Fourier Series

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Abstract

These notes introduce Fourier series and discuss some applications.

1 Introduction

Joseph Fourier (1768-1830) who gave his name to Fourier series, was not the first to use Fourier series neither did he answer all the questions about them. These series had already been studied by Euler, d'Alembert, Bernoulli and others before him. Fourier also thought wrongly that any function could be represented by Fourier series. However, these series bear his name because he studied them extensively. The first concise study of these series appeared in Fourier's publications in 1807, 1811 and 1822 in his *Théorie analytique de la chaleur*. He applied the technique of Fourier series to solve the heat equation. He had the insight to see the power of this new method. His work set the path for techniques that continue to be developed even today.

Fourier Series, like Taylor series, are special types of expansion of functions. With Taylor series, we are interested in expanding a function in terms of the special set of functions 1, x, x^2, x^3, ... or more generally in terms of 1, $(x-a)$, $(x-a)^2$, $(x-a)^3$, ... You will remember from calculus that if a function f has a power series representation at a then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{1}$$

With Fourier series, we are interested in expanding a function f in terms of the special set of functions 1, $\cos x$, $\cos 2x$, $\cos 3x$, ..., $\sin x$, $\sin 2x$, $\sin 3x$, ... Thus, a Fourier series expansion of a function is an expression of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

After reviewing periodic functions, we will focus on learning how to represent a function by its Fourier series. We will only partially answer the question regarding which functions have a Fourier series representation. We will finish these notes by discussing some applications.

2 Even, Odd and Periodic Functions

In this section, we review some results about even, odd and periodic functions. These results will be needed for the remaining sections.

Definition 1 (Even and Odd) *Let f be a function defined on an interval I (finite or infinite) centered at $x = 0$.*

1. *f is said to be even if $f(-x) = f(x)$ for every x in I.*
2. *f is said to be odd if $f(-x) = -f(x)$ for every x in I.*

The graph of an even function is symmetric with respect to the y-axis. The graph of an odd function is symmetric with respect to the origin. For example, 5, x^2, x^n where n is even, $\cos x$ are even functions while x, x^3, x^n where n is odd, $\sin x$ are odd.

You will recall from calculus the following important theorem about integrating even and odd functions over an interval of the form $[-a, a]$ where $a > 0$.

Theorem 2 *Let f be a function which domain includes $[-a, a]$ where $a > 0$.*

1. *If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$*
2. *If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$*

There are several useful algebraic properties of even and odd functions as shown in the theorem below.

Theorem 3 *When adding or multiplying even and odd functions, the following is true:*

- *even + even = even*
- *even × even = even*
- *odd + odd = odd*
- *odd × odd = even*
- *even × odd = odd*

Definition 4 (Periodic) *Let $T > 0$.*

1. *A function f is called T-**periodic** or simply **periodic** if*

$$f(x + T) = f(x) \tag{2}$$

   for all x.

2. *The number T is called a **period** of f.*

3. *If f is non-constant, then the smallest positive number T with the above property is called the* ***fundamental period*** *or simply the* ***period*** *of f.*

Let us first remark that if T is a period for f, then nT is also a period for any integer $n > 0$. This is easy to see using equation 2 repeatedly:

$$
\begin{aligned}
f(x) &= f(x+T) \\
&= f((x+T)+T) = f(x+2T) \\
&= f((x+2T)+T) = f(x+3T) \\
&\vdots \\
&= f((x+(n-1)T)+T) = f(x+nT)
\end{aligned}
$$

Classical examples of periodic functions are $\sin x$, $\cos x$ and other trigonometric functions. $\sin x$ and $\cos x$ have period 2π. $\tan x$ has period π. We will see more examples below.

Because the values of a periodic function of period T repeat every T units, it is enough to know such a function on any interval of length T. Its graph is obtained by repeating the portion over any interval of length T. Consequently, to define a T-periodic function, it is enough to define it over any interval of length T. Since different intervals may be chosen, the same function may be defined different ways.

Example 5 *Describe the 2-periodic function shown in figure 1 in two different ways:*

1. *By considering its values on the interval $0 \leq x < 2$;*
2. *By considering its values on the interval $-1 \leq x < 1$.*

Solution 1. *On the interval $0 \leq x < 2$, the function is a portion of the line $y = -x + 1$ thus $f(x) = -x + 1$ if $0 \leq x < 2$. The relation $f(x+2) = f(x)$ describes f for all other values of x.*

2. *On the interval $-1 \leq x < 1$, the function consists of two lines. So we have*

$$
f(x) = \begin{cases} -x-1 & \text{if} \quad -1 \leq x < 0 \\ -x+1 & \text{if} \quad 0 \leq x, \leq 1 \end{cases}
$$

The relation $f(x+2) = f(x)$ describes f for all other values of x.

Although we have different formulas, they describe the same function. Of course, in practice, we use common sense to select the most appropriate formula.

Next, we look at an important theorem concerning integration of periodic functions over one period.

Theorem 6 (Integration Over One Period) *Suppose that f is T-periodic. Then for any real number a, we have*

$$
\int_0^T f(x)\,dx = \int_a^{a+T} f(x)\,dx \tag{3}
$$

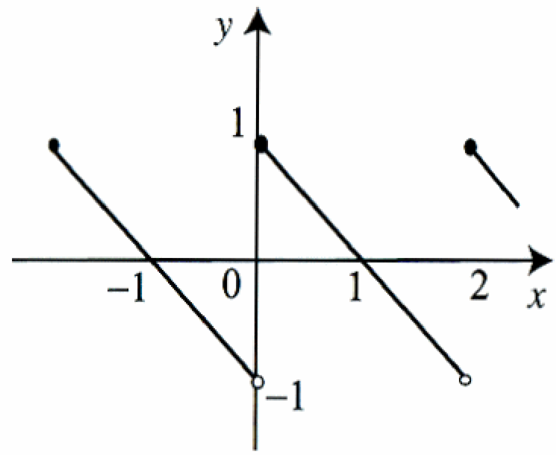


Figure 1: A function of period 2

Proof. *Define $F(a) = \int_a^{a+T} f(x)\,dx$. By the fundamental theorem of calculus, $F'(a) = f(a+T) - f(a) = 0$ since f is T-periodic. Hence, $F(a)$ is a constant for all a. In particular, $F(0) = F(a)$ which implies the theorem.* ■

We illustrate this theorem with an example.

Example 7 *Let f be the 2-periodic function shown in figure 1. Compute the integrals below:*

1. $\int_{-1}^{1} [f(x)]^2\,dx$

2. $\int_{-N}^{N} [f(x)]^2\,dx$ *where N is any positive integer.*

Solution 8 1. *We described this function earlier and noticed that its simplest expression was not over the interval $[-1,1]$ but over the interval $[0,2]$. We should also note that if f is 2-periodic, so is $[f(x)]^2$ (why?). Using theorem 6, we have*

$$
\begin{aligned}
\int_{-1}^{1} [f(x)]^2\,dx &= \int_0^2 [f(x)]^2\,dx \\
&= \int_0^2 (-x+1)^2\,dx \\
&= \frac{-1}{3}(-x+1)^3\Big|_0^2 \\
&= \frac{2}{3}
\end{aligned}
$$

2. *We break* $\int_{-N}^{N}[f(x)]^2\,dx$ *into the sum of* N *integrals over intervals of length* 2.

$$\int_{-N}^{N}[f(x)]^2\,dx=\int_{-N}^{-N+2}[f(x)]^2\,dx+\int_{-N+2}^{-N+4}[f(x)]^2\,dx+...+\int_{N-2}^{N}[f(x)]^2\,dx$$

By theorem 6, each integral is $\frac{2}{3}$. *Thus*

$$\int_{-N}^{N}[f(x)]^2\,dx=\frac{2N}{3}$$

The following result about combining periodic functions is important.

Theorem 9 *When combining periodic functions, the following is true:*

1. *If* $f_1, f_2, ..., f_n$ *are* T*-periodic, then* $a_1f_1+a_2f_2+...+a_nf_n$ *is also* T*-periodic.*
2. *If* f *and* g *are two* T*-periodic functions so is* $f(x)\,g(x)$.
3. *If* f *and* g *are two* T*-periodic functions so is* $\frac{f(x)}{g(x)}$ *where* $g(x)\neq 0$.
4. *If* f *has period* T *and* $a>0$ *then* $f\left(\frac{x}{a}\right)$ *has period* aT *and* $f(ax)$ *has period* $\frac{T}{a}$.
5. *If* f *has period* T *and* g *is any function (not necessarily periodic) then the composition* $g\circ f$ *has period* T.

   Proof. *See problems.* ■

The functions in the 2π-periodic trigonometric system

$$1, \cos x, \cos 2x, ..., \cos mx, ..., \sin x, \sin 2x, ..., \sin nx, ...$$

are among the most important periodic functions. The reader will verify that they are indeed 2π-periodic. They share another important property.

Definition 10 (Orthogonal Functions) *Two functions* f *and* g *are said to be* ***orthogonal*** *over the interval* $[a,b]$ *if*

$$\int_a^b f(x)\,g(x)\,dx=0 \tag{4}$$

The notion of orthogonality is very important in many areas of mathematics.

Theorem 11 *The functions in the trigonometric system* $1, \cos x, \cos 2x, ..., \cos mx, ..., \sin x, \sin 2x, ..., \sin nx, ...$ *are orthogonal over the interval* $[-\pi, \pi]$ *in other words, if* m *and* n *are two non-negative integers, then*

$$\begin{aligned}\int_{-\pi}^{\pi} \cos mx \cos nx dx &= 0 \textit{ if } m \neq n \qquad (5)\\ \int_{-\pi}^{\pi} \cos mx \sin nx dx &= 0 \ \forall m, n\\ \int_{-\pi}^{\pi} \sin mx \sin nx dx &= 0 \textit{ if } m \neq n\end{aligned}$$

Proof. *There are different ways to prove this theorem. One way involves using the identities*

$$\begin{aligned}\sin\alpha\cos\beta &= \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]\\ \cos\alpha\sin\beta &= \frac{1}{2}\left[\sin(\alpha+\beta)-\sin(\alpha-\beta)\right]\\ \sin\alpha\sin\beta &= \frac{1}{2}\left[\cos(\alpha+\beta)-\cos(\alpha-\beta)\right]\\ \cos\alpha\cos\beta &= \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]\end{aligned}$$

We illustrate the technique by proving $\int_{-\pi}^{\pi} \cos mx \cos nx dx = 0$ *if* $m \neq n$. *We see that* $\cos mx \cos nx = \frac{1}{2}\left[\cos(m+n)x + \cos(m-n)x\right]$. *Therefore*

$$\begin{aligned}\int_{-\pi}^{\pi} \cos mx \cos nx dx &= \frac{1}{2}\int_{-\pi}^{\pi} \left[\cos(m+n)x + \cos(m-n)\right] dx\\ &= \frac{1}{2}\left[\frac{1}{m+n}\sin(m+n)x + \frac{1}{m-n}\sin(m-n)x\right]\Bigg|_{-\pi}^{\pi}\\ &= 0\end{aligned}$$

■

Remark 12 *We also have the useful identities*

$$\int_{-\pi}^{\pi} \cos^2 mx dx = \int_{-\pi}^{\pi} \sin^2 mx dx = \pi \textit{ for all } m \neq 0 \qquad (6)$$

We finish this section by looking at another example of a periodic function, which does not involve trigonometric function but rather the **greatest integer** function, also known as the **floor** function, denoted $\lfloor x \rfloor$. $\lfloor x \rfloor$ represents the greatest integer not larger than x. For example, $\lfloor 5.2 \rfloor = 5$, $\lfloor 5 \rfloor = 5$, $\lfloor -5.2 \rfloor = -6$, $\lfloor -5 \rfloor = -5$. Its graph is shown in figure 2.

Example 13 *Let* $f(x) = x - \lfloor x \rfloor$. *This gives the fractional part of* x. *For*

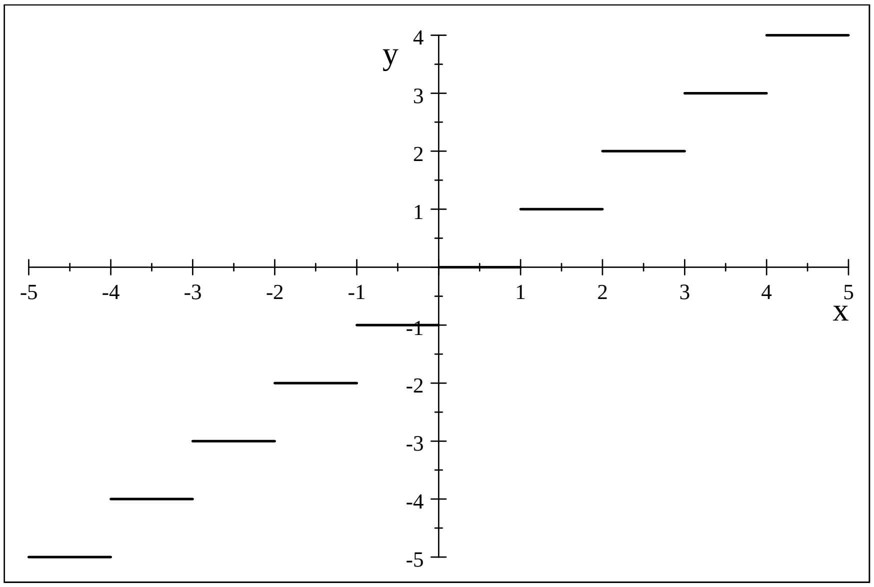


Figure 2: Graph of $\lfloor x \rfloor$

$0 \leq x < 1$, $\lfloor x \rfloor = 0$, so $f(x) = x$. Also, since $\lfloor x+1 \rfloor = 1 + \lfloor x \rfloor$, we get

$$\begin{aligned} f(x+1) &= x + 1 - \lfloor x+1 \rfloor \\ &= x + 1 - 1 - \lfloor x \rfloor \\ &= x - \lfloor x \rfloor \\ &= f(x) \end{aligned}$$

So, f is periodic with period 1. Its graph is obtained by repeating the portion of its graph over the interval $0 \leq x < 1$. Its graph is shown in figure

The practice problem will explore further properties of periodic functions.

2.1 Practice Problems

1. Prove theorem 2.
2. Prove theorem 3.
3. **Sums of periodic functions.** Show that if $f_1, f_2, ..., f_n$ are T-periodic, then $a_1 f_1 + a_2 f_2 + ... + a_n f_n$ is also T-periodic.
4. **Sums of periodic functions. Let** $f(x) = \cos x + \cos \pi x$.
   (a) Show that the equation $f(x) = 2$ has a unique solution.

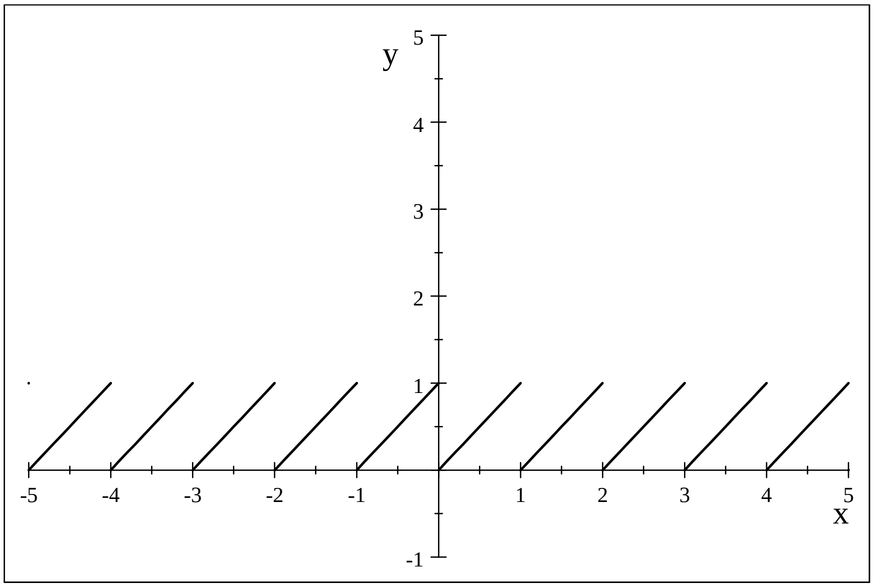


Figure 3: Graph of $x - \lfloor x \rfloor$

(b) Conclude from part a that f is not periodic. Does this contradict the previous problem?

5. Finish proving theorem 11.

6. **Operations on periodic functions.**

   (a) Show that if f and g are two T-periodic functions so is $f(x)\,g(x)$.

   (b) Show that if f and g are two T-periodic functions so is $\frac{f(x)}{g(x)}$ where $g(x) \neq 0$.

   (c) Show that if f has period T and $a > 0$ then $f\left(\frac{x}{a}\right)$ has period aT and $f(ax)$ has period $\frac{T}{a}$.

   (d) Show that if f has period T and g is any function (not necessarily periodic) then the composition $g \circ f$ has period T.

7. Using the previous problem, find the period of the functions below.

   (a) $\sin 2x$

   (b) $\cos \frac{1}{2}x + 3\sin 2x$

   (c) $\frac{1}{2+\sin x}$

   (d) $e^{\cos x}$

8. **Antiderivative of periodic functions.** Suppose that f is 2π-periodic and let a be a fixed real number. Define

$$F(x) = \int_a^x f(t)\,dt \text{ for all } x$$

Show that F is 2π-periodic if and only if $\int_0^{2\pi} f(t)\,dt = 0$. (hint: use theorem 6)

3 Fourier Series of 2π-Periodic Functions

As noted earlier, Fourier Series are special expansions of functions of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{7}$$

To be able to use Fourier Series, we need to know:

1. Which functions have Fourier series expansions?
2. If a function has a Fourier series expansion, how do we compute the coefficients $a_0, a_1, ..., b_0, b_1, ...$?

A general answer to the first question is beyond the scope of these notes. In this section, we will answer the second question. In these notes, we will give conditions which are sufficient for functions to have a Fourier Series Expansion.

3.1 Euler Formulas for the Coefficients

The coefficients which appear in the Fourier series were known to Euler before Fourier, hence they bear his name. We will derive them the same way Fourier did. This technique is worth remembering.

- **Computation of** a_0. Starting with equation 7, we integrate each side over the interval $[-\pi, \pi]$ and assuming term by term integration is legitimate, we obtain

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos nx dx + b_n \int_{-\pi}^{\pi} \sin nx dx \right]$$

Since $\int_{-\pi}^{\pi} \cos nx dx = \int_{-\pi}^{\pi} \sin nx dx = 0$ for $n = 1, 2, 3, ...$, we have

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\,dx &= a_0 \int_{-\pi}^{\pi} dx \\ &= 2\pi a_0 \end{aligned}$$

Thus

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

- **Computation of** a_n. Again, starting with equation 7, we multiply each side by $\cos mx$ for a fixed integer $m \geq 1$ and integrate each side as before. We obtain

$$\int_{-\pi}^{\pi} f(x) \cos mx dx = a_0 \int_{-\pi}^{\pi} \cos mx dx + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos nx \cos mx dx + b_n \int_{-\pi}^{\pi} \sin nx \cos mx dx \right]$$

  Now, from equation 5, we have: $\int_{-\pi}^{\pi} \cos mx dx = 0$, $\int_{-\pi}^{\pi} \sin nx \cos mx dx = 0$ and $\int_{-\pi}^{\pi} \cos nx \cos mx dx = \begin{cases} 0 & \text{if} \quad m \neq n \\ \pi & \text{if} \quad m = n \end{cases}$. So, we are left with

$$\int_{-\pi}^{\pi} f(x) \cos mx dx = \pi a_m$$

  Thus

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx dx \text{ for } m \geq 1$$

- **Computation of** b_n. We proceed in a similar way. Starting with equation 7, we multiply each side by $\sin mx$ for a fixed integer $m \geq 1$ and integrate each side as before. We leave it to the reader to verify that

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx dx \text{ for } m \geq 1$$

We summarize our findings in the following proposition.

Proposition 14 *Suppose that the 2π-periodic function f has the Fourier series representation*

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Then the coefficients a_0, a_n, b_n for $n = 1, 2, ...$ are called the Fourier coefficients of f and are given by the Euler's formulas

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \tag{8}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \text{ for } n = 1, 2, ... \tag{9}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \text{ for } n = 1, 2, ... \tag{10}$$

Definition 15 *For a positive integer N, we denote the N^{th} partial sum of the Fourier series of f by $S_N(x)$. So, we have*

$$S_N(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

We now illustrate what we did with some examples.

Example 16 *Find the Fourier series of $f(x) = \sin x$.*
Using the formulas above along with equation 7, we find that

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin x dx = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin x \cos nx dx = 0 \textit{ for all } n$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin x \sin nx dx = 0 \textit{ except when } n = 1$$

When $n = 1$,we have $a_1 = 1$. Thus, a Fourier series of $\sin x$ is $\sin x$. Of course, this was to be expected.

Example 17 *Find the Fourier series of $f(x) = \left|\sin \frac{x}{2}\right|$.*
Clearly, this function is 2π-periodic. Its graph is shown in figure 4.

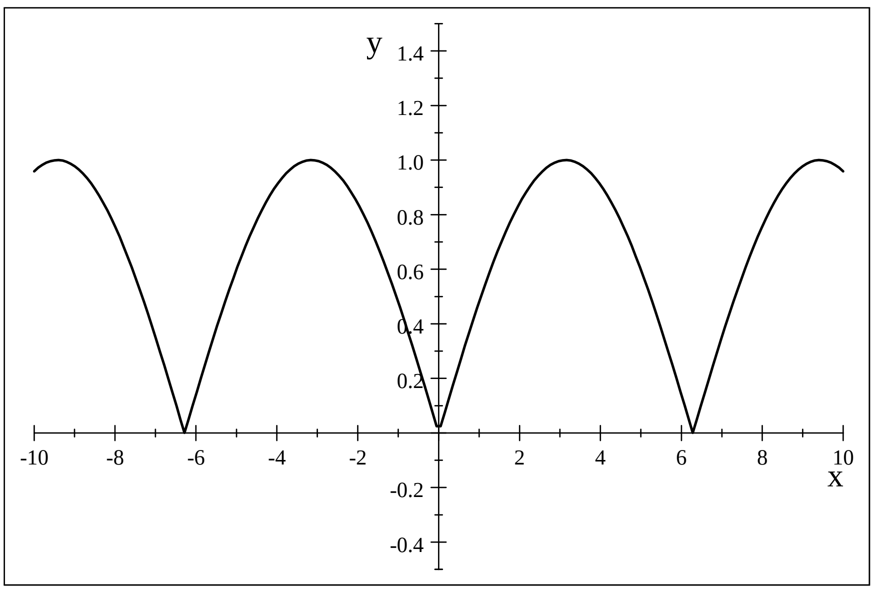


Figure 4: Graph of $|\sin x|$

1. ***Computation of** a_0. Using the formulas above along with equation 7,*

we find that

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sin\frac{x}{2}\right|dx \\ &= \frac{1}{\pi}\int_0^{\pi}\sin\frac{x}{2}dx \text{ since } \left|\sin\frac{x}{2}\right| \text{ is even and } \sin\frac{x}{2}\geq 0 \text{ on } [0,\pi] \\ &= \frac{2}{\pi} \end{aligned}$$

2. ***Computation of*** a_n.

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left|\sin\frac{x}{2}\right|\cos nx dx \\ &= \frac{2}{\pi}\int_0^{\pi}\sin\frac{x}{2}\cos nx dx \\ &= \frac{1}{\pi}\int_0^{\pi}\left[\sin\frac{2n+1}{2}x-\sin\frac{n-1}{2}x\right]dx \text{ if } n\geq 1 \\ &= \frac{1}{\pi}\left[\frac{-2}{2n+1}\cos\frac{2n+1}{2}x+\frac{2}{2n-1}\cos\frac{2n-1}{2}x\right]\Bigg|_0^{\pi} \\ &= \frac{1}{\pi}\left[\frac{2}{2n+1}-\frac{2}{2n-1}\right] \\ &= \frac{-4}{\pi\left(4n^2-1\right)} \end{aligned}$$

3. ***Computation of*** b_n.

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}|\sin x|\sin nx dx \\ &= 0 \text{ since } \int_{-\pi}^{\pi}|\sin x|\sin nx \text{ is odd} \end{aligned}$$

4. ***In conclusion***

$$\left|\sin\frac{x}{2}\right| = \frac{2}{\pi}+\sum_{n=1}^{\infty}\frac{-4}{\pi\left(4n^2-1\right)}\cos nx$$

To see how this series compares to the function, we will plot some of the partial sums. Let

$$S_N(x) = \frac{2}{\pi}+\sum_{n=1}^{N}\frac{-4}{\pi\left(4n^2-1\right)}\cos nx$$

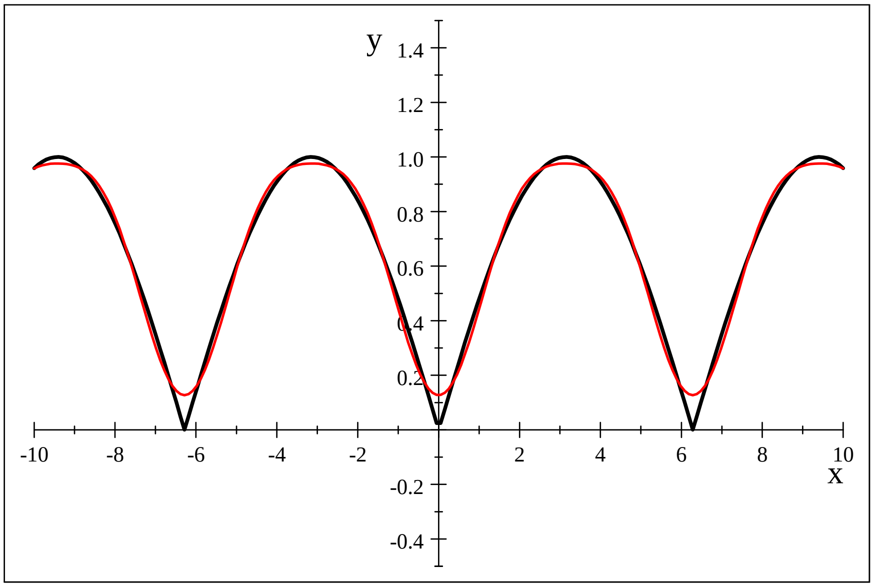


Figure 5: Graph of $\left|\sin \frac{x}{2}\right|$ and $S_2(x)$

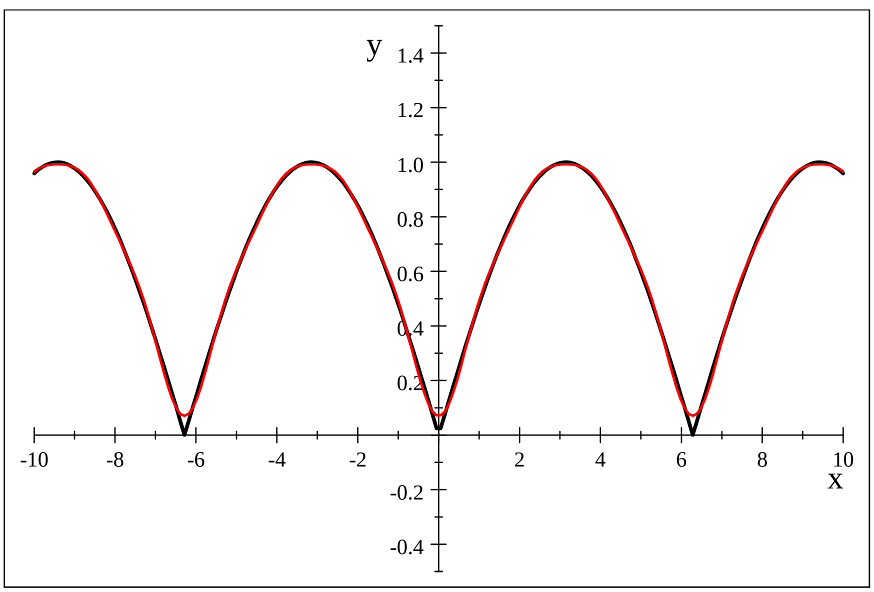


Figure 6: Graph of $\left|\sin \frac{x}{2}\right|$ and $S_4(x)$

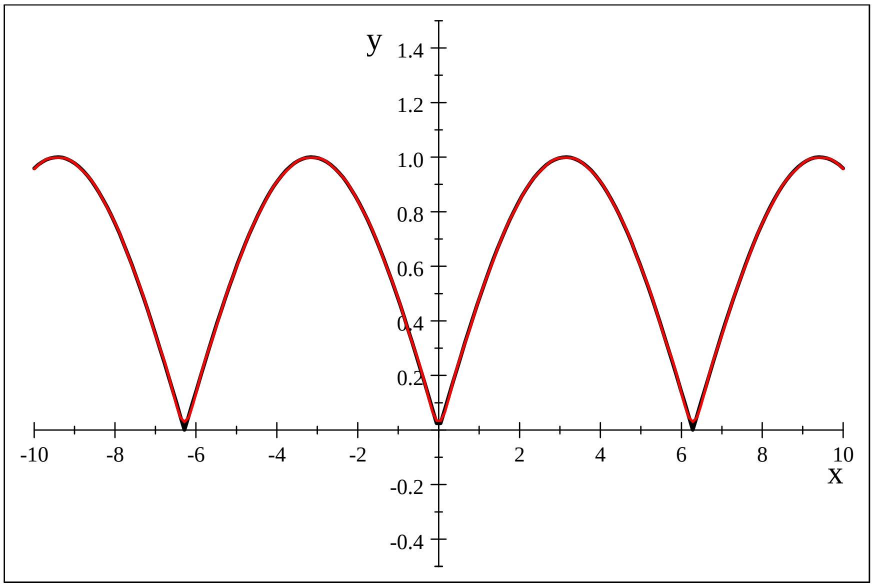


Figure 7: Graph of $\left|\sin \frac{x}{2}\right|$ and $S_{10}(x)$

Example 18 *We now look at a 2π-periodic function with discontinuities and derive its Fourier series using the formulas of this section (assuming it is legitimate). This function is called the sawtooth function. It is defined by*

$$g(x) = \begin{cases} \frac{1}{2}(\pi - x) & \text{if} \quad 0 < x \leq 2\pi \\ g(x + 2\pi) & \text{otherwise} \end{cases}$$

Find the Fourier series for this function. Plot this function as well as $S_1(x), S_7(x), S_{20}(x)$ where $S_N(x)$ is the N^{th} partial sum of its Fourier series.
Since f is described between 0 and 2π, we can use theorem 6 to compute the Fourier coefficients integrating between 0 and 2π.

1. ***Computation of** a_0.*

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \\ &= \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)\,dx \\ &= 0 \end{aligned}$$

2. ***Computation of*** a_n.

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx \textit{ for } n = 1, 2, ... \\ &= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\cos nx dx \\ &= \frac{1}{2\pi}\left[\pi\int_0^{2\pi}\cos nx dx - \int_0^{2\pi} x\cos nx dx\right] \end{aligned}$$

The first integral is 0*. The second can be evaluated by parts.*

$$\begin{aligned} \int_0^{2\pi} x\cos nx dx &= \frac{x}{n}\sin nx\Big|_0^{2\pi} - \frac{1}{n}\int_0^{2\pi}\sin nx dx \\ &= \frac{1}{n^2}\cos nx\Big|_0^{2\pi} \\ &= 0 \end{aligned}$$

so

$$a_n = 0$$

3. ***Computation of*** b_n.

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx \textit{ for } n = 1, 2, ... \\ &= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\sin nx dx \\ &= \frac{1}{2\pi}\left[\pi\int_0^{2\pi}\sin nx dx - \int_0^{2\pi} x\sin nx dx\right] \end{aligned}$$

The first integral is 0*. The second can be done by parts.*

$$\begin{aligned} \int_0^{2\pi} x\sin nx dx &= \frac{-x}{n}\cos nx\Big|_0^{2\pi} + \frac{1}{n}\int_0^{2\pi}\cos nx dx \\ &= \frac{-2\pi}{n} + \frac{1}{n^2}\sin nx\Big|_0^{2\pi} \\ &= \frac{-2\pi}{n} \end{aligned}$$

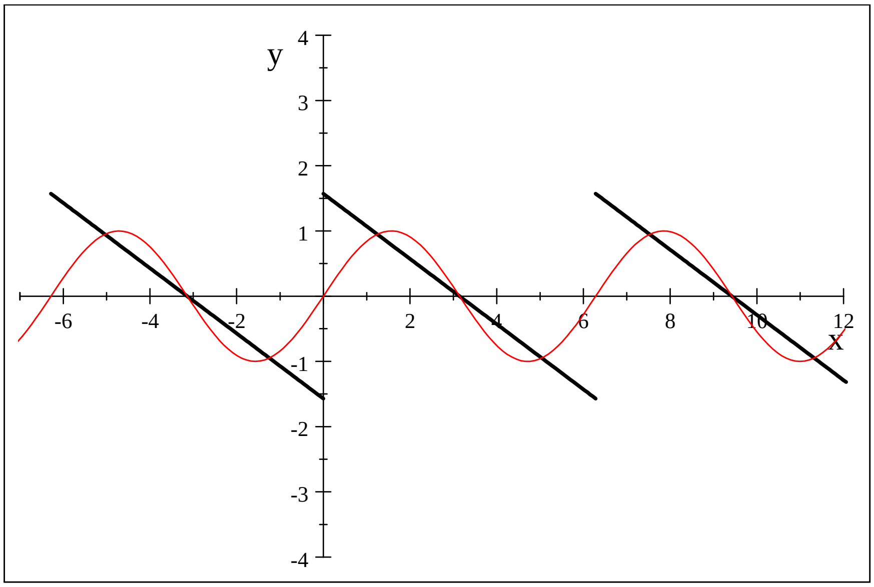
Therefore

$$\begin{aligned} b_n &= \frac{1}{2\pi}\left[0 - \frac{-2\pi}{n}\right] \\ &= \frac{1}{n} \end{aligned}$$

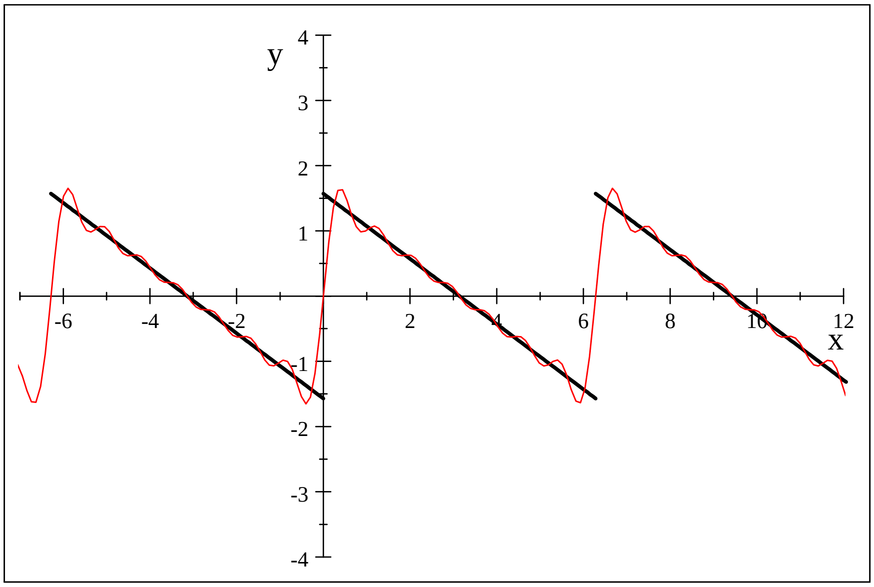
4. ***Conclusion***. *The Fourier series of the sawtooth function is*

$$g(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

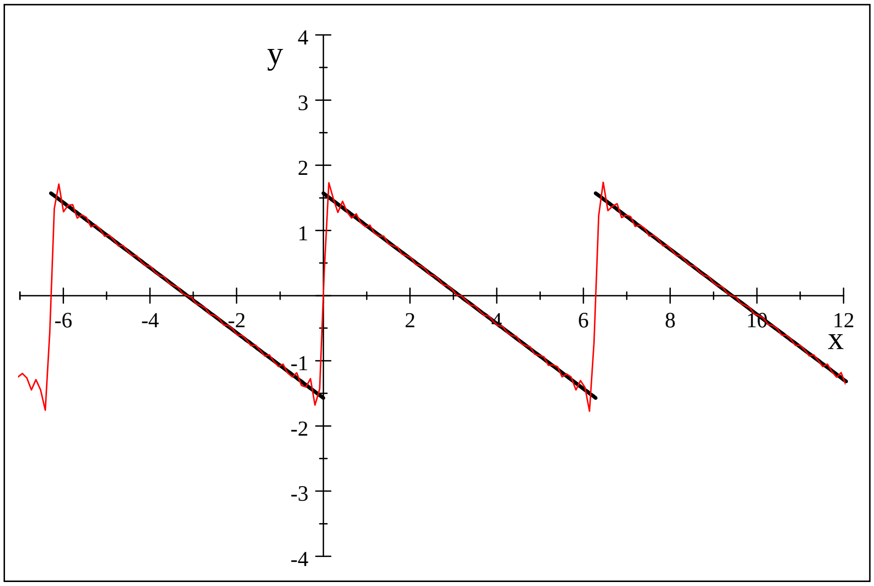
Below, we show the graphs of $S_1(x)$, $S_7(x)$, $S_{20}(x)$.



Graph of the sawtooth function (black) and $S_1(x)$ *(red)*



Graph of the sawtooth function (black) and $S_7(x)$ *(red)*



Graph of the sawtooth function (black) and $S_{20}(x)$ (red)

Remark 19 *Several important facts are worth noticing here.*

1. *The Fourier series seems to agree with the function, except at the points of discontinuity.*

2. *At the points of discontinuity, the series converges to* 0*, which is the average value of the function from the left and from the right.*

3. *Near the points of discontinuity, the Fourier series overshoots its limiting values. This is a well known phenomenon, known as* ***Gibbs phenomenon***. *To see a simulation of this phenomenon, visit the* **??**

3.2 Piecewise Continuous and Piecewise Smooth Functions

After defining some useful concepts, we give a sufficient condition for a function to have a Fourier series representation.

Notation 20 *We will denote* $f(c-) = \lim_{x\to c^-} f(x)$ *and* $f(c+) = \lim_{x\to c^+} f(x)$

Remembering that a function f is continuous at c if $\lim_{x\to c} f(x) = f(c)$, we see that a function f is continuous at c if and only if

$$f(c-) = f(c+) = f(c)$$

Definition 21 (Piecewise Continuous) *A function f is said to be piecewise continuous on the interval $[a, b]$ if the following are satisfied:*

1. *$f(a+)$ and $f(b-)$ exist.*
2. *f is defined and continuous on (a, b) except at a finite number of points in (a, b) where the left and right limit at these points exist.*

Definition 22 (Piecewise Smooth) *A function f, defined on $[a, b]$ is said to be piecewise smooth if f and f' are piecewise continuous on $[a, b]$ that is if the following are satisfied:*

1. *f is piecewise continuous on $[a, b]$*
2. *f' exists on (a, b) except possibly at finitely many points in (a, b) where theone sided limits of f' at these points exists.*
3. *$\lim_{x \to a^+} f'(x)$ and $\lim_{x \to b^-} f'(x)$ exists.*

The sawtooth function is piecewise smooth. A simple example of a function which is not piecewise smooth is $x^{\frac{1}{3}}$ for $-1 \leq x \leq 1$. Its derivative does not exist at 0, neither do the one sided limits of its derivative at 0.

Definition 23 *The average of f at c is defined to be*

$$\frac{f(c-) + f(c+)}{2}$$

Clearly, if f is continuous at c, then its average at c is $f(c)$.

We are now ready to state a fundamental result in the theory of Fourier series.

Theorem 24 *Suppose that $f$is a 2π-periodic piecewise smooth function. Then, for all x, we have*

$$\frac{f(x-) + f(x+)}{2} = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{11}$$

where the coefficients are given by equations 8, 9, and 10. In particular, if f is piecewise smooth and continuous at x, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{12}$$

Thus, at points where f is continuous, the Fourier series converges to the function. At points of discontinuity, the series converges to the average of the function at these points. This was the case in the example with the sawtooth function.

We do one more example.

Example 25 (Triangular Wave) *The 2π-periodic triangular wave is given on the interval $[-\pi, \pi]$ by*

$$h(x) = \begin{cases} \pi + x & \text{if } -\pi \le x \le 0 \\ \pi - x & \text{if } 0 \le x \le \pi \end{cases}$$

1. *Find its Fourier series.*
2. *Plot $h(x)$ as well as some partial sums of its Fourier series.*
3. *Show how this series could be used to approximate π $\left(\text{actually } \pi^2\right)$.*

Solution 26 1. *We begin by plotting $h(x)$ We see the function is piecewise*

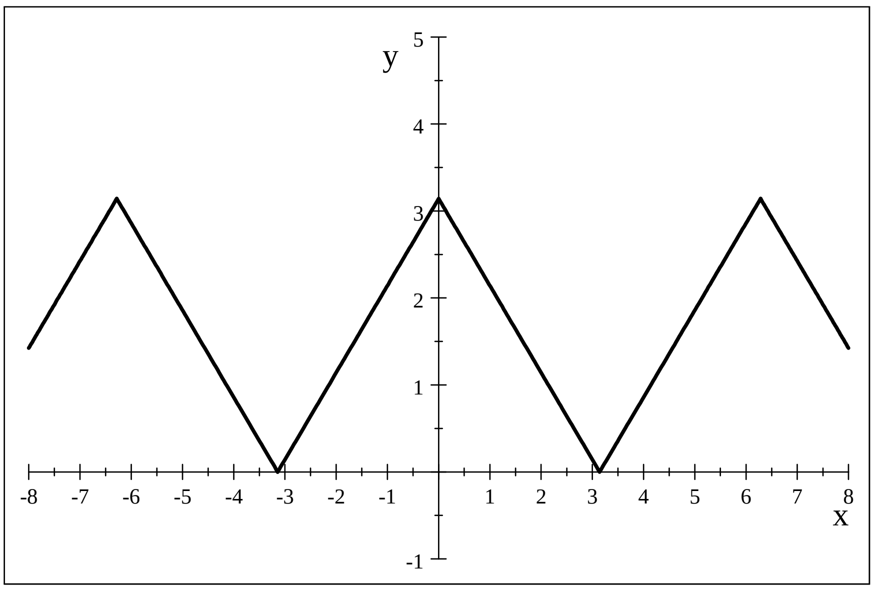


Figure 8: Plot of the triangular wave

smooth and continuous for all x.

- ***Computation of*** a_0.

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ &= \frac{1}{2\pi}\pi^2 \\ &= \frac{\pi}{2} \end{aligned}$$

- ***Computation of*** a_n.

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0} (\pi + x)\cos nx dx + \int_{0}^{\pi} (\pi - x)\cos nx dx\right] \\
&= \frac{2}{\pi}\int_{0}^{\pi} (\pi - x)\cos nx dx \ \textit{replacing } x \textit{ by } -x \textit{ in the first integral} \\
&= \frac{2}{\pi}\left[\frac{\pi - x}{n}\sin nx\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\sin nx dx\right] \\
&= \frac{2}{\pi}\left[\frac{-1}{n^2}\cos nx\Big|_0^{\pi}\right] \\
&= \frac{2}{\pi}\left[\frac{1}{n^2} + \frac{\cos n\pi}{n^2}\right] \\
&= \frac{2}{\pi}\left[\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right] \\
&= \begin{cases} 0 & \textit{if} \quad n \textit{ even} \\ \frac{4}{\pi n^2} & \textit{if} \quad n \textit{ odd} \end{cases}
\end{aligned}$$

- ***Computation of*** b_n.

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \\
&= 0 \ \textit{since the integrand is odd}
\end{aligned}$$

- ***Conclusion***.

$$h(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}$$

2. Let $S_N(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=0}^{N}\frac{\cos(2n+1)x}{(2n+1)^2}$. *We plot* $S_1(x)$, $S_5(x)$

3. From $h(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}$, *if we let* $x = 0$, *we get*

$$\begin{aligned}
\pi &= \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \\
\frac{\pi}{2} &= \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \\
\frac{\pi^2}{8} &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \\
&= 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots
\end{aligned}$$

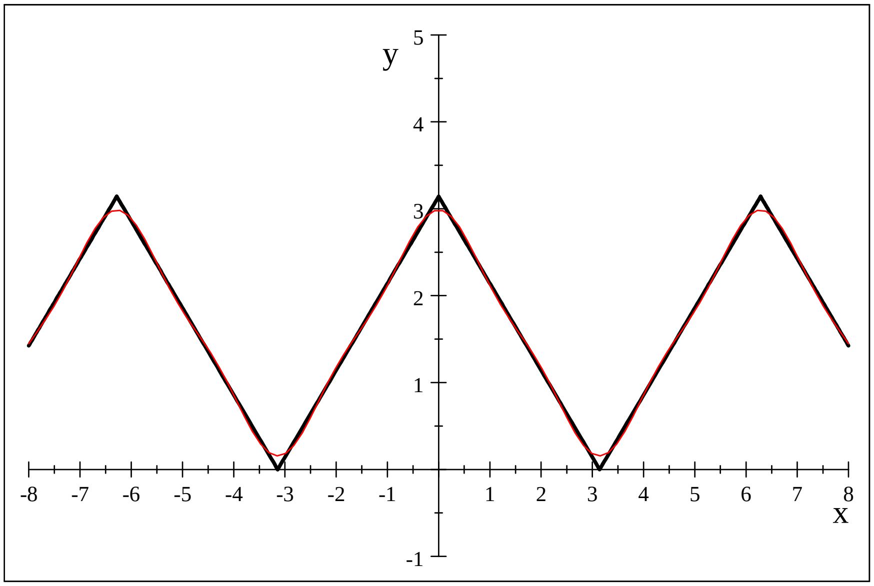


Figure 9: Plot of the triangular wave and $S_1(x)$

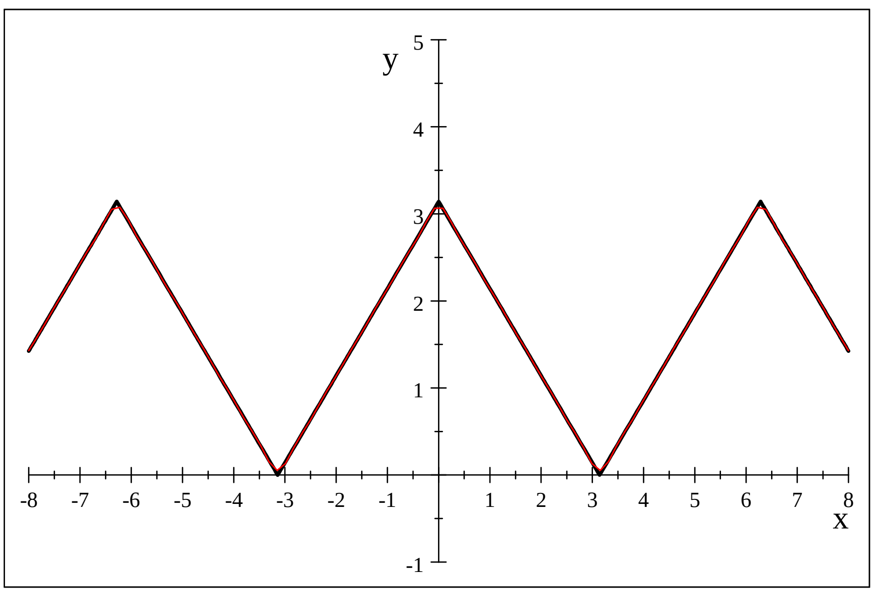


Figure 10: Plot of the triangular wave and $S_5(x)$

This allows us to approximate π^2.

3.3 Practice Problems

1. Show that another way to compute the Fourier coefficients is

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx \text{ for } n = 1, 2, \ldots$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx \text{ for } n = 1, 2, \ldots$$

2. In the problems below, you are given a 2π-periodic function and its Fourier series. For each function, (a) Derive the Fourier series, (b) sketch the graph of the function and some of the partial sums of its Fourier series on the interval $[-2\pi, 2\pi]$.

   (a) $f(x) = |x|$ if $-\pi \leq x < \pi$, Fourier series: $\frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)x}{(2n+1)^2}$

   (b) $f(x) = \begin{cases} \frac{x}{\pi} & \text{if } \quad 0 \leq x < \pi \\ 0 & \text{if } \quad -\pi < x \leq 0 \end{cases}$, Fourier series: $\frac{1}{4} - \frac{1}{\pi^2} \sum_{n=1}^{\infty} \left[\left(\frac{1}{n^2} - \frac{(-1)^n}{n^2} \right) \cos nx + \frac{\pi(-1)^n}{n} \sin nx \right.$

   (c) $f(x) = x^2$ if $-\pi \leq x \leq \pi$, Fourier series: $\frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$

   (d) $f(x) = x$ if $-\pi < x < \pi$, Fourier series: $2 \sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n} \sin nx$

3. Use the Fourier series of 2c) to obtain

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

4. Use the Fourier series of 2d) to obtain

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

4 Fourier Series of Functions with Arbitrary Periods

So far, we've worked with 2π-periodic functions for convenience. Similar results exist for functions having any period. These results can be obtained in a similar manner. However, there is an easier way, one calculus II students are familiar with: substitution. One way to obtain a new series representation is to perform a substitution in a known series. Suppose that f is a function with period $T = 2p > 0$ for which we want to find a Fourier series. In other words $f(x + 2p) = f(x)$. If we let $t = \dfrac{\pi x}{p}$ and we define

$$g(t) = f(x) = f\left(\frac{pt}{\pi}\right)$$

Then, g has period $\dfrac{2p}{\frac{p}{\pi}}$ that is 2π. So, g has a Fourier series representation

$$g(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos nt + b_n \sin nt\right]$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)\, dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \cos ntdt$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \sin tdt$$

Using the substitution $t = \dfrac{\pi x}{p}$, we obtain

$$g\left(\frac{\pi x}{p}\right) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p}\right]$$

that is

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p}\right]$$

Using the same substitution, we can express the Fourier coefficients in terms of x and f. We do it for a_0. If $t = \dfrac{\pi x}{p}$ then $dt = \dfrac{\pi}{p} dx$ so

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)\, dt \\ &= \frac{1}{2\pi} \frac{\pi}{p} \int_{-p}^{p} g\left(\frac{\pi x}{p}\right) dx \\ &= \frac{1}{2p} \int_{-p}^{p} f(x)\, dx \end{aligned}$$

So, we have the following:

Theorem 27 *Suppose that f is a $2p$-periodic piecewise smooth function. The Fourier series of ff is given by*

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} \right] \tag{13}$$

where

$$a_0 = \frac{1}{2p} \int_{-p}^{p} f(x)\, dx \tag{14}$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi x}{p} dx \tag{15}$$

and

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi x}{p} dx \tag{16}$$

The Fourier series converges to $f(x)$ if f is continuous at x and to $\dfrac{f(x-) + f(x+)}{2}$ otherwise.

We finish this section by noticing that in the special cases that f is either even or odd, the series simplifies greatly. If f is even, then $\int_{-p}^{p} f(x) \sin \dfrac{n\pi x}{p}$ is odd so that $b_n = 0$ and the series is simply a cosine series. Similarly, if f is odd, then $\int_{-p}^{p} f(x) \cos \dfrac{n\pi x}{p}$ is odd and $a_n = 0$ and the series is simply a sine series. We summarize this in a theorem.

Theorem 28 *Suppose that f is $2p$-periodic and has the Fourier series representation*

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} \right]$$

Then:

1. *f is even if and only if $b_n = 0$ for all n and in this case*

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{p}$$

2. *f is odd if and only if $a_n = 0$ for all n and in this case*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{p}$$

5 Some Applications

In the examples, we saw how we could use Fourier series to approximate π. One of the main uses of Fourier series is in solving some of the differential equations from mathematical physics such as the wave equation or the heat equation. Fourier developed his theory by working on the heat equation. Fourier series also have applications in music synthesis and image processing. All these will be presented in another talk. We will mention the relationship between sound (music) and Fourier series.

When we represent a signal $f(t)$ by its Fourier series $f(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi t}{p} + b_n \sin \frac{n\pi t}{p} \right]$, we are finding the contribution of each frequency $\frac{n\pi}{p}$ to the signal. The value of the corresponding coefficients give us that contribution. The n^{th} term of the partial sum of the Fourier series, $a_n \cos \frac{n\pi t}{p} + b_n \sin \frac{n\pi t}{p}$, is called the n^{th} **harmonic** of f. Its **amplitude** is given by $\sqrt{a_n^2 + b_n^2}$. Conversely, we can create a signal by using the Fourier series $a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi t}{p} + b_n \sin \frac{n\pi t}{p} \right]$ for a given value of p and playing with the value of the coefficients.

Audio signals describe air pressure variations captured by our ears and perceived as sounds. We will focus here on periodic audio signals also known as **tones**. Such signals can be represented by Fourier series.

A **pure tone** can be written as $x(t) = a\cos(\omega t + \phi)$ where $a > 0$ is the amplitude , $\omega > 0$ is the frequency in radians/seconds and ϕ is the phase angle. An alternative way to represent the frequency is in Hertz. The frequency f in Hertz is given by $f = \frac{\omega}{2\pi}$.

The **pitch** of a pure tone is logarithmically related to the frequency. An **octave** is a frequency range between f and $2f$ for a given frequency f in Hertz. Tones separated by an octave are perceived by our ears to be very similar. In western music, an octave is divided into 12 **notes** equally spaced on the logarithmic scale. The ordering of notes in the octave beginning at the frequency 220 Hz are shown below

Note	A	A#	B	C	C#	D	D#	E	F	F#	G	G#	A
Frequency (Hz)	220	233	247	262	277	294	311	330	349	370	392	414	440

A more complicated tone can be represented by a Fourier series of the form

$$x(t) = a_1 \cos(\omega t + \phi_1) + a_2 \cos(\omega t + \phi_2) + ...$$

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