 0, \quad t > 0$$
$$u(x,0) = g(x), \quad 0 < x < 1$$

Define $w(x,t) = u_1(x,t) - u_2(x,t)$.
(a) Show that w satisfies the initial value problem

$$w_t = w_{xx} - w, \quad 0 < x < 1, \; t > 0$$

$$w(x,0) = 0, \quad 0 < x < 1$$

(b) Define $E(t) = \int_0^1 w^2(x,t)dx \geq 0$ for all $t \geq 0$. Show that $E'(t) \leq 0$. Hence, $0 \leq E(t) \leq E(0)$ for all $t > 0$.
(c) Show that $E(t) = 0, \; w(x,t) = 0$. Hence, conclude that $u_1 = u_2$.

Exercise 18.11
Consider the heat induction in a bar where the left end temperature is maintained at 0, and the right end is perfectly insulated. We assume $k = 1$ and $L = 1$.
(a) Derive the boundary conditions of the temperature at the endpoints.
(b) Following the separation of variables approach, derive the ODEs for X and T.
(c) Consider the equation in $X(x)$. What are the values of $X(0)$ and $X(1)$? Show that solutions of the form $X(x) = \sin\sqrt{-\lambda}x$ satisfy the ODE and one of the boundary conditions. Can you choose a value of λ so that the other boundary condition is also satisfied?

Exercise 18.12
Using the method of separation of variables find the solution of the heat equation

$$u_t = ku_{xx}$$

satisfying the following boundary and initial conditions:
(a) $u(0,t) = u(L,t) = 0, \; u(x,0) = 6\sin\left(\frac{9\pi x}{L}\right)$
(b) $u(0,t) = u(L,t) = 0, \; u(x,0) = 3\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right)$

Exercise 18.13
Using the method of separation of variables find the solution of the heat equation

$$u_t = ku_{xx}$$

satisfying the following boundary and initial conditions:
(a) $u_x(0,t) = u_x(L,t) = 0,\ u(x,0) = \cos\left(\frac{\pi x}{L}\right) + 4\cos\left(\frac{5\pi x}{L}\right).$
(b) $u_x(0,t) = u_x(L,t) = 0,\ u(x,0) = 5.$

Exercise 18.14
Find the solution of the following heat conduction partial differential equation

$$u_t = 8u_{xx}, \quad 0 < x < 4\pi, \quad t > 0$$

$$u(0,t) = u(4\pi,t) = 0, \quad t > 0$$

$$u(x,0) = 6\sin x, \quad 0 < x < 4\pi.$$

19 Elliptic Type: Laplace's Equations in Rectangular Domains

Boundary value problems are of great importance in physical applications. Mathematically, a boundary-value problem consists of finding a function which satisfies a given partial differential equation and particular boundary conditions. Physically speaking, the problem is independent of time, involving only space coordinates.
Just as initial-value problems are associated with hyperbolic PDE, boundary value problems are associated with PDE of elliptic type. In contrast to initial-value problems, boundary-value problems are considerably more difficult to solve.
The main model example of an elliptic type PDE is the **Laplace equation**

$$\Delta u = u_{xx} + u_{yy} = 0 \tag{19.1}$$

where the symbol Δ is referred to as the **Laplacian**. Solutions of this equation are called **harmonic functions.**

Example 19.1
Show that, for all $(x, y) \neq (0, 0)$, $u(x, y) = \ln(x^2 + y^2)$ is a harmonic function.

Solution.
We have

$$\begin{aligned} u_x =& \frac{2x}{x^2+y^2} \\ u_{xx} =& \frac{2y^2-2x^2}{(x^2+y^2)^2} \\ u_y =& \frac{2y}{x^2+y^2} \\ u_{yy} =& \frac{2x^2-2y^2}{(x^2+y^2)^2} \end{aligned}$$

Plugging these expressions into the equation we find $u_{xx} + u_{yy} = 0$. Hence, $u(x, y)$ is harmonic ■

The Laplace equation is arguably the most important differential equation in

all of applied mathematics. It arises in an astonishing variety of mathematical and physical systems, ranging through fluid mechanics, electromagnetism, potential theory, solid mechanics, heat conduction, geometry, probability, number theory, and on and on.
There are two main modifications of the Laplace equation: the **Poisson equation** (a non-homogeneous Laplace equation):

$$\Delta u = f(x, y)$$

and the **eigenvalue problem** (the **Helmholtz equation**):

$$\Delta u = \lambda u, \quad \lambda \in \mathbb{R}.$$

Solving Laplace's Equation (19.1)
Note first that both independent variables are spatial variables and each variable occurs in a 2nd order derivative and so we will need two boundary conditions for each variable a total of four boundary conditions.
Consider (19.1) in the rectangle

$$\Omega = \{(x, y) : 0 \le x \le a, 0 \le y \le b\}$$

with the Dirichlet boundary conditions

$$u(0, y) = f_1(y),\ u(a, y) = f_2(y),\ u(x, 0) = g_1(x),\ u(x, b) = g_2(x)$$

where $0 \le x \le a$ and $0 \le y \le b$.
The separation of variables method is most successful when the boundary conditions are homogeneous. Thus, solving the Laplace's equation in Ω requires solving four initial boundary conditions problems, where in each problem three of the four conditions are homogeneous. The four problems to be solved are

$$(I)\left\{\begin{array}{c} u_{xx} + u_{yy} = 0 \\ u(0, y) = f_1(y), \\ u(a, y) = u(x, 0) = u(x, b) = 0 \end{array}\right. \qquad (II)\left\{\begin{array}{c} u_{xx} + u_{yy} = 0 \\ u(a, y) = f_2(y), \\ u(0, y) = u(x, 0) = u(x, b) = 0 \end{array}\right.$$

$$(III)\left\{\begin{array}{c} u_{xx} + u_{yy} = 0 \\ u(x, 0) = g_1(x), \\ u(0, y) = u(a, y) = u(x, b) = 0 \end{array}\right. \qquad (IV)\left\{\begin{array}{c} u_{xx} + u_{yy} = 0 \\ u(x, b) = g_2(x), \\ u(0, y) = u(a, y) = u(x, 0) = 0 \end{array}\right.$$

If we let $u_i(x,y)$, $i = 1,2,3,4$, denote the solution of each of the above problems, then the solution to our original system will be

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y).$$

In each of the above problems, we will apply separation of variables to (19.1) and find a product solution that will satisfy the differential equation and the three homogeneous boundary conditions. Using the Principle of Superposition we will find a solution to the problem and then apply the final boundary condition to determine the value of the constant(s) that are left in the problem. The process is nearly identical in many ways to what we did when we were solving the heat equation.
We will illustrate how to find $u(x,y) = u_4(x,y)$. So let's assume that the solution can be written in the form $u(x,y) = X(x)Y(y)$. Substituting in (19.1), we obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Assuming $X(x)Y(y)$ is nonzero, dividing for $X(x)Y(y)$ and subtracting $\frac{Y''(y)}{Y(y)}$ from both sides, we find:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

The left hand side is a function of x while the right hand side is a function of y. This says that they must equal to a constant. That is,

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

where λ is a constant. This results in the following two ODEs

$$X'' - \lambda X = 0 \text{ and } Y'' + \lambda Y = 0.$$

As far as the boundary conditions, we have for all $0 \le x \le a$ and $0 \le y \le b$

$$u(0,y) = 0 = X(0)Y(y) \Longrightarrow X(0) = 0$$

$$u(a,y) = 0 = X(a)Y(y) \Longrightarrow X(a) = 0$$
$$u(x,0) = 0 = X(x)Y(0) \Longrightarrow Y(0) = 0$$
$$u(x,b) = g_2(x) = X(x)Y(b).$$

Note that X and Y are not the zero functions for otherwise $u \equiv 0$ and this contradicts our assumption that u is the non-trivial solution.
Consider the first equation: since $X'' - \lambda X = 0$ the solution depends on the sign of λ. If $\lambda = 0$ then $X(x) = Ax+B$. Now, the conditions $X(0) = X(a) = 0$ imply $A = B = 0$ and so $u \equiv 0$. So assume that $\lambda \neq 0$. If $\lambda > 0$ then $X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$. Now, the conditions $X(0) = X(a) = 0,\ \lambda \neq 0$ imply $A = B = 0$ and hence the solution is the trivial solution. Hence, in order to have a nontrivial solution we must have $\lambda < 0$. In this case,

$$X(x) = A\cos\sqrt{-\lambda}x + B\sin\sqrt{-\lambda}x.$$

The condition $X(0) = 0$ implies $A = 0$. The condition $X(a) = 0$ implies $B\sin\sqrt{-\lambda}a = 0$. We must have $B \neq 0$ otherwise $X(x) = 0$ and this leads to the trivial solution. Since $B \neq 0$, we obtain $\sin\sqrt{-\lambda}a = 0$ or $\sqrt{-\lambda}a = n\pi$ where $n \in \mathbb{N}$. Solving for λ we find $\lambda_n = -\frac{n^2\pi^2}{a^2}$. Thus, we obtain infinitely many solutions given by

$$X_n(x) = \sin\frac{n\pi}{a}x, \quad n \in \mathbb{N}.$$

Now, solving the equation

$$Y'' + \lambda Y = 0$$

we obtain

$$Y_n(y) = a_n e^{\sqrt{-\lambda_n}y} + b_n e^{-\sqrt{-\lambda_n}y} = A_n\cosh\sqrt{-\lambda_n}y + B_n\sinh\sqrt{-\lambda_n}y,\ n \in \mathbb{N}.$$

Using the boundary condition $Y(0) = 0$ we obtain $A_n = 0$ for all $n \in \mathbb{N}$. Hence, the functions

$$u_n(x,y) = B_n\sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right),\ n \in \mathbb{N}$$

satisfy (19.1) and the boundary conditions $u(0,y) = u(a,y) = u(x,0) = 0$.
Now, in order for these solutions to satisfy the boundary value condition $u(x,b) = g_2(x)$, we invoke the superposition principle of linear PDE to write

$$u(x,y) = \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right). \tag{19.2}$$

To determine the unknown constants B_n we use the boundary condition $u(x,b) = g_2(x)$ in (19.2) to obtain

$$g_2(x) = \sum_{n=1}^{\infty}\left(B_n\sinh\left(\frac{n\pi}{a}b\right)\right)\sin\left(\frac{n\pi}{a}x\right).$$

Since the right-hand side is the Fourier sine series of g_2 on the interval $[0, a]$, the coefficients B_n are given by

$$B_n = \left[\frac{2}{a}\int_0^a g_2(x)\sin\left(\frac{n\pi}{a}x\right)dx\right][\sinh\left(\frac{n\pi}{a}b\right)]^{-1}. \tag{19.3}$$

Thus, the solution to the Laplace's equation is given by (19.1) with the B_n's calculated from (19.3).

Example 19.2
Solve

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0, y) = f_1(y), \\ u(a, y) = u(x, 0) = u(x, b) = 0 \end{cases}$$

Solution.
Assume that the solution can be written in the form $u(x, y) = X(x)Y(y)$. Substituting in (19.1), we obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Assuming $X(x)Y(y)$ is nonzero, dividing for $X(x)Y(y)$ and subtracting $\frac{Y''(y)}{Y(y)}$ from both sides, we find:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

The left hand side is a function of x while the right hand side is a function of y. This says that they must equal to a constant. That is,

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

where λ is a constant. This results in the following two ODEs

$$X'' - \lambda X = 0 \text{ and } Y'' + \lambda Y = 0.$$

As far as the boundary conditions, we have for all $0 \le x \le a$ and $0 \le y \le b$

$$u(0, y) = f_1(y) = X(0)Y(y)$$

$$u(a, y) = 0 = X(a)Y(y) \Longrightarrow X(a) = 0$$

$$u(x,0) = 0 = X(x)Y(0) \Longrightarrow Y(0) = 0$$

$$u(x,b) = 0 = X(x)Y(b) \Longrightarrow Y(b) = 0$$

Note that X and Y are not the zero functions for otherwise $u \equiv 0$ and this contradicts our assumption that u is the non-trivial solution.
Consider the second equation: since $Y'' + \lambda Y = 0$ the solution depends on the sign of λ. If $\lambda = 0$ then $Y(y) = Ay + B$. Now, the conditions $Y(0) = Y(b) = 0$ imply $A = B = 0$ and so $u \equiv 0$. So assume that $\lambda \neq 0$. If $\lambda < 0$ then $Y(y) = Ae^{\sqrt{-\lambda}y} + Be^{-\sqrt{-\lambda}y}$. Now, the condition $Y(0) = Y(b) = 0$ imply $A = B = 0$ and hence the solution is the trivial solution. Hence, in order to have a nontrivial solution we must have $\lambda > 0$. In this case,

$$Y(y) = A\cos\sqrt{\lambda}y + B\sin\sqrt{\lambda}y.$$

The condition $Y(0) = 0$ implies $A = 0$. The condition $Y(b) = 0$ implies $B\sin\sqrt{\lambda}b = 0$. We must have $B \neq 0$ otherwise $Y(y) = 0$ and this leads to the trivial solution. Since $B \neq 0$, we obtain $\sin\sqrt{\lambda}b = 0$ or $\sqrt{\lambda}b = n\pi$ where $n \in \mathbb{N}$. Solving for λ we find $\lambda_n = \frac{n^2\pi^2}{b^2}$. Thus, we obtain infinitely many solutions given by

$$Y_n(y) = \sin\left(\frac{n\pi}{b}y\right), \quad n \in \mathbb{N}.$$

Now, solving the equation

$$X'' - \lambda X = 0, \ \lambda > 0$$

we obtain

$$X_n(x) = a_n e^{\sqrt{\lambda_n}x} + b_n e^{-\sqrt{\lambda_n}x} = A_n\cosh\left(\frac{n\pi}{b}x\right) + B_n\sinh\left(\frac{n\pi}{b}x\right), \ n \in \mathbb{N}.$$

However, this is not really suited for dealing with the boundary condition $X(a) = 0$. So, let's also notice that the following is also a solution.

$$X_n(x) = A_n\cosh\left(\frac{n\pi}{b}(x-a)\right) + B_n\sinh\left(\frac{n\pi}{b}(x-a)\right), \ n \in \mathbb{N}.$$

Now, using the boundary condition $X(a) = 0$ we obtain $A_n = 0$ for all $n \in \mathbb{N}$. Hence, the functions

$$u_n(x,y) = B_n\sin\left(\frac{n\pi}{b}y\right)\sinh\left(\frac{n\pi}{b}(x-a)\right), \ n \in \mathbb{N}$$

satisfy (19.1) and the boundary conditions $u(a,y) = u(x,0) = u(x,b) = 0$. Now, in order for these solutions to satisfy the boundary value condition $u(0,y) = f_1(y)$, we invoke the superposition principle of linear PDE to write

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{b}y\right)\sinh\left(\frac{n\pi}{b}(x-a)\right). \tag{19.4}$$

To determine the unknown constants B_n we use the boundary condition $u(0,y) = f_1(y)$ in (19.4) to obtain

$$f_1(y) = \sum_{n=1}^{\infty}\left(B_n \sinh\left(-\frac{n\pi}{b}a\right)\right)\sin\left(\frac{n\pi}{b}y\right).$$

Since the right-hand side is the Fourier sine series of f_1 on the interval $[0,b]$, the coefficients B_n are given by

$$B_n = \left[\frac{2}{b}\int_0^b f_1(y)\sin\left(\frac{n\pi}{b}y\right)dy\right]\left[\sinh\left(-\frac{n\pi}{b}a\right)\right]^{-1}. \tag{19.5}$$

Thus, the solution to the Laplace's equation is given by (19.4) with the B_n's calculated from (19.5) ■

Example 19.3
Solve

$$\begin{gathered}
u_{xx} + u_{yy} = 0, \quad 0 < x < L,\ 0 < y < H \\
u(0,y) = u(L,y) = 0, \quad 0 < y < H \\
u(x,0) = u_y(x,0), \quad u(x,H) = f(x), \quad 0 < x < L.
\end{gathered}$$

Solution.
Using separation of variables we find

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda.$$

We first solve

$$\left\{\begin{array}{ll} X'' - \lambda X = 0 & 0 < x < L \\ X(0) = X(L) = 0 & \end{array}\right.$$

We find $\lambda_n = -\frac{n^2\pi^2}{L^2}$ and

$$X_n(x) = \sin\frac{n\pi}{L}x, \quad n \in \mathbb{N}.$$

Next we need to solve

$$\begin{cases} Y'' + \lambda Y = 0 & 0 < y < H \\ Y(0) - Y'(0) = 0 \end{cases}$$

The solution of the ODE is

$$Y_n(y) = A_n \cosh\left(\frac{n\pi}{L}y\right) + B_n \sinh\left(\frac{n\pi}{L}y\right), \ n \in \mathbb{N}.$$

The boundary condition $Y(0) - Y'(0) = 0$ implies

$$A_n - B_n\frac{n\pi}{L} = 0.$$

Hence,

$$Y_n = B_n\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}y\right) + B_n \sinh\left(\frac{n\pi}{L}y\right), \ n \in \mathbb{N}.$$

Using the superposition principle and the results above we have

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x\left[\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}y\right) + \sinh\left(\frac{n\pi}{L}y\right)\right].$$

Substituting in the condition $u(x,H) = f(x)$ we find

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x\left[\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}H\right) + \sinh\left(\frac{n\pi}{L}H\right)\right].$$

Recall the Fourier sine series of f on $[0,L]$ given by

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x$$

where

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx.$$

Thus, the general solution is given by

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x\left[\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}y\right) + \sinh\left(\frac{n\pi}{L}y\right)\right].$$

with the B_n satisfying

$$B_n\left[\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}H\right) + \sinh\left(\frac{n\pi}{L}H\right)\right] = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx \ \blacksquare$$

Practice Problems

Exercise 19.1
Solve

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(a,y) = f_2(y), \\ u(0,y) = u(x,0) = u(x,b) = 0. \end{cases}$$

Exercise 19.2
Solve

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x,0) = g_1(x), \\ u(0,y) = u(a,y) = u(x,b) = 0. \end{cases}$$

Exercise 19.3
Solve

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x,0) = u(0,y) = 0, \\ u(1,y) = 2y, u(x,1) = 3\sin \pi x + 2x \end{cases}$$

where $0 \le x \le 1$ and $0 \le y \le 1$. Hint: Define $U(x,y) = u(x,y) - 2xy$.

Exercise 19.4
Show that $u(x,y) = x^2 - y^2$ and $u(x,y) = 2xy$ are harmonic functions.

Exercise 19.5
Solve

$$u_{xx} + u_{yy} = 0, \quad 0 \le x \le L, \quad -\frac{H}{2} \le y \le \frac{H}{2}$$

subject to

$$u(0,y) = u(L,y) = 0, \quad -\frac{H}{2} < y < \frac{H}{2}$$

$$u(x, -\frac{H}{2}) = f_1(x), \quad u(x, \frac{H}{2}) = f_2(x), \ 0 \le x \le L.$$

Exercise 19.6
Consider a complex valued function $f(z) = u(x,y) + iv(x,y)$ where $i = \sqrt{-1}$. We say that f is **holomorphic** or **analytic** if and only if f can be expressed as a power series in z, i.e.

$$u(x,y) + iv(x,y) = \sum_{n=0}^{\infty} a_n z^n.$$

(a) By differentiating with respect to x and y show that

$$u_x = v_y \text{ and } u_y = -v_x$$

These are known as the Cauchy-Riemann equations.
(b) Show that $\Delta u = 0$ and $\Delta v = 0$.

Exercise 19.7
Show that Laplace's equation in polar coordinates is given by

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

Exercise 19.8
Solve

$$u_{xx} + u_{yy} = 0, \quad 0 \le x \le 2, \ 0 \le y \le 3$$

subject to

$$u(x,0) = 0, \quad u(x,3) = \frac{x}{2}$$

$$u(0,y) = \sin\left(\frac{4\pi}{3}y\right), \quad u(2,y) = 7.$$

Exercise 19.9
Solve

$$u_{xx} + u_{yy} = 0, \quad 0 \le x \le L, \ 0 \le y \le H$$

subject to

$$u_y(x,0) = 0, \quad u(x,H) = 0$$

$$u(0,y) = u(L,y) = 4\cos\left(\frac{\pi y}{2H}\right).$$

Exercise 19.10
Solve

$$u_{xx} + u_{yy} = 0, \quad x > 0, \ 0 \le y \le H$$

subject to

$$u(0,y) = f(y), |u(x,0)| < \infty$$

$$u_y(x,0) = u_y(x,H) = 0.$$

Sample Exam Questions

Exercise 19.11
Consider Laplace's equation inside a rectangle

$$u_{xx} + u_{yy} = 0,\ 0 \le x \le L,\ 0 \le y \le H$$

subject to the boundary conditions

$$u(0,y) = 0,\ u(L,y) = 0,\ u(x,0) - u_y(x,0) = 0,\ u(x,H) = 20\sin\left(\frac{\pi x}{L}\right) - 5\sin\left(\frac{3\pi x}{L}\right).$$

Find the solution $u(x,y)$.

Exercise 19.12
Solve Laplace'e equation $u_{xx} + u_{yy} = 0$ in the rectangle $0 < x, y < 1$ subject to the conditions

$$u(0,y) = u(1,y) = 0, \quad 0 < y < 1$$
$$u(x,0) = \sin(2\pi x), \quad u_x(x,0) = -2\pi\sin(2\pi x), \quad 0 < x < 1.$$

Exercise 19.13
Find the solution to Laplace's equation on the rectangle $0 < x < 1, 0 < y < 1$ with boundary conditions

$$u(x,0) = 0, \quad u(x,1) = 1$$

$$u_x(0,y) = u_x(1,y) = 0.$$

Exercise 19.14
Solve Laplace's equation on the rectangle $0 < x < a,\ 0 < y < b$ with the boundary conditions

$$u_x(0,y) = -a, \quad u_x(a,y) = 0$$

$$u_y(x,0) = b, \quad u_y(x,b) = 0.$$

Exercise 19.15
Solve Laplace's equation on the rectangle $0 < x < \pi,\ 0 < y < 2$ with the boundary conditions

$$u(0,y) = u(\pi,y) = 0$$
$$u_y(x,0) = 0, \quad u_y(x,2) = 2\sin 3x - 5\sin 10x.$$

20 Laplace's Equations in Circular Regions

In the previous section we solved the Dirichlet problem for Laplace's equation on a rectangular region. However, if the domain of the solution is a disc, an annulus, or a circular wedge, it is useful to study the two-dimensional Laplace's equation in polar coordinates.
It is well known in calculus that the cartesian coordinates (x, y) and the polar coordinates (r, θ) of a point are related by the formulas

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

where $r = (x^2 + y^2)^{\frac{1}{2}}$ and $\tan\theta = \frac{y}{x}$. Using the chain rule we obtain

$$\begin{aligned}
u_x =& u_r r_x + u_\theta \theta_x = \cos\theta u_r - \frac{\sin\theta}{r} u_\theta \\
u_{xx} =& u_{xr} r_x + u_{x\theta}\theta_x \\
=& \left(\cos\theta u_{rr} + \frac{\sin\theta}{r^2} u_\theta - \frac{\sin\theta}{r} u_{r\theta}\right)\cos\theta \\
&+ \left(-\sin\theta u_r + \cos\theta u_{r\theta} - \frac{\cos\theta}{r} u_\theta - \frac{\sin\theta}{r} u_{\theta\theta}\right)\left(-\frac{\sin\theta}{r}\right) \\
u_y =& u_r r_y + u_\theta\theta_y = \sin\theta u_r + \frac{\cos\theta}{r} u_\theta \\
u_{yy} =& u_{yr} r_y + u_{y\theta}\theta_y \\
=& \left(\sin\theta u_{rr} - \frac{\cos\theta}{r^2} u_\theta + \frac{\cos\theta}{r} u_{r\theta}\right)\sin\theta \\
&+ \left(\cos\theta u_r + \sin\theta u_{r\theta} - \frac{\sin\theta}{r} u_\theta + \frac{\cos\theta}{r} u_{\theta\theta}\right)\left(\frac{\cos\theta}{r}\right)
\end{aligned}$$

Substituting these equations into $\Delta u = 0$ we obtain

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0. \tag{20.1}$$

Example 20.1
Find the solution to

$$\Delta u = 0, \quad x^2 + y^2 < a^2$$

subject to
(i) Boundary condition: $u(a, \theta) = f(\theta), \quad -\pi \le \theta \le \pi.$
(ii) Boundedness at the origin: $|u(0, \theta)| < \infty.$
(iii) Periodicity: $u(r, \theta + 2\pi) = u(r, \theta), \quad -\pi \le \theta \le \pi.$

Solution.
First, note that (iii) implies that $u(r,-\pi) = u(r,\pi)$ and $u_\theta(r,-\pi) = u_\theta(r,\pi)$. Next, we will apply the method of separation of variables to (21.1). Suppose that a solution $u(r,\theta)$ of (21.1) can be written in the form $u(r,\theta) = R(r)\Theta(\theta)$. Substituting in (21.1) we obtain

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0.$$

Dividing by $R\Theta$ (under the assumption that $R\Theta \neq 0$) we obtain

$$\frac{\Theta''(\theta)}{\Theta(\theta)} = -r^2\frac{R''(r)}{R(r)} - r\frac{R'(r)}{R(r)}.$$

The left-hand side is independent of r whereas the right-hand side is independent of θ so that there is a constant λ such that

$$-\frac{\Theta''(\theta)}{\Theta(\theta)} = r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = \lambda.$$

This results in the following ODEs

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0 \tag{20.2}$$

and

$$r^2R''(r) + rR'(r) - \lambda R(r) = 0. \tag{20.3}$$

The second equation is known as **Euler's equation.** Both of these equations are easily solvable. To solve (20.2), we only have to add the appropriate boundary conditions. From (iii), we have $\Theta(-\pi) = \Theta(\pi)$ and $\Theta'(-\pi) = \Theta'(\pi)$. If $\lambda > 0$ then $\Theta(\theta) = A\cos(\sqrt{\lambda}\theta) + B\cos(\sqrt{\lambda}\theta)$. Using the condition $\Theta(-\pi) = \Theta(\pi)$ we obtain $2B\sin(\sqrt{\lambda}\pi) = 0$. Using the condition $\Theta'(-\pi) = \Theta'(\pi)$ we obtain $2\sqrt{\lambda}A\sin(\sqrt{\lambda}\pi) = 0$. If $\sin(\sqrt{\lambda}\pi) \neq 0$ then $A = B = 0$ and we get the trivial solution. Therefore, we require $\sin(\sqrt{\lambda}\pi) = 0$ and this leads to $\lambda_n = n^2$ for $n = 1, 2, \cdots$. Note that we start with $n = 1$ since $\lambda > 0$. Hence,

$$\Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta.$$

If $\lambda = 0$ then $\Theta(\theta) = A + B\theta$ and the conditions $\Theta(-\pi) = \Theta(\pi)$ and $\Theta'(-\pi) = \Theta'(\pi)$ leads to $\Theta(\theta) = A$. If $\lambda < 0$ then

$$\Theta(\theta) = A\cosh(\sqrt{-\lambda}\theta) + B\sinh(\sqrt{-\lambda}\theta)$$

and applying the conditions $\Theta(-\pi) = \Theta(\pi)$ and $\Theta'(-\pi) = \Theta'(\pi)$ we find $A = B = 0$. In summary, we have

$$\Theta_n(\theta) = A'_n \cos n\theta + B'_n \sin n\theta, n = 0, 1, 2 \cdots$$

The equation in R is of Euler type and its solution must be of the form $R(r) = r^\alpha$. Substituting into (20.3) and using $\lambda = n^2$, we find

$$\alpha(\alpha - 1)r^\alpha + \alpha r^\alpha - n^2 r^\alpha = 0.$$

Solving this equation we find $\alpha = \pm n$. Hence, we let

$$R_n(r) = C_n r^n + D_n r^{-n}, n \in \mathbb{N}.$$

For $n = 0$, $R = 1$ is a solution. To find a second solution, we solve the equation

$$r^2 R'' + rR' = 0.$$

This can be done by dividing through by r and using the substitution $S = R'$ to obtain $rS' + S = 0$. Solving this by noting that the left-hand side is just $(rS)'$ we find $S = \frac{c}{r}$. Hence, $R' = \frac{c}{r}$ and this implies $R(r) = C \ln r$. Thus, $R = 1$ and $R = \ln r$ form a couple of linearly independent solutions of (20.3) and so a general solution is given by

$$R_0(r) = C_0 + D_0 \ln r.$$

By assumption (ii), $u(r, \theta)$ must be bounded near $r = 0$, and so does R_n. Since r^{-n} and $\ln r$ are unbounded near $r = 0$, we must set $D_0 = D_n = 0$. In this case, the solutions to Euler's equation are given by

$$R_n(r) = C_n r^n, \; n = 0, 1, 2, \cdots.$$

Using the superposition principle, and combining the results obtained above, we find

$$u(r, \theta) = C_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta).$$

Now, using the boundary condition $u(a, \theta) = f(\theta)$ we can write

$$f(\theta) = C_0 + \sum_{n=1}^{\infty} (a^n A_n \cos n\theta + a^n B_n \sin n\theta)$$

which is usually written in a more convenient equivalent form by

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta).$$

It is obvious that a_n and b_n are the Fourier coefficients, and therefore can be determined by the formulas

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta d\theta, \quad n = 0, 1, \cdots$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta d\theta, \quad n = 1, 2, \cdots.$$

Finally, the general solution to our problem is given by

$$u(r, \theta) = C_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

where

$$\begin{aligned}
C_0 =& \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) d\theta \\
A_n =& \frac{a_n}{a^n} = \frac{1}{a^n \pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta d\theta, \quad n = 1, 2, \cdots \\
B_n =& \frac{b_n}{a^n} = \frac{1}{a^n \pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta d\theta, \quad n = 1, 2, \cdots
\end{aligned}$$

Example 20.2
Find a $2\pi-$periodic function solution

$$\Delta u = 0, \quad -\pi \le \theta < \pi, \ 1 \le r \le 2$$

subject to

$$u(1, \theta) = u(2, \theta) = \sin\theta, \quad -\pi \le \theta < \pi.$$

Solution.
Use separation of variables. First, solving for $\Theta(\theta))$, we see that in order to ensure that the solution is $2\pi-$periodic in θ, the eigenvalues are $\lambda = n^2$. When solving the equation for $R(r)$, we do NOT need to throw out solutions

which are not bounded as $r \to 0$. This is because we are working in the annulus where r is bounded away from 0 and ∞. Therefore, we obtain the general solution

$$u(r,\theta) = (C_0 + C_1 \ln r) + \sum_{n=1}^{\infty} [(C_n r^n + D_n r^{-n}) \cos n\theta + (A_n r^n + B_n r^{-n}) \sin n\theta].$$

But

$$C_0 + \sum_{n=1}^{\infty} [(C_n + D_n) \cos n\theta + (A_n + B_n) \sin n\theta] = \sin\theta$$

and

$$C_0 + C_1 \ln 2 + \sum_{n=1}^{\infty} [(C_n 2^n + D_n 2^{-n}) \cos n\theta + (A_n 2^n + B_n 2^{-n}) \sin n\theta] = \sin\theta$$

Hence, comparing coefficients we must have

$$\begin{aligned}
C_0 =& 0\\
C_n + D_n =& 0\\
A_n + B_n =& 0 \quad n \neq 1\\
A_1 + B_1 =& 1\\
C_n 2^n + D_n 2^{-n} =& 0\\
A_n 2^n + B_n 2^{-n} =& 0 \quad n \neq 1\\
2A_1 + 2^{-1} B_1 =& 1
\end{aligned}$$

Solving these equations we find $C_0 = C_n = D_n = 0, A_1 = \frac{1}{3}, B_1 = \frac{2}{3}$, and $A_n = B_n = 0$ for $n \neq 1$. Hence, the solution to the problem is

$$u(r,\theta) = \frac{1}{3}\left(r + \frac{2}{r}\right) \sin\theta \blacksquare$$

Example 20.3
Solve Laplace's equation inside a 60° wedge of radius a subject to the boundary conditions:
(1) $u(a,\theta) = \frac{1}{3}\cos 9\theta - \frac{1}{9}\cos 3\theta$.
(2) $u_\theta(r,0) = 0,\ u_\theta(r,\frac{\pi}{3}) = 0$.
(3) $|u(0,\theta)| < \infty$.

Solution.
Letting $u(r,\theta) = R(r)\Theta(\theta)$ and separating the variables we obtain the eigenvalue problem

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0.$$

As above, one can easily see that the solution is of the form

$$\Theta(\theta) = A\cos\sqrt{\lambda}\theta + B\sin\sqrt{\lambda}\theta.$$

The condition $\Theta'(0) = 0$ implies $B = 0$. The condition $\Theta'\left(\frac{\pi}{3}\right) = 0$ implies $\lambda_n = (3n)^2,\ n = 0, 1, 2, \cdots$. Thus, the angular solution is

$$\Theta_n(\theta) = A'_n \cos 3n\theta, \quad n = 0, 1, 2, \cdots$$

The corresponding solutions of the radial problem are

$$R_n(r) = A_n r^{3n} + B_n r^{-3n},\ n = 0, 1, \cdots.$$

To obtain a solution that remains bounded as $r \to 0$ we take $B_n = 0$. Hence,

$$u_n(r,\theta) = \sum_{n=0}^{\infty} C_n r^{3n} \cos 3n\theta, \quad n = 0, 1, 2, \cdots$$

Using the boundary condition

$$u(a,\theta) = \frac{1}{3}\cos 9\theta - \frac{1}{9}\cos 3\theta$$

we obtain $C_1 a^3 = -\frac{1}{9}$ and $C_3 a^9 = \frac{1}{3}$ and 0 otherwise. Thus,

$$u(a,\theta) = \frac{1}{3}\left(\frac{r}{a}\right)^9 \cos 9\theta - \frac{1}{9}\left(\frac{r}{a}\right)^3 \cos 3\theta \ \blacksquare$$

Practice Problems

Exercise 20.1
Solve the Laplace's equation in the unit disk with $u(1,\theta) = 3\sin 5\theta$.

Exercise 20.2
Solve the Laplace's equation in the upper half of the unit disk with $u(1,\theta) = \pi - \theta$.

Exercise 20.3
Solve the Laplace's equation in the unit disk with $u_r(1,\theta) = 2\cos 2\theta$.

Exercise 20.4
Consider

$$u(r,\theta) = C_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

with

$$\begin{aligned}
C_0 =& \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)d\phi \\
A_n =& \frac{a_n}{a^n} = \frac{1}{a^n\pi}\int_{-\pi}^{\pi} f(\phi)\cos n\phi d\phi, \quad n = 1, 2, \cdots \\
B_n =& \frac{b_n}{a^n} = \frac{1}{a^n\pi}\int_{-\pi}^{\pi} f(\phi)\sin n\phi d\phi, \quad n = 1, 2, \cdots
\end{aligned}$$

Using the trigonometric identity

$$\cos a \cos b + \sin a \sin b = \cos(a - b)$$

show that

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta - \phi)\right] d\phi.$$

Exercise 20.5
(a) Using Euler's formula from complex analysis $e^{it} = \cos t + i\sin t$ show that

$$\cos t = \frac{1}{2}(e^{it} + e^{-it}),$$

where $i = \sqrt{-1}$.
(b) Show that

$$1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta - \phi) = 1 + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\phi)} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{-in(\theta-\phi)}.$$

(c) Let $q_1 = \frac{r}{a}e^{i(\theta-\phi)}$ and $q_2 = \frac{r}{a}e^{-i(\theta-\phi)}$. It is defined in complex analysis that the absolute value of a complex number $z = x+iy$ is given by $|z| = (x^2+y^2)^{\frac{1}{2}}$. Using these concepts, show that $|q_1| < 1$ and $|q_2| < 1$.

Exercise 20.6
(a)Show that

$$\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\phi)} = \frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}}$$

and

$$\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{-in(\theta-\phi)} = \frac{re^{-i(\theta-\phi)}}{a - re^{-i(\theta-\phi)}}$$

Hint: Each sum is a geoemtric series with a ratio less than 1 in absolute value so that these series converges.
(b) Show that

$$1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta - \phi) = \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \phi) + r^2}.$$

Exercise 20.7
Show that

$$u(r,\theta) = \frac{a^2 - r^2}{2\pi}\int_{-\pi}^{\pi}\frac{f(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2}d\phi.$$

This is known as the **Poisson formula** in polar coordinates.

Exercise 20.8
Solve

$$u_{xx} + u_{yy} = 0, \quad x^2 + y^2 < 1$$

subject to

$$u(1,\theta) = \theta, \quad -\pi \le \theta \le \pi.$$

Exercise 20.9
The vibrations of a symmetric circular membrane where the displacement $u(r,t)$ depends on r and t only can be describe by the one-dimensional wave equation in polar coordinates

$$u_{tt} = c^2(u_{rr} + \frac{1}{r}u_r), \quad 0 < r < a, \ t > 0$$

with initial condition

$$u(a,t) = 0, \quad t > 0$$

and boundary conditions

$$u(r,0) = f(r), \quad u_t(r,0) = g(r), \quad 0 < r < a.$$

(a) Show that the assumption $u(r,t) = R(r)T(t)$ leads to the equation

$$\frac{1}{c^2}\frac{T''}{T} = \frac{1}{R}R'' + \frac{1}{r}\frac{R'}{R} = \lambda.$$

(b) Show that $\lambda < 0$.

Exercise 20.10
Cartesian coordinates and cylindrical coordinates are shown in Figure 20.1 below.

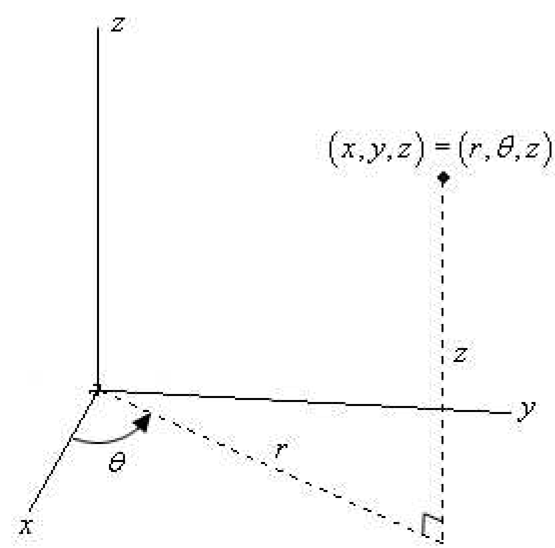


Figure 20.1

(a) Show that $x = r\cos\theta,\ y = r\sin\theta,\quad z = z.$
(b) Show that

$$u_{xx} + u_{yy} + u_{zz} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}.$$

Sample Exam Questions

Exercise 20.11
An important result about harmonic functions is the so-called the **maximum principle** which states: Any harmonic function $u(x,y)$ defined in a domain Ω satisfies the inequality

$$\min_{(x,y)\in\partial\Omega} u(x,y) \le u(x,y) \le \max_{(x,y)\in\partial\Omega} u(x,y), \quad \forall (x,y)\in\Omega\cup\partial\Omega$$

where $\partial\Omega$ denotes the boundary of Ω.
Let u be harmonic in $\Omega=\{(x,y): x^2+y^2<1\}$ and satisfies $u(x,y)=2-x$ for all $(x,y)\in\partial\Omega$. Show that $u(x,y)>0$ for all $(x,y)\in\Omega$.

Exercise 20.12
Let u be harmonic in $\Omega=\{(x,y): x^2+y^2<1\}$ and satisfies $u(x,y)=1+3x$ for all $(x,y)\in\partial\Omega$. Determine
(i) $\max_{(x,y)\in\Omega} u(x,y)$
(ii) $\min_{(x,y)\in\Omega} u(x,y)$
without solving $\Delta u=0$.

Exercise 20.13
Let $u_1(x,y)$ and $u_2(x,y)$ be harmonic functions on a smooth domain Ω such that

$$u_1|_{\partial\Omega}=g_1(x,y) \text{ and } u_2|_{\partial\Omega}=g_3(x,y)$$

where g_1 and g_2 are continuous functions satisfying

$$\max_{(x,y)\in\partial\Omega} g_1(x,y) < \min_{(x,y)\in\partial\Omega} g_1(x,y).$$

Prove that $u_1(x,y)<u_2(x,y)$ for all $(x,y)\in\Omega\cup\partial\Omega$.

Exercise 20.14
Show that $r^n\cos(n\theta)$ and $r^n\sin(n\theta)$ satisfy Laplace's equation in polar coordinates.

Exercise 20.15
Solve the Dirichlet problem

$$\Delta u=0, \quad 0\le r<a, \quad -\pi\le\theta\le\pi$$
$$u(a,\theta)=\sin^2\theta.$$

Exercise 20.16
Solve Laplace's equation

$$u_{xx} + u_{yy} = 0$$

outside a circular disk ($r \geq a$) subject to the boundary condition

$$u(a, \theta) = \ln 2 + 4\cos 3\theta.$$

You may assume that the solution remains bounded as $r \to \infty$.

The Laplace Transform Solutions for PDEs

If in a partial differential equation the time t is one of the independent variables of the searched-for function, we say that the PDE is an **evolution equation.** Examples of evolutions equations are the heat equation and the wave equation. In contrast, when the equation involves only spatial independent variables then the equation is called a **stationary equation.** Examples of stationary equations are the Laplace's equations and Poisson equations.
There are classes of methods that can be used for solving the initial value or initial boundary problems for evolution equations. We refer to these methods as the methods of **integral transforms.** The fundamental ones are the Laplace and the Fourier transforms. In this chapter we will just consider the Laplace transform.

21 Essentials of the Laplace Transform

Laplace transform has been introduced in an ODE course, and is used especially to solve linear ODEs with constant coefficients, where the equations are transformed to algebraic equations. This idea can be easily extended to PDEs, where the transformation leads to the decrease of the number of independent variables. PDEs in two variables are thus reduced to ODEs. In this section we review the Laplace transform and its properties.
Laplace transform is yet another operational tool for solving constant coefficients linear differential equations. The process of solution consists of three main steps:
• The given "hard" problem is transformed into a "simple" equation.
• This simple equation is solved by purely algebraic manipulations.
• The solution of the simple equation is transformed back to obtain the so-

lution of the given problem.
In this way the Laplace transformation reduces the problem of solving a differential equation to an algebraic problem. The third step is made easier by tables, whose role is similar to that of integral tables in integration.
The above procedure can be summarized by Figure 21.1

Figure 21.1

The Laplace transform is defined in the following way. Let $f(t)$ be defined for $t \geq 0$. Then the **Laplace transform** of f, which is denoted by $\mathcal{L}[f(t)]$ or by $F(s)$, is defined by the following equation

$$\mathcal{L}[f(t)] = F(s) = \lim_{T\to\infty}\int_0^T f(t)e^{-st}dt = \int_0^\infty f(t)e^{-st}dt$$

The integral which defines a Laplace transform is an improper integral. An improper integral may **converge** or **diverge**, depending on the integrand. When the improper integral is convergent then we say that the function $f(t)$ possesses a Laplace transform. So what types of functions possess Laplace transforms, that is, what type of functions guarantees a convergent improper integral.

Example 21.1
Find the Laplace transform, if it exists, of each of the following functions

$$(a)\ f(t) = e^{at} \quad (b)\ f(t) = 1 \quad (c)\ f(t) = t \quad (d)\ f(t) = e^{t^2}$$

Solution.
(a) Using the definition of Laplace transform we see that

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-(s-a)t}dt = \lim_{T\to\infty}\int_0^T e^{-(s-a)t}dt.$$

But

$$\int_0^T e^{-(s-a)t}dt = \begin{cases} T & \text{if } s = a \\ \frac{1-e^{-(s-a)T}}{s-a} & \text{if } s \neq a. \end{cases}$$

For the improper integral to converge we need $s > a$. In this case,

$$\mathcal{L}[e^{at}] = F(s) = \frac{1}{s-a}, \quad s > a.$$

(b) In a similar way to what was done in part (a), we find

$$\mathcal{L}[1] = \int_0^\infty e^{-st}dt = \lim_{T\to\infty}\int_0^T e^{-st}dt = \frac{1}{s},\ s > 0.$$

(c) We have

$$\mathcal{L}[t] = \int_0^\infty te^{-st}dt = \left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^\infty = \frac{1}{s^2},\ s > 0.$$

(d) Again using the definition of Laplace transform we find

$$\mathcal{L}[e^{t^2}] = \int_0^\infty e^{t^2-st}dt.$$

If $s \le 0$ then $t^2 - st \ge 0$ so that $e^{t^2-st} \ge 1$ and this implies that $\int_0^\infty e^{t^2-st}dt \ge \int_0^\infty dt$. Since the integral on the right is divergent, by the comparison theorem of improper integrals (see Theorem 23.1 below) the integral on the left is also divergent. Now, if $s > 0$ then $\int_0^\infty e^{t(t-s)}dt \ge \int_s^\infty dt$. By the same reasoning the integral on the left is divergent. This shows that the function $f(t) = e^{t^2}$ does not possess a Laplace transform ■

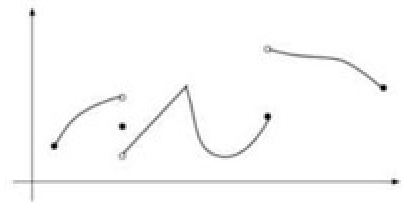
The above example raises the question of what class or classes of functions possess a Laplace transform. To answer this question we introduce few mathematical concepts.

A function f that satisfies

$$|f(t)| \le Me^{at}, \quad t \ge C \tag{21.1}$$

is said to be a function with an **exponential order a.** If $C = 0$ in (21.1) then the function is said to be **exponentially bounded.** Clearly, if $\lim_{t\to\infty} e^{-at}f(t) = 0$ for some $a > 0$ then f is of exponential order a.

A function f is called **piecewise continuous** on an interval if the interval can be broken into a finite number of subintervals on which the function is continuous on each open subinterval (i.e. the subinterval without its endpoints) and has a finite limit at the endpoints (**jump discontinuities** and no vertical asymptotes) of each subinterval. Below is a sketch of a piecewise continuous function.



Note that a piecewise continuous function is a function that has a finite number of breaks in it and doesn't blow up to infinity anywhere. A function defined for $t \geq 0$ is said to be **piecewise continuous on the infinite interval** if it is piecewise continuous on $0 \leq t \leq T$ for all $T > 0$. Also, note that a bounded continuous function is piecewise continuous.

Example 21.2
Show that the following functions are piecewise continuous and exponentially bounded for $t \geq 0$.

$$\text{(a) } f(t) = t^n \quad \text{(b) } f(t) = t^n \sin at$$

Solution.
(a) Since $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \geq \frac{t^n}{n!}$, $t^n \leq n!e^t$ for all $t \geq 0$. Hence, t^n is continuous (and hence piecewise consitnuous) and exponentially bounded.
(b) Since $|t^n \sin at| \leq n!e^t$, $t^n \sin at$ is piecewise continuous and exponentially bounded ■

The following is an existence result of Laplace transform.

Theorem 21.1
Suppose that $f(t)$ is piecewise continuous on $t \geq 0$ and has an exponential order a. Then the Laplace transform

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

exists as long as $s > a$. Note that the two conditions above are sufficient, but not necessary, for $F(s)$ to exist.

In what follows, we will denote the class of all piecewise continuous functions with an exponential order by $\mathcal{PE}$. The next theorem shows that any linear combination of functions in $\mathcal{PE}$ is also in $\mathcal{PE}$. The same is true for the product of two functions in $\mathcal{PE}$.

Theorem 21.2
Suppose that $f(t)$ and $g(t)$ are two elements of $\mathcal{PE}$ with

$$|f(t)| \le M_1 e^{a_1 t}, \quad t \ge C_1 \quad \text{and} \quad |g(t)| \le M_2 e^{a_1 t}, \quad t \ge C_2.$$

(i) For any constants α and β the function $\alpha f(t) + \beta g(t)$ is also a member of $\mathcal{PE}$. Moreover

$$\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)].$$

(ii) The function $h(t) = f(t)g(t)$ is an element of $\mathcal{PE}$.

We next discuss the problem of how to determine the function $f(t)$ if $F(s)$ is given. That is, how do we invert the transform. The following result on uniqueness provides a possible answer. This result establishes a one-to-one correspondence between the set $\mathcal{PE}$ and its Laplace transforms. Alternatively, the following theorem asserts that the Laplace transform of a member in $\mathcal{PE}$ is unique.

Theorem 21.3
Let $f(t)$ and $g(t)$ be two elements in $\mathcal{PE}$ with Laplace transforms $F(s)$ and $G(s)$ such that $F(s) = G(s)$ for some $s > a$. Then $f(t) = g(t)$ for all $t \ge 0$ where both functions are continuous.

With the above theorem, we can now officially define the inverse Laplace transform as follows: For a function $f \in \mathcal{PE}$ whose Laplace transform is F, we call f the **inverse Laplace transform** of F and write $f = \mathcal{L}^{-1}[F(s)]$. Symbolically

$$f(t) = \mathcal{L}^{-1}[F(s)] \Longleftrightarrow F(s) = \mathcal{L}[f(t)].$$

Example 21.3
Find $\mathcal{L}^{-1}\left(\frac{1}{s-1}\right), \quad s > 1.$

Solution.
From Example 23.1(a), we have that $\mathcal{L}[e^{at}] = \frac{1}{s-a}, \quad s > a$. In particular, for $a = 1$ we find that $\mathcal{L}[e^t] = \frac{1}{s-1}, \ s > 1$. Hence, $\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = e^t, \ t \geq 0$ ■.

The above theorem states that if $f(t)$ is continuous and has a Laplace transform $F(s)$, then there is no other function that has the same Laplace transform. To find $\mathcal{L}^{-1}[F(s)]$, we can inspect tables of Laplace transforms of known functions to find a particular $f(t)$ that yields the given $F(s)$.
When the function $f(t)$ is not continuous, the uniqueness of the inverse Laplace transform is not assured. The following example addresses the uniqueness issue.

Example 21.4
Consider the two functions $f(t) = H(t)H(3-t)$ and $g(t) = H(t) - H(t-3)$, where H is the Heaviside function defined by

$$H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

(a) Are the two functions identical?
(b) Show that $\mathcal{L}[f(t)] = \mathcal{L}[g(t)$.

Solution.
(a) We have

$$f(t) = \begin{cases} 1, & 0 \leq t \leq 3 \\ 0, & t > 3 \end{cases}$$

and

$$g(t) = \begin{cases} 1, & 0 \leq t < 3 \\ 0, & t \geq 3 \end{cases}$$

Since $f(3) = 1$ and $g(3) = 0$, f and g are not identical.
(b) We have

$$\mathcal{L}[f(t)] = \mathcal{L}[g(t)] = \int_0^3 e^{-st} dt = \frac{1 - e^{-3s}}{s}, s > 0.$$

Thus, both functions $f(t)$ and $g(t)$ have the same Laplace transform even though they are not identical. However, they are equal on the interval(s) where they are both continuous ■

The inverse Laplace transform possesses a linear property as indicated in the following result.

Theorem 21.4
Given two Laplace transforms $F(s)$ and $G(s)$ then

$$\mathcal{L}^{-1}[aF(s)+bG(s)] = a\mathcal{L}^{-1}[F(s)] + b\mathcal{L}^{-1}[G(s)]$$

for any constants a and b.

Convolution integrals are useful when finding the inverse Laplace transform of products. They are defined as follows: The **convolution** of two scalar piecewise continuous functions $f(t)$ and $g(t)$ defined for $t \geq 0$ is the integral

$$(f*g)(t) = \int_0^t f(t-s)g(s)ds.$$

Example 21.5
Find $f*g$ where $f(t) = e^{-t}$ and $g(t) = \sin t$.

Solution.
Using integration by parts twice we arrive at

$$\begin{array}{rcl}(f*g)(t) &=& \int_0^t e^{-(t-s)}\sin s ds\\ &=& \frac{1}{2}\left[e^{-(t-s)}(\sin s - \cos s)\right]_0^t\\ &=& \frac{e^{-t}}{2} + \frac{1}{2}(\sin t - \cos t) \blacksquare\end{array}$$

Next, we state several properties of convolution product, which resemble those of ordinary product.

Theorem 21.5
Let $f(t), g(t)$, and $k(t)$ be three piecewise continuous scalar functions defined for $t \geq 0$ and c_1 and c_2 are arbitrary constants. Then
(i) $f*g = g*f$ (Commutative Law)
(ii) $(f*g)*k = f*(g*k)$ (Associative Law)
(iii) $f*(c_1g+c_2k) = c_1f*g + c_2f*k$ (Distributive Law)

Example 21.6
Express the solution to the initial value problem $y' + \alpha y = g(t),\ y(0) = y_0$ in terms of a convolution integral.

Solution.
Solving this initial value problem by the method of integrating factor we find

$$y(t) = e^{-\alpha t}y_0 + \int_0^t e^{-\alpha(t-s)}g(s)ds = e^{-\alpha t}y_0 + (e^{-\alpha t} * g)(t) \blacksquare$$

The following theorem, known as the Convolution Theorem, provides a way for finding the Laplace transform of a convolution integral and also finding the inverse Laplace transform of a product.

Theorem 21.6
If $f(t)$ and $g(t)$ are elements in $\mathcal{PE}$ then

$$\mathcal{L}[(f * g)(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)] = F(s)G(s).$$

Thus, $(f * g)(t) = \mathcal{L}^{-1}[F(s)G(s)]$.0

Example 21.7
Use the convolution theorem to find the inverse Laplace transform of

$$P(s) = \frac{1}{(s^2+a^2)^2}.$$

Solution.
Note that

$$P(s) = \left(\frac{1}{s^2+a^2}\right)\left(\frac{1}{s^2+a^2}\right).$$

So, in this case we have, $F(s) = G(s) = \frac{1}{s^2+a^2}$ so that $f(t) = g(t) = \frac{1}{a}\sin(at)$. Thus,

$$(f * g)(t) = \frac{1}{a^2}\int_0^t \sin(at - as)\sin(as)ds = \frac{1}{2a^3}(\sin(at) - at\cos(at)) \blacksquare$$

The next example provides a solution method for solving ordinary differential equations.

Example 21.8
Solve the initial value problem

$$4y'' + y = g(t), \quad y(0) = 3, \ y'(0) = -7$$

Solution.
Take the Laplace transform of all the terms and plug in the initial conditions to obtain

$$4(s^2Y(s) - 3s + 7) + Y(s) = G(s)$$

or

$$(4s^2 + 1)Y(s) - 12s + 28 = G(s).$$

Solving for $Y(s)$ we find

$$\begin{aligned}
Y(s) =&\frac{12s - 28}{4\left(s^2 + \frac{1}{4}\right)} + \frac{G(s)}{4\left(s^2 + \frac{1}{4}\right)}\\
=&\frac{3s}{s^2 + \left(\left(\frac{1}{2}\right)^2\right.} - 7\frac{\left(\frac{1}{2}\right)^2}{s^2 + \left(\frac{1}{2}\right)^2} + \frac{1}{4}G(s)\frac{\left(\frac{1}{2}\right)^2}{s^2 + \left(\frac{1}{2}\right)^2}
\end{aligned}$$

Hence,

$$y(t) = 3\cos\left(\frac{t}{2}\right) - 7\sin\left(\frac{t}{2}\right) + \frac{1}{2}\int_0^t \sin\left(\frac{s}{2}\right)g(t - s)ds.$$

So, once we decide on a $g(t)$ all we need to do is to evaluate the integral and we'll have the solution ■

We conclude this section with the following table of Laplace transform pairs.

f(t)	F(s)
$H(t)=\begin{cases} 1, & t\geq 0 \\ 0, & t<0 \end{cases}$	$\frac{1}{s},\ s>0$
$t^n,\ n=1,2,\cdots$	$\frac{n!}{s^{n+1}},\ s>0$
$e^{\alpha t}$	$\frac{1}{s-\alpha},\ s>\alpha$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2},\ s>0$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2},\ s>0$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2},\ s>\lvert\omega\rvert$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2},\ s>\lvert\omega\rvert$
$e^{\alpha t}f(t),\ with\ \lvert f(t)\rvert\leq Me^{at}$	$F(s-\alpha),\ s>\alpha+a$
$e^{\alpha t}t^n,\ n=1,2,\cdots$	$\frac{n!}{(s-\alpha)^{n+1}},\ s>\alpha$
$e^{\alpha t}\sin(\omega t)$	$\frac{\omega}{(s-\alpha)^2+\omega^2},\ s>\alpha$
$e^{\alpha t}\cos(\omega t)$	$\frac{s-\alpha}{(s-\alpha)^2+\omega^2},\ s>\alpha$
$f(t-\alpha)H(t-\alpha),\ \alpha\geq 0$ $with\ \lvert f(t)\rvert\leq Me^{at}$	$e^{-\alpha s}F(s),\ s>a$
$H(t-\alpha),\ \alpha\geq 0$	$\frac{e^{-\alpha s}}{s},\ s>0$
$tf(t)$	-$F'(s)$
$\frac{t}{2\omega}\sin\omega t$	$\frac{s}{(s^2+\omega^2)^2},\ s>0$
$\frac{1}{2\omega^3}[\sin\omega t-\omega t\cos\omega t]$	$\frac{1}{(s^2+\omega^2)^2},\ s>0$
$f'(t),\ with\ f(t)\ continuous$ $and\ \lvert f'(t)\rvert\leq Me^{at}$	$sF(s)-f(0)$ $s>\max\{a,0\}+1$
$f''(t),\ with\ f'(t)\ continuous$ $and\ \lvert f''(t)\rvert\leq Me^{at}$	$s^2F(s)-sf(0)-f'(0)$ $s>\max\{a,0\}+1$
$f^{(n)}(t),\ with\ f^{(n-1)}(t)\ continuous$ $and\ \lvert f^{(n)}(t)\rvert\leq Me^{at}$	$s^nF(s)-s^{n-1}f(0)-\cdots$ -$sf^{(n-2)}(0)-f^{(n-1)}(0)$ $s>\max\{a,0\}+1$
$\int_0^t f(u)du,\ with\ \lvert f(t)\rvert\leq Me^{at}$	$\frac{F(s)}{s},\ \ s>\max\{a,0\}+1$

Table $\mathcal{L}$

Practice Problems

Exercise 21.1
Determine whether the integral $\int_0^\infty \frac{1}{1+t^2} dt$ converges. If the integral converges, give its value.

Exercise 21.2
Determine whether the integral $\int_0^\infty \frac{t}{1+t^2} dt$ converges. If the integral converges, give its value.

Exercise 21.3
Determine whether the integral $\int_0^\infty e^{-t} \cos(e^{-t}) dt$ converges. If the integral converges, give its value.

Exercise 21.4
Using the definition, find $\mathcal{L}[e^{3t}]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Exercise 21.5
Using the definition, find $\mathcal{L}[t-5]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Exercise 21.6
Using the definition, find $\mathcal{L}[e^{(t-1)^2}]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Exercise 21.7
Using the definition, find $\mathcal{L}[(t-2)^2]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Exercise 21.8
Using the definition, find $\mathcal{L}[f(t)]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

$$f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t-1, & t \ge 1 \end{cases}$$

Exercise 21.9
Using the definition, find $\mathcal{L}[f(t)]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

$$f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t-1, & 1 \le t < 2 \\ 0, & t \ge 2. \end{cases}$$

Exercise 21.10
Let n be a positive integer. Using integration by parts establish the reduction formula

$$\int t^n e^{-st} dt = -\frac{t^n e^{-st}}{s} + \frac{n}{s}\int t^{n-1}e^{-st}dt, \quad s > 0.$$

Exercise 21.11
For $s > 0$ and n a positive integer evaluate the limits

$$\text{(a) } \lim_{t\to 0} t^n e^{-st} \qquad \text{(b) } \lim_{t\to\infty} t^n e^{-st}$$

Exercise 21.12
Use the linearity property of Laplace transform to find $\mathcal{L}[5e^{-7t} + t + 2e^{2t}]$. Find the domain of $F(s)$.

Exercise 21.13
Find $\mathcal{L}^{-1}\left(\frac{3}{s-2}\right)$.

Exercise 21.14
Find $\mathcal{L}^{-1}\left(-\frac{2}{s^2} + \frac{1}{s+1}\right)$.

Exercise 21.15
Find $\mathcal{L}^{-1}\left(\frac{2}{s+2} + \frac{2}{s-2}\right)$.

Exercise 21.16
Use Table $\mathcal{L}$ to find $\mathcal{L}[2e^t + 5]$.

Exercise 21.17
Use Table $\mathcal{L}$ to find $\mathcal{L}[e^{3t-3}H(t-1)]$.

Exercise 21.18
Use Table $\mathcal{L}$ to find $\mathcal{L}[\sin^2 \omega t]$.

Exercise 21.19
Use Table $\mathcal{L}$ to find $\mathcal{L}[\sin 3t \cos 3t]$.

Exercise 21.20
Use Table $\mathcal{L}$ to find $\mathcal{L}[e^{2t} \cos 3t]$.

Exercise 21.21
Use Table $\mathcal{L}$ to find $\mathcal{L}[e^{4t}(t^2 + 3t + 5)]$.

Exercise 21.22
Use Table $\mathcal{L}$ to find $\mathcal{L}^{-1}[\frac{10}{s^2+25} + \frac{4}{s-3}]$.

Exercise 21.23
Use Table $\mathcal{L}$ to find $\mathcal{L}^{-1}[\frac{5}{(s-3)^4}]$.

Sample Exam Questions

Exercise 21.24
Use Table $\mathcal{L}$ to find $\mathcal{L}^{-1}[\frac{e^{-2s}}{s-9}]$.

Exercise 21.25
Using the partial fraction decomposition find $\mathcal{L}^{-1}\left[\frac{12}{(s-3)(s+1)}\right]$.

Exercise 21.26
Using the partial fraction decomposition find $\mathcal{L}^{-1}\left[\frac{24e^{-5s}}{s^2-9}\right]$.

Exercise 21.27
Use Laplace transform technique to solve the initial value problem

$$y' + 4y = g(t), \quad y(0) = 2$$

where

$$g(t) = \begin{cases} 0, & 0 \le t < 1 \\ 12, & 1 \le t < 3 \\ 0, & t \ge 3 \end{cases}$$

Exercise 21.28
Use Laplace transform technique to solve the initial value problem

$$y'' - 4y = e^{3t}, \quad y(0) = 0, \quad y'(0) = 0.$$

Exercise 21.29
Consider the functions $f(t) = e^t$ and $g(t) = e^{-2t}$, $t \ge 0$. Compute $f * g$ in two different ways.
(a) By directly evaluating the integral.
(b) By computing $\mathcal{L}^{-1}[F(s)G(s)]$ where $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$.

Exercise 21.30
Consider the functions $f(t) = \sin t$ and $g(t) = \cos t$, $t \ge 0$. Compute $f * g$ in two different ways.
(a) By directly evaluating the integral.
(b) By computing $\mathcal{L}^{-1}[F(s)G(s)]$ where $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$.

Exercise 21.31
Compute $t * t * t$.

Exercise 21.32
Compute $H(t) * e^{-t} * e^{-2t}$.

Exercise 21.33
Compute $t * e^{-t} * e^{t}$.

22 Solving PDEs Using Laplace Transform

The same idea for solving linear ODEs using Laplace transform can be exploited when solving PDEs for functions in two variables $u = u(x,t)$. The transformation will be done with respect to the time variable $t \geq 0$, the spatial variable x will be treated as a parameter unaffected by this transform. In particular we define the Laplace transform of $u(x,t)$ by the formula

$$\mathcal{L}(u(x,t)) = U(x,s) = \int_0^\infty u(x,\tau)e^{-s\tau}d\tau.$$

The time derivatives are transformed in the same way as in the case of functions in one variable, that is, for example

$$\mathcal{L}(u_t)(x,t) = sU(x,s) - u(x,0)$$

and

$$\mathcal{L}(u_{tt})(x,s) = s^2U(x,s) - su(x,0) - u_t(x,0).$$

The spatial derivatives remain unchanged, for example,

$$\mathcal{L}u_x(x,t) = \int_0^\infty u_x(x,\tau)e^{-s\tau}d\tau = \frac{\partial}{\partial x}\int_0^\infty u(x,\tau)e^{-s\tau}d\tau = U_x(x,s).$$

Likewise, we have

$$\mathcal{L}u_{xx}(x,t) = U_{xx}(x,s).$$

Thus, applying the Laplace transform to a PDE in two variables x and t we obtain an ODE in the variable x and with the parameter s.

Example 22.1

Let $u(x,t)$ be the concentration of a chemical contaminant dissolved in a liquid on a half-infinte domain $x > 0$. Let us assume that at time $t = 0$ the concentration is 0 and on the boundary $x = 0$, constant unit concentration of the contaminant is kept for $t > 0$. The behaviour of this problem is described by the following mathematical model

$$\begin{cases} u_t - u_{xx} = 0 & ,\ x > 0,\ t > 0 \\ u(x,0) = 0, & \\ u(0,t) = 1, & \\ |u(x,t)| < \infty, & \forall x > 0, t > 0. \end{cases}$$

Find $u(x,t)$.

Solution.
Applying Laplace transform to both sides of the equation we obtain

$$sU(x,s) - u(x,0) - U_{xx}(x,s) = 0$$

or

$$U_{xx}(x,s) - sU(x,s) = 0.$$

This is a second order linear ODE in the variable x and positive parameter s. Its general solution is

$$U(x,s) = A(s)e^{-\sqrt{s}x} + B(s)e^{\sqrt{s}x}.$$

Since $U(x,s)$ is bounded in the variable x, we must have $B(s) = 0$ and in this case we obtain

$$U(x,s) = A(s)e^{-\sqrt{s}x}.$$

Next, we apply Laplace transform to the boundary condition obtaining

$$U(0,s) = \mathcal{L}(1) = \frac{1}{s}.$$

This leads to $A(s) = \frac{1}{s}$ and the transformed solution becomes

$$U(x,s) = \frac{1}{s}e^{-\sqrt{s}x}.$$

Thus,

$$u(x,t) = \mathcal{L}^{-1}\left(\frac{1}{s}e^{-\sqrt{s}x}\right).$$

One can use a software package to find the expression for $\mathcal{L}^{-1}\left(\frac{1}{s}e^{-\sqrt{s}x}\right)$ ■

Example 22.2
Solve the following initial boundary value problem

$$\begin{cases} u_t - u_{xx} = 0 & , \ x > 0, \ t > 0 \\ u(x,0) = 0, \\ u(0,t) = f(t), \\ |u(x,t)| < \infty, \quad \forall x > 0, t > 0. \end{cases}$$

Solution.
Following the argument of the previous example we find

$$U(x,s) = F(s)e^{-\sqrt{s}x}, \quad F(s) = \mathcal{L}f(t).$$

Thus, using Theorem 21.6 we can write

$$u(x,t) = \mathcal{L}^{-1}\left(F(s)e^{-\sqrt{s}x}\right) = f * \mathcal{L}^{-1}(e^{-\sqrt{s}x}).$$

It can be shown that

$$\mathcal{L}^{-1}(e^{-\sqrt{s}x}) = \frac{x}{\sqrt{4\pi t^3}}e^{-\frac{x^2}{4t}}.$$

Hence,

$$u(x,t) = \int_0^t \frac{x}{\sqrt{4\pi(t-s)^3}}e^{-\frac{x^2}{4(t-s)}}f(s)ds \blacksquare$$

Example 22.3
Solve the wave equation

$$\begin{cases} u_{tt} - c^2u_{xx} = 0 & , \ x > 0, \ t > 0 \\ u(x,0) = u_t(x,0) = 0, & \\ u(0,t) = f(t), & \\ |u(x,t)| < \infty, & \forall x > 0, t > 0. \end{cases}$$

Solution.
Applying Laplace transform to both sides of the equation we obtain

$$s^2U(x,s) - su(x,0) - u_t(x,0) - c^2U_{xx}(x,s) = 0$$

or

$$c^2U_{xx}(x,s) - s^2U(x,s) = 0.$$

This is a second order linear ODE in the variable x and positive parameter s. Its general solution is

$$U(x,s) = A(s)e^{-\frac{s}{c}x} + B(s)e^{\frac{s}{c}x}.$$

Since $U(x,s)$ is bounded in the variable x, we must have $B(s) = 0$ and in this case we obtain

$$U(x,s) = A(s)e^{-\frac{s}{c}x}.$$

Next, we apply Laplace transform to the boundary condition obtaining

$$U(0,s) = \mathcal{L}(f(t)) = F(s).$$

This leads to $A(s) = F(s)$ and the transformed solution becomes

$$U(x,s) = F(s)e^{-\frac{s}{c}x}.$$

Thus,

$$u(x,t) = \mathcal{L}^{-1}\left(F(s)e^{-\frac{x}{c}s}\right) = H\left(t - \frac{x}{c}\right) f\left(t - \frac{x}{c}\right) \blacksquare$$

Remark 22.1
Laplace transforms are useful in solving parabolic and some hyperbolic PDEs. They are not in general useful in solving elliptic PDEs.

Practice Problems

Exercise 22.1
Solve by Laplace transform

$$\begin{cases} u_t + u_x = 0 & , \ x > 0, \ t > 0 \\ u(x,0) = \sin x, & \\ u(0,t) = 0 & \end{cases}$$

Hint: Method of integrating factor of ODEs.

Exercise 22.2
Solve by Laplace transform

$$\begin{cases} u_t + u_x = -u & , \ x > 0, \ t > 0 \\ u(x,0) = \sin x, & \\ u(0,t) = 0 & \end{cases}$$

Exercise 22.3
Solve

$$u_t = 4u_{xx}$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = 2\sin \pi x + 3\sin 2\pi x.$$

Hint: A particular solution of a second order ODE must be found using the method of variation of parameters.

Exercise 22.4
Solve by Laplace transform

$$\begin{cases} u_t - u_x = u & , \ x > 0, \ t > 0 \\ u(x,0) = e^{-5x}, & \\ |u(x,t)| < \infty & \end{cases}$$

Exercise 22.5
Solve by Laplace transform

$$\begin{cases} u_t + u_x = t & , \ x > 0, \ t > 0 \\ u(x,0) = 0, & \\ u(0,t) = t^2 & \end{cases}$$

Exercise 22.6
Solve by Laplace transform

$$\begin{cases} xu_t + u_x = 0 & , \ x > 0, \ t > 0 \\ u(x,0) = 0, & \\ u(0,t) = t & \end{cases}$$

Exercise 22.7
Solve by Laplace transform

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & , \ x > 0, \ t > 0 \\ u(x,0) = u_t(x,0) = 0, & \\ u(0,t) = \sin x, & \\ |u(x,t)| < \infty & \end{cases}$$

Exercise 22.8
Solve by Laplace transform

$$u_{tt} - 9u_{xx} = 0, \ 0 \le x \le \pi, \ t > 0$$

$$u(0,t) = u(\pi,t) = 0,$$

$$u_t(x,0) = 0, \quad u(x,0) = 2\sin x.$$

Exercise 22.9
Solve by Laplace transform

$$\begin{cases} u_{xy} = 1 & , \ x > 0, \ y > 0 \\ u(x,0) = 1, & \\ u(0,y) = y + 1. & \end{cases}$$

Exercise 22.10
Solve by Laplace transform

$$\begin{cases} u_{tt} = c^2 u_{xx} & , \ x > 0, \ t > 0 \\ u(x,0) = u_t(x,0) = 0, & \\ u_x(0,t) = f(t), & \\ |u(x,t)| < \infty. & \end{cases}$$

Sample Exam Questions

Exercise 22.11
Solve by Laplace transform

$$\begin{cases} u_t + u_x = u & , \; x > 0, \; t > 0 \\ u(x,0) = \sin x, & \\ u(0,t) = 0 & \end{cases}$$

Exercise 22.12
Solve by Laplace transform

$$\begin{cases} u_t - c^2 u_{xx} = 0 & , \; x > 0, \; t > 0 \\ u(x,0) = T, & \\ u(0,t) = 0, & \\ |u(x,t)| < \infty & \end{cases}$$

Exercise 22.13
Solve by Laplace transform

$$u_t - 3u_{xx} = 0, \; 0 \le x \le 2, \; t > 0$$

$$u(0,t) = u(2,t) = 0,$$

$$u(x,0) = 5\sin(\pi x)$$

Exercise 22.14
Solve by Laplace transform

$$u_t - 4u_{xx} = 0, \; 0 \le x \le \pi, \; t > 0$$

$$u_x(0,t) = u(\pi,t) = 0,$$

$$u(x,0) = 40\cos\frac{x}{2}$$

Exercise 22.15
Solve by Laplace transform

$$u_{tt} - 4u_{xx} = 0, \; 0 \le x \le 2, \; t > 0$$

$$u(0,t) = u(2,t) = 0,$$

$$u_t(x,0) = 0, \quad u(x,0) = 3\sin\pi x.$$

The Fourier Transform Solutions for PDEs

In the previous chapter we discussed one class of integral transform methods, the Laplace transfom. In this chapter, we consider a second fundamental class of integral transform methods, the so-called **Fourier transform**.
Fourier series are designed to solve boundary value problems on bounded intervals. The extension of Fourier methods to the entire real line leads naturally to the Fourier transform, an extremely powerful mathematical tool for the analysis of non-periodic functions. The Fourier transform is of fundamental importance in a broad range of applications, including both ordinary and partial differential equations, quantum mechanics, signal processing, control theory, and probability, to name but a few.

23 Complex Version of Fourier Series

We have seen in Section 15 that a $2L-$periodic function $f : \mathbb{R} \to \mathbb{R}$ that is piecewise smooth on $[-L, L]$ can be expanded in a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right)$$

at all points of continuity of f. In the context of Fourier analysis, this is referred to as the **real form** of the Fourier series. It is often convenient to recast this series in **complex form** by means of Euler formula

$$e^{ix} = \cos x + i \sin x.$$

It follows from this formula that

$$e^{ix} + e^{-ix} = 2\cos x \text{ and } e^{ix} - e^{-ix} = 2i\sin x$$

or

$$\cos x = \tfrac{e^{ix}+e^{-ix}}{2} \text{ and } \sin x = \tfrac{e^{ix}-e^{-ix}}{2i}.$$

Hence the Fourier expansion of f can be rewritten as

$$\begin{aligned} f(x) =& \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\left(\frac{e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}}}{2}\right)\right. \\ & \left. + b_n\left(\frac{e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}}}{2i}\right)\right] \end{aligned}$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}} \tag{23.1}$$

where $c_0 = \frac{a_0}{2}$ and for $n \in \mathbb{N}$ we have

$$c_n = \frac{a_n - ib_n}{2}$$

$$c_{-n} = \frac{a_n + ib_n}{2}.$$

It follows that if $n \in \mathbb{N}$ then

$$a_n = c_n + c_{-n} \quad \text{and} \quad b_n = i(c_n - c_{-n}). \tag{23.2}$$

That is, a_n and b_n can be easily found once we have formulas for c_n. In order to find these formulas, we need to evaluate the following integral

$$\begin{aligned} \int_{-L}^{L} e^{\frac{in\pi x}{L}} e^{-\frac{im\pi x}{L}} dx =& \int_{-L}^{L} e^{\frac{i(n-m)\pi x}{L}} dx \\ =& \frac{L}{i(n-m)\pi} \left. e^{\frac{i(n-m)\pi x}{L}} \right]_{-L}^{L} \\ =& -\frac{iL}{(n-m)\pi}[\cos[(n-m)\pi] + i\sin[(n-m)\pi] \\ & - \cos[-(n-m)\pi] - i\sin[-(n-m)\pi]] \\ =& 0 \end{aligned}$$

if $n \neq m$. If $n = m$ then

$$\int_{-L}^{L} e^{\frac{in\pi x}{L}} e^{-\frac{in\pi x}{L}} dx = 2L.$$

Now, if we multiply (23.1) by $e^{-\frac{in\pi x}{L}}$ and integrate from $-L$ to L and apply the last result we find

$$\int_{-L}^{L} f(x) e^{-\frac{in\pi x}{L}} dx = 2Lc_n$$

which yields the formula for coefficients of the complex form of the Fourier series:

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-\frac{in\pi x}{L}} dx, \quad n = 0, \pm 1, \pm 2, \cdots.$$

Example 23.1
Find the complex Fourier coefficients of the function

$$f(x) = x, \quad -\pi \leq x \leq \pi$$

extended to be periodic of period 2π.

Solution.
Using integration by parts and the fact that $e^{i\pi} = e^{-i\pi} = -1$ we find

$$\begin{aligned}
c_n =& \frac{1}{2\pi}\int_{-\pi}^{\pi} x e^{-inx} dx \\
=& \frac{1}{2\pi}\left[\left(\frac{ix}{n}\right) e^{-inx}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\left(\frac{i}{n}\right) e^{-inx} dx\right] \\
=& \frac{1}{2\pi}\left[\left(\frac{i\pi}{n}\right) e^{-in\pi} + \left(\frac{i\pi}{n}\right) e^{in\pi}\right] \\
+& \frac{1}{2\pi}\left[\frac{1}{n^2} e^{-in\pi} - \frac{1}{n^2} e^{in\pi}\right] \\
=& \frac{1}{2\pi}\left[2i\frac{\pi}{n}(-1)^n\right] + \frac{1}{2\pi}(0) = \frac{(-1)^n i}{n} \blacksquare
\end{aligned}$$

Remark 23.1
It is often the case that the complex form of the Fourier series is far simpler to calculate than the real form. One can then use (23.2) to find the real form of the Fourier series. For example, the Fourier coefficients of the real form of the previous function are given by

$$a_n = (c_n + c_{-n}) = 0 \text{ and } b_n = i(c_n - c_{-n}) = \tfrac{2}{n}(-1)^{n+1}, \quad n \in \mathbb{N}$$

Practice Problems

Exercise 23.1
Find the complex Fourier coefficients of the function

$$f(x) = x, \quad -1 \leq x \leq 1$$

extended to be periodic of period 2.

Exercise 23.2
Let

$$f(x) = \begin{cases} 0 & -\pi < x < \frac{-\pi}{2} \\ 1 & \frac{-\pi}{2} < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

be $2\pi-$periodic. Find its complex series representation.

Exercise 23.3
Find the complex Fourier series of the $2\pi-$periodic function $f(x) = e^{ax}$ over the interval $(-\pi, \pi)$.

Exercise 23.4
Find the complex Fourier series of the $2\pi-$periodic function $f(x) = \sin x$ over the interval $(-\pi, \pi)$.

Exercise 23.5
Find the complex Fourier series of the $2\pi-$periodic function defined

$$f(x) = \begin{cases} 1 & 0 < x < T \\ 0 & T < x < 2\pi \end{cases}$$

Exercise 23.6
Let $f(x) = x^2, \ -\pi < x < \pi$, be $2\pi-$periodic.
(a) Calculate the complex Fourier series representation of f.
(b) Using the complex Fourier series found in (a), recover the real Fourier series representation of f.

Exercise 23.7
Let $f(x) = \sin n\pi x, \ -\frac{1}{2} < x < \frac{1}{2}$, be of period 1.
(a) Calculate the coefficients a_n, b_n and c_n.
(b) Find the complex Fourier series representation of f.

Exercise 23.8
Let $f(x) = 2 - x, \ -2 < x < 2$, be of period 2.
(a) Calculate the coefficients a_n, b_n and c_n.
(b) Find the complex Fourier series representation of f.

Exercise 23.9
Suppose that the coefficients c_n of the complex Fourier series are given by

$$c_n = \begin{cases} \frac{2}{i\pi n} & \text{if } |n| \text{ is odd} \\ 0 & \text{if } |n| \text{ is even.} \end{cases}$$

Find $a_n, \ n = 0, 1, 2, \cdots$ and $b_n, \ n = 1, 2, \cdots$.

Exercise 23.10
Recall that any complex number z can be written as $z = Re(z) + iIm(z)$ where $Re(z)$ is called the real part of z and $Im(z)$ is called the **imaginary part**. The **complex conjugate** of z is the complex number $\overline{z} = Re(z) - iIm(z)$. Using these definitions show that $a_n = 2Re(c_n)$ and $b_n = -2Im(c_n)$.

Sample Exam Questions

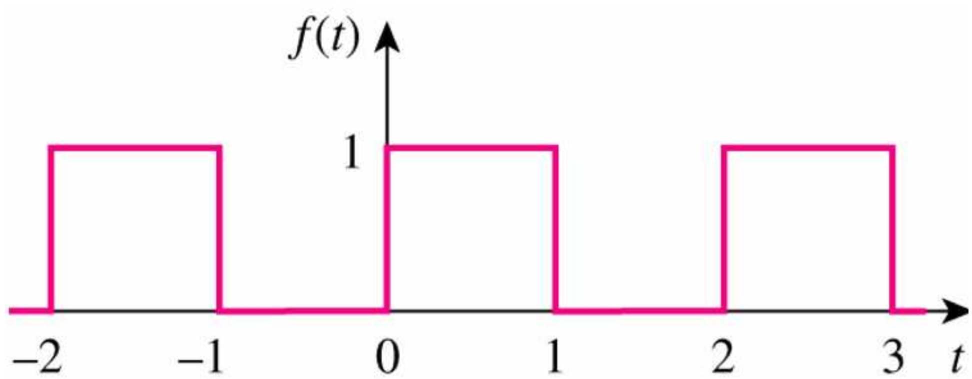
Exercise 23.11
Suppose that

$$c_n = \begin{cases} \frac{i}{2\pi n}[e^{-inT} - 1] & \text{if } n \neq 0 \\ \frac{T}{2\pi} & \text{if } n = 0. \end{cases}$$

Find a_n and b_n.

Exercise 23.12
Find the complex Fourier series of the function $f(x) = e^x$ on $[-2, 2]$.

Exercise 23.13
Consider the wave form



(a) Write $f(x)$ explicitly. What is the period of f.
(b) Determine a_0 and a_n for $n \in \mathbb{N}$.
(c) Determine b_n for $n \in \mathbb{N}$.
(d) Determine c_0 and c_n for $n \in \mathbb{N}$.

Exercise 23.14
If z is a complex number we define $\sin z = \frac{1}{2}(e^{iz} - e^{-iz})$. Find the complex form of the Fourier series for $\sin 3x$ without evaluating any integrals.

Exercise 23.15
Find c_n for the $2\pi-$periodic function

$$f(x) = \begin{cases} 1 & \text{if } s \leq x \leq s + h \\ 0 & \text{elsewhere in } [-\pi, \pi] \end{cases}$$

24 The One Dimensional Fourier Transform

One of the problems with the theory of Fourier series discussed so far is that it applies only to periodic functions. There are many times when one would like to divide a function which is not periodic into a superposition of sines and cosines. The Fourier transform is the tool often used for this purpose. Like the Laplace transform, the Fourier transform is often an effective tool in finding explicit solutions to differential equations.
To start with, let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise continuous function that vanishes outside an interval of the form $[-\pi L, \pi L]$. This function can be extended to a periodic function, still denoted by f, of period $2\pi L$. From the previous section we can find the complex Fourier series of f to be

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{inx}{L}} \tag{24.1}$$

where

$$c_n = \frac{1}{2\pi L} \int_{-\pi L}^{\pi L} f(x) e^{-\frac{inx}{L}} dx.$$

Let $\xi \in \mathbb{R}$. Multiply both sides of (24.1) by $e^{-i\xi x}$ and then integrate both sides from $-\pi L$ to πL. Assuming integration and summation can be interchanged we find

$$\int_{-\pi L}^{\pi L} f(x) e^{-i\xi x} dx = \sum_{n=-\infty}^{\infty} c_n \int_{-\pi L}^{\pi L} e^{-i\xi x} e^{\frac{inx}{L}} dx.$$

It can be shown that the RHS converges, say to $\hat{f}(\xi)$, as $L \to \infty$. Hence, we find

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx. \tag{24.2}$$

Thus, for a piecewise continuous function f, we define the **Fourier transform** of f to be the function $\hat{f}$ given by (24.2). We will use the notation $\mathcal{F}[f(x)] = \hat{f}(\xi)$.
Now, letting $\xi = \frac{n}{L}$ in (24.2) we find

$$\hat{f}\left(\frac{n}{L}\right) = \int_{-\pi L}^{\pi L} f(x) e^{-\frac{inx}{L}} dx = 2\pi L c_n.$$

Hence, (24.1) can be written in the form

$$f(x) = \frac{1}{2\pi L} \sum_{n=-\infty}^{\infty} \hat{f}\left(\frac{n}{L}\right) e^{\frac{inx}{L}}.$$

In the limit as $L \to \infty$, it can be shown that this last sum approaches an improper integral, and our formula becomes

$$\mathcal{F}^{-1}[\hat{f}(\xi)] = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi \qquad (24.3)$$

Equation (24.3) is called the **Fourier inversion formula**. If we make use of Euler's formula, we can write the Fourier inversion formula in terms of sines and cosines,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) \cos \xi x d\xi + \frac{i}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) \sin \xi x d\xi$$

a superposition of sines and cosines of various frequencies.
Equations (24.2) and (24.3) allow one to pass back and forth between a given function and its representation as a superposition of oscillations of various frequencies.

Example 24.1
Find the Fourier transform of the function $f(x)$ defined by

$$f(x) = \begin{cases} e^{-ax} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

for some $a > 0$.

Solution.
We have

$$\begin{aligned} \hat{f}(\xi) &= \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx = \int_{0}^{\infty} e^{-ax} e^{-i\xi x} dx \\ &= \int_{0}^{\infty} e^{-ax - i\xi x} dx = \left. \frac{e^{-x(a+i\xi)}}{-(a+i\xi)} \right|_{0}^{\infty} \\ &= \frac{1}{a+i\xi} \blacksquare \end{aligned}$$

The following theorem lists the basic properties of Fourier transform

Theorem 24.1
Let f, g, be piecewise continuous functions. Then we have the following properties:
(1) **Linearity:** $\mathcal{F}[\alpha f(x)+\beta g(x)] = \alpha\mathcal{F}[f(x)]+\beta\mathcal{F}[g(x)]$, where α and β are arbitrary numbers.
(2) **Shifting**: $\mathcal{F}[f(x-\alpha)] = e^{-i\alpha\xi}\mathcal{F}[f(x)]$.
(3) **Scaling:** $\mathcal{F}[f\left(\frac{x}{\alpha}\right)] = \alpha\mathcal{F}[f(\alpha x)]$.
(4) **Continuity:** If $\int_{-\infty}^{\infty}|f(x)|dx < \infty$ then $\hat{f}$ is continuous in ξ.
(5) **Differentiation:** $\mathcal{F}[f^{(n)}(x)] = (i\xi)^n\mathcal{F}[f(x)).$
(6) **Integration:** $\mathcal{F}\left[\int_0^x f(s)ds\right] = -\frac{1}{i\xi}\mathcal{F}[f(x)]$.
(7) **Parseval's Relation:** $\int_{-\infty}^{\infty}|f(x)|^2dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{f}(\xi)|^2d\xi$.
(8) **Duality:** $\mathcal{F}[\mathcal{F}[f(x)]] = 2\pi f(-x)$.
(9) **Multiplication by** $x^n : \mathcal{F}[x^n f(x)] = i^n\hat{f}^{(n)}(\xi)$.
(10) **Gaussians:** $\mathcal{F}[e^{-\alpha x^2}] = \sqrt{\frac{\pi}{\alpha}}e^{-\frac{\xi^2}{4\alpha}}$.
(11) **Product:** $\mathcal{F}[(f(x)g(x)] = \frac{1}{2\pi}\mathcal{F}[f(x)] * \mathcal{F}[g(x)]$.
(12) **Convolution:** $\mathcal{F}[(f*g)(x)] = \mathcal{F}[f(x)]\cdot\mathcal{F}[g(x)]$.

Example 24.2
Determine the Fourier transform of the Gaussian $u(x) = e^{-\alpha x^2},\ \alpha > 0$.

Solution.
We have
$$\hat{u}(\xi) = \int_{-\infty}^{\infty} e^{-\alpha x^2}e^{-i\xi x}dx.$$
If we differentiate this relation with respect to the variable ξ and then integrate by parts we obtain

$$\begin{aligned}\hat{u}'(\xi) =& -i\int_{-\infty}^{\infty} xe^{-\alpha x^2}e^{-i\xi x}dx\\ =&\frac{i}{2\alpha}\int_{-\infty}^{\infty}\frac{d}{dx}(e^{-\alpha x^2})e^{-i\xi x}dx\\ =&\frac{i\xi}{2\alpha}\int_{-\infty}^{\infty}(e^{-\alpha x^2})e^{-i\xi x}dx = -\frac{\xi}{2\alpha}\hat{u}(\xi)\end{aligned}$$

Thus we have arrived at the ODE $\hat{u}'(\xi) = -\frac{\xi}{2\alpha}\hat{u}(\xi)$ whose general solution has the form
$$\hat{u}(\xi) = Ce^{-\frac{\xi^2}{4\alpha}}$$

Since

$$\hat{u}(0) = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} = C$$

we find

$$\hat{u}(\xi) = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\xi^2}{4\alpha}} \blacksquare$$

Example 24.3
Prove

$$\mathcal{F}[f(-x)] = \hat{f}(-\xi).$$

Solution.
Using a change of variables we find

$$\mathcal{F}[f(-x)] = \int_{-\infty}^{\infty} f(-x)e^{-i\xi x}dx = \int_{-\infty}^{\infty} f(x)e^{i\xi x}dx = \hat{f}(-\xi) \blacksquare$$

Example 24.4
Prove

$$\mathcal{F}[\mathcal{F}[f(x)]] = 2\pi f(-x).$$

Solution.
We have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi x}d\xi$$

Thus,

$$2\pi f(-x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{-i\xi x}d\xi = \mathcal{F}[\hat{f}(\xi)] = \mathcal{F}[\mathcal{F}[f(x)]] \blacksquare$$

The following theorem lists the properties of inverse Fourier transform

Theorem 24.2
Let f and g be piecewise continuous functions.
(1') **Linearity:** $\mathcal{F}^{-1}[\alpha\hat{f}(\xi) + \beta\hat{g}(\xi)] = \alpha\mathcal{F}^{-1}[\hat{f}(\xi)] + \beta\mathcal{F}^{-1}[\hat{g}(\xi)]$.
(2') **Derivatives:** $\mathcal{F}^{-1}[\hat{f}^{(n)}(\xi)] = (-ix)^n f(x)$.
(3') **Multiplication by** $\xi^n : \mathcal{F}^{-1}[\xi^n\hat{f}(\xi)] = (-i)^n f^{(n)}(x)$.
(4') **Multiplication by** $e^{-i\xi\alpha} : \mathcal{F}^{-1}[e^{-i\xi\alpha}\hat{f}(\xi)] = f(x-\alpha)$.
(5') **Gaussians:** $\mathcal{F}^{-1}[e^{-\alpha\xi^2}] = \frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{x^2}{4\alpha}}$.
(6') **Product:** $\mathcal{F}^{-1}[\hat{f}(\xi)\hat{g}(\xi)] = f(x) * g(x)$.
(7') **Convolution:** $\mathcal{F}^{-1}[\hat{f} * \hat{g}(\xi)] = 2\pi(fg)(x)$.

Remark 24.1
It is important to mention that there exists no established convention how to define the Fourier transform. In the literature, we can meet an equivalent definition of (24.2) with the constant $\frac{1}{\sqrt{2\pi}}$ or $\frac{1}{2\pi}$ in front of the integral. There also exist definitions with positive sign in the exponent. The reader should keep this fact in mind while working with various sources or using the transformation tables.

Practice Problems

Exercise 24.1
Find the Fourier transform of the function

$$f(x) = \begin{cases} 1 & \text{if } -1 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 24.2
Obtain the transformed problem when applying the Fourier transform with respect to the spatial variable to the equation and initial condition

$$u_t + cu_x = 0$$

$$u(x,0) = f(x).$$

Exercise 24.3
Obtain the transformed problem when applying the Fourier transform with respect to the spatial variable to the equation and both initial conditions

$$u_{tt} = c^2 u_{xx},\ x \in \mathbb{R},\ t > 0$$

$$u(x,0) = f(x)$$

$$u_t(x,0) = g(x).$$

Exercise 24.4
Obtain the transformed problem when applying the Fourier transform with respect to the spatial variable to the equation and both initial conditions

$$\Delta u = u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R},\ 0 < y < L$$

$$u(x,0) = 0$$

$$u(x,L) = \begin{cases} 1 & \text{if } -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

Exercise 24.5
Find the Fourier transform of $f(x) = e^{-|x|\alpha}$, where $\alpha > 0$.

Exercise 24.6
Prove that

$$\mathcal{F}[e^{-x}H(x)] = \frac{1}{1+i\xi}$$

where

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 24.7
Prove that

$$\mathcal{F}\left[\frac{1}{1+ix}\right] = 2\pi e^{\xi}H(-\xi).$$

Exercise 24.8
Prove

$$\mathcal{F}[f(x-\alpha)] = e^{-i\xi\alpha}\hat{f}(\xi).$$

Exercise 24.9
Prove

$$\mathcal{F}[e^{i\alpha x}f(x)] = \hat{f}(\xi-\alpha).$$

Exercise 24.10
Prove the following

$$\mathcal{F}[\cos(\alpha x)f(x)] = \frac{1}{2}[\hat{f}(\xi+\alpha) + \hat{f}(\xi-\alpha)]$$

$$\mathcal{F}[\sin(\alpha x)f(x)] = \frac{i}{2}[\hat{f}(\xi+\alpha) - \hat{f}(\xi-\alpha)]$$

Sample Exam Questions

Exercise 24.11
Prove

$$\mathcal{F}[f'(x)] = (i\xi)\hat{f}(\xi).$$

Exercise 24.12
Find the Fourier transform of $f(x) = 1 - |x|$ for $-1 \le x \le 1$ and 0 otherwise.

Exercise 24.13
Find, using the definition, the Fourier transform of

$$f(x) = \begin{cases} -1 & -a < x < 0 \\ 1 & 0 < x < a \\ 0 & \text{otherwise} \end{cases}$$

Exercise 24.14
Find the inverse Fourier transform of $\hat{f}(\xi) = e^{-\frac{\xi^2}{2}}$.

Exercise 24.15
Find $\mathcal{F}^{-1}\left(\frac{1}{a+i\xi}\right)$.

25 Applications of Fourier Transforms to PDEs

Fourier transform is a useful tool for solving differential equations. In this section, we apply Fourier transforms in solving various PDE problems. Contrary to Laplace transform, which usually uses the time variable, the Fourier transform is applied to the spatial variable on the whole real line.
The Fourier transform will be applied to the spatial variable x while the variable t remains fixed. The PDE in the two variables x and t passes under the Fourier transform to an ODE in the $t-$variable. We solve this ODE to obtain the transformed solution $\hat{u}$ which can be converted to the original solution u by means of the inverse Fourier transform. We illustrate these ideas in the examples below.

First Order Transport Equation
Consider the initial value problem

$$u_t + cu_x = 0$$

$$u(x,0) = f(x).$$

Let $\hat{u}(\xi,t)$ be the Fourier transform of u in x. Performing the Fourier transform on both the PDE and the initial condition, we reduce the PDE into an ODE in t

$$\frac{\partial \hat{u}}{\partial t} + i\xi c\hat{u} = 0$$

$$\hat{u}(\xi,0) = \hat{f}(\xi).$$

Solution of the ODE gives

$$\hat{u}(\xi,t) = \hat{f}(\xi)e^{-i\xi ct}.$$

Thus,

$$u(x,t) = \mathcal{F}^{-1}[\hat{u}(\xi,t)] = f(x-ct)$$

which is exactly the same as we obtained by using the method of characteristics.

Second Order Wave Equation
Consider the two dimensional wave equation

$$u_{tt} = c^2u_{xx},\ x \in \mathbb{R},\ t > 0$$

$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x).$$

Again, by performing the Fourier transform of u in x, we reduce the PDE problem into an ODE problem in the variable t:

$$\frac{\partial^2 \hat{u}}{\partial t^2} = -c^2\xi^2\hat{u}$$

$$\hat{u}(\xi,0) = \hat{f}(\xi)$$
$$\hat{u}_t(\xi,0) = \hat{g}(\xi).$$

General solution to the ODE is

$$\hat{u}(\xi,t) = \Phi(\xi)e^{-i\xi ct} + \Psi(\xi)e^{i\xi ct}$$

where Φ and Ψ are two arbitrary functions of ξ. Performing the inverse transformation and making use of the translation theorem, we get the general solution

$$u(x,t) = \phi(x-ct) + \psi(x+ct)$$

where $\hat{\phi} = \Phi$ and $\hat{\psi} = \Psi$. But

$$\Phi(\xi) = \frac{1}{2}\left[\hat{f}(\xi) - \frac{1}{i\xi c}\hat{g}(\xi)\right]$$

$$\Psi(\xi) = \frac{1}{2}\left[\hat{f}(\xi) + \frac{1}{i\xi c}\hat{g}(\xi)\right].$$

By using the integration property, we find the inverse transforms of Φ and Ψ

$$\phi(x) = \frac{1}{2}\left[f(x) + \frac{1}{c}\int_0^x g(s)ds\right]$$

$$\psi(x) = \frac{1}{2}\left[f(x) - \frac{1}{c}\int_0^x g(s)ds\right].$$

Application of the translation property then yields directly the D'Alambert solution

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

Second Order Heat Equation
Next, we consider the heat equation

$$u_t = ku_{xx}, \quad x \in \mathbb{R},\ t > 0$$

$$u(x,0) = f(x).$$

Performing Fourier Transform in x for the PDE and the initial condition, we obtain

$$\frac{\partial \hat{u}}{\partial t} = -k\xi^2 \hat{u}$$

$$\hat{u}(\xi,0) = \hat{f}(\xi).$$

Treating ξ as a parameter, we obtain the solution to the above ODE problem

$$\hat{u}(\xi,t) = \hat{f}(\xi)e^{-k\xi^2 t}.$$

Application of the convolution theorem yields

$$\begin{aligned} u(x,t) =& f(x) * \mathcal{F}^{-1}[e^{-k\xi^2 t}] \\ =& f(x) * \left[\frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}\right] \\ =& \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f(s) e^{-\frac{(x-s)^2}{4kt}} ds \end{aligned}$$

Laplace's Equation in 2D
Consider the problem

$$\Delta u = u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R},\ 0 < y < L$$

$$u(x,0) = 0$$

$$u(x,L) = \begin{cases} 1 & \text{if } -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

Performing Fourier Transform in x for the PDE we obtain the second order PDE in y

$$\hat{u}_{yy} = \xi^2 \hat{u}.$$

The general solution is given by

$$\hat{u}(\xi,y) = A(\xi)\sinh \xi y + B(\xi)\cosh \xi y.$$

Using the boundary condition $\hat{u}(\xi,0)=0$ we find $B(\xi)=0$. Using the second boundary condition we find

$$\begin{aligned}\hat{u}(\xi,L)=&\int_{-\infty}^{\infty}u(x,L)e^{-i\xi x}dx\\=&\int_{-a}^{a}e^{-i\xi x}dx=\int_{-a}^{a}\cos\xi x dx\\=&\frac{2\sin\xi a}{\xi}\end{aligned}$$

Hence,

$$A(\xi)\sinh\xi L=\frac{2\sin\xi a}{\xi}$$

and this implies

$$A(\xi)=\frac{2\sin\xi a}{\xi\sinh\xi L}.$$

Thus,

$$\hat{u}(\xi,y)=\frac{2\sin\xi a}{\xi\sinh\xi L}\sinh\xi y.$$

Taking inverse Fourier transform we find

$$u(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin\xi a}{\xi\sinh\xi L}\sinh\xi y e^{i\xi x}d\xi.$$

Noting that the integrand is an even function in ξ, we can simplify a little to obtain

$$u(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin\xi a}{\xi\sinh\xi L}\sinh\xi y\cos\xi x d\xi \blacksquare$$

Practice Problems

Exercise 25.1
Solve, by using Fourier transform

$$u_t + cu_x = 0$$
$$u(x,0) = e^{-\frac{x^2}{4}}.$$

Exercise 25.2
Solve, by using Fourier transform

$$u_t = ku_{xx} - \alpha u, \quad x \in \mathbb{R}$$
$$u(x,0) = e^{-\frac{x^2}{\gamma}}.$$

Exercise 25.3
Solve the heat equation

$$u_t = ku_{xx}$$

subject to the initial condition

$$u(x,0) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 25.4
Use Fourier transform to solve the heat equation

$$u_t = u_{xx} + u, \quad -\infty < x < \infty < \quad t > 0$$
$$u(x,0) = f(x).$$

Exercise 25.5
Prove that

$$\int_{-\infty}^{\infty} e^{-|\xi|y} e^{i\xi x} d\xi = \frac{2y}{x^2 + y^2}.$$

Exercise 25.6
Solve the Laplace's equation in the half plane

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \ 0 < y < \infty$$

subject to the boundary condition

$$u(x,0) = f(x), \ |u(x,y)| < \infty.$$

Exercise 25.7
Use Fourier transform to find the transformed equation of

$$u_{tt} + (\alpha + \beta)u_t + \alpha\beta u = c^2 u_{xx}$$

where $\alpha, \beta > 0$.

Exercise 25.8
Solve the initial value problem

$$u_t + 3u_x = 0$$

$$u(x, 0) = e^{-x}$$

using the Fourier transform.

Exercise 25.9
Solve the initial value problem

$$u_t = ku_{xx}$$

$$u(x, 0) = e^{-x}$$

using the Fourier transform.

Sample Exam Questions

Exercise 25.10
Solve the initial value problem

$$u_t = ku_{xx}$$

$$u(x,0) = e^{-x^2}$$

using the Fourier transform.

Exercise 25.11
Solve the initial value problem

$$u_t + cu_x = 0$$

$$u(x,0) = x^2$$

using the Fourier transform.

Exercise 25.12
Solve, by using Fourier transform

$$\Delta u = 0$$

$$u_y(x,0) = f(x)$$

$$\lim_{x^2+y^2\to\infty} u(x,y) = 0.$$

Answers and Solutions

Section 1

1.1
(a) $y^3 \cos(xy)$
(b) $e^{x^2y}(2y + 4x^2y^2)$
(c) 0

1.2
(a) $f_x(x,y) = 4x^3,\ f_y(x,y) = \frac{3}{\sqrt{y}}$
(b) $f_x(x,y,z) = 2xy + 43,\ f_y(x,y,z) = x^2 - 20yz^3 - \frac{28}{1+16y^2},\ f_z(x,y,z) = -30y^2z^2$
(c) $f_s(s,t) = \frac{2t^7}{s} - \frac{4}{7}s^{-\frac{3}{7}},\ f_t(s,t) = 7t^6 \ln(s^2) - \frac{27}{t^4}$
(d) $f_x(x,y) = \frac{4}{x^2}\sin\left(\frac{4}{x}\right)e^{x^2y-5y^3} + \cos\left(\frac{4}{x}\right)e^{x^2y-5y^3}(2xy),\ f_y(x,y) = \cos\left(\frac{4}{x}\right)e^{x^2y-5y^3}(x^2 - 15y^2)$
(e) $f_u(u,v) = \frac{9(u^2+5v)-9u(2u)}{(u^2+5v)^2} = \frac{-9u^2+45v}{(u^2+5v)^2},\ f_v(u,v) = \frac{-45u}{(u^2+5v)^2}$
(f) $f_x(x,y,z) = \frac{\sin y}{z^2},\ f_y(x,y,z) = \frac{x\cos y}{z^2},\ f_z(x,y,z) = -2\frac{x\sin y}{z^3}$
(g) $f_x(x,y) = \frac{1}{2}\left(2x + \frac{5}{5x-3y^2}\right)(x^2 + \ln(5x-3y^2))^{-\frac{1}{2}},\ f_y(x,y) = -\frac{3y}{5x-3y^2}(x^2 + \ln(5x-3y^2))^{-\frac{1}{2}}$

1.3 3

1.4
$\frac{\partial z}{\partial s} = t^2e^{st^2}\sin(s^2t) + 2ste^{st^2}\cos(s^2t)$
$\frac{\partial z}{\partial t} = 2ste^{st^2}\sin(s^2t) + s^2e^{st^2}\cos(s^2t)$

1.5 u is the depedent variable whereas x and y are the independent variables.

1.6 We have
$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx.$$
By the change of variable $u = -x$ we find
$$\int_{-a}^{0} f(u)du = -\int_{a}^{0} f(-u)du = -\int_{0}^{a} f(u)du.$$
Hence, the result follows.

1.7 We have
$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx.$$
By the change of variable $u = -x$ we find
$$\int_{-a}^{0} f(u)du = -\int_{a}^{0} f(-u)du = \int_{0}^{a} f(u)du.$$
Hence, the result follows.

1.8 By the product rule of derivatives we have
$$(uv)' = u'v + uv'.$$
Integrate both sides to obtain
$$uv = \int u'vdx + \int uv'dx.$$
Now subtract $\int u'vdx$ from both sides to obtain the desired result.

1.9
$u_{tt} = -\sin\left(\frac{x}{\epsilon}\right)\sin\left(\frac{t}{\epsilon}\right)$
$u_{xx} = -\sin\left(\frac{x}{\epsilon}\right)\sin\left(\frac{t}{\epsilon}\right).$

1.10
$u_{tt} = \sin\left(\frac{x}{\epsilon}\right)\sinh\left(\frac{t}{\epsilon}\right)$
$u_{xx} = -\sin\left(\frac{x}{\epsilon}\right)\sinh\left(\frac{t}{\epsilon}\right).$

1.11 $\epsilon^2 \sup\{\left|\sinh\left(\frac{t}{\epsilon}\right)\right|\}$

1.12 (a) We have $\sup\{|u_n(x,0)-1| : x \in \mathbb{R}\} = \frac{1}{n}\sup\{|\sin nx| : x \in \mathbb{R}\} = \frac{1}{n}$.
(b) We have $\sup\{|u_n(x,t)-1| : x \in \mathbb{R}\} = \frac{e^{n^2 t}}{n}$ ∎

Section 2

2.1 (a) For all $0 \le x < 1$ we have $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0$. Also, $\lim_{n\to\infty} f_n(1) = 1$. Hence, the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f.
(b) Suppose the contrary. Let $\epsilon = \frac{1}{2}$. Then there exists a positive integer N such that for all $n \ge N$ we have

$$|f_n(x) - f(x)| < \frac{1}{2}$$

for all $x \in [0,1]$. In particular, we have

$$|f_N(x) - f(x)| < \frac{1}{2}$$

for all $x \in [0,1]$. Choose $(0.5)^{\frac{1}{N}} < x < 1$. Then $|f_N(x) - f(x)| = x^N > 0.5 = \epsilon$ which is a contradiction. Hence, the given sequence does not converge uniformly.

2.2 For every real number x, we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx + x^2}{n^2} = \lim_{n\to\infty} \frac{x}{n} + \lim_{n\to\infty} \frac{x^2}{n^2} = 0$$

Thus, $\{f_n\}_{n=1}^{\infty}$ converges pointwise to the zero function on $\mathbb{R}$.

2.3 For every real number x, we have

$$-\frac{1}{\sqrt{n+1}} \le f_n(x) \le \frac{1}{\sqrt{n+1}}.$$

Moreover,

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0.$$

Applying the squeeze rule for sequences, we obtain

$$\lim_{n\to\infty} f_n(x) = 0$$

for all x in $\mathbb{R}$. Thus, $\{f_n\}_{n=1}^{\infty}$ converges pointwise to the zero function on $\mathbb{R}$.

2.4 First of all, observe that $f_n(0) = 0$ for every n in $\mathbb{N}$. So the sequence $\{f_n(0)\}_{n=1}^{\infty}$ is constant and converges to zero. Now suppose $0 < x < 1$ then $n^2x_n = n^2e^{n\ln x}$. But $\ln x < 0$ when $0 < x < 1$, it follows that

$$\lim_{n\to\infty} f_n(x) = 0 \text{ for } 0 < x < 1$$

Finally, $f_n(1) = n^2$ for all n. So,

$$\lim_{n\to\infty} f_n(1) = \infty.$$

Therefore, $\{f_n\}_{n=1}^{\infty}$ is not pointwise convergent on $[0, 1]$.

2.5 For $-\frac{\pi}{2} \le x < 0$ and $0 < x \le \frac{\pi}{2}$ we have

$$\lim_{n\to\infty} (\cos x)^n = 0.$$

For $x = 0$ we have $f_n(0) = 1$ for all n in $\mathbb{N}$. Therefore, $\{f_n\}_{n=1}^{\infty}$ converges pointwise to

$$f(x) = \begin{cases} 0 & \text{if } -\frac{\pi}{2} \le x < 0 \text{ and } 0 < x \le \frac{\pi}{2} \\ 1 & \text{if } x = 0. \end{cases}$$

2.6 (a) Let $\epsilon > 0$ be given. Let N be a positive integer such that $N > \frac{1}{\epsilon}$. Then for $n \ge N$

$$\left| x - \frac{x^n}{n} - x \right| = \frac{|x|^n}{n} < \frac{1}{n} \le \frac{1}{N} < \epsilon.$$

Thus, the given sequence converges uniformly (and pointwise) to the function $f(x) = x$.
(b) Since $\lim_{n\to\infty} f_n'(x) = 1$ for all $x \in [0, 1)$, the sequence $\{f_n'\}_{n=1}^{\infty}$ converges pointwise to $f'(x) = 1$. However, the convergence is not uniform. To see this, let $\epsilon = \frac{1}{2}$ and suppose that the convergence is uniform. Then there is a positive integer N such that for $n \ge N$ we have

$$|1 - x^{n-1} - 1| = |x|^{n-1} < \frac{1}{2}.$$

In particular, if we let $n = N + 1$ we must have $x^N < \frac{1}{2}$ for all $x \in [0, 1)$. But $x = \left(\frac{1}{2}\right)^{\frac{1}{N}} \in [0, 1)$ and $x^N = \frac{1}{2}$ which contradicts $x^N < \frac{1}{2}$. Hence, the convergence is not uniform.

2.7 (a) The pointwise limit is

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ \frac{1}{2} & \text{if } x = 1 \\ 1 & \text{if } 1 < x \le 2 \end{cases}$$

(b) The convergence cannot be uniform because if it were f would have to be continuous.

2.8 (a) Let $\epsilon > 0$ be given. Note that

$$|f_n(x) - \frac{1}{2}| = \left|\frac{2\cos x - \sin^2 x}{2(2n + \sin^2 x)}\right| \le \frac{3}{4n}.$$

Since $\lim_{n\to\infty} \frac{3}{4n} = 0$ we can find a positive integer N such that if $n \ge N$ then $\frac{3}{4n} < \epsilon$. Thus, for $n \ge N$ and all $x \in \mathbb{R}$ we have

$$|f_n(x) - \frac{1}{2}| \le \frac{3}{4n} < \epsilon.$$

This shows that $f_n \to \frac{1}{2}$ uniformly on $\mathbb{R}$ and also on $[2, 7]$.
(b) We have

$$\lim_{n\to\infty} \int_2^7 f_n x dx = \int_2^7 \lim_{n\to\infty} f_n x dx = \int_2^7 \frac{1}{2} dx = \frac{5}{2}.$$

2.9 We have proved earlier that this sequence converges pointwise to the discontinuous function

$$f(x) = \begin{cases} 0 & \text{if } -\frac{\pi}{2} \le x < 0 \text{ and } 0 < x \le \frac{\pi}{2} \\ 1 & \text{if } x = 0 \end{cases}$$

Therefore, uniform convergence cannot occur for this given sequence.

2.10 (a) Using the squeeze rule we find

$$\lim_{n\to\infty} \sup\{|f_n(x)| : 2 \le x \le 5\} = 0.$$

Thus, $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the zero function.
(b) We have

$$\lim_{n\to\infty}\int_2^5 f_n(x)dx = \int_2^5 0dx = 0.$$

Section 3.

3.1 $y = \frac{1}{2}(1 - e^{-t^2})$.

3.2 $y(t) = \frac{3t-1}{9} + e^{-2t} + Ce^{-3t}$.

3.3 $y(t) = 3\sin t + \frac{3\cos t}{t} + \frac{C}{t}$.

3.4 $y(t) = \frac{1}{13}(3\sin(3t) + 2\cos(3t)) + Ce^{-2t}$.

3.5 $y(t) = Ce^{-\sin t} - 3$.

3.6 $\alpha = -2$.

3.7 $p(t) = 2$ and $g(t) = 2t + 3$.

3.8 $y_0 = y(0) = -1$ and $g(t) = 2e^t + \cos t + \sin t$.

3.9 1.

3.10 $u(x, y) = f(bx - ay)e^{-\frac{c}{a^2+b^2}(ax+by)}$.

3.11 $y(t) = t\ln|t| + 7t$.

3.12 Since $p(t) = a$ we find $\mu(t) = e^{at}$. Suppose first that $a = \lambda$. Then

$$y' + ay = be^{-at}$$

and the corresponding general solution is

$$y(t) = bte^{-at} + Ce^{-at}$$

Thus,

$$\begin{array}{rcl}\lim_{t\to\infty} y(t) & = & \lim_{t\to\infty}(\frac{bt}{e^{at}} + \frac{C}{e^{at}}) \\ & = & \lim_{t\to\infty}\frac{b}{ae^{at}} = 0\end{array}$$

Now, suppose that $a \neq \lambda$ then

$$y(t) = \frac{b}{a-\lambda} e^{-\lambda t} + Ce^{-at}$$

Thus,

$$\lim_{t\to\infty} y(t) = 0.$$

3.13 $y(t) = (-te^t + e^t)^{-1}$.

3.14 $y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{t^2}{2} + \frac{1}{12t^2}$.

3.15 $y(t) = tSi(t) + (3 - Si(1))t$.

Section 4

4.1 $y(t) = \left(\frac{3}{2}e^{t^2} + C\right)^{\frac{1}{3}}$.

4.2 $y(t) = Ce^{\frac{t^2}{2} - 2t}$.

4.3 $y(t) = Ct^2 + 4$.

4.4 $y(t) = \frac{2Ce^{4t}}{1+Ce^{4t}}$.

4.5 $y(t) = \sqrt{5 - 4\cos(2t)}$.

4.6 $y(t) = -\sqrt{(-2\cos t + 4)}$.

4.7 $y(t) = e^{1-t} - 1$.

4.8 $y(t) = \frac{2}{\sqrt{-4t^2+1}}$.

4.9 $y(t) = \tan(t + \pi) = -\cot t$.

4.10 $y(t) = \frac{3-e^{-t^2}}{3+e^{-t^2}}$.

4.11 $\alpha = \frac{1}{2},\ y_0 = \frac{1}{2}$ and $n = 3$.

4.12 $u(x, y) = F(y)e^{-3x} + G(x)$ where $F(y) = \int f(y)dy$.

4.13 $y^2 + \cos y + \cos t + \frac{t^2}{2} = 2.$

4.14 $3y^2y' + \cos y + 2t = 0, \quad y(2) = 0.$

4.15 The ODE is not separable.

Section 5

5.1 $y(t) = -2e^t + e^{3t}$. $\lim_{t\to-\infty} y(t) = 0$ and $\lim_{t\to\infty} y(t) = \infty$.

5.2 $y(t) = -2\sqrt{2}e^{(-2-\sqrt{2})t} + 2\sqrt{2}e^{(-2+\sqrt{2})t}$. $\lim_{t\to-\infty} y(t) = -\infty$ and $\lim_{t\to\infty} y(t) = 0$.

5.3 $y(t) = -2e^{-\frac{\sqrt{2}}{2}t}$. $\lim_{t\to-\infty} y(t) = -\infty$ and $\lim_{t\to\infty} y(t) = 0$.

5.4 $y'' - y' - 2y = 0.$

5.5 $y(t) = e^{\frac{t}{3}-1}(1-t).$

5.6 $y(t) = e^{-\frac{2t}{5}}(t-1).$

5.7 $y(0) = 2$ and $y'(0) = -2$.

5.8 $y(t) = c_1e^{3t} + c_2te^{3t}.$

5.9 $y(t) = 3e^{-t}\cos t + 2e^{-t}\sin t.$

5.10 $y(t) = -e^{\frac{1}{2}(t+\pi)}(3\cos\frac{t}{2} + \sin\frac{t}{2}).$

5.11 $y(t) = y_h(t) + y_p(t) = e^{\frac{1}{2}t}(c_1\cos\frac{\sqrt{3}}{2}t + c_2\sin\frac{\sqrt{3}}{2}t) + \frac{6}{73}\cos 3t - \frac{16}{73}\sin 3t.$

5.12 $y(t) = y_h(t) + y_p(t) = c_1e^{(-2-\sqrt{6})t} + c_2e^{(-2+\sqrt{6})t} - t^2 - \frac{5}{2}t - 9.$

5.13 $y = Ax^4 + Bx^4\ln x.$

5.14 $y = x^{-1}(A\cos(\sqrt{3}\ln x) + B\sin(\sqrt{3}\ln x)).$

5.15 (a) $\lambda_n = n^2, \quad y_n(x) = \sin nx, \quad n = 1, 2, \cdots$.
(b) $\lambda_n = \left[\left(n - \frac{1}{2}\right)\frac{\pi}{L}\right]^2$ and $y_n = \sin\left(\frac{\pi}{L}\left(n - \frac{1}{2}\right)x\right), \quad n = 1, 2, 3, \cdots$.
(c) $\lambda_n = \left(\pi\left(n - \frac{1}{2}\right)\right)^2, \quad y_n(x) = \cos\left(\pi\left(n - \frac{1}{2}\right)\right)x, \quad n = 1, 2, \cdots$.

5.16 We consider first the cases (a) and (b). Multiply the equation by $y'(x)$ and integrate in x from 0 to L.

$$\int_0^L (ky'(x))'y(x)dx + \int_0^L \lambda y^2(x)dx = 0.$$

Use integration by parts in the first integral

$$[ky'(x)y(x)]_0^L - \int_0^L k(y'(x))^2 dx + \int_0^L \lambda y^2(x)dx = 0.$$

The boundary term vanishes because of the boundary conditions. We solve the above equation for λ and obtain

$$\lambda = \frac{\int_0^L k(y'(x))^2 dx}{\int_0^L y^2(x)dx} \geq 0.$$

For the case (c), we repeat the above argument but by integrating from $-L$ to L.

5.17 $y(t) = 2e^{\frac{t}{2}}$.

5.18 $y(t) = c_1 + c_2e^t - \frac{1}{10}\cos(2t) + \frac{1}{5}\sin(2t) + 5te^t$ ■

5.19 $y(t) = \frac{17}{15}e^t + \frac{1}{6}e^{-2t} - \frac{1}{2}t - \frac{1}{4} - \frac{3}{20}\sin 2t - \frac{1}{20}\cos 2t$.

Section 6

6.1 (a) ODE (b) PDE (c) ODE.

6.2 $u_{ss} = 0$.

6.3 $u_{ss} + u_{tt} = 0$.

6.4 (a) Order 3, nonlinear (b) Order 1, linear, homogeneous (c) Order 2, linear, nonhomogeneous.

6.5 (a) Linear, homogeneous, order 3.
(b) Linear, nonhomogeneous, order 3. The inhomogeneity is $-\sin y$.
(c) Nonlinear, order 2. The nonlinear term is uu_x.
(d) Nonlinear, order 3. The nonlinear terms are $u_x u_{xxy}$ and uu_y.
(e) Linear, nonhomogeneous, order 2. The inhomogeneity is $f(x, y, t)$.

6.6 (a) Linear. (b) Linear. (c) Nonlinear. (d) Nonlinear.

6.7 (a) PDE, linear, second order, homogeneous.
(b) PDE, linear, second order, homogeneous.
(c) PDE, nonlinear, fourth order.
(d) ODE, linear, second order, nonhomogeneous.
(e) PDE, linear, second order, nonhomogeneous.
(f) PDE, quasilinear, second order.

6.8 $A(x, y, z)u_{xx}+B(x, y, z)u_{xy}+C(x, y, z)u_{yy}+E(x, y, z)u_{xz}+F(x, y, z)u_{yz}+G(x, y, z)u_{zz}+H(x, y, z)u_x+I(x, y, z)u_y+J(x, y, z)u_z+K(x, y, z)u = L(x, y, z)$.

6.9 (a) Order 3, linear, homogeneous.
(b) Order 1, nonlinear.
(c) Order 4, linear, nonhomogeneous
(d) Order 2, nonlinear.
(e) Order 2, linear, homogeneous.

6.10 $u_{ww} = 0$.

6.11 $u_{vw} = 0$.

6.12 $u_{vw} = 0$.

6.13 $u_s = 0$.

6.14 $u_s = \frac{1}{2}$.

6.15 $u_w = u$.

Section 7

7.1 $a = b = 0$.

7.2 Substituting into the differential equation we find

$$tX''T - XT' = 0$$

or

$$\frac{X''}{X} = \frac{T'}{tT}.$$

The LHS is a function of x only whereas the RHS is a function of t only. This is true only when both sides are constant. That is, there is λ such that

$$\frac{X''}{X} = \frac{T'}{tT} = \lambda$$

and this leads to the two ODEs $X'' = \lambda X$ and $T' = \lambda tT$.

7.3 We have $xu_x + (x+1)yu_y = \frac{x}{y}(e^x + xe^x) + (x+1)y\left(-\frac{xe^x}{y^2}\right) = 0$ and $u(1,1) = e$.

7.4 We have $u_x + u_y + 2u = e^{-2y}\cos(x-y) - 2e^{-2y}\sin(x-y) - e^{-2y}\cos(x-y) + 2e^{-2y}\sin(x-y) = 0$ and $u(x,0) = \sin x$.

7.5 (a) The general solution to this equation is $u(x) = C$ where C is an arbitrary constant.
(b) The general solution is $u(x,y) = f(y)$ where f is an arbitrary function of y.

7.6 (a) The general solution to this equation is $u(x) = C_1x + C_2$ where C_1 and C_2 are arbitrary constants.
(b) We have $u_y = f(y)$ where f is an arbitrary function of y. Hence, $u(x,y) = \int_a^y f(t)dt$.

7.7 Let $v(x,y) = y + 2x$. Then

$$\begin{aligned} u_x =& 2f_v(v) + g(v) + 2xg_v(v) \\ u_{xx} =& 4f_{vv}(v) + 4g_v(v) + 4xg_{vv}(v) \\ u_y =& f_v(v) + xg_v(v) \\ u_{yy} =& f_{vv}(v) + xg_{vv}(v) \\ u_{xy} =& 2f_{vv}(v) + g_v(v) + 2xg_{vv}(v) \end{aligned}$$

Hence,

$$\begin{aligned} u_{xx} - 4u_{xy} + 4u_{yy} =& 4f_{vv}(v) + 4g_v(v) + 4xg_{vv}(v) \\ & -8f_{vv}(v) - 4g_v(v) - 8xg_{vv}(v) \\ & +4f_{vv}(v) + 4xg_{vv}(v) = 0. \end{aligned}$$

7.8 $u_{tt} = c^2 u_{xx}$.

7.9 Let $v = x + p(u)t$. Using the chain rule we find

$$u_t = f_v \cdot v_t = f_v \cdot (p(u) + p_u u_t t).$$

Thus

$$(1 - tf_v p_u)u_t = f_v p.$$

If $1 - tf_v p_u \equiv 0$ on any $t-$interval I then $f_v p \equiv 0$ on I which implies that $f_v \equiv 0$ or $p \equiv 0$ on I. But either condition will imply that $tf_v p_u \equiv 0$ and this will imply that $1 = 1 - tf_v p_u = 0$, a contradiction. Hence, we must have $1 - tf_v p_u \neq 0$. In this case,

$$u_t = \frac{f_v p}{1 - tf_v p_u}.$$

Likewise,

$$u_x = f_v \cdot (1 + p_u u_x t)$$

or

$$u_x = \frac{f_v}{1 - tf_v p_u}.$$

It follows that $u_t = p(u)u_x$.
If $u_t = (\sin u)u_x$ then $p(u) = \sin u$ so that the general solution is given by

$$u(x,t) = f(x + t\sin u)$$

where f is an arbitrary differentiable function in one variable.

7.10 $u(x,y) = xf(x-y) + g(x-y)$.

7.11 Using integration by parts, we compute

$$\begin{aligned}\int_0^L u_{xx}(x,t)u(x,t)dx = u_x(x,t)u(x,t)|_{x=0}^L - \int_0^L u_x^2(x,t)dx \\ =& u_x(L,t)u(L,t) - u_x(0,t)u(0,t) - \int_0^L u_x^2(x,t)dx \\ =& -\int_0^L u_x^2(x,t)dx \le 0\end{aligned}$$

Note that we have used the boundary conditions $u(0,t) = u(L,t) = 0$ and the fact that $u_x^2(x,t) \geq 0$ for all $x \in [0,L]$.

7.12 (a) This can be done by plugging in the equations.
(b) Plug in.
(c) We have $\sup\{|u_n(x,0) - 1| : x \in \mathbb{R}\} = \frac{1}{n}\sup\{|\sin nx| : x \in \mathbb{R}\} = \frac{1}{n}$.
(d) We have $\sup\{|u_n(x,t) - 1| : x \in \mathbb{R}\} = \frac{e^{n^2t}}{n}$.
(e) We have $\lim_{t\to\infty}\sup\{|u_n(x,t) - 1| : x \in \mathbb{R}, t > 0\} = \lim_{t\to\infty}\frac{e^{n^2t}}{n} = \infty$. Hence, the solution is unstable and thus the problem is ill-posed.

7.13 (a) $u(x,y) = x^3 + xy^2 + f(y)$, where f is an arbitrary function.
(b) $u(x,y) = \frac{x^3y^2}{6} + F(x) + g(y)$, where $F(x) = \int f(x)dx$.
(c) $u(x,t) = \frac{1}{18}e^{2x+3t} + t\int_a^x f(s)ds + \int_a^x g(s)ds$.

7.14 (b) $u(x,y) = xf(y-2x) + g(y-2x)$.

7.15 We have

$$\begin{aligned}u_t =& cu_v - cu_w \\ u_{tt} =& c^2u_{vv} - 2c^2u_{wv} + c^2u_{ww} \\ u_x =& u_v + u_w \\ u_{xx} =& u_{vv} + 2u_{vw} + u_{ww}\end{aligned}$$

Substituting we find $u_{vw} = 0$ and solving this equation we find $u_v = f(v)$ and $u(v,w) = F(v) + G(w)$ where $F(v) = \int f(v)dv$.

Finally, using the fact that $v = x + ct$ and $w = x - ct$; we get d'Alembert's solution to the one-dimensional wave equation:

$$u(x,t) = F(x+ct) + G(x-ct)$$

where F and G are arbitrary differentiable functions.

Section 8

8.1 (a) Linear (b) Quasi-linear, nonlinear (c) Nonlinear (d) Semi-linear, nonlinear.

8.2 Let $w = 2x - y$. Then $u_x + 2u_y - u = e^x f(w) + 2e^x f_w(w) - 2e^x f_w(w) - e^x f_w(w) = 0$.

8.3 We have $xu_x - yu_y = x(\sqrt{xy} + \frac{xy}{2\sqrt{xy}}) - y\frac{x^2}{2\sqrt{xy}} = x\sqrt{xy} = u$. Also, $u(y,y) = y^2$.

8.4 We have $-yu_x + xu_y = -2xy\sin(x^2+y^2) + 2xy\sin(x^2+y^2) = 0$. Moreover, $u(0,y) = \cos y^2$.

8.5 We have $\frac{1}{x}u_x + \frac{1}{y}u_y = \frac{1}{x}(-x) + \frac{1}{y}(1+y) = \frac{1}{y}$. Moreover, $u(x,1) = \frac{1}{2}(3-x^2)$.

8.6 $3a - 7b = 0$.

8.7 Plug $u = av + w$ into the equation. Using the linearity of L and the assumptions on v and w, obtain

$$L(u) = L(av+w) = aL(v) + L(w) = 0 + f = f$$

for any constant a. Therefore, u solves the nonhomogeneous equation for any a.

8.8 $u_s + \frac{cu}{a^2+b^2} = 0$.

8.9 $u(x,t) = \frac{1}{2}(x+y) + f(x-y)$.

8.10 We have

$$\begin{aligned}u_x =& -4e^{-4x}f(2x-3y)+2e^{-4x}f'(2x-3y)\\ u_y =& -3e^{-4x}f'(2x-3y)\end{aligned}$$

Thus,

$$\begin{aligned}3u_x+2u_y+12u =& -12e^{-4x}f(2x-3y)+6e^{-4x}f'(2x-3y)\\ &-6e^{-4x}f'(2x-3y)+12e^{-4x}f(2x-3y)=0.\end{aligned}$$

8.11 $u(x,y)=f(ax-bt)e^{\frac{t}{a}}$.

8.12 $u(x,y)=f(bx-ay)$.

8.13 $u_w+\lambda u=f(v+cw,w)$.

8.14 $vw_v(v)=Aw(v)$.

Section 9

9.1 $u(x,t)=\sin(x-3t)$.

9.2 $u(x,y)=e^{-\frac{c(ax+by)}{a^2+b^2}}f(bx-ay)$.

9.3 $u(x,y)=x\cos(2x-y)+f(y-2x)$.

9.4 The change of coordinates $v=x+t$ and $w=x-t$ reduces the original equation to the equation $u_v=\frac{v+w}{4}$ whose solution is given by $u(v,w)=\frac{v^2}{8}+\frac{wv}{4}+g(w)$ or $u(x,t)=\frac{(x+t)^2}{8}+\frac{x^2-t^2}{4}+g(x-t)$. But $u(x,x)=1$ so that $1=\frac{x^2}{2}+g(0)$ or $g(0)=1-\frac{x^2}{2}$ which is impossible since $g(0)$ is a constant. Hence, the given initial value problem has no solution.

9.5 $u(x,t)=\frac{e^{-3t}}{1+(x-2t)^2}$.

9.6 $u(x,t)=e^{3t}\left[(x-t)^2+\frac{1}{9}\right]-\frac{1}{3}t-\frac{1}{9}$.

9.7 Using the chain rule we find $w_t=u_te^{\lambda t}+\lambda ue^{\lambda t}$ and $w_x=u_xe^{\lambda t}$. Substituting these equations into the original equation we find

$$w_te^{-\lambda t}-\lambda u+cw_xe^{-\lambda t}+\lambda u=0$$

or

$$w_t + cw_x = 0$$

9.8 $u(x,y) = \frac{h(x-y)}{1-yh(x-y)}$.

9.9 (a) $w(x,t)$ is a solution to the equation follows from the principle of superposition. Moreover, $w(x,0) = u(x,0) - v(x,0) = f(x) - g(x)$.
(b) $w(x,t) = f(x-ct) - g(x-ct)$.
(c) From (b) we see that

$$\max_{x,t}\{|u(x,t) - v(x,t)|\} = \max_{x}\{|f(x) - g(x)|\}.$$

Thus, small changes in the initial data produces small changes in the solution. Hence, the problem is a well-posed problem.

9.10

$$u(x,t) = \begin{cases} g\left(t - \frac{x}{c}\right) e^{-\frac{\lambda}{c}x} & \text{if } x < ct \\ 0 & \text{if } x \geq ct. \end{cases}$$

9.11 $u(x,t) = \sin\left(\frac{2x-3t}{2}\right)$.

9.12 $u(x,y) = \frac{1}{2}(x+y) + f(x-y)$.

9.13 (a) $a = 1,\ b = 0, c = B$, and $d = -A$ (b) $u(x,y) = f(Bx - Ay)e^{-\frac{C}{A}x}$.

9.14 $u(x,y) = f(x-y)e^{-x}$.

9.15 $u(x,y) = f(x-y)e^{-x} + x + y - 2$.

Section 10

10.1 The characteristics are hyperbolas: $xy = k$.

10.2 The characteristics are circles centered at the origin: $x^2 + y^2 = k$.

10.3 The characteristics are parallel lines with common slope equals to $1 : x - y = k$.

10.4 $f\left(\frac{y}{x}, xe^{-\arctan u}\right) = 0$ where f is an arbitrary differentiable function.

10.5 $f(y+u, y\ln(y+u)-x) = 0$ or $u = -y + g(y\ln(y+u)-x)$ where f and g are arbitrary differentiable functions.

10.6 $f\left(\frac{y}{x}, \frac{u}{x}\right) = 0$ or $u = xg\left(\frac{y}{x}\right)$ where f and g are arbitrary differentiable functions.

10.7 $f\left(\frac{y}{x}, \frac{u}{x^n}\right) = 0$ or $u = x^n g\left(\frac{y}{x}\right)$ where f and g are arbitrary differentiable functions.

10.8 $f(x+y+z, xyu) = 0$ or $u = \frac{g(x+y+z)}{xy}$ where f and g are arbitrary differentiable functions.

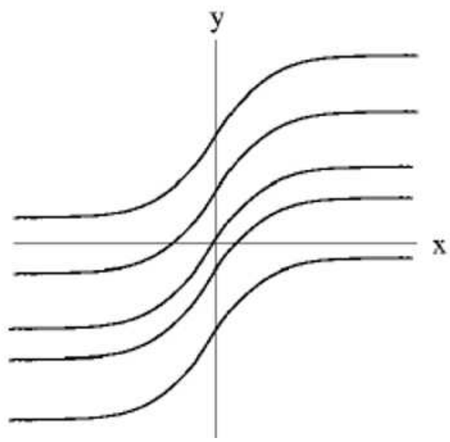
10.9 $f(xy, x^4 - u^4 - 2xyu^2) = 0$, where f is an arbitrary differentiable function.

10.10 $f(x^2+y^2-u^2, 2xy+u^2) = 0$ where f is an arbitrary differentiable function.

10.11 $f(\frac{y}{u}, x^2+y^2+u^2) = 0$ where f is an arbitrary differentiable function.

10.12 $u(x,y) = e^x f(y-2x)$.

10.13 $u(x,y) = f(y - \arctan x)$ for any differentiable function f. The characteristics are shown below.



10.14 $u = e^x f(ye^{-x})$ where f is an arbitrary differential function.

10.15 The characteristics are solutions to the DE $\frac{dy}{dx} = x$. Solving this ODE we find $y = \frac{x^2}{2} + C$.

10.16 $u = e^{-\frac{x^2}{2}} f(ye^{-x})$ where f is an arbitrary differentiable function of one variable.

10.17 Solving $\frac{dy}{dx} = \frac{x}{y}$ by the separation of variables we find $x^2 - y^2 = k$, where k is a constant.

10.18 Solving $\frac{dy}{dx} = \frac{x}{y}$ by the separation of variables we find $x^2 - y^2 = k$, where k is a constant.

Section 11

11.1 $u(x, y) = \frac{1-xy}{x+y}, \quad x + y \neq 0.$

11.2 $u(x, y) = (x + y)(x^2 - y^2).$

11.3 $2xyu + x^2 + y^2 - 2u + 2 = 0.$

11.4 $u(x, y) = \ln\left(x + 1 - \frac{y}{x}\right).$

11.5 $u(x,y) = f(xe^{-y})$.

11.6 $u(t,x) = f(x - at)$.

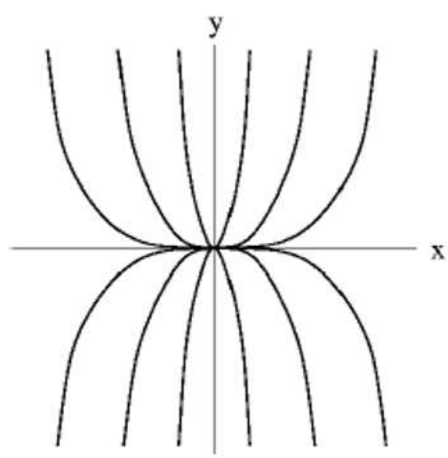
11.7 $u(x,y) = \frac{1}{\sec(x-ay)-y}$.

11.8 $u(x,y) = h\left(y - \frac{(x-1)^2}{2} - (x-1)\right)e^{x-1}$.

11.9 $u(x,y) = f(x - uy)$.

11.10 $u(x,y) = y - \sin^{-1} x$.

11.11 (i) $y = Cx^2$. The characteristics are parobolas in the plane centered at the origin. See figure below.



(ii) $u(x,y) = e^{yx^{-2}}$.
(iii) In the first case, we cannot substitute $x = 0$ into yx^{-2} (the argument of the function f, above) because x^{-2} is not defined at 0. Similarly, in the second case, we'd need to find a function f so that $f(0) = h(x)$. If h is not constant, it is not possible to satisfy this condition for all $x \in \mathbb{R}$.
(iv) All characteristics intersect at $(0,0)$. Since the solution is constant along any characteristic, if the solution is not exactly constant for all (x,y), then the limit of $u(x,y)$ as $(x,y) \to (0,0)$ is different if we approach $(0,0)$ along different characteristics. Therefore, the method doesn't work at that point.

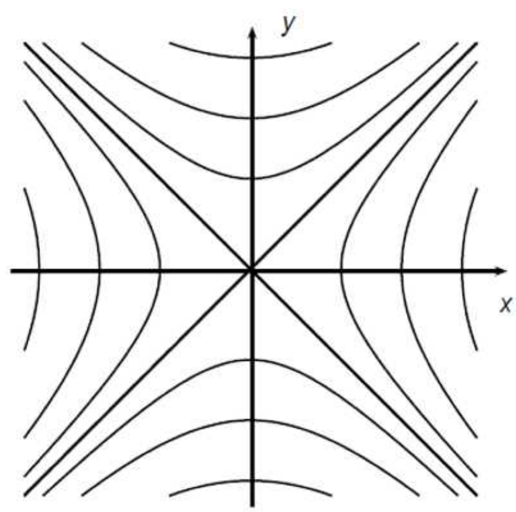
11.12 $u(x,y) = e^y \cos(x-y)$.

11.13 (a) $u = e^x f(ye^{-x})$ where f is an arbitrary differential function.
(b) We want $2 = u(x,3x) = e^x f(3e^x e^{-x}) = e^x f(3)$. This equation is impossible so this Cauchy problem has no solutions.
(c) We want $e^x = e^x f(e^x e^{-x}) \Longrightarrow f(1) = 1$. In this case, there are infinitely many solutions to this Cauchy problem, namely, $u(x,y) = e^x f(ye^{-x})$ where f is an arbitrary function satisfying $f(1) = 1$.

11.14 $u(x,y) = -1 + 2e^{\frac{x^2}{2}} e^{-\frac{(4x-y)^2}{2}}$.

11.15 The Cauchy problem has no solutions.

11.16 (a) The characteristics satisfy the ODE $\frac{dy}{dx} = \frac{x}{y}$. Solving this equation we find $x^2 - y^2 = C$. Thus, the characteristics are hyperbolas.
(b)



(c) The general solution to the PDE is $u(x,y) = f(x^2 - y^2)$ where f is an arbitrary differentiable function. Since $u(0,y) = e^{-y^2}$ we find $f(y) = e^y$. Hence, $u(x,y) = e^{x^2-y^2}$.
(d) This solution is only defined in the region covered by characteristics that cross the y axis: $y^2 - x^2 > 0$. The solution in the region $y^2 - x^2 < 0$ can be any function of the form $u(x,y) = f(x^2 - y^2)$.

11.17 (a) Solving the ODE $\frac{dy}{dx} = y$ we find the characteristics $ye^{-x} = C$. Thus, $u(x, y) = f(ye^{-x})$. If $u(x, 0) = 1$ then we choose f to be any arbitrary differentiable function satisfying $f(0) = 1$.
(b) The line $y = 0$ is a characteristic so that u has to be constant there. Hence, there is no solution satisfying the condition $u(x, 0) = x$.

Section 12

12.1 (a) Hyperbolic (b) Parabolic (c) Elliptic.

12.2 (a) Ellitpic (b) Parabolic (c) Hyperbolic.

12.3 • The PDE is of hyperbolic type if $4y^2(x^2 + x + 1) > 0$. This is true for all $y \neq 0$. Graphically, this is the $xy-$plane with the $x-$axis removed,
• The PDE is of parabolic type if $4y^2(x^2 + x + 1) = 0$. Since $x^2 + x + 1 > 0$ for all $x \in \mathbb{R}$, we must have $y = 0$. Graphically, this is $x-$axis.
• The PDE is of elliptic type if $4y^2(x^2 + x + 1) < 0$ which can not happen.

12.4 We have

$$\begin{aligned} u_x(x,t) &= -\sin x \sin t, \\ u_{xx}(x,t) &= -\cos x \sin t, \\ u_t(x,t) &= \cos x \cos t, \\ u_{tt}(x,t) &= -\cos x \sin t. \end{aligned}$$

Thus,

$$\begin{aligned} u_{xx}(x,t) &= -\cos x \sin t = u_{tt}(x,t), \\ u(x,0) &= \cos x \sin 0 = 0, \\ u_t(x,0) &= \cos x \cos 0 = \cos x, \\ u_x(0,t) &= -\sin 0 \sin t = 0. \end{aligned}$$

12.5 (a) Quasi-linear (b) Semi-linear (c) Linear (d) Nonlinear.

12.6 We have

$$\begin{aligned} u_x =& \frac{2x}{x^2+y^2} \\ u_{xx} =& \frac{2y^2-2x^2}{(x^2+y^2)^2} \\ u_y =& \frac{2y}{x^2+y^2} \\ u_{yy} =& \frac{2x^2-2y^2}{(x^2+y^2)^2} \end{aligned}$$

Plugging these expressions into the equation we find $u_{xx}+u_{yy}=0$. Similar argument holds for the second part of the problem.

12.7 Multiplying the equation by u and integrating, we obtain

$$\begin{aligned} \lambda\int_0^L u^2(x)dx =& \int_0^L uu_{xx}(x)dx \\ =& [u(L)u_x(L)-u(0)u_x(0)] - \int_0^L u_x^2(x)dx \\ =& -\left[k_Lu(L)^2+k_0u(0)^2+\int_0^L u_x^2(x)dx\right] \end{aligned}$$

For $\lambda > 0$, because $k_0, k_L > 0$, the right-hand side is nonpositive and the left-hand side is nonnegative. Therefore, both sides must be zero, and there can be no solution other than $u \equiv 0$, which is the trivial solution.

12.8 Substitute $u(x,y)=f(x)g(y)$ into the left side of the equation to obtain $f(x)g(y)(f(x)g(y))_{xy}=f(x)g(y)f'(x)g'(y)$. Now, substitute the same thing into the right side to obtain $(f(x)g(y))_x(f(x)g(y))_y=f'(x)g(y)f(x)g'(y)=f(x)g(y)f'(x)g'(y)$. So the sides are equal, which means $f(x)g(y)$ is a solution.

12.9 We have

$$(u_n)_{xx}=-n^2\sin nx\sinh ny \text{ and } (u_n)_{yy}=n^2\sin nx\sinh ny$$

Hence, $\Delta u_n=0$.

12.10 $u(x,y)=\frac{x^2y^2}{4}+F(x)+G(y)$, where $F(x)=\int f(x)dx$.

12.11 (a) We have $A=2,\ B=-4,\ C=7$ so $B^2-4AC=16-56=-40<0$. So this equation is elliptic everywhere in $\mathbb{R}^2$.
(b) We have $A=1,\ B=-2\cos x,\ C=-\sin^2 x$ so $B^2-4AC=4\cos^2 x+4\sin^2 x=4>0$. So this equation is hyperbolic everywhere in $\mathbb{R}^2$.
(c) We have $A=y,\ B=2(x-1),\ C=-(y+2)$ so $B^2-4AC=4(x-1)^2+4y(y+2)=4[(x-1)^2+(y+1)^2-4]$. The equation is parabolic if $(x-1)^2+(y+1)^2=4$. It is hyperbolic if $(x-1)^2+(y+1)^2>4$ and elliptic if $(x-1)^2+(y+1)^2<4$.

12.12 Using the chain rule we find

$$\begin{aligned}
u_t(x,t)=&\frac{1}{2}(cf'(x+ct)-cf'(x-ct))+\frac{1}{2c}[g(x+ct)(c)-g(x-ct)(-c))\\
=&\frac{c}{2}(f'(x+ct)-f'(x-ct))+\frac{1}{2}(g(x+ct)+g(x-ct))\\
u_{tt}=&\frac{c^2}{2}(f''(x+ct)+f''(x-ct))+\frac{c}{2}(g'(x+ct)-g'(c-xt))\\
u_x(x,t)=&\frac{1}{2}(f'(x+ct)+f'(x-ct))+\frac{1}{2c}[g(x+ct)-g(x-ct)]\\
u_{xx}(x,t)=&\frac{1}{2}(f''(x+ct)+f''(x-ct))+\frac{1}{2c}[g'(x+ct)-g'(x-ct)]
\end{aligned}$$

By substitutition we see that $c^2u_{xx}=u_{tt}$. Moreover,

$$u(x,0)=\frac{1}{2}(f(x)+f(x))+\frac{1}{2c}\int_x^x g(s)ds=f(x)$$

and

$$u_t(x,0)=g(x).$$

12.13 (a) $1+4x^2y>0$, (b) $1+4x^2y=0$, (c) $1+4x^2y<0$.

12.14 $u(x,y)=f(y-3x)+g(x+y)$.

12.15 $u(x,y)=f(y-3x)+g(x+y)=\frac{10x^2+y^2-7xy+6}{6}$.

Section 13

13.1 Let $z(x,t) = \alpha v(x,t) + \beta w(x,t)$. Then we have

$$\begin{aligned} c^2 z_{xx} =& c^2\alpha v_{xx} + c^2\beta w_{xx} \\ =&\alpha v_{tt} + \beta v_{tt} \\ =& z_{tt}. \end{aligned}$$

13.2 Indeed we have $c^2 u_{xx}(x,t) = 0 = u_{tt}(x,t)$.

13.3 $u(x,t) = 0$.

13.4 $u(x,t) = \frac{1}{2}(\cos(x-3t) + \cos(x+3t))$.

13.5 $u(x,t) = \frac{1}{2}\left[\frac{1}{1+(x+t)^2} + \frac{1}{1+(x-t)^2}\right]$.

13.6 $u(x,t) = 1 + \frac{1}{8\pi}[\sin(2\pi x + 4\pi t) - \sin(2\pi x - 4\pi t)]$.

13.7

$$u(x,t) = \begin{cases} 1 & \text{if } x-5t<0 \text{ and } x+5t<0 \\ \frac{1}{2} & \text{if } x-5t<0 \text{ and } x+5t>0 \\ \frac{1}{2} & \text{if } x-5t>0 \text{ and } x+5t<0 \\ 0 & \text{if } x-5t>0 \text{ and } x+5t>0 \end{cases} \blacksquare$$

13.8 $u(x,t) = \frac{1}{2}[e^{-(x+ct)^2} + e^{-(x-ct)^2}] + \frac{t}{2} + \frac{1}{4c}\cos(2x)\sin(2ct)$.

13.9 Just plug the translated/differentiated/dialated solution into the wave equation and check that it is a solution.

13.10 $v(r) = A\cos(nr) + B\sin(nr)$.

13.11 $u(x,t) = \frac{1}{2}[e^{x-ct} + e^{x+ct} + \frac{1}{c}(\cos(x-ct) - \cos(x+ct))]$.

13.12 (a) We have

$$\begin{aligned}\frac{dE}{dt}(t) =& \int_0^L u_t u_{tt} dx + \int_0^L c^2 u_x u_{xt} dx\\ =& \int_0^L u_t u_{tt} dx + c^2 u_t(L,t)u_x(L,t) - c^2 u_t(0,t)u_x(0,t) - c^2 \int_0^L u_t u_{xx} dx\\ =& c^2 u_t(L,t)u_x(L,t) - c^2 u_t(0,t)u_x(0,t) + \int_0^L u_t(u_{tt} - c^2 u_{xx})dx\\ =& c^2(u_t(L,t)u_x(L,t) - u_t(0,t)u_x(0,t))\end{aligned}$$

since $u_{tt} - c^2 u_{xx} = 0$.
(b) Since the ends are fixed, we have $u_t(0,t) = u_t(L,t) = 0$. From (a) we have

$$\frac{dE}{dt}(t) = c^2(u_t(L,t)u_x(L,t) - u_t(0,t)u_x(0,t)) = 0.$$

(c) Assuming free ends boundary conditions, that is $u_x(0,t) = u_x(L,t) = 0$, we find $\frac{dE}{dt}(t) = 0$.

13.13 Using the previous exercise, we find

$$\frac{dE}{dt}(t) = -d \int_0^L (u_t)^2 dx.$$

The right-hand side is nonpositive, so the energy either decreases or is constant. The latter case can occur only if $u_t(x,t)$ is identically zero, which means that the string is at rest.

13.14 (a) By the chain rule we have $u_t(x,t) = -cR'(x-ct)$ and $u_{tt}(x,t) = c^2 R''(x-ct)$. Likewise, $u_x(x,t) = R'(x-ct)$ and $u_{xx} = R''(x-ct)$. Thus, $u_{tt} = c^2 u_{xx}$.
(b) We have

$$\frac{1}{2}\int_0^L (u_t)^2 dx = \int_0^L \frac{c^2}{2}[R'(x-ct)]^2 dx = \int_0^L \frac{c^2}{2}(u_x)^2 dx.$$

13.15 $u(x,t) = x^2 + 4t^2 + \frac{1}{4}\sin 2x \sin 4t$.

Section 14

14.1 Let $z(x,t) = \alpha u(x,t) + \beta v(x,t)$. Then we have

$$\begin{aligned} kz_{xx} =& k\alpha u_{xx} + k\beta v_{xx} \\ =& \alpha u_t + \beta v_t \\ =& z_t. \end{aligned}$$

14.2 Indeed we have $ku_{xx}(x,t) = 0 = u_t(x,t)$.

14.3 $u(x,t) = T_0 + \frac{T_L - T_0}{L} x$.

14.4 Let $\overline{u}$ be the solution to (14.1) that satisfies $\overline{u}(0,t) = \overline{u}(L,t) = 0$. Let $w(x,t)$ be the time independent solution to (14.1) that satisfies $w(0,t) = T_0$ and $w(L,t) = T_L$. That is, $w(x,t) = T_0 + \frac{T_L - T_0}{L} x$. From Exercise 14.1, the function $u(x,t) = \overline{u}(x,t) + w(x,t)$ is a solution to (14.1) that satisfies $u(0,t) = T_0$ and $u(L,t) = T_L$.

14.5 $u(x,t) = 0$.

14.6 Substituting $u(x,t) = X(x)T(t)$ into (14.1) we obtain

$$k\frac{X''}{X} = \frac{T'}{T}.$$

Since X only depends on x and T only depends on t, we must have that there is a constant λ such that

$$k\tfrac{X''}{X} = \lambda \text{ and } \tfrac{T'}{T} = \lambda.$$

This gives the two ordinary differential equations

$$X'' - \tfrac{\lambda}{k} X = 0 \text{ and } T' - \lambda T = 0.$$

14.7 (a) Letting $\alpha = \frac{\lambda}{k} > 0$ we obtain the ODE $X'' - \alpha X = 0$ whose general solution is given by $X(x) = Ae^{x\sqrt{\alpha}} + Be^{-x\sqrt{\alpha}}$ for some constants A and B.
(b) The condition $u(0,t) = 0$ implies that $X(0) = 0$ which in turn implies $A + B = 0$. Likewise, the condition $u(L,t) = 0$ implies $Ae^{\sqrt{L\alpha}} + Be^{-L\sqrt{\alpha}} = 0$. Hence, $A(e^{L\sqrt{\alpha}} - e^{-L\sqrt{\alpha}}) = 0$.
(c) If $A = 0$ then $B = 0$ and $u(x,t)$ is the trivial solution which contradicts

the assumption that u is non-trivial. Hence, we must have $A \neq 0$.
(d) Using (b) and (c) we obtain $e^{L\sqrt{\alpha}} = e^{-L\sqrt{\alpha}}$ or $e^{2L\sqrt{\alpha}} = 1$. This equation is impossible since $2L\sqrt{\alpha} > 0$. Hence, we must have $\lambda < 0$ so that $X(x) = A\cos(x\sqrt{-\alpha}) + B\sin(x\sqrt{-\alpha}$.

14.8 (a)Now, write $\beta = \sqrt{-\frac{\lambda}{k}}$. Then we obtain the equation $X'' + \beta^2 X = 0$ whose general solution is given by

$$X(x) = c_1 \cos\beta x + c_2 \sin\beta x.$$

(b) Using $X(0) = 0$ we obtain $c_1 = 0$. Since $c_2 \neq 0$ we must have $\sin\beta L = 0$. Thus, $\lambda = -\frac{kn^2\pi^2}{L^2}$, where n is an integer.

14.9 For each integer $n \geq 0$ we have $u_n(x,t) = \frac{c_n}{T(0)}T(0)e^{\frac{kn^2\pi^2}{L^2}t}\sin\left(\frac{n\pi}{L}\right)x$ is a solution to (14.1). By superposition, $u(x,t)$ is also a solution to (14.1). Moreover, $u(0,t) = u(L,t) = 0$ since $u_n(0,t) = u_n(L,t) = 0$.

14.10 (i) $u(0,t) = 0$ and $u(a,t) = 100$ for $t > 0$.
(ii) $u_x(0,t) = u_x(a,t) = 0$ for $t > 0$.

14.11 Solving this problem we find $u(x,t) = e^{-t}\sin x$. We have

$$E(t) = \int_0^\pi [e^{-2t}\sin^2 x + e^{-2t}\cos^2 x]dx = \int_0^\pi e^{-2t}dx = \pi e^{-2t}.$$

Thus, $E'(t) = -2\pi e^{-2t} < 0$ for all $t > 0$.

14.12 $E(t) = \int_0^L f(x)dx + (1+4L)t$.

14.13 $v(x) = x + 2$.

14.14 (a) $v(x) = \frac{T}{L}x$.
(b) $v(x) = T$.
(c) $v(x) = \alpha x + T$.

14.15 (a) $E(t) = \int_0^L c\rho u(x,t)dx$.
(b) We integrate the equation in x from 0 to L :

$$\int_0^L c\rho u_t(x,t)dx = \int_0^L K u_{xx}dx = Ku_x(x,t)|_0^L = 0,$$

since $u_x(0,t) = u_x(L,t) = 0$. The left-hand side can also be written as

$$\frac{d}{dt}\int_0^L c\rho u(x,t)dx = E'(t).$$

Thus, we have shown that $E'(t) = 0$ so that $E(t)$ is constant.

14.16 (a) The total thermal energy is

$$E(t) = \int_0^L u(x,t)dx.$$

We have

$$\frac{dE}{dt} = \int_0^L u_t(x,t)dx = u_x|_0^L + \int_0^L xdx = (7-\beta) + \frac{L^2}{2}.$$

(b) The steady solution (equilibrium) is possible if the right-hand side vanishes:

$$(7-\beta) + \frac{L^2}{2} = 0$$

Solving this equation for β we find $\beta = 7 + \frac{L^2}{2}$.
(c) By integrating the equation $u_{xx} + x = 0$ we find the steady solution

$$u(x) = -\frac{x^3}{6} + C_1 x + C_2$$

From the condition $u_x(0) = \beta$ we find $C_1 = \beta$. The steady solution should also have the same value of the total energy as the initial condition. This means

$$\int_0^L \left(-\frac{x^3}{6} + \beta x + C_2\right) dx = \int_0^L f(x)dx = E(0).$$

Performing the integration and then solving for C_2 we find

$$C_2 = \frac{1}{L}\int_0^L f(x)dx + \frac{L^3}{24} - \beta\frac{L}{2}.$$

Therefore, the steady-state solution is

$$u(x) = \frac{1}{L}\int_0^L f(x)dx + \frac{L^3}{24} - \beta\frac{L}{2} + \beta x - \frac{x^3}{6}.$$

Section 15

15.1 (a) We have $(fg)(x+T) = f(x+T)g(x+T) = f(x)g(x) = (fg)(x)$.
(b) We have $(c_1f+c_2g)(x+T) = c_1f(x+T)+c_2g(x+T) = c_1f(x)+c_2g(x) = (c_1f+c_2g)(x)$.

15.2 (a) For $n \neq m$ we have

$$\begin{aligned}\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx =& -\frac{1}{2}\int_{-L}^{L}\left[\cos\left(\frac{(m+n)\pi}{L}x\right) - \cos\left(\frac{(m-n)\pi}{L}x\right)\right]dx\\ =& -\frac{1}{2}\left[\frac{L}{(m+n)\pi}\sin\left(\frac{(m+n)\pi}{L}x\right)\right.\\ &\left.-\frac{L}{(m-n)\pi}\sin\left(\frac{(m-n)\pi}{L}x\right)\right]_{-L}^{L}\\ =&0\end{aligned}$$

where we used the trigonometric identiy

$$\sin a \sin b = \frac{1}{2}[-\cos(a+b) + \cos(a-b)].$$

(b) For $n \neq m$ we have

$$\begin{aligned}\int_{-L}^{L} \cos\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx =& \frac{1}{2}\int_{-L}^{L}\left[\sin\left(\frac{(m+n)\pi}{L}x\right) - \sin\left(\frac{(m-n)\pi}{L}x\right)\right]dx\\ =& \frac{1}{2}\left[-\frac{L}{(m+n)\pi}\cos\left(\frac{(m+n)\pi}{L}x\right)\right.\\ &\left.+\frac{L}{(m-n)\pi}\cos\left(\frac{(m-n)\pi}{L}x\right)\right]_{-L}^{L}\\ =&0\end{aligned}$$

where we used the trigonometric identiy

$$\cos a \sin b = \frac{1}{2}[\sin(a+b) - \sin(a-b)].$$

15.3 (a) L (b) L (c) 0.

15.4

$$\begin{aligned}
a_0 =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0\\
a_n =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx\\
=& -\int_{-\pi}^{0}\cos nx dx + \int_0^{\pi}\cos nx dx = 0\\
b_n =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx\\
=& -\int_{-\pi}^{0}\sin nx dx + \int_0^{\pi}\sin nx dx\\
=& \frac{2}{n}[1-(-1)^n]
\end{aligned}$$

15.5 $f(x) = -\frac{1}{6} + \sum_{n=1}^{\infty}\frac{4}{(n\pi)^2}(-1)^n\cos{(n\pi x)}$.

15.6 $f(x) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right]\sin\left(\frac{nx}{2}\right)$.

15.7 $f(x) = \sum_{n=1}^{\infty}\frac{4}{(n\pi)^2}[1-(-1)^n]\cos\left(\frac{n\pi}{2}x\right)$.

15.8 Since the sided limits at the point of discontinuity $x = 0$ do not exist, the function is not piecewise continuous in $[-1, 1]$.

15.9 Define the function

$$g(a) = \int_{-L+a}^{L+a} f(x)dx.$$

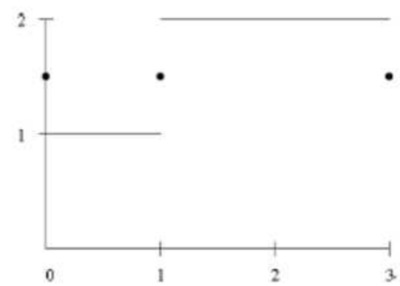
Using the fundamental theorem of calculus, we have

$$\begin{aligned}
\frac{dg}{da} =& \frac{d}{da}\int_{-L+a}^{L+a} f(x)dx\\
=& f(L+a) - f(-L+a) = f(-L+a+2L) - f(-L+a)\\
=& f(-L+a) - f(-L+a) = 0
\end{aligned}$$

Hence, g is a constant function, and in particular we can write $g(a) = g(0)$ for all $a \in \mathbb{R}$ which gives the desired result.

15.10 (i) $f(x) = \frac{10}{3} + \sum_{n=1}^{\infty} \left[-\frac{1}{n\pi} \sin\left(\frac{2n\pi}{3}\right) \cos\left(\frac{2n\pi x}{3}\right) - \frac{1}{n\pi} \left(-\cos\left(\frac{2n\pi}{3}\right) + 1 \right) \sin\left(\frac{2n\pi x}{3}\right) \right].$
(ii) Using the theorem discussed in class, because this function and its derivative are piecewise continuous, the Fourier series will converge to the function at each point of continuity. At any point of discontinuity, the Fourier series will converge to the average of the left and right limits.
(iii)

f(x)



limit of Fourier series

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15.11 (a) $a_0 = 2,\ a_n = b_n = 0$ for $n \in \mathbb{N}$.
(b) $a_0 = 4,\ a_n = 0,\ b_1 = 1$, and $b_n = 0$.
(c) $a_0 = 1,\ a_n = 0,\ b_n = \frac{1}{\pi n}[1 - (-1)^n],\ n \in \mathbb{N}$.
(d) $a_0 = a_n = 0,\ b_n = \frac{2L}{\pi n}(-1)^{n+1},\ n \in \mathbb{N}$.

15.12 -1

15.13 $a_n = 0$ for all $n \in \mathbb{N}$.

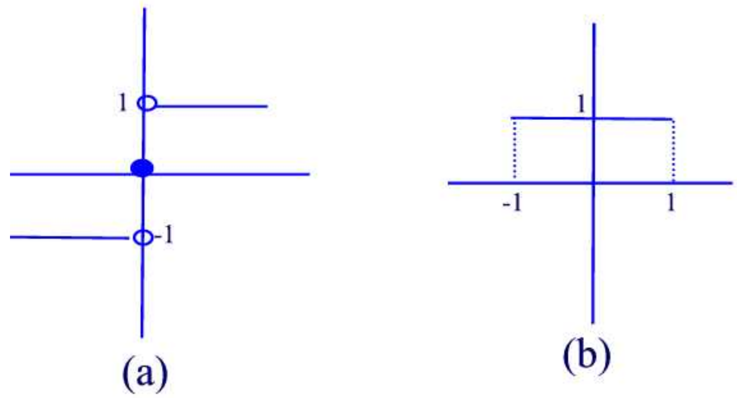
15.14 $\frac{f(0^-)+f(0^+)}{2} = \frac{-\pi+\pi}{2} = 0.$

15.15 (a) $f(x) = \frac{3}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}.$
(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}.$

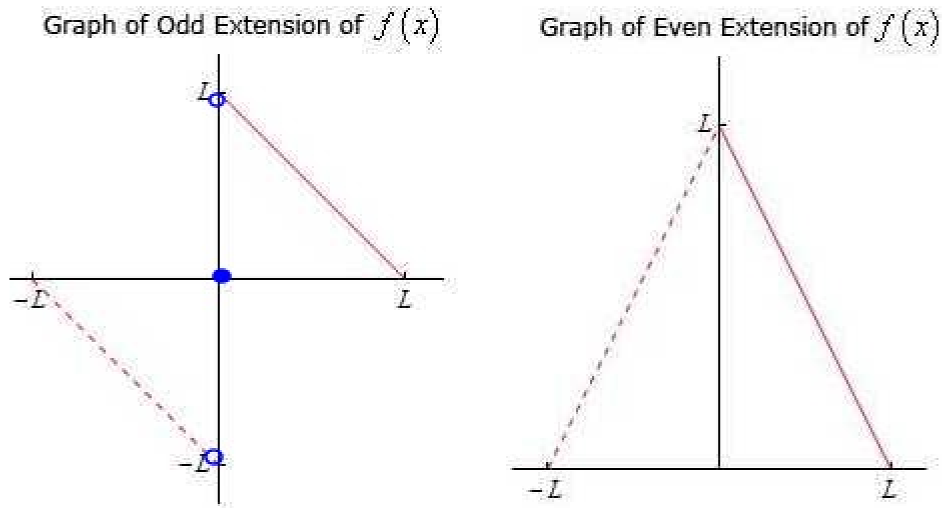
Section 16

16.1 $f(x) = 0.$

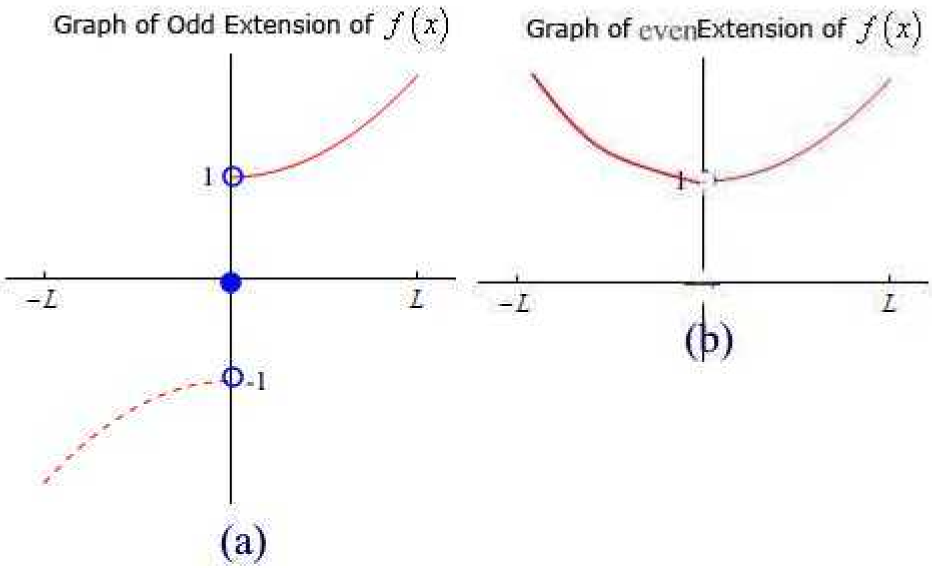
16.2


1
-1
(a)
1
-1
1
(b)

16.3


Graph of Odd Extension of $f(x)$
L
$-L$
L
$-L$
Graph of Even Extension of $f(x)$
L
$-L$
L

16.4



16.5 $f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}[2\cos(n\pi/2) - 1 - (-1)^n]\cos nx.$

16.6 $f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}[(-1)^n - 1]\cos nx.$

16.7 $f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}[1 - (-1)^n]\sin nx.$

16.8 $f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} n\left(\frac{1-(-1)^n}{n^2-1}\right)\sin nx.$

16.9 $f(x) = \frac{1}{2}(e^2 - 1) + \sum_{n=1}^{\infty} \frac{4[(-1)^n e^2 - 1]}{4+n^2\pi^2}\cos(n\pi x).$

16.10 (a) If $f(x) = \sin\left(\frac{2\pi}{L}x\right)$ then $b_n = 0$ if $n \neq 2$ and $b_2 = 1$.
(b) If $f(x) = 1$ then

$$b_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right)dx = \frac{2}{n\pi}[1 - (-1)^n].$$

(c) If $f(x) = \cos\left(\frac{\pi}{L}x\right)$ then

$$b_1 = \frac{2}{L}\int_0^L \cos\left(\frac{\pi}{L}x\right)\sin\left(\frac{\pi}{L}x\right)dx = 0$$

and for $n \neq 1$ we have

$$\begin{aligned}
b_n =&\frac{2}{L}\int_0^L \cos\left(\frac{\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx\\
=&\frac{1}{2}\frac{2}{L}\int_0^L \left[\sin\left(\frac{\pi x}{L}\right)(1+n) - \sin\left(\frac{\pi x}{L}\right)(1-n)\right]dx\\
=&\frac{1}{L}\left[-\frac{L}{(1+n)\pi}\cos\left(\frac{\pi x}{L}\right)(1+n) + \frac{L}{(1-n)\pi}\cos\left(\frac{\pi x}{L}\right)(1-n)\right]_0^L\\
=&\frac{2n}{(n^2-1)\pi}[1-(-1)^n].
\end{aligned}$$

16.11 (a) $a_0 = 10$ and $a_1 = 1$, and $a_n = 0$ for $n \neq 1$.
(b) $a_0 = L$ and $a_n = \frac{2L}{(\pi n)^2}[(-1)^n - 1], \quad n \in \mathbb{N}$.
(c) $a_0 = 1$ and $a_n = \frac{2}{\pi n}\sin\left(\frac{\pi n}{2}\right), \ n \in \mathbb{N}$.

16.12 By definition of Fourier sine coefficients,

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

The symmetry around $x = \frac{L}{2}$ can be written as

$$f\left(\frac{L}{2}+x\right) = f\left(\frac{L}{2}-x\right)$$

for all $x \in \mathbb{R}$. To use this symmetry it is convenient to make the change of variable $x - \frac{L}{2} = u$ in the above integral to obtain

$$b_n = \int_{-\frac{L}{2}}^{\frac{L}{2}} f\left(\frac{L}{2}+u\right)\sin\left[\frac{n\pi}{L}\left(\frac{L}{2}+u\right)\right]du.$$

Since $f\left(\frac{L}{2}+u\right)$ is even in u and for n even $\sin\left[\frac{n\pi}{L}\left(\frac{L}{2}+u\right)\right] = \sin\left(\frac{n\pi u}{L}\right)$ is odd in u, the integrand of the above integral is odd in u for n even. Since the intergral is from $-\frac{L}{2}$ to $\frac{L}{2}$ we must have $b_{2n} = 0$ for $n = 0, 1, 2, \cdots$ ■

16.13 By definition of Fourier cosine coefficients,

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

The anti-symmetry around $x = \frac{L}{2}$ can be written as

$$f\left(\frac{L}{2} - y\right) = -f\left(\frac{L}{2} + y\right)$$

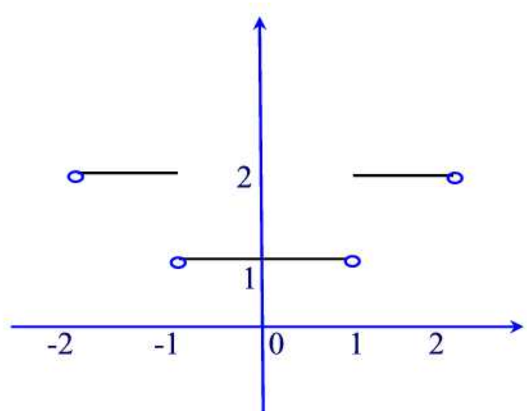
for all $y \in \mathbb{R}$. To use this symmetry it is convenient to make the change of variable $x = \frac{L}{2} + y$ in the above integral to obtain

$$a_n = \int_{-\frac{L}{2}}^{\frac{L}{2}} f\left(\frac{L}{2} + y\right) \cos\left[\frac{n\pi}{L}\left(\frac{L}{2} + y\right)\right] dy.$$

Since $f\left(\frac{L}{2} + y\right)$ is odd in y and for n even $\cos\left[\frac{n\pi}{L}\left(\frac{L}{2} + y\right)\right] = \pm\cos\left(\frac{n\pi y}{L}\right)$ is even in y, the integrand of the above integral is odd in y for n even. Since the intergral is from $-\frac{L}{2}$ to $\frac{L}{2}$ we must have $a_{2n} = 0$ for all $n = 0, 1, 2, \cdots$.

16.14 $\sin\left(\frac{n\pi x}{L}\right) = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{1-(-1)^n}{n^2-1} \cos\left(\frac{n\pi x}{L}\right)$.

16.15 (a)



(b) $a_0 = \frac{2}{2}\int_0^2 f(x)dx = 3.$

(c) We have

$$\begin{aligned}
a_n =&\frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx\\
=&\int_0^1 \cos\left(\frac{n\pi x}{2}\right)dx+\int_1^2 2\cos\left(\frac{n\pi x}{2}\right)dx\\
=&\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1+2\,\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_1^2\\
=&-\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right).
\end{aligned}$$

(d) $b_n = 0$ since $f(x)\sin\left(\frac{n\pi x}{2}\right)$ is odd in$-2 \le x \le 2$.
(e)

$$f(x) = \frac{3}{2}+\sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right)\cos\left(\frac{n\pi x}{2}\right).$$

Section 17

17.1 We look for a solution of the form $u(x,y) = X(x)Y(y)$. Substituting in the given equation, we obtain

$$X''Y + XY'' + \lambda XY = 0.$$

Assuming $X(x)Y(y)$ is nonzero, dividing for $X(x)Y(y)$ and subtract both sides for $\frac{X''(x)}{X(x)}$, we find:

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \lambda.$$

The left hand side is a function of x while the right hand side is a function of y. This says that they must equal to a constant. That is,

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \lambda = \delta.$$

where δ is a constant. This results in the following two ODEs

$$X'' + \delta X = 0 \text{ and } Y'' + (\lambda - \delta)Y = 0.$$

- If $\delta > 0$ and $\lambda - \delta > 0$ then

$$\begin{aligned}X(x) =& A\cos\delta x + B\sin\delta x\\ Y(y) =& C\cos{(\lambda-\delta)y} + D\sin{(\lambda-\delta)y}\end{aligned}$$

- If $\delta > 0$ and $\lambda - \delta < 0$ then

$$\begin{aligned}X(x) =& A\cos\delta x + B\sin\delta x\\ Y(y) =& Ce^{-\sqrt{-(\lambda-\delta)}y} + De^{\sqrt{-(\lambda-\delta)}y}\end{aligned}$$

- If $\delta = \lambda > 0$ then

$$\begin{aligned}X(x) =& A\cos\delta x + B\sin\delta x\\ Y(y) =& Cy + D\end{aligned}$$

- If $\delta = \lambda < 0$ then

$$\begin{aligned}X(x) =& Ae^{-\sqrt{-\delta}x} + Be^{\sqrt{-\delta}x}\\ Y(y) =& Cy + D\end{aligned}$$

- If $\delta < 0$ and $\lambda - \delta > 0$ then

$$\begin{aligned}X(x) =& Ae^{-\sqrt{-\delta}x} + Be^{\sqrt{-\delta}x}\\ Y(y) =& C\cos{(\lambda-\delta)y} + D\sin{(\lambda-\delta)y}\end{aligned}$$

- If $\delta < 0$ and $\lambda - \delta < 0$ then

$$\begin{aligned}X(x) =& Ae^{-\sqrt{-\delta}x} + Be^{\sqrt{-\delta}x}\\ Y(y) =& Ce^{-\sqrt{(\lambda-\delta)}y} + De^{\sqrt{(\lambda-\delta)}y}.\end{aligned}$$

17.2 Let's assume that the solution can be written in the form $u(x,t) = X(x)T(t)$. Substituting into the heat equation we obtain

$$\frac{X''}{X} = \frac{T'}{kT}.$$

Since X only depends on x and T only depends on t, we must have that there is a constant λ such that

$$\tfrac{X''}{X} = \lambda \text{ and } \tfrac{T'}{kT} = \lambda.$$

This gives the two ordinary differential equations

$$X'' - \lambda X = 0 \text{ and } T' - k\lambda T = 0.$$

Next, we consider the three cases of the sign of λ.
Case 1: $\lambda = 0$
In this case, $X'' = 0$ and $T' = 0$. Solving these equations we find $X(x) = ax + b$ and $T(t) = c$.

Case 2: $\lambda > 0$
In this case, $X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$ and $T(t) = Ce^{k\lambda t}$.

Case 3: $\lambda < 0$
In this case, $X(x) = A\cos\sqrt{-\lambda}x + B\sin\sqrt{-\lambda}x$ and and $T(t) = Ce^{k\lambda t}$.

17.3 $r^2R''(r) + rR'(r) - \lambda R(r) = 0$ and $\Theta''(\theta) + \lambda\Theta(\theta) = 0$.

17.4 $X'' = (2+\lambda)X,\ T'' = \lambda T, X(0) = 0,\ X(1) = 0$.

17.5 $X'' - \lambda X = 0,\ T' = k\lambda T, X'(0) = 0 = X'(L)$.

17.6 $u(x,t) = Ce^{\lambda(x-t)}$.

17.7 $5X''' - 7X'' - \lambda X = 0$ and $3Y'' - \lambda Y' = 0$.

17.8 $u(x,y) = Ce^{\lambda x - \frac{y}{\lambda}}$.

17.9 $u(x,y) = Ce^{\lambda x}y^{\lambda}$.

17.10 We look for a solution of the form $u(x,y) = X(x)T(t)$. Substituting in the wave equation, we obtain

$$X''(x)T(t) - X(x)T''(t) = 0.$$

Assuming $X(x)T(t)$ is nonzero, dividing for $X(x)T(t)$ we find:

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)}.$$

The left hand side is a function of x while the right hand side is a function of t. This says that they must equal to a constant. That is,

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \lambda$$

where λ is a constant. This results in the following two ODEs

$$X'' - \lambda X = 0 \text{ and } T'' - \lambda T = 0.$$

The solutions of these equations depend on the sign of λ.

• If $\lambda > 0$ then the solutions are given

$$\begin{aligned} X(x) =& Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x} \\ T(t) =& Ce^{\sqrt{\lambda}t} + De^{-\sqrt{\lambda}t} \end{aligned}$$

where A, B, C, and D are constants. In this case,

$$u(x,t) = k_1 e^{\sqrt{\lambda}(x+t)} + k_2 e^{\sqrt{\lambda}(x-t)} + k_3 e^{-\sqrt{\lambda}(x+t)} + k_4 e^{-\sqrt{\lambda}(x-t)}.$$

• If $\lambda = 0$ then

$$\begin{aligned} X(x) =& Ax + B \\ T(t) =& Ct + D \end{aligned}$$

where A, B, and C are arbitrary constants. In this case,

$$u(x,t) = k_1 xt + k_2 x + k_3 t + k_4.$$

• If $\lambda < 0$ then

$$\begin{aligned} X(x) =& A\cos\sqrt{-\lambda}x + B\sin\sqrt{-\lambda}x \\ T(t) =& A\cos\sqrt{-\lambda}t + B\sin\sqrt{-\lambda}t \end{aligned}$$

where A, B, C, and D are arbitrary constants. In this case,

$$\begin{aligned} u(x,t) =& k_1\cos\sqrt{-\lambda}x\cos\sqrt{-\lambda}t + k_2\cos\sqrt{-\lambda}x\sin\sqrt{-\lambda}t \\ &+ k_3\sin\sqrt{-\lambda}x\cos\sqrt{-\lambda}t + k_4\sin\sqrt{-\lambda}x\sin\sqrt{-\lambda}t. \end{aligned}$$

17.11 (a) $u(r,t) = R(r)T(t),\ T'(t) = k\lambda T,\quad r(rR')' = \lambda R.$

(b) $u(x,t) = X(x)T(t),\ T' = \lambda T, \quad kX'' - (\alpha+\lambda)X = 0.$
(c) $u(x,t) = X(x)T(t),\ T' = \lambda T, \quad kX'' - aX' = \lambda X.$
(d) $u(x,t) = X(x)Y(y),\ X'' = \lambda X, \quad Y'' = -\lambda Y.$
(e) $u(x,t) = X(x)T(t),\ T' = k\lambda T,\ X'''' = \lambda X.$

17.12 $u(x,y) = Ce^{\lambda(x+y)}.$

17.13 $X'' = \lambda X, \quad Y' - Y'' + Y = \lambda Y.$

Section 18

18.1 $u(x,t) = \sin\left(\frac{\pi}{2}x\right)e^{-\frac{\pi^2 k}{4}t} + 3\sin\left(\frac{5\pi}{2}x\right)e^{-\frac{25\pi^2 k}{4}t}.$

18.2 $u(x,t) = \frac{8d}{\pi^3}\sum_{n=1}^\infty \frac{1}{(2n-1)^3}\sin\left(\frac{(2n-1)\pi}{L}x\right)e^{-\frac{k(2n-1)^2\pi^2}{L^2}t}.$

18.3 $u(x,t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^\infty \frac{1}{(4n^2-1)}\cos\left(\frac{2n\pi}{L}x\right)e^{-k\frac{4n^2\pi^2}{L^2}t}.$

18.4 $u(x,t) = \sum_{n=1}^\infty C_n\sin\left(\frac{n\pi}{L}x\right)e^{-\frac{n^2\pi^2}{L^2}t}$ where

$$C_n = \begin{cases} -\frac{4}{n\pi} & n = 2, 6, 10, \cdots \\ 0 & n = 4, 8, 12, \cdots \\ \frac{6}{n\pi} & n \text{ is odd.} \end{cases}$$

18.5 $u(x,t) = 6\sin\left(\frac{9\pi}{L}x\right)e^{\frac{-81\pi^2}{L^2}t}.$

18.6 $u(x,t) = \frac{1}{2} + \sum_{n=1}^\infty C_n\cos\left(\frac{n\pi}{L}x\right)e^{-\frac{n^2\pi^2}{L^2}t}$ where

$$C_n = \begin{cases} -\frac{2}{n\pi} & n = 1, 5, 9, \cdots \\ \frac{2}{n\pi} & n = 3, 7, 11, \cdots \\ 0 & n \text{ is even} \end{cases}$$

18.7 $u(x,t) = 6 + 4\cos\left(\frac{3\pi}{L}x\right)e^{-\frac{9\pi^2}{L^2}t}.$

18.8 $u(x,t) = -3\cos\left(\frac{8\pi}{L}x\right)e^{-\frac{64\pi^2}{L^2}t}.$

18.9

$$u(x,t) = \sum_{n=0}^\infty a_n\cos\left(\frac{n\pi}{L}x\right)e^{-\left(1+\frac{n^2\pi^2}{L^2}\right)t}.$$

As $t \to \infty$, $e^{-\left(1+\frac{n^2\pi^2}{L^2}\right)t} \to 0$ for each $n \in \mathbb{N}$. Hence, $u(x,t) \to 0$.

18.10 (b) We have

$$\begin{aligned}
E'(t) =&2\int_0^1 w(x,t)w_t(x,t)dx\\
=&2\int_0^1 w(x,t)[w_{xx}(x,t) - w(x,t)]dx\\
=& 2w(x,t)w_x(x,t)|_0^1 - 2\left[\int_0^1 w_x^2(x,t)dx + \int_0^1 w^2(x,t)dx\right]\\
=&-2\left[\int_0^1 w_x^2(x,t)dx + \int_0^1 w^2(x,t)dx\right] \le 0
\end{aligned}$$

Hence, E is decreasing, and $0 \le E(t) \le E(0)$ for all $t > 0$.
(c) Since $w(x,0) = 0$, we must have $E(0) = 0$. Hence, $E(t) = 0$ for all $t \ge 0$. This implies that $w(x,t) = 0$ for all $t > 0$ and all $0 < x < 1$. Therefore $u_1(x,t) = u_2(x,t)$. This means that the given problem has a unique solution.

18.11 (a) $u(0,t) = 0$ and $u_x(1,t) = 0$.
(b) Let's assume that the solution can be written in the form $u(x,t) = X(x)T(t)$. Substituting into the heat equation we obtain

$$\frac{X''}{X} = \frac{T'}{kT}.$$

Since X only depends on x and T only depends on t, we must have that there is a constant λ such that

$$\tfrac{X''}{X} = \lambda \text{ and } \tfrac{T'}{kT} = \lambda.$$

This gives the two ordinary differential equations

$$X'' - \lambda X = 0 \text{ and } T' - k\lambda T = 0.$$

As far as the boundary conditions, we have

$$u(0,t) = 0 = X(0)T(t) \Longrightarrow X(0) = 0$$

and

$$u_x(1,t) = 0 = X'(1)T(t) \Longrightarrow X'(1) = 0.$$

Note that T is not the zero function for otherwise $u \equiv 0$ and this contradicts our assumption that u is the non-trivial solution.
(c) We have $X' = \sqrt{-\lambda}\cos\sqrt{-\lambda}x$ and $X'' = \lambda \sin\sqrt{-\lambda}x$. Thus, $X''-\lambda X = 0$. Moreover $X(0) = 0$. Now, $X'(1) = 0$ implies $\cos\sqrt{-\lambda} = 0$ or $\sqrt{-\lambda} = \left(n - \frac{1}{2}\right)\pi, \quad n \in \mathbb{N}$. Hence, $\lambda = -\left(n - \frac{1}{2}\right)^2 \pi^2$.

18.12 (a) Let's assume that the solution can be written in the form $u(x,t) = X(x)T(t)$. Substituting into the heat equation we obtain

$$\frac{X''}{X} = \frac{T'}{kT}.$$

Since the LHS only depends on x and the RHS only depends on t, there must be a constant λ such that

$$\tfrac{X''}{X} = \lambda \text{ and } \tfrac{T'}{kT} = \lambda.$$

This gives the two ordinary differential equations

$$X'' - \lambda X = 0 \text{ and } T' - k\lambda T = 0.$$

As far as the boundary conditions, we have

$$u(0,t) = 0 = X(0)T(t) \Longrightarrow X(0) = 0$$

and

$$u(L,t) = 0 = X(L)T(t) \Longrightarrow X(L) = 0.$$

Note that T is not the zero function for otherwise $u \equiv 0$ and this contradicts our assumption that u is the non-trivial solution.
Next, we consider the three cases of the sign of λ.

Case 1: $\lambda = 0$
In this case, $X'' = 0$. Solving this equation we find $X(x) = ax + b$. Since $X(0) = 0$ we find $b = 0$. Since $X(L) = 0$ we find $a = 0$. Hence, $X \equiv 0$ and $u(x,t) \equiv 0$. That is, u is the trivial solution.

Case 2: $\lambda > 0$
In this case, $X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$. Again, the conditions $X(0) = X(L) = 0$ imply $A = B = 0$ and hence the solution is the trivial solution.

Case 3: $\lambda < 0$
In this case, $X(x) = A\cos\sqrt{-\lambda}x + B\sin\sqrt{-\lambda}x$. The condition $X(0) = 0$ implies $A = 0$. The condition $X(L) = 0$ implies $B\sin\sqrt{-\lambda}L = 0$. We must have $B \neq 0$ otherwise $X(x) = 0$ and this leads to the trivial solution. Since $B \neq 0$, we obtain $\sin\sqrt{-\lambda}L = 0$ or $\sqrt{-\lambda}L = n\pi$ where $n \in \mathbb{N}$. Solving for λ we find $\lambda = -\frac{n^2\pi^2}{L^2}$. Thus, we obtain infinitely many solutions given by

$$X_n(x) = A_n \sin\frac{n\pi}{L}x, \quad n \in \mathbb{N}.$$

Now, solving the equation

$$T' - \lambda k T = 0$$

by the method of separation of variables we obtain

$$T_n(t) = B_n e^{-\frac{n^2\pi^2}{L^2}kt},\ n \in \mathbb{N}.$$

Hence, the functions

$$u_n(x,t) = C_n \sin\left(\frac{n\pi}{L}x\right)e^{-\frac{n^2\pi^2}{L^2}kt},\ n \in \mathbb{N}$$

satisfy $u_t = ku_{xx}$ and the boundary conditions $u(0,t) = u(L,t) = 0$.
Now, in order for these solutions to satisfy the initial value condition $u(x,0) = 6\sin\left(\frac{9\pi x}{L}\right)$, we invoke the superposition principle of linear PDE to write

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right)e^{-\frac{n^2\pi^2}{L^2}kt}. \tag{1}$$

To determine the unknown constants C_n we use the initial condition $u(x,0) = 6\sin\left(\frac{9\pi x}{L}\right)$ in (1) to obtain

$$6\sin\left(\frac{9\pi x}{L}\right) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right).$$

By equating coefficients we find $C_9 = 6$ and $C_n = 0$ if $n \neq 9$. Hence, the solution to the problem is given by

$$u(x,t) = 6\sin\left(\frac{9\pi x}{L}\right)e^{-\frac{81\pi^2}{L^2}kt}.$$

(b) Similar to (a), we find

$$u(x,t) = 3\sin\left(\frac{\pi}{L}x\right)e^{-\frac{\pi^2 kt}{L^2}} - \sin\left(\frac{3\pi}{L}x\right)e^{-\frac{9\pi^2 kt}{L^2}}$$

18.13 $u(x,t) = \cos\left(\frac{\pi x}{L}\right)e^{-\frac{pi^2 kt}{L^2}} + 4\cos\left(\frac{5\pi x}{L}\right)e^{-\frac{25pi^2 kt}{L^2}}$.
(b) $u(x,t) = 5$.

18.14 $u(x,t) = 6\sin xe^{-8t}$.

Section 19

19.1 $u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{b}y\right)\sinh\left(\frac{n\pi}{b}x\right)$ where

$$B_n = \left[\frac{2}{b}\int_0^b f_2(y)\sin\left(\frac{n\pi}{b}y\right)dy\right]\left[\sinh\left(\frac{n\pi}{b}a\right)\right]^{-1}.$$

19.2 $u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{a}x\sinh\left(\frac{n\pi}{a}(y-b)\right)$ where

$$B_n = \left[\frac{2}{a}\int_0^a g_1(x)\sin\left(\frac{n\pi}{a}x\right)dx\right][\sinh\left(-\frac{n\pi}{a}b\right)]^{-1}.$$

19.3 $u(x,y) = 2xy + \frac{3}{\sinh\pi}\sin\pi x\sinh\pi y$.

19.4 If $u(x,y) = x^2 - y^2$ then $u_{xx} = 2$ and $u_{yy} = -2$ so that $\Delta u = 0$. If $u(x,y) = 2xy$ then $u_{xx} = u_{yy} = 0$ so that $\Delta u = 0$.

19.5

$$u(x,y) = \sum_{n=1}^{\infty}[A_n\cosh\left(\frac{n\pi}{L}y\right) + B_n\sinh\left(\frac{n\pi}{L}y\right)]\sin\frac{n\pi}{L}x.$$

where

$$A_n = \left[\frac{2}{L}\int_0^L (f_1(x)+f_2(x))\sin\frac{n\pi}{L}xdx\right]\left[\cosh\left(\frac{n\pi H}{2L}\right)\right]^{-1}$$

and

$$B_n = \left[\frac{2}{L}\int_0^L (f_2(x)-f_1(x))\sin\frac{n\pi}{L}xdx\right]\left[\sinh\left(\frac{n\pi H}{2L}\right)\right]^{-1}$$

19.6 (a) Differentiating term by term with respect to x we find

$$u_x + iv_x = \sum_{n=0}^{\infty} na_n(x+iy)^{n-1}.$$

Likewise, differentiating term by term with respect to y we find

$$u_y + iv_y = \sum_{n=0}^{\infty} na_n i(x+iy)^{n-1}.$$

Multiply this equation by i we find

$$-iu_y + v_y = \sum_{n=0}^{\infty} na_n(x+iy)^{n-1}.$$

Hence, $u_x + iv_x = v_y - iu_y$ which implies $u_x = v_y$ and $v_x = -u_y$.
(b) We have $u_{xx} = (v_y)_x = (v_x)_y = -u_{yy}$ so that $\Delta u = 0$. Similar argument for $\Delta v = 0$.

19.7 Polar and Cartesian coordinates are related by the expressions $x = r\cos\theta$ and $y = r\sin\theta$ where $r = (x^2+y^2)^{\frac{1}{2}}$ and $\tan\theta = \frac{y}{x}$. Using the chain rule we obtain

$$\begin{aligned}
u_x =& u_r r_x + u_\theta \theta_x = \cos\theta u_r - \frac{\sin\theta}{r} u_\theta\\
u_{xx} =& u_{xr} r_x + u_{x\theta}\theta_x\\
=& \left(\cos\theta u_{rr} + \frac{\sin\theta}{r^2} u_\theta - \frac{\sin\theta}{r} u_{r\theta}\right)\cos\theta\\
&+ \left(-\sin\theta u_r + \cos\theta u_{r\theta} - \frac{\cos\theta}{r} u_\theta - \frac{\sin\theta}{r} u_{\theta\theta}\right)\left(-\frac{\sin\theta}{r}\right)\\
u_y =& u_r r_y + u_\theta \theta_y = \sin\theta u_r + \frac{\cos\theta}{r} u_\theta\\
u_{yy} =& u_{yr} r_y + u_{y\theta}\theta_y\\
=& \left(\sin\theta u_{rr} - \frac{\cos\theta}{r^2} u_\theta + \frac{\cos\theta}{r} u_{r\theta}\right)\sin\theta\\
&+ \left(\cos\theta u_r + \sin\theta u_{r\theta} - \frac{\sin\theta}{r} u_\theta + \frac{\cos\theta}{r} u_{\theta\theta}\right)\left(\frac{\cos\theta}{r}\right)
\end{aligned}$$

Substituting these equations into (21.1) we obtain the dersired equation.

19.8 $u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y)$ where

$$u_1(x,y) = 0$$

$$u_2(x,y) = \sum_{n=1}^{\infty}\left[-\frac{2}{n\pi}\cdot\frac{(-1)^n}{\sinh\left(\frac{3n\pi}{2}\right)}\right]\sin\frac{n\pi}{2}x\sinh\left(\frac{n\pi}{2}y\right)$$

$$u_3(x,y) = \frac{1}{\sinh\left(\frac{8\pi}{3}\right)}\sinh\left(\frac{4\pi(x-2)}{3}\right)\sin\left(\frac{4\pi}{3}y\right)$$

$$u_4(x,y) = \sum_{n=1}^{\infty}\frac{14(1-(-1)^n)}{n\pi\sinh\left(\frac{2n\pi}{3}\right)}\sin\left(\frac{n\pi}{3}y\right)\sinh\left(\frac{n\pi}{3}x\right).$$

19.9

$$u(x,y) = \frac{4}{\sinh\left(\frac{\pi L}{2H}\right)}\left\{\sinh\left(\frac{\pi x}{2H}\right) - \sinh\left(\frac{\pi(x-L)}{2H}\right)\right\}\cos\frac{\pi y}{2H}.$$

19.10 $u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\sqrt{\lambda_n}x}\cos\sqrt{\lambda_n}y$ where

$$A_0 = \frac{1}{H}\int_0^H f(y)dy$$

$$A_n = \frac{2}{H}\int_0^H f(y)\cos\frac{n\pi}{H}ydy.$$

19.11 $u(x,y) = \frac{20}{Y_1(H)}Y_1(y)\sin\left(\frac{\pi x}{L}\right) - \frac{5}{Y_3(H)}\sin\left(\frac{3\pi x}{L}\right).$

19.12 $u(x,y) = \sin(2\pi x)e^{-2\pi y}.$

19.13 $u(x,y) = y.$

19.14 $u(x,y) = \frac{1}{2}x^2 - \frac{1}{2}y^2 - ax_by + C$ where C is an arbitrary constant.

19.15 $u(x,y) = \frac{2\cosh 3y\sin 3x}{\cosh 6} - \frac{5\cosh 10y\sin 10x}{\cosh 20}.$

Section 20

20.1 $u(r,\theta) = 3r^5 \sin 5\theta$.

20.2 $u(r,\theta) = \frac{\pi}{4} + \sum_{n=1}^{\infty} r^n \left[\frac{1-(-1)^n}{n^2\pi}\cos n\theta + \frac{\sin n\theta}{n}\right]$.

20.3 $u(r,\theta) = C_0 + r^2 \cos 2\theta$.

20.4 Substituting C_0, A_n, and B_n into the right-hand side of $u(r,\theta)$ we find

$$\begin{aligned}
u(r,\theta) =& \frac{1}{2\pi}\int_0^{2\pi} f(\phi)d\phi + \sum_{n=1}^{\infty} \frac{r^n}{\pi a^n}\int_0^{2\pi} f(\phi)\left[\cos n\phi \cos n\theta + \sin n\phi \sin n\theta\right] d\phi \\
=& \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta - \phi)\right] d\phi.
\end{aligned}$$

20.5 (a) We have $e^{it} = \cos t + i \sin t$ and $e^{-it} = \cos t - i \sin t$. The result follows by adding these two equalities and dividing by 2.
(b) This follows from the fact that

$$\cos n(\theta - \phi) = \frac{1}{2}(e^{in(\theta-\phi)} + e^{-in(\theta-\phi)}).$$

(c) We have $|q_1| = \frac{r}{a}\sqrt{\cos{(\theta-\phi)}^2 + \sin{(\theta-\phi)}^2} = \frac{r}{a} < 1$ since $0 < r < a$. A similar argument shows that $|q_2| < 1$.

20.6 (a) The first sum is a convergent geometric series with ratio q_1 and sum

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\phi)} =& \frac{\frac{r}{a}e^{i(\theta-\phi)}}{1-q_1} \\
=& \frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}}
\end{aligned}$$

Similar argument for the second sum.
(b) We have

$$\begin{aligned}1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n(\theta-\phi) =&1+\frac{re^{i(\theta-\phi)}}{a-re^{i(\theta-\phi)}}\\&+\frac{re^{-i(\theta-\phi)}}{a-re^{-i(\theta-\phi)}}\\&=1+\frac{r}{ae^{-i(\theta-\phi)}-r}+\frac{r}{ae^{-i(\theta-\phi)}-r}\\&=1+\frac{r}{a\cos(\theta-\phi)-r-ai\sin(\theta-\phi)}\\&+\frac{r}{a\cos(\theta-\phi)-r+ai\sin(\theta-\phi)}\\&=1+\frac{r[a\cos(\theta-\phi)-r+ai\sin(\theta-\phi)]}{a^2+2ar\cos(\theta-\phi)+r^2}\\&+\frac{r[a\cos(\theta-\phi)-r-ai\sin(\theta-\phi)]}{a^2-2ar\cos(\theta-\phi)+r^2}\\&=\frac{a^2-r^2}{a^2-2ar\cos(\theta-\phi)+r^2}.\end{aligned}$$

20.7 We have

$$\begin{aligned}u(r,\theta)=&\frac{1}{2\pi}\int_0^{2\pi}f(\phi)\left[1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n(\theta-\phi)\right]d\phi\\=&\frac{1}{2\pi}\int_0^{2\pi}f(\phi)\frac{a^2-r^2}{a^2-2ar\cos(\theta-\phi)+r^2}d\phi\\=&\frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{f(\phi)}{a^2-2ar\cos(\theta-\phi)+r^2}d\phi.\end{aligned}$$

20.8 $u(r,\theta)=2\sum_{n=1}^{\infty}(-1)^{n+1}r^n\frac{\sin n\theta}{n}$.

20.9 (a) Differentiating $u(r,t)=R(r)T(t)$ with respect to r and t we find

$$u_{tt}=RT'' \text{ and } u_r=R'T \text{ and } u_{rr}=R''T.$$

Substituting these into the given PDE we find

$$RT''=c^2\left(R''T+\frac{1}{r}R'T\right)$$

Dividing both sides by c^2RT we find

$$\frac{1}{c^2}\frac{T''}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R}.$$

Since the RHS of the above equation depends on r only, and the LHS depends on t only, they must equal to a constant λ.
(b) The given boundary conditions imply

$$u(a,t) = 0 = R(a)T(t) \Longrightarrow R(a) = 0$$

$$u(r,0) = f(r) = R(r)T(0)$$

$$u_t(r,0) = g(r) = R(r)T'(0).$$

If $\lambda = 0$ then $R'' + \frac{1}{r}R' = 0$ and this implies $R(r) = C\ln r$. Using the condition $R(a) = 0$ we find $C = 0$ so that $R(r) = 0$ and hence $u \equiv 0$. If $\lambda > 0$ then $T'' - \lambda c^2 T = 0$. This equation has the solution

$$T(t) = A\cos(c\sqrt{\lambda}t) + B\sin(c\sqrt{\lambda}t).$$

The condition $u(r,0) = f(r)$ implies that $A = f(r)$ which is not possible. Hence, $\lambda < 0$.

20.10 (a) Follows from the figure and the definitions of trigonometric functions in a right triangle.
(b) The result follows from equation (20.1).

20.11 By the maximum principle we have

$$\min_{(x,y)\in\partial\Omega} u(x,y) \le u(x,y) \le \max_{(x,y)\in\partial\Omega} u(x,y), \quad \forall (x,y) \in \Omega$$

But $\min_{(x,y)\in\partial\Omega} u(x,y) = u(1,0) = 1$ and $\max_{(x,y)\in\partial\Omega} u(x,y) = u(-1,0) = 3$. Hence,

$$1 \le u(x,y) \le 3$$

and this implies that $u(x,y) > 0$ for all $(x,y) \in \Omega$.

20.12 (i) $u(1,0) = 4$ (ii) $u(-1,0) = -2$.

20.13 Using the maximum principle and the hypothesis on g_1 and g_2, for all $(x,y) \in \Omega \cup \partial\Omega$ we have

$$\begin{aligned}\min_{(x,y)\in\partial\Omega} u_1(x,y) = & \min_{(x,y)\in\partial\Omega} g_1(x,y) \\ & \leq u_1(x,y) \leq \max_{(x,y)\in\partial\Omega} u_1(x,y) \\ & = \max_{(x,y)\in\partial\Omega} g_1(x,y) < \max_{(x,y)\in\partial\Omega} g_2(x,y) \\ & \leq \min_{(x,y)\in\partial\Omega} g_1(x,y) = \min_{(x,y)\in\partial\Omega} u_2(x,y) \\ & \leq u_2(x,y) \leq \max_{(x,y)\in\partial\Omega} u_2(x,y) = \max_{(x,y)\in\partial\Omega} g_2(x,y).\end{aligned}$$

20.14 We have

$$\begin{aligned}\Delta(r^n \cos(n\theta)) = & \frac{\partial^2}{\partial r^2}(r^n \cos(n\theta)) + \frac{1}{r}\frac{\partial}{\partial r}(r^n \cos(n\theta)) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(r^n \cos(n\theta)) \\ = & n(n-1)r^{n-2}\cos(n\theta) + nr^{n-2}\cos(n\theta) - r^{n-2}n^2\cos(n\theta) = 0\end{aligned}$$

Likewise, $\Delta(r^n \sin(n\theta)) = 0$.

20.15 $u(r,\theta) = \frac{1}{2} - \frac{r^2}{2a^2}\cos 2\theta$.

20.16 $u(r,\theta) = \ln 2 + 4\left(\frac{a}{r}\right)^3 \cos 3\theta$.

Section 21

21.1 Convergent.

21.2 Divergent.

21.3 Convergent.

21.4 $\frac{1}{s-3}, \; s > 3$.

21.5 $\frac{1}{s^2} - \frac{5}{s}, \; s > 0$.

21.6 $f(t) = e^{(t-1)^2}$ does not have a Laplace transform.

21.7 $\frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3},\ s > 0.$

21.8 $\frac{e^{-s}}{s^2},\ s > 0.$

21.9 $-\frac{e^{-2s}}{s} + \frac{1}{s^2}(e^{-s} - e^{-2s}),\ s \neq 0.$

21.10 $-\frac{t^n e^{-st}}{s} + \frac{n}{s}\int t^{n-1}e^{-st}dt,\ s > 0.$

21.11 (a) 0 (b) 0.

21.12 $\frac{5}{s+7} + \frac{1}{s^2} + \frac{2}{s-2},\ s > 2.$

21.13 $3e^{2t},\ t \geq 0.$

21.14 $-2t + e^{-t},\ t \geq 0.$

21.15 $2(e^{-2t} + e^{2t}),\ t \geq 0.$

21.16 $\frac{2}{s-1} + \frac{5}{s},\ s > 1.$

21.17 $\frac{e^{-s}}{s-3},\ s > 3.$

21.18 $\frac{1}{2}\left(\frac{1}{s} - \frac{s^2}{s^2+4\omega^2}\right),\ s > 0.$

21.19 $\frac{3}{s^2+26},\ s > 0.$

21.20 $\frac{s-3}{(s-3)^2+9},\ s > 3.$

21.21 $\frac{2}{(s-4)^3} + \frac{3}{(s-4)^2} + \frac{5}{s-4},\ s > 4.$

21.22 $2\sin 5t + 4e^{3t},\ t \geq 0.$

21.23 $\frac{5}{6}e^{3t}t^3,\ t \geq 0.$

21.24

$$\begin{cases} 0, & 0 \leq t < 2 \\ e^{9(t-2)}, & t \geq 2. \end{cases}$$

21.25 $3e^{3t} - 3e^{-t},\ t \geq 0.$

21.26 $4[e^{3(t-5)} - e^{-3(t-5)}]H(t-5),\ t \geq 0.$

21.27 $y(t) = 2e^{-4t} + 3[H(t-1) - H(t-3)] - 3[e^{-4(t-1)}H(t-1) - e^{-4(t-3)}H(t-3)],\ t \geq 0.$

21.28 $\frac{1}{5}e^{3t} + \frac{1}{20}e^{-2t} - \frac{1}{4}e^{2t},\ t \geq 0.$

21.29 $\frac{e^t - e^{-2t}}{3}.$

21.30 $\frac{t}{2}\sin t.$

21.31 $\frac{t^5}{120}.$

21.32 $\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}.$

21.33 $-t + \frac{e^t}{2} - \frac{e^{-t}}{2}.$

Section 22

22.1 $u(x,t) = \sin(x-t) - H(t-x)\sin(x-t).$

22.2 $u(x,t) = [\sin(x-t) - H(t-x)\sin(x-t)]e^{-t}.$

22.3 $u(x,t) = 2e^{-4\pi^2 t}\sin \pi x + 6e^{-16\pi^2 t}\sin 2\pi x.$

22.4 $u(x,t) = [\sin(x-t) - H(t-x)\sin(x-t)]e^{t}.$

22.5 $u(x,t) = t^2 e^{-x} - te^{-x} + t.$

22.6 $u(x,t) = \left(t - \frac{1}{2}x^2\right) H\left(t - \frac{1}{2}x^2\right).$

22.7 $u(x,t) = \mathcal{L}^{-1}\left(\frac{e^{-\frac{s}{c}x}}{s^2+1}\right) = H\left(t - \frac{x}{c}\right)\sin\left(t - \frac{x}{c}\right).$

22.8 $u(x,t) = 2\sin x \cos 3t.$

22.9 $u(x,y) = y(x+1) + 1.$

22.10 $u(x,t) = \mathcal{L}^{-1}\left(\frac{e^{-\frac{s}{c}x}}{s^2+1}\right) = h\left(t - \frac{x}{c}\right)\sin\left(t - \frac{x}{c}\right).$

22.11 $u(x,t) = e^{-5x}e^{-4t}H(t).$

22.12 $u(x,t) = \mathcal{L}^{-1}\left(-\frac{T}{s}e^{-\frac{\sqrt{s}}{c}x} + \frac{T}{s}\right).$

22.13 $u(x,t) = 5e^{-3\pi^2 t}\sin(\pi x).$

22.14 $u(x,t) = 40e^{-t}\cos\frac{x}{2}.$

22.15 $u(x,t) = 3\sin \pi x \cos 2\pi t.$

Section 23

23.1 $\frac{(-1)^n i}{n\pi}.$

23.2 $f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)(e^{inx} + e^{-inx}).$

23.3 $f(x) = \frac{\sinh a\pi}{\pi}\sum_{n=-\infty}^{\infty} \frac{(-1)^n(a+in)}{(a^2+n^2)}e^{inx}.$

23.4 $f(x) = \frac{e^{ix} - e^{-ix}}{2i}.$

23.5 $f(x) = \frac{1}{2\pi}\left\{T + \sum_{n=-\infty}^{-1} \frac{i}{n}[e^{-int} - 1]e^{int} + \sum_{n=1}^{\infty} \frac{i}{n}[e^{-int} - 1]e^{int}\right\}.$

23.6 (a) $f(x) = \frac{\pi^2}{3} + \sum_{n=-\infty}^{-1} \frac{2}{n^2}(-1)^n e^{inx} + \sum_{n=1}^{\infty} \frac{2}{n^2}(-1)^n e^{inx}.$
(b) $f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx.$

23.7 (a)

$$
\begin{aligned}
a_0 =& 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \sin n\pi x dx = -\frac{2}{\pi}[\cos\frac{\pi}{2} - \cos -\frac{\pi}{2}] = 0 \\
a_n =& 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \sin n\pi x \cos 2n\pi x dx = 0 \\
b_n =& 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \sin n\pi x \sin 2n\pi x dx = \frac{8(-1)^n n}{\pi - 4n^2\pi}
\end{aligned}
$$

(b) $f(x) = \frac{4}{\pi}\sum_{n=-\infty}^{\infty} \frac{(-1)^n n}{i(1-4n^2)} e^{2n\pi i x}$.

23.8 (a)

$$
\begin{aligned}
a_0 =& \frac{1}{2}\int_{-2}^{2}(2-x)dx = 4 \\
a_n =& \frac{1}{2}\int_{-2}^{2}(2-x)\cos\left(\frac{n\pi}{2}x\right)dx = 0 \\
b_n =& \frac{1}{2}\int_{-2}^{2}(2-x)\sin\left(\frac{n\pi}{2}x\right)dx = \frac{4(-1)^n}{n\pi}
\end{aligned}
$$

(b) $f(x) = 2 + \sum_{n=-\infty}^{-1} \frac{2(-1)^{n+1}i}{n\pi} e^{\left(\frac{in\pi}{2}x\right)} + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}i}{n\pi} e^{\left(\frac{in\pi}{2}x\right)}$.

23.9 $a_n = c_n + c_{-n} = 0$. We have for $|n|$ odd $b_n = i\frac{4}{in\pi} = \frac{4}{n\pi}$ and for $|n|$ even $b_n = 0$.

23.10 Note that for any complex number z we have $z + \overline{z} = 2Re(z)$ and $z - \overline{z} = -2iRe(z)$. Thus,

$$c_n + \overline{c_n} = a_n$$

which means that $a_n = 2Re(c_n)$. Likewise, we have

$$c_n - \overline{c_n} = ib_n$$

That is $ib_n = -2iIm(c_n)$. Hence, $b_n = -2Im(c_n)$.

23.11 $a_n = 2Re(c_n) = \frac{1}{\pi n}\sin(nT)$ and $b_n = \frac{1-\cos(nT)}{n\pi}$.

23.12 $f(x) = i\sum_{n=-\infty}^{\infty} \frac{i\sin{(2-in\pi)}}{2-in\pi} e^{\frac{in\pi}{2}x}$.

23.13 (a) We have

$$f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & 1 < t < 2 \end{cases}$$

and $f(t+2) = f(t)$ for all $t \in \mathbb{R}$.
(b) We have

$$\begin{aligned} a_0 =& \frac{2}{L}\int_0^L f(x)dx = \int_0^2 dx = \int_0^1 dx = 1 \\ a_n =& \int_0^1 \cos n\pi x dx = \frac{\sin n\pi}{n\pi} = 0. \end{aligned}$$

(c) We have

$$b_n = \int_0^1 \sin n\pi x dx = \frac{1-\cos n\pi}{n\pi} = \frac{1-(-1)^n}{n\pi}.$$

Hence,

$$b_n = \begin{cases} \frac{2}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

(d) We have $c_0 = \frac{a_0}{2} = \frac{1}{2}$ and for $n \in \mathbb{N}$ we have

$$c_n = \frac{a_n - ib_n}{2} = \begin{cases} -\frac{i}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

23.14 $\sin 3x = \frac{1}{2}(e^{3ix} - e^{-3ix})$.

23.15 $e^{-ins}\left(\frac{1-e^{-inh}}{2\pi in}\right)$.

Section 24

24.1

$$\hat{f}(\xi) = \begin{cases} 2\frac{\sin\xi}{\xi} & \text{if } \xi \neq 0 \\ 2 & \text{if } \xi = 0. \end{cases}$$

24.2

$$\frac{\partial \hat{u}}{\partial t} + i\xi c\hat{u} = 0$$

$$\hat{u}(\xi, 0) = \hat{f}(\xi).$$

24.3

$$\frac{\partial^2 \hat{u}}{\partial t^2} = -c^2\xi^2\hat{u}$$
$$\hat{u}(\xi, 0) = \hat{f}(\xi)$$
$$\hat{u}_t(\xi, 0) = \hat{g}(\xi).$$

24.4

$$\hat{u}_{yy} = \xi^2\hat{u}$$
$$\hat{u}(\xi, 0) = 0, \ \hat{u}(\xi, L) = \frac{2\sin\xi a}{\xi}.$$

24.5 $\frac{1}{\alpha - i\xi} + \frac{1}{\alpha + i\xi} = \frac{2\alpha}{\alpha^2+\xi^2}$.

24.6 We have

$$\begin{aligned}\mathcal{F}[e^{-x}H(x)] &= \int_{-\infty}^{\infty} e^{-x}H(x)e^{-i\xi x}dx \\ &= \int_0^{\infty} e^{-x(1+i\xi)}dx = -\left.\frac{e^{-x(1+i\xi)}}{1+i\xi}\right|_0^{\infty} = \frac{1}{1+i\xi}.\end{aligned}$$

24.7 Using the duality property, we have

$$\mathcal{F}\left[\frac{1}{1+ix}\right] = \mathcal{F}[\mathcal{F}[e^{-\xi}H(\xi)]] = 2\pi e^{\xi}H(-\xi).$$

24.8 We have

$$\begin{aligned}\mathcal{F}[f(x-\alpha)] =& \int_{-\infty}^{\infty} f(x-\alpha)e^{-i\xi x}dx \\ =& e^{-i\xi\alpha}\int_{-\infty}^{\infty} f(u)e^{-i\xi u}du \\ =& e^{-i\xi\alpha}\hat{f}(\xi)\end{aligned}$$

where $u = x - \alpha$.

24.9 We have

$$\mathcal{F}[e^{i\alpha x}f(x)] = \int_{-\infty}^{\infty} e^{i\alpha x}f(x)e^{-i\xi x}dx = \int_{-\infty}^{\infty} e^{ix(\alpha-\xi}f(x)e^{-i\xi x}dx = \hat{f}(\xi - \alpha).$$

24.10 We will just prove the first one. We have

$$\begin{aligned}\mathcal{F}[\cos{(\alpha x)}f(x)] =& \mathcal{F}[\frac{f(x)e^{i\alpha x}}{2} + f(x)\frac{e^{-i\alpha x}}{2}\\ =& \frac{1}{2}[\mathcal{F}[f(x)e^{i\alpha x}] + \mathcal{F}[f(x)e^{-i\alpha x}]]\\ =& \frac{1}{2}[\hat{f}(x-\alpha) + \hat{f}(x+\alpha)].\end{aligned}$$

24.11 Using the definition and integration by parts we find

$$\begin{aligned}\mathcal{F}[f'(x)] =& \int_{-\infty}^{\infty} f'(x)e^{-i\xi x}dx\\ =& f(x)e^{-i\xi x}\Big|_{-\infty}^{\infty} + (i\xi)\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx\\ =& f(x)\cos\xi x - if(x)\sin\xi x + (i\xi)\hat{f}(\xi) = (i\xi)\hat{f}(\xi)\end{aligned}$$

where we used the fact that $\lim_{x\to\infty} f(x) = 0$.

24.12 $\frac{2}{\xi^2}(1 - \cos\xi)$.

24.13 $\frac{2}{i\xi}(1 - \cos\xi a)$.

24.14 $\mathcal{F}^{-1}[\hat{f}(\xi)] = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$.

24.15 $\mathcal{F}^{-1}\left(\frac{1}{a+i\xi}\right) = e^{-ax}, \quad x \geq 0$.

Section 25

25.1 $u(x,t) = f(x) * \mathcal{F}^{-1}[-\frac{1}{|\xi|}e^{-|\xi|y}]$.

25.2 $u(x,t) = \mathcal{F}^{-1}[u(\xi,t)] = e^{-\frac{(x-ct)^2}{4}}$.

25.3

$$\begin{aligned} u(x,t) =& \sqrt{\frac{\gamma}{4\pi}} e^{-\alpha t} \mathcal{F}^{-1}[e^{-\xi^2(kt+\frac{\gamma}{4})}] \\ =& \sqrt{\frac{\gamma}{4\pi}} e^{-\alpha t} \cdot \sqrt{\frac{\pi}{kt+\gamma/4}} \cdot e^{-\frac{x^2}{4(kt+\gamma/4)}} \\ =& \sqrt{\gamma} 4kt + \gamma e^{-\frac{x^2}{4kt+\gamma}} e^{-\alpha t}. \end{aligned}$$

25.4 $u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-\frac{(x-s)^2}{4kt}} ds$.

25.5

$$\begin{aligned} u(x,t) =& e^t \mathcal{F}^{-1}[e^{-\xi^2 t}] \\ =& e^{-\alpha t} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}. \end{aligned}$$

25.6 We have

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-|\xi|y} e^{i\xi x} d\xi =& \int_{-\infty}^{0} e^{\xi y} e^{i\xi x} d\xi + \int_0^\infty e^{-\xi y} e^{i\xi x} d\xi \\ =& \frac{1}{y+ix} e^{\xi(y+ix)} \Big|_{-\infty}^{0} + \frac{1}{-y+ix} e^{\xi(-y+ix)} \Big|_0^\infty \\ =& \frac{1}{y+ix} + \frac{1}{-y+ix} = \frac{2y}{x^2+y^2}. \end{aligned}$$

25.7

$$\begin{aligned} u(x,y) =& \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-|\xi|y} e^{i\xi x} d\xi \\ =& \frac{1}{2\pi} f(x) * \left[\frac{2y}{x^2+y^2}\right] \\ =& \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \frac{2y}{(x-\xi)^2+y^2} d\xi. \end{aligned}$$

25.8 $\hat{u}_{tt} + (\alpha + \beta)\hat{u}_t + \alpha\beta\hat{u} = -c^2\xi^2\hat{u}$.

25.9 $u(x, t) = e^{-(x-3t)}$.

25.10 $u(x, t) = e^{-(x-kt)}$.

25.11 $u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-s^2 - \frac{(x-s)^2}{4kt}} ds$.

25.12 $u(x, t) = (x - ct)^2$.

25.13 $u(x, t) = f(x) * \mathcal{F}^{-1}[-\frac{1}{|\xi|} e^{-|\xi| y}]$.