TRIPLE INTEGRALS

Topics Covered

- Theory
- Meaning
- Rules for Solving
- Examples!

Theory

As the name implies, triple integrals are 3 successive integrations, used to calculate a volume, or to integrate in a 4th dimension, over 3 other independent dimensions. Examples:

$$\iiint_{0}^{V} f(x,y,z)dV$$

$$\int_{1}^{5}\int_{0}^{3x-2}\int_{\sin(x+y)}^{\cos(x+y)}(2x+e^{yz})\,dz\,dy\,dx$$

$$\int_{0}^{2\pi}\int_{0}^{2}\int_{-3}^{3} r\,dz\,dr\,d\theta$$

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- To understand triple integrals, it is extremely important to have an understanding of double integrals, coordinate geometry in 3 dimensions, and polar (cylindrical) coordinates. Sums of triple integrals are based on these topics and cannot be solved without this prior knowledge.

Meaning

- Just as a single integral over a curve represents an area (2D), and a double integral over a curve represents a volume (3D), a triple integral represents a summation in a hypothetical 4^{th} dimension. To understand this, imagine a slightly different scenario, where the first 3 dimensions are space, space, and time, and that the fourth dimension is the third space dimension: suppose you have an integral of z where z is a function of x, y, and t, and x and y are also functions of t. Integrating z with respect to x and y gives us the volume as a function of t. If we plug in a value for t, this gives us the numerical volume at a particular instance of time.

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Meaning (Contd.)

- However, as the volume is a function of t, it changes with respect to time. Thus, if we integrate this result with respect to t, we are adding up all the different volumes over a period of time. For example, imagine a balloon that is being inflated. At any particular instant of time, we can use a double integral to calculate its volume. However, if we wanted to sum up all the volumes over the entire inflation process, we must use a triple integral as described above.

Meaning (Contd.)

- Triple Integrals can also be used to represent a volume, in the same way that a double integral can be used to represent an area. In the triple integral

$$\int_c^d \int_a^b \int_0^z f(x, y, z)\, dz\, dy\, dx$$

If $f(x, y, z) = 1$ then this triple integral is the same as

$$\int_c^d \int_a^b z(x, y)\, dydx$$

which is simply the volume under the surface represented by *z(x,y)*.

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Rules for Solving

In general, if you are evaluating a triple integral of *f(x,y,z)* over a volume V, by properly choosing the limits, you can integrate with respect to the 3 variables in any order. For example, if V represents a rectangular box, then for x, y, and z, the limits of integration will all be constants and the order does not matter at all.

$$\int_e^f \int_c^d \int_a^b f \, dz \, dy \, dx = \int_c^d \int_a^b \int_e^f f \, dx \, dz \, dy = \ldots$$

and so on, for 6 different possible orders of integration.

Now consider the case where you want to find the triple integral

$$\iiint_{0}^{V} f(x,y,z)dV$$

where V is the volume under the plane

$z + 2x + 3y = 6$, in the first quadrant.

Since V is in the first quadrant, it is bounded by the 3 planes $x = 0, y = 0, z = 0$ in addition to the plane whose equation is given above.

Clearly, the shape of V is a tetrahedron with the vertices (3,0,0), (0,2,0), (0,0,6), and (0,0,0).

One way to set up the integral is to first consider the z direction. In this direction we enter through $z = 0$ and leave through $z = 6 - 2x - 3y$. If we do this first, then we can consider the x or y direction next. If y, we enter through $y = 0$, and leave through $y = -\frac{2}{3}x + 2$. Thus, x must be given the limits 0 to 3, and our triple integral is:

$$\int_0^3 \int_0^{-\frac{2}{3}x+2} \int_0^{6-2x-3y} f(x, y, z)\, dz\, dy\, dx$$

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Consider the same volume, but now first we will go through in the x direction. In this direction we enter through $\mathrm{x} = 0$ and leave through $x = \frac{1}{2}(6 - z - 3y)$. If we do this first, then we can consider the z or y direction next. If y, we enter through $y = 0$, and leave through $y = -\frac{1}{3}z + 2$, and z must be from 0 to 6. Thus the integral would be

$$\int_0^6 \int_0^{-\frac{1}{3}z+2} \int_0^{\frac{1}{2}(6-z-3y)} f(x, y, z)\, dx\, dy\, dz$$

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Both these integrals are correct and either one can be used to solve the sum and obtain the same value. It is important to see that choosing the order of integration depends on how the limits are chosen. However, this is entirely independent of the function $f(x, y, z)$. In the first case, z must be integrated first, since the limits were chosen such that z is dependent on (x,y). After this, since y is chosen to be dependent on x, y is integrated next. Lastly, x as the most independent variable is integrated. It is important to note that the third (outermost) integral MUST always have limits of integration that are constants, and the result should be a number. If any variables are surviving after the integral is evaluated, most probably you have made a mistake somewhere.

POLAR (CYLINDRICAL) COORDINATES:

Triple integrals can also be used with polar coordinates in the exact same way to calculate a volume, or to integrate over a volume. For example:

$$\int_{0}^{2\pi}\int_{0}^{2}\int_{-3}^{3} r\,dz\,dr\,d\theta$$

is the triple integral used to calculate the volume of a cylinder of height 6 and radius 2. With polar coordinates, usually the easiest order of integration is z, then r then θ as shown above, though it is not necessary to do it in this order.

NOTE: It is very important to remember that in polar and cylindrical coordinates, there is an extra r in the integral, just like in double integrals. It is very easy to forget this r which will lead to a wrong answer.

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Examples!

1. Integrate the function $f(x,y,z) = xy$ over the volume enclose by the planes $z = x + y$ and $z = 0$, and between the surfaces $y = x^2$ and $x = y^2$.

SOLUTION:

Since z is expressed as a function of *(x,y),* we should integrate in the z direction first. After this we consider the xy-plane. The two curves meet at (0,0) and (1,1). We can integrate in x or y first. If we choose y, we can see that the region begins at $y = x^2$ and ends at $y = \sqrt{x}$, and so x is between 0 and 1. Thus the integral is:

$$\int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} f(x,y,z)\, dz\, dy\, dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} xy \, dz \, dy \, dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} xy\left({}_{z=0}^{z=x+y}[z] \right) dy \, dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} xy(x+y) dy \, dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2y + y^2x) dy \, dx$$

$$= \int_0^1 {}_{y=x^2}^{y=\sqrt{x}}\left(\frac{x^2y^2}{2} + \frac{y^3x}{3} \right) dx$$

$$= \frac{1}{2}\int_0^1 \left(x^2\sqrt{x}^2 - x^2(x^2)^2 \right) dx + \frac{1}{3}\int_0^1 \left(\sqrt{x}^3 x - (x^2)^3 x \right) dx$$

$$= \frac{1}{2}\int_0^1 (x^3 - x^6) dx + \frac{1}{3}\int_0^1 \left(x^{\frac{5}{2}} - x^7 \right) dx$$

$$= \frac{1}{2}\left(\frac{1}{4} - \frac{1}{7} \right) + \frac{1}{3}\left(\frac{2}{7} - \frac{1}{8} \right) = \boxed{\frac{3}{28}}$$

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2. In the following integral, exchange the order of integration of y and z:

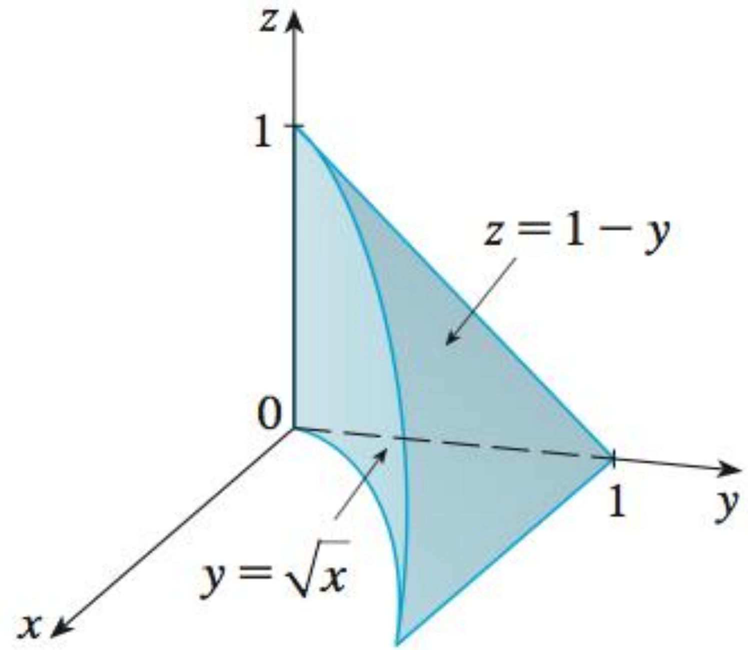
$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx$$

SOLUTION:

With a problem like this, it helps to draw the figure enclose by the surfaces. From the integrals, it can be seen that *z* enters the volume at $z = 0$ and leaves through the plane $z = 1 - y$. In the xy plane, y enters through $y = \sqrt{x}$ and leaves through $y = 1$. These 2 conditions alone are enough to draw the figure and see that $0 \leq x \leq 1$.

The shape is like a tetrahedron with the vertices A(0,0,0), B(0,0,1), C(1,1,0), and D(0,1,0), but with a curved surface between vertices A, B, C.

Rewriting the equation of the plane as $y = 1 - z$, and looking at this figure, clearly, $\sqrt{x} \leq y \leq 1 - z$. Consider the projection of this surface in the zx plane. It is a 3-sided figure bounded by $x = 0, z = 0,$ and $z = 1 - \sqrt{x}$. Again, it can be seen that these conditions are sufficient to obtain the limits of x as $0 \leq x \leq 1$. Thus, the integral is:



$$\int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x, y, z)\, dy\, dz\, dx$$

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3. Evaluate $\iiint_E x dV$ where E is enclosed by $z = 0, z = x + y + 5, x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

SOLUTION: We will use cylindrical coordinates to easily solve this sum. Converting the given outer limits of E we get: $z = 0, z = r\cos\theta + r\sin\theta + 5, r = 2$ and $r = 3$. Since there is no limitation on the values of θ, we assume it has the values of 0 to 2π. This is also in accordance with the diagram which is obtained by drawing these surfaces. Also, we substitute $x = r\cos\theta$ in the argument of the integral.

Thus the integral is:

$$\int_0^{2\pi}\int_2^3\int_0^{r\cos\theta+r\sin\theta+5}(r\cos\theta)r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_2^3(r^2\cos\theta)(r\cos\theta+r\sin\theta+5)\,dr\,d\theta$$

$$=\int_0^{2\pi}\left[\frac{r^4}{4}\cos^2\theta+\frac{r^4}{4}\cos\theta\sin\theta+\frac{5}{3}r^3\cos\theta\right]_{r=2}^{r=3}d\theta$$

$$=\int_0^{2\pi}\left[\frac{65}{8}(1+\cos 2\theta)+\frac{65}{8}\sin(2\theta)+\frac{95}{3}\cos\theta\right]d\theta$$

$$=\frac{65}{8}\left(2\pi+0-\frac{1}{2}+0-0+\frac{1}{2}\right)+\frac{95}{3}(0)$$

$$=\boxed{\frac{65\pi}{4}}$$

References

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