Lecture Notes
on
Multivariate Calculus

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April 25, 2003

§0. Preface

The objective of these notes is to present the basic aspects of multivariate calculus. As in the univariate setting, there are many computational rules here. These rules are indeed important, but computer software packages allow many of these calculations to be done by machine. The emphasis in these notes is on the concepts behind the computational formulas. It is these concepts that give multivariate calculus its power and importance.

The specific objectives are the following.

(1) Develop a solid understanding of functions and the geometric and algebraic meaning of the graph of a function. Develop the ability to translate geometric properties of the graph of a function into equations, and vice-versa.

(2) Develop understanding, not just algorithms.

(3) Develop an intuitive and formal understanding of derivatives.

(4) Devlop an intuitive and formal understanding of integrals.

(5) Develop an understanding of linearity and its use in the calculus.

(6) Develop the the ability to translate a verbal description of a problem into a mathematical description, and vice-versa.

Throughout these notes are various exercises and problems. The reader should attempt to work all of these. Solutions, sometimes in the form of hints, are provided for most of the problems.

These notes begin with a brief review of univariate calculus.

§1. A Brief Review

There are two central ideas in the theory of calculus of functions of a single variable.

The first core idea is that every function can be closely approximated by a linear function over small pieces of the domain of the function. Stated more formally, for any function f and any point a on the x axis the graph of f near $(a, f(a))$ is approximately a straight line. This idea led to the development of the notion of derivative and is formally captured in the definition $f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$. Intuitively, the definition means that for small values of h, $f(a + h) = f(a) + f'(a)h$, approximately.

The second core idea arises by considering the slopes of *all* of the approximating lines of a *single* given function f. Graphically the slopes of all of these lines together with a single point on the graph of the function f can be used to completely reconstruct f. This idea is formally captured in the Fundamental Theorem of Calculus: $f(b) = f(a) + \int_a^b f'(t)\,dt$. The importance of the Fundamental Theorem stems from the fact that in many applications geometric or physical reasoning lead to a formula for the derivative f' of the function of interest. The function f can then be constructed from the knowledge of f' by using the Fundamental Theorem.

The objective is to extend these two ideas to the context of functions of several variables. In this discussion there are 5 key ingredients: points, functions, integrals, lines, and derivatives. The development begins with a discussion of sets, which are the mathematical objects used to represent collections of points.

Problems

Problem 1–1. Suppose $f(x) = 2x^2 - 5$. Compute $f'(3)$ using the familiar formulas of calculus. True or False: $f'(3) = \lim_{h\to 0} \dfrac{2(3+h)^2 - 18}{h}$.

Problem 1–2. Compute $\lim_{h\to 0} \dfrac{(5+h)^2 - 25}{h}$.

Problem 1–3. True or False: $f'(b) = \lim_{b\to a} \dfrac{f(b) - f(a)}{b - a}$.

Problem 1–4. If $f(t) = e^{\sin t}$, compute $\int_0^{\pi} f'(t)\,dt$.

Problem 1–5. Suppose $A(r)$ is the area of a circle of radius r and $C(r)$ is the circumference of the circle of radius r. Argue geometrically that $A'(r) = C(r)$.

Problem 1–6. Suppose $V(r)$ is the volume of a sphere of radius r and $A(r)$ is the surface area of a sphere of radius r. What is the relationship between $V(r)$ and $A(r)$?

Solutions to Problems

Problem 1–1. Using the rules, $f'(x) = 4x$ so $f'(3) = 12$. From the definition $f'(3) = \lim_{h \to 0} \dfrac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \dfrac{2(3+h)^2 - 18}{h}$, so the second assertion is true.

Problem 1–2. Defining $f(x) = x^2$, this limit is nothing more than $f'(5) = 10$.

Problem 1–3. False. The limit should be as $a \to b$.

Problem 1–4. Using the Fundamental Theorem of Calculus, $\int_0^\pi f'(t)\,dt = f(\pi) - f(0) = e^{\sin \pi} - e^{\sin 0} = 1 - 1 = 0$.

Problem 1–5. For small values of h, $A(r+h) - A(r)$ is the area of a thin ring shaped region whose area is approximately $hC(r)$.

Problem 1–6. As in the preceding problem, for small values of h, $V(r+h) - V(r)$ is the volume of a thin shell whose volume is approximately $A(r)h$. Thus $V'(r) = A(r)$.

§2. Sets

Multivariate calculus has a strong geometric flavor. Geometric space is nothing more than a collection of points, and geometric objects in space consist of some sub-collection of points in space. The mathematical language of sets is used to provide an accurate description of geometric objects.

A **set** is simply a collection of objects. One might speak of the set of students in this classroom, the set of bicycles on campus, and so on. An individual object in a set is called an **element** of the set.

In mathematics, the sets of interest often consist of numbers. One of most often used sets is the set of real numbers. The collection of all numbers which can be written in decimal form (repeating or not) is the set of **real numbers**, and is denoted by **R**.

Giving sets a visual representation is often very useful. The visual representation of the set of real numbers **R** is as a straight, infinite, line. The individual numbers (elements) are located along this line.

Often, additional requirements are made which narrows the set of possible values to a piece, that is, a **subset**, of the original set. The subset is specified notationally by giving the condition required to be an element of the subset.

Example 2–1. The set of real numbers which are at least 3 is written notationally as $\{x \in \mathbf{R} : x \geq 3\}$. The notation is read as "the set of x in the real numbers such that x is greater than or equal to 3." In this notation the colon is read as "such that" or "with the property that." The notation $x \in \mathbf{R}$, which is read as "x is an element of **R**," means that the number x is an element of the set of real numbers. The inequality following the colon gives the additional property required to be a member of this particular subset. This same set could be written $\{x : x \in \mathbf{R} \text{ and } x \geq 3\}$.

Exercise 2–1. Give this set a visual interpretation by graphing it on a number line.

Exercise 2–2. Translate into words: $\{x : x \in \mathbf{R} \text{ and } x \leq 1/2\}$.

Exercise 2–3. Graph the set in the previous exercise.

Exercise 2–4. Is $5 \in \left\{x \in \mathbf{R} : x^2 > x\right\}$?

As the example suggests, subsets of the real line are often connected with algebraic problems involving a single variable. In many cases the relationship of interest will be between two or more variables. A higher dimensional set is used as the backdrop for visualizing such relationships.

For a situation involving two variables the subsets will be visualized in a two dimensional plane. Denote by $\mathbf{R}^2$ the set $\left\{\begin{pmatrix} x \\ y \end{pmatrix} : x \in \mathbf{R} \text{ and } y \in \mathbf{R}\right\}$. This is the set of ordered pairs $\begin{pmatrix} x \\ y \end{pmatrix}$ in which each member of the pair is a real number. The two numbers are called the **coordinates** of the point. Visually, $\mathbf{R}^2$ is a two dimensional plane.

Example 2–2. The set $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2 : x = 3 \text{ and } y = 5\right\}$ can be visualized easily. This set consists of a single point. As a notational convenience, this point is written as $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

Example 2–3. A basic way of visualizing a more complicated set, such as $A = \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2 : y = 2x\right\}$ is to first rewrite this set as $\left\{\begin{pmatrix} x \\ 2x \end{pmatrix} : x \in \mathbf{R}\right\}$ and then plot several individual points in the set, hoping to see a pattern. (This method is tedious for humans, but easy for computers.)

Exercise 2–5. What familiar geometric object is the set in the previous example?

Exercise 2–6. Is the set $B = \left\{\begin{pmatrix} 2x \\ 4x \end{pmatrix} : x \in \mathbf{R}\right\}$ the same as the set A?

The previous exercise illustrates that the same set can have many different descriptions. Showing that two descriptions are describing the same set is accomplished by taking an arbitrary element meeting the first description and showing that this element also meets the second description, and vice-versa.

Example 2–4. Are the sets $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2 : x - y = 5\right\}$ and $\left\{\begin{pmatrix} t \\ t-5 \end{pmatrix} : t \in \mathbf{R}\right\}$ the same? Suppose $\begin{pmatrix} x \\ y \end{pmatrix}$ is in the first set. Using the condition gives $y = x - 5$, so in fact $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x-5 \end{pmatrix}$, and since x is a real number, this point meets the requirement to be an element of the second set. On the other hand, suppose $\begin{pmatrix} t \\ t-5 \end{pmatrix}$ is in the second set. Since $t - (t - 5) = 5$, this point meets the requirement to be in the first set. Thus the two descriptions are describing the same set.

Exercise 2–7. Is the set $\left\{\begin{pmatrix} s+5 \\ s \end{pmatrix} : s \in \mathbf{R}\right\}$ the same as the set in the example?

Sometimes a set has no elements. The set with no elements is called the **empty set** and is denoted by $\varnothing$.

Example 2–5. The set $\left\{x \in \mathbf{R} : x^2 = -5\right\}$ has no elements, so $\left\{x \in \mathbf{R} : x^2 = -5\right\} = \varnothing$.

Spaces of dimension larger than 2 are often useful as well. Denote by $\mathbf{R}^d$ the set $\left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} : x_i \in \mathbf{R} \text{ for } 1 \le i \le d \right\}$. This is the set of ordered d-tuples of real numbers. Most of the work here will involve the spaces $\mathbf{R}^2$ and $\mathbf{R}^3$. The results and ideas can be easily carried over into spaces of higher dimension.

Problems

Problem 2–1. Find the set $\left\{x \in \mathbf{R} : \sqrt{x^2} = x\right\}$ and graph it.

Problem 2–2. Find the set $\{x \in \mathbf{R} : 2x + 3 = 5\}$ and graph it.

Problem 2–3. Find the set $\left\{x \in \mathbf{R} : (x + 2)^2 = x^2 + 4x + 4\right\}$ and graph it.

Problem 2–4. Write in set notation: the set of real numbers between 4 and 7, exclusive. What geometric object is this set?

Problem 2–5. Write in set notation: the set of points in the plane for which the second coordinate is 2 more than the first coordinate. What geometric object is this set?

Problem 2–6. True or False: $7 \in \left\{x \in \mathbf{R} : x^2 - 2x + 5 > 3\right\}$.

Problem 2–7. True or False: The point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is an element of the set

$$\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2 : xy - 5x = 7\right\}.$$

Problem 2–8. Are the sets

$$\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2 : 2x + 3y = 7 \text{ and } 4x + 6y = 14\right\}$$

and

$$\left\{\begin{pmatrix} s \\ t \end{pmatrix} \in \mathbf{R}^2 : 6s + 9t = 21\right\}$$

the same? Geometrically, what are these sets?

Problem 2–9. Consider the region $S = \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2 : 0 \leq x \leq 2, 0 \leq y \leq x + 4\right\}$. Sketch the region S, and label each corner of the region with the coordinates of the corner point. What is the area of the region S?

Problem 2–10. Graph the set $\left\{\begin{pmatrix} x \\ y \end{pmatrix} : 0 \leq x \leq 2, 0 \leq y \leq x^2\right\}$.

Solutions to Problems

Problem 2–1. $\left\{x \in \mathbf{R} : \sqrt{x^2} = x\right\} = \{x \in \mathbf{R} : x \geq 0\}$. The graph is the half line beginning at the 0 and extending to the right.

Problem 2–2. $\{x \in \mathbf{R} : 2x + 3 = 5\} = \{x \in \mathbf{R} : x = 1\}$ which graphs as a single point.

Problem 2–3. $\left\{x \in \mathbf{R} : (x+2)^2 = x^2 + 4x + 4\right\} = \mathbf{R}$. The equation $(x+2)^2 = x^2 + 4x + 4$ is an example of an **identity**, since equality holds for all values of x for which both sides are defined.

Problem 2–4. $\{x \in \mathbf{R} : 4 < x < 7\}$. This set is a line segment, without its endpoints.

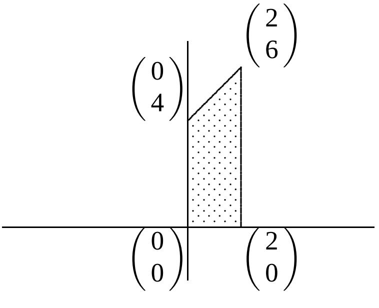
Problem 2–5. $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2 : y = x + 2\right\}$ or $\left\{\begin{pmatrix} x \\ x+2 \end{pmatrix} : x \in \mathbf{R}\right\}$. This set is a line.

Problem 2–6. Here 7 is a real number and $7^2 - 2 \times 7 + 5 = 49 - 14 + 5 = 40 > 3$, so the answer is true.

Problem 2–7. Since $2 \times 3 - 5 \times 2 = -4 \neq 7$ the answer is false.

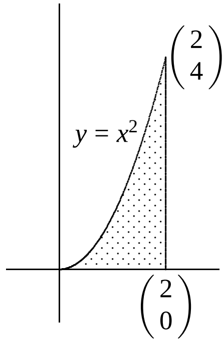
Problem 2–8. Yes. If $\begin{pmatrix} x \\ y \end{pmatrix}$ is in the first set, then $2x + 3y = 7$ and by multiplication the second requirement is also met. Multiplying by 3 shows that $6x + 9y = 21$, so the point $\begin{pmatrix} x \\ y \end{pmatrix}$ meets the requirement to be in the second set. If $\begin{pmatrix} s \\ t \end{pmatrix}$ is in the second set, then $6s + 9t = 21$ and division by 3 shows that $2s + 3t = 7$ while multiplication by $2/3$ shows that $4s + 6t = 14$ also holds. So the requirements for $\begin{pmatrix} s \\ t \end{pmatrix}$ to be in the first set are satisfied. Thus the two sets are the same. Geometrically, the sets are a line in $\mathbf{R}^2$.

Problem 2–9.



The area is the area of a rectangle plus the area of a triangle, and is $4 \times 2 + (1/2)2 \times 2 = 10$.

Problem 2–10. This set is the region in the plane bounded above by the parabola $y = x^2$, the x-axis, and the line $x = 2$.



Solutions to Exercises

Exercise 2–2. The set of real numbers that are less than or equal to 1/2.

Exercise 2–3. The graph is an infinite ray which extends from the point 1/2 to the left.

Exercise 2–4. The number 5 is a real number and $5^2 = 25 > 5$, so the answer is yes.

Exercise 2–5. A line through the origin.

Exercise 2–6. Yes, each point in B has a second coordinate which is twice its first coordinate.

Exercise 2–7. Yes. If $\begin{pmatrix} s+5 \\ s \end{pmatrix}$ is in this new set, then since $s + 5 - s = 5$, this point meets the requirement to be in the first set of the example. On the other hand, if $\begin{pmatrix} x \\ y \end{pmatrix}$ is in the first set of the example, then $x = y + 5$ so that $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y+5 \\ y \end{pmatrix}$, and since y is a real number this point meets the requirement to be in the new set. Thus the new set and the first set of the example are the same. Can you give the argument to show that the new set and the second set of the example are the same?

§3. Functions

Multivariate calculus involves the study of functions. A function consists of three parts.

(1) A set called the **domain** of the function.

(2) A set called the **range** of the function.

(3) A rule which assigns to each element of the domain one and only one element of the range.

Often a function is specified just by giving the rule. In such cases the domain is then understood to be the largest set on which the rule makes sense, and the range is the set of output values that results by applying the rule to all of the elements in the domain.

Example 3–1. The rule $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x + y$ defines a function with domain $\mathbf{R}^2$ and range space $\mathbf{R}^1$. (The symbol $\mapsto$ is read 'maps to'.)

Usually functions are given a symbolic name which is attached to the rule.

Example 3–2. A function f can be defined by the formula $f\begin{pmatrix} x \\ y \end{pmatrix} = x + y$.

Exercise 3–1. What is the domain and range of the function $g\begin{pmatrix} x \\ y \end{pmatrix} = \sqrt{xy}$?

Example 3–3. What is the domain and range of the function $h\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x - y \end{pmatrix}$? The formula defining h makes sense for any pair of input values, so the domain is $\mathbf{R}^2$. Any pair of output values are also possible, so the range is $\mathbf{R}^2$ as well.

Exercise 3–2. What pair of input values x and y would produce an output value of $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ for the function h?

Exercise 3–3. What is the domain and range of the function $k\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ x - y \end{pmatrix}$?

The **graph** of a function is the set of all possible pairs of elements for which the first element of the pair lies in the domain of the function and the second element of the pair is the corresponding output value. The graph must therefore lie in a space whose dimension is the sum of the dimensions of the domain and the range of the function.

Example 3–4. The graph of the function of the previous example is

$$\left\{ \begin{pmatrix} x \\ y \\ f\begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2 \right\}.$$

Notice that the graph is a subset of 3 dimensional space.

Example 3–5. In what space does the graph of the function $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ x-y \end{pmatrix}$ lie?

In most cases here, the functions of interest will have $\mathbf{R}^2$ or $\mathbf{R}^3$ as their domain, and either $\mathbf{R}$, $\mathbf{R}^2$, or $\mathbf{R}^3$ as their range. Notation such as $f : \mathbf{R}^2 \to \mathbf{R}^3$ is used to denote a function whose domain is a subset of $\mathbf{R}^2$ and whose range is a subset of $\mathbf{R}^3$. Functions whose range is a higher dimensional space have a nice structure which allows some simplification of their study.

Example 3–6. Consider the function $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ xy \end{pmatrix}$. The domain of g is $\mathbf{R}^2$. The **component functions** of g are the two function with domain $\mathbf{R}^2$ and range $\mathbf{R}$ defined by the formulas $g_1\begin{pmatrix} x \\ y \end{pmatrix} = x + y$ and $g_2\begin{pmatrix} x \\ y \end{pmatrix} = xy$. Most of the properties of g of interest here can be obtained by studying the component functions of g.

Because of the availability of component functions, the preliminary parts of the discussion here can consider functions whose range is $\mathbf{R}$.

Sometimes the output of one function can be used as input to another. If the range of the function g is contained in the domain of the function f, the **composition** of f with g, denoted $f \circ g$, is the function defined by the formula $(f \circ g)(v) = f(g(v))$. Notice that the function which is applied first is the one farthest to the right in the notation.

Example 3–7. Suppose f has domain and range $\mathbf{R}$ and is given by $f(x) = 2x - 3$ while g has domain $\mathbf{R}$ and is given by $g(x) = x^2$. Then $(f \circ g)(x) = 2x^2 - 3$. Notice that $(g \circ f)(x) = (2x - 3)^2$.

Occasionally, for a given function f there will be another function which 'undoes what f does.' This function is called the **inverse function** of f and is denoted by f^{-1}. The requirements for the inverse function are that

$f^{-1}(f(x)) = x$ for all x in the domain of f, and
$f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} (which is the range of f).

Example 3–8. If $f(x) = 2x - 3$ then $f^{-1}(x) = (x + 3)/2$. Simple substitution shows that the two requirements are met.

Problems

Problem 3–1. Write the graph of the function $f(x) = x^2$ in set notation.

Problem 3–2. True or False: If $f : \mathbf{R}^3 \to \mathbf{R}^2$ the graph of f is a subset of $\mathbf{R}^4$.

Problem 3–3. Consider the function $g(t) = \begin{pmatrix} t+5 \\ t^2 - 7t \end{pmatrix}$. What is the domain of g? What are the component functions $g_1(t)$ and $g_2(t)$?

Problem 3–4. True or False: If $f : \mathbf{R}^2 \to \mathbf{R}^3$ it is possible that $f\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$.

Problem 3–5. Suppose $g(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$ and $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ x+y \end{pmatrix}$. What is $f \circ g$? What is $g \circ f$?

Problem 3–6. Suppose $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ x+y \end{pmatrix}$. Does f^{-1} exist? If so, what is $f^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$?

Solutions to Problems

Problem 3–1. The graph is $\left\{ \begin{pmatrix} s \\ t \end{pmatrix} \in \mathbf{R}^2 : t = s^2 \right\}$. This could also be written as $\left\{ \begin{pmatrix} x \\ x^2 \end{pmatrix} : x \in \mathbf{R} \right\}$.

Problem 3–2. False. The graph is a subset of $\mathbf{R}^5$.

Problem 3–3. The domain of g is $\mathbf{R}$. The component functions are $g_1(t) = t + 5$ and $g_2(t) = t^2 - 7t$.

Problem 3–4. False. The output value of f must be a point in 3 dimensional space.

Problem 3–5. $(f \circ g)(t) = \begin{pmatrix} t - t^2 \\ t + t^2 \end{pmatrix}$, while $g \circ f$ does not make sense.

Problem 3–6. Simple computations show that $f^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (x+y)/2 \\ (y-x)/2 \end{pmatrix}$.

Solutions to Exercises

Exercise 3–1. The domain is $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} : xy \geq 0 \right\}$; the range is $\{x \in \mathbf{R} : x \geq 0\}$.

Exercise 3–2. The input values must satisfy $x + y = 2$ and $x - y = 3$, from which $x = 5/2$ and $y = -1/2$.

Exercise 3–3. The domain is $\mathbf{R}^2$, but the range is the line with equation $y = x$ in $\mathbf{R}^2$.

Exercise 3–3. The graph lies in 4 dimensional space.

§4. Multiple Integrals

One interpretation of the integral $\int_a^b f(t)\,dt$ is as the signed area of the region bounded by the graph of the function f over the interval $[a, b]$. A similar geometric interpretation can be attached to the integral of a real valued function of two or more variables.

Example 4–1. Suppose U is the **unit square** in the plane, that is, U is the region $\left\{\begin{pmatrix} x \\ y \end{pmatrix} : 0 \le x \le 1, 0 \le y \le 1\right\}$. If $f\begin{pmatrix} x \\ y \end{pmatrix}$ is a function of two variables which takes real values the geometric interpretation of the **double integral** $\int_U f\begin{pmatrix} x \\ y \end{pmatrix} dx \times dy$ is as the signed volume of the solid bounded by the graph of f over the square U.

Operating at an intuitive level, this volume is well defined. How should this volume be computed? The method of slicing is applied once again!

Example 4–2. As a more specific example, $\int_U xy^2\, dx \times dy$ will be computed. The graph of the function xy^2 as x and y range over the points in the unit square U is a sheet. There are two different ways in which slicing can be done to compute the volume represented by the integral. One way of slicing is to make the slices parallel to the y axis. Denote by $V(x)$ the volume of the solid to the left of x. Then simple geometric argument shows that for small h, $V(x + h) = V(x) + h \int_0^1 xy^2\, dy$, approximately. Using the definition of derivative then gives $V'(x) = \int_0^1 xy^2\, dy = x/3$. Finally, $\int_U xy^2\, dx \times dy = V(1) - V(0) = \int_0^1 x/3\, dx = 1/6$. Here the Fundamental Theorem of Calculus has been used.

Exercise 4–1. What does the computation look like if slicing is done parallel to the x axis?

Notice that in working out this particular example the first integral that was computed treated one of the variables temporarily as though it were a number. The second integration then was carried out using this same variable as the variable of integration. For this reason, in this context the integrations produced by the slicing method are called **iterated integrals**.

Example 4–3. In the previous example, the slicing method produced the iterated integral $\int_0^1 \left(\int_0^1 xy^2\, dy\right) dx$. Notice that when evaluating iterated integrals, work proceeds from the inner most integral outwards. Since this order of computing is understood, the parentheses are usually omitted and the iterated integral is written simply as $\int_0^1 \int_0^1 xy^2\, dy\, dx$.

Exercise 4–2. What is the iterated integral of the previous exercise?

It may seem geometrically obvious that the two iterated integrals must always give the same value. However, this is not true! If the integrand is sometimes positive and sometimes negative, the two iterated integrals can give different results.

Example 4–4. The function $f\begin{pmatrix} x \\ y \end{pmatrix}$ takes the value $1/y^2$ if $0 < x < y < 1$ and the value $-1/x^2$ if $0 < y < x < 1$. Then for any value of y between 0 and 1, $\int_0^1 f\begin{pmatrix} x \\ y \end{pmatrix} dx = \int_0^y 1/y^2\, dx - \int_y^1 1/x^2\, dx = 1/y + (1 - 1/y) = 1$, so that $\int_0^1 \left(\int_0^1 f\begin{pmatrix} x \\ y \end{pmatrix} dx \right) dy = 1$. On the other hand, for any x between 0 and 1, $\int_0^1 f\begin{pmatrix} x \\ y \end{pmatrix} dy = \int_0^x -1/x^2\, dy + \int_x^1 1/y^2\, dy = -1/x + (-1 + 1/x) = -1$, so that $\int_0^1 \left(\int_0^1 f\begin{pmatrix} x \\ y \end{pmatrix} dy \right) dx = -1$.

The phenonmenon observed in the example is caused by the fact that the integrand f has both arbitrarily large positive and negative values on the region of integration. An important theorem due to Fubini and Tonelli states that if *one* of the iterated integrals of the absolute value of the integrand is finite, then both iterated integrals of the function itself are finite and equal. Practically speaking, this means that unless the integrand has both arbitrarily large positive and negative values on the region of integration, the two iterated integrals must give the same value.

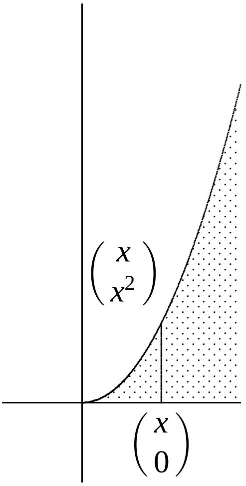
Technically, the question of the existence of a double integral can also be raised. As long as the set of discontinuities of the integrand is of lower dimension than the dimension of the space of the domain of the integrand, the double integral will exist. This is in accord with geometric intuition that the volume of a lower dimensional set must be zero. For example, the 2 dimensional volume (area) of a line is zero; the 3 dimensional volume of a 2 dimensional square is also zero.

The method of slicing can be used to compute integrals over somewhat more complicated regions.

Example 4–5. Suppose T is the triangular shaped region

$$T = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : 0 \le y \le x^2, x \ge 0 \right\}$$

in the x-y plane. What is $\int_T e^{-xy}\, dx \times dy$? In this case, the iterated integral obtained by slicing parallel to the y axis is $\int_0^\infty \int_0^{x^2} e^{-xy}\, dy\, dx$. To understand how the limits of integration are determined in this case, notice that a slice parallel to the y axis is made by first fixing a value of x through which the vertical slice will be made. From the definition of T, the value of x can be any positive number. This gives the limits for the *outermost* integral, the integral with respect to x. Once a value of x is given, the definition of the region T implies that y must lie between 0 and x^2. This gives the limits for the *innermost* integral. This is illustrated in the picture below. The region T is shaded.



Sketching the region of integration often helps in determing the limits of integration.

Exercise 4–3. What is the iterated integral obtained by slicing parallel to the x axis?

The computations of the example and exercise can be summarized as follows. When finding the limits of integration for an iterated integral, work from the outermost integral in; when computing the value of an iterated integral, work from the innermost integral out.

Notice that when the region of integration is not a rectangle, the iterated integrals can have quite different limits of integration. Notice too, that the integrand is always the same! Geometrically this is because the integrand is specifying the height of the solid, which does not change with the direction of the slicing.

The slicing method can also be used to evaluate triple integrals, that is, integrals over a region in three dimensional space. Notice that in this case, each slice will be a two dimensional integral. Higher dimensional integrals can be handled similarly.

Example 4–6. Suppose C is the three dimensional unit cube

$$C = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1 \right\}.$$

What is $\int_C xyz\, dx \times dy \times dz$? Slicing parallel to the x-y plane gives the iterated integral $\int_0^1 \int_0^1 \int_0^1 xyz\, dx\, dy\, dz$.

Example 4–7. How many other iterated integrals are there in this case, and what are they?

Example 4–8. When the region of integration is not a 3 dimensional box, the iterated integrals are often more challenging to discover. Suppose the region of integration is the set $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 0 \le x \le y \le z \le 1 \right\}$. One iterated integral for $\int_W x\, dx \times dy \times dz$ is $\int_0^1 \int_0^z \int_0^y x\, dx\, dy\, dz$. To reason out the limits of integration in this case, notice that z must lie between 0 and 1 because of the definition of the set W. Once z is fixed, y must lie between 0 and z, and once z and y are known, x must lie between 0 and y.

Exercise 4–4. What limits of integration would be used for the iterated integral with the order $dy\, dz\, dx$?

Example 4–9. One use of multiple integrals is to compute the mass of a solid with non-constant density. Suppose the density of a solid at the point $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is $\delta \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and the solid occupies the region S in space. Then the mass of the solid is $\int_S \delta \begin{pmatrix} x \\ y \\ z \end{pmatrix} dx \times dy \times dz$.

Multiple integrals are useful, but do not include all types of integration that may be of interest.

Example 4–10. Suppose L is the line segment $L = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : y = x, 0 \le x \le 1 \right\}$. Then $\int_L f \begin{pmatrix} x \\ y \end{pmatrix} dx \times dy = 0$ no matter what the function f. This is because the one dimensional set L has no area.

Exercise 4–5. What iterated integral would be used to compute the double integral of the last example?

Such integrals over lower dimensional subsets of higher dimensional spaces are useful in physics and many other areas. One of the major objectives here is to define such integrals in a meaningful way.

The slicing method allows the use of iterated integrals to compute multiple integrals. The region of integration in a multiple integral must be a set of the same dimension as the number of variables.

Problems

Problem 4–1. Is $\int_0^{\pi} e^{\sin x}\,dx$ positive or negative? Briefly justify your answer.

Problem 4–2. True or False: It is possible to change the value of $f\begin{pmatrix} x \\ y \end{pmatrix}$ at infinitely many points in the two dimensional set D *without* affecting the value of $\int_D f\begin{pmatrix} x \\ y \end{pmatrix} dx \times dy$.

Problem 4–3. Suppose $R = \left\{\begin{pmatrix} x \\ y \end{pmatrix} : 0 \le x \le 3, 0 \le y \le 5\right\}$. Write the two iterated integrals for $\int_R xy\,dx \times dy$ and evaluate both of them.

Problem 4–4. Suppose $T = \left\{\begin{pmatrix} x \\ y \end{pmatrix} : 0 \le x \le 1, 0 \le y \le x\right\}$. Write the two iterated integrals for $\int_T xy\,dx \times dy$ and evaluate both of them.

Problem 4–5. Suppose $L = \left\{\begin{pmatrix} x \\ y \end{pmatrix} : 0 \le x \le 2, 0 \le y \le e^x\right\}$. Write the two iterated integrals for $\int_L xy\,dx \times dy$ and evaluate one of them.

Problem 4–6. For which set R in the plane does $\int_R xy^2\,dx \times dy = \int_1^2 \int_x^{x^2} xy^2\,dy\,dx$? Write your answer in set notation.

Problem 4–7. Sketch the region of integration for the integral $\int_0^1 \int_{x^2}^1 1\,dy\,dx$, write an equivalent iterated integral with the order of integration reversed, and evaluate the double integral.

Problem 4–8. Sketch the region of integration for the integral $\int_1^2 \int_1^x 1\,dy\,dx$, write an equivalent iterated integral with the order of integration reversed, and evaluate the double integral.

Problem 4–9. Suppose $B = \left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} : 4y^2 + 4z^2 \le x \le 4\right\}$. Compute $\int_B x\,dx \times dy \times dz$.

Problem 4–10. Suppose $A = \left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} : 0 \le y \le 2, 0 \le y \le x, x \le z \le 2x\right\}$. Compute $\int_A e^{-x}\,dx \times dy \times dz$.

Problem 4–11. The unit sphere in $\mathbf{R}^3$ is the set $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x^2 + y^2 + z^2 \leq 1 \right\}$. Write a triple integral representing the volume of this sphere, and evaluate the integral to compute the volume.

Problem 4–12. The unit sphere of the previous problem has center at the origin and radius 1. What is the volume of a sphere with the center at the origin and radius $r > 0$? What is the surface area of such a sphere?

Problem 4–13. A settling tank is used to separate out waste material. Suppose the tank is 30 meters wide, 70 meters long and 5 meters deep. Suppose also that the density of the material at a depth of d meters is $3d^2 + 1$. What is the mass of the material in the tank?

Solutions to Problems

Problem 4–1. Since the values of the exponential function are always positive, the graph of $e^{\sin x}$ is always above the horizontal axis. Since this integral represents the signed area under the graph of this function, the integral is positive.

Problem 4–2. True. As long as the infinitely many points lie in a one dimensional subset of D, the changes in the value of f have no effect on the value of the integral.

Problem 4–3. One iterated integral is $\int_0^3 \int_0^5 xy\,dy\,dx = \int_0^3 25x/2\,dx = 225/4$. The other iterated integral is $\int_0^5 \int_0^3 xy\,dx\,dy = 225/4$.

Problem 4–4. One iterated integral is $\int_0^1 \int_0^x xy\,dy\,dx = \int_0^1 x^3/2\,dx = 1/8$. The other is $\int_0^1 \int_y^1 xy\,dx\,dy = \int_0^1 (y-y^3)/2\,dy = 1/4 - 1/8 = 1/8$.

Problem 4–5. One iterated integral is $\int_0^2 \int_0^{e^x} xy\,dy\,dx = \int_0^2 xe^{2x}/2\,dx = (3e^4+1)/8$. The other iterated integral has two parts: $\int_0^1 \int_0^2 xy\,dx\,dy + \int_1^{e^2} \int_{\ln y}^2 xy\,dx\,dy$. The value is the same as before.

Problem 4–6. The region is $R = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : 1 \le x \le 2, x \le y \le x^2 \right\}$.

Problem 4–7. The region of integration is below the parabola $y = x^2$ for $0 \le x \le 1$. The other iterated integral is $\int_0^1 \int_0^{\sqrt{y}} 1\,dx\,dy$.

Problem 4–8. The region is below the line $y = x$ and above the line $y = 1$ with $1 \le x \le 2$. The other iterated integral is $\int_1^2 \int_y^2 1\,dx\,dy$.

Problem 4–9. From the definition of B, x must lie between 0 and 4. Once x is fixed, y and z lie within a circle of radius $\sqrt{x/4}$ centered at the origin. One iterated integral is $\int_0^4 \int_{-\sqrt{x/4}}^{\sqrt{x/4}} \int_{-\sqrt{x/4-y^2}}^{\sqrt{x/4-y^2}} x\,dz\,dy\,dx = 16\pi/3$.

Problem 4–10. From the definition of A, y must lie between 0 and 2; once y is known, x must be at least y; once y and x are known, z must lie between x and $2x$. This gives one iterated integral is $\int_0^2 \int_y^\infty \int_x^{2x} e^{-x}\,dz\,dx\,dy = 2 - 4e^{-2}$.

Problem 4–11. The volume is $\int_S dx \times dy \times dz$, and one iterated integral is given by $\int_{-1}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-z^2-y^2}}^{\sqrt{1-z^2-y^2}} dx\,dy\,dz$. Geometrically, the two inside integrals give the area of a circle of radius $\sqrt{1-z^2}$, thus the iterated integral is $\int_{-1}^{1} \pi(1-z^2)\,dz = 4\pi/3$.

Problem 4–12. The volume is $4\pi r^3/3$ and the surface area is the derivative of this with respect to r, namely $4\pi r^2$.

Problem 4–13. One iterated integral for the mass is

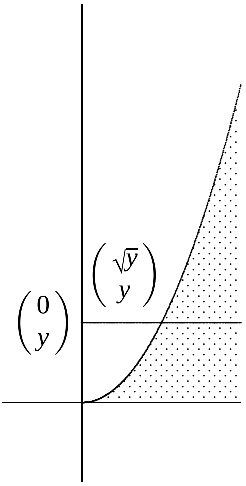
$$\int_{-5}^{0} \int_{0}^{70} \int_{0}^{30} (3z^2 + 1)\,dx\,dy\,dz.$$

Solutions to Exercises

Exercise 4–1. In this case let $V(y)$ be the volume of the solid below level y. Then $V'(y) = \int_0^1 xy^2\,dx = y^2/2$, and $\int_U xy^2\,dx \times dy = V(1) - V(0) = \int_0^1 y^2/2\,dy = 1/6$, the same numerical result as before.

Exercise 4–2. Here the iterated integral is $\int_0^1 \int_0^1 xy^2\,dx\,dy$.

Exercise 4–3. This iterated integral is $\int_0^\infty \int_{\sqrt{y}}^\infty e^{-xy}\,dx\,dy$. In this case, y can be any positive value; after y is determined, the values of x that are used begin with the x coordinate of the intersection of the *horizontal* line through $\begin{pmatrix} 0 \\ y \end{pmatrix}$ and the parabola, which is $\begin{pmatrix} \sqrt{y} \\ y \end{pmatrix}$, and extend indefinitely to the right.



Exercise 4–3. There are five others. One of them is $\int_0^1 \int_0^1 \int_0^1 xyz\,dy\,dz\,dx$.

Exercise 4–4. In this case, x must lie between 0 and 1; once x is known, z must lie between x and 1; once x and z are known, y must lie between x and z. So the iterated integral in this order is $\int_0^1 \int_x^1 \int_x^z x\,dy\,dz\,dx$.

Exercise 4–5. One iterated integral would be $\int_0^1 \int_x^x f\begin{pmatrix} x \\ y \end{pmatrix} dy\,dx$, which is clearly zero.

§5. Geometry in Higher Dimensions

As a prelude to the study of differentiation of functions of several variables, the geometry of higher dimensional spaces will be examined in greater detail.

Interpreting sets as geometric objects allows some degree of visualization of the set. The possibility of geometric interpretation is enhanced when the algebraic operations of addition of two points and multiplication of a single point by a number can be defined. When these two operations on points can be defined, the points are then usually referred to as vectors. These algebraic operations also have geometric interpretations.

Example 5–1. In $\mathbf{R}^2$, if $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} w \\ z \end{pmatrix}$ are points an operation of addition can be defined by the formula $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} x+w \\ y+z \end{pmatrix}$. Also if c is a number, multiplication of the point $\begin{pmatrix} x \\ y \end{pmatrix}$ by the number c can be defined by $c\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ cy \end{pmatrix}$.

Exercise 5–1. What is $\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 7 \end{pmatrix}$? What is $3\begin{pmatrix} 1 \\ 4 \end{pmatrix}$?

Notice that any two points in $\mathbf{R}^2$ can be added and the result will again be a point in $\mathbf{R}^2$. Also, any point in $\mathbf{R}^2$ can be multiplied by a number and the result will again be a point in $\mathbf{R}^2$. The set $\mathbf{R}^2$ together with these two operations constitutes a **vector space**. The individual points in $\mathbf{R}^2$ are called **vectors**. This new terminology reflects the fact that these two algebraic operations have been defined and are available for use. An individual point is still just a point, too!

Example 5–2. The set $\mathbf{R}^3$ is a vector space when addition and scalar multiplication are defined componentwise, as in the case of $\mathbf{R}^2$ above.

Exercise 5–2. What is $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \\ 6 \end{pmatrix}$? What is $-5\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$? What is $\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$?

Exercise 5–3. What is $\begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \end{pmatrix}$?

The previous exercise shows that once addition is defined, so is subtraction. The **zero vector**, which in $\mathbf{R}^2$ is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, is the unique element of the set which has the property that the sum of a given vector and the zero vector is just the given vector.

Technically, the interaction between the addition operation and scalar multiplication must obey the familiar rules, such as the distributive and associative laws, and the fact that addition is commutative. In all of the examples here these rules will hold, just as in the case of the addition and multiplication of numbers.

One of the central ideas is the interplay between algebra and geometry. The algebraic operations on vectors have important geometric interpretations.

Example 5–3. A single vector v in $\mathbf{R}^2$ can be interpreted as specifying both a magnitude and a direction. The magnitude determined by v is the distance from the point v to the origin. The direction determined by v is the direction one would walk along a straight line path *starting* at the origin and heading toward the point v.

The magnitude of a vector v is called the **norm** of the vector v and is denoted by $\| v \|$.

Example 5–4. If $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathbf{R}^2$ then $\| v \| = \sqrt{1^2 + 2^2}$. This is nothing more than the distance from the origin to the point v. Similarly, if $w = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ then $\| w \| = \sqrt{1^2 + 2^2 + 3^2}$.

Exercise 5–4. What is the magnitude and direction determined by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$?

Closely related to the norm of a vector is the **dot product** of two vectors. The dot product on $\mathbf{R}^2$ is defined by $\begin{pmatrix} a \\ b \end{pmatrix} \bullet \begin{pmatrix} c \\ d \end{pmatrix} = ac + bd$; on $\mathbf{R}^3$ the dot product is defined by the formula $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} d \\ e \\ f \end{pmatrix} = ad + be + cf$. Similar formulas are used on higher dimensional spaces. A general fact is that $u \bullet v = \| u \| \, \| v \| \cos \theta$ where $0 \le \theta \le \pi$ is the angle formed by the points u, the origin, and v, with the vertex of the angle at the origin. The most important consequence of this fact is that the angle between u and v is a right angle if and only if $u \bullet v = 0$.

Exercise 5–5. What is the angle between $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$?

Example 5–5. Given two vectors in u and v in $\mathbf{R}^2$, the 4 vectors 0, u, $u + v$ and v are the vertices of a parallelogram.

Example 5–6. Verify that the points $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, and $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ are the vertices of a parallelogram.

Example 5–7. Given a vector v in $\mathbf{R}^2$ and a number c, the vectors 0, v and cv lie on a line.

Exercise 5–6. Show that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $3\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ lie on a line.

The previous exercise shows that scalar multiplication can be visualized as stretching (if $c > 1$) or shrinking ($0 < c < 1$) the original vector without changing its direction. When the scalar c is negative, the direction is reversed, along with the stretching or shrinking.

Problems

Problem 5–1. True or False: The vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ are perpendicular.

Problem 5–2. True or False: The distance from $\binom{1}{2}$ to $\binom{3}{4}$ is $\left\| \binom{1}{2} - \binom{3}{4} \right\|$.

Problem 5–3. What geometric object is $\left\{ c\binom{1}{2} : c \in \mathbf{R} \right\}$?

Problem 5–4. What geometric object is $\left\{ c\binom{1}{2} + \binom{3}{5} : c \in \mathbf{R} \right\}$?

Problem 5–5. Is the set $\left\{ \binom{x}{y} \in \mathbf{R}^2 : y = 2x - 1 \right\}$ the same as the set of the preceding problem?

Problem 5–6. What geometric object is $\left\{ c\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} : c \in \mathbf{R} \right\}$?

Problem 5–7. Is the set $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{R}^3 : y = 2x - 3 \text{ and } z = 3x - 6 \right\}$ the same as the set of the preceding problem?

Problem 5–8. True or False: If v is a non-zero vector then the vector $v/\|v\|$ has norm 1.

Solutions to Problems

Problem 5–1. Since $\begin{pmatrix}1\\2\\3\end{pmatrix}\bullet\begin{pmatrix}3\\-1\\1\end{pmatrix} = 4 \neq 0$, the answer is false.

Problem 5–2. True. In general, the distance from the point u to the point v is $\| u - v \|$.

Problem 5–3. This set is the line with equation $y = 2x$.

Problem 5–4. This is the line $y = 2x - 1$.

Problem 5–5. A point in the first set has the form $\begin{pmatrix}c+3\\2c+5\end{pmatrix}$ and $2(c+3)-1 = 2c+5$, so such a point is in the second set. A point $\begin{pmatrix}x\\y\end{pmatrix}$ in the second set has $y = 2x - 1$ so $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\2x-1\end{pmatrix} = x\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}0\\-1\end{pmatrix}$. Writing $x = c + 3$ now gives the point in the form $c\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}3\\5\end{pmatrix}$, which is in the second set. So the two sets are the same.

Problem 5–6. A line in $\mathbf{R}^3$ which does not contain the origin.

Problem 5–7. A point in the first set has the form $\begin{pmatrix}c+4\\2c+5\\3c+6\end{pmatrix}$ and these coordinates satisfy the 2 equations describing the second set. The equations describing the second set show that a point in the second set has coordinates of the form $\begin{pmatrix}x\\2x-3\\3x-6\end{pmatrix} = x\begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}0\\3\\6\end{pmatrix}$, where x is a real number. Now write $x = c + 4$ to see that this vector has the same form as one in the other set. So the two sets are the same.

Problem 5–8. True.

Solutions to Exercises

Exercise 5–1. The results are $\begin{pmatrix} 6 \\ 10 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 12 \end{pmatrix}$.

Exercise 5–2. The first two results are $\begin{pmatrix} -1 \\ 6 \\ 9 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ -10 \\ -15 \end{pmatrix}$. The last sum doesn't make sense. The summands must come from the same set, and $\mathbf{R}^2$ and $\mathbf{R}^3$ are different sets.

Exercise 5–3. $\begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Exercise 5–4. The magnitude is $\sqrt{(1-0)^2 + (1-0)^2} = \sqrt{2}$, by the distance formula. The direction might be called northeast.

Exercise 5–5. Since $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$, the angle between these two vectors is 90°.

Exercise 5–5. The distance formula shows that the opposite sides have the same length, and computing slopes shows that the opposite sides are also parallel.

Exercise 5–6. Computing slopes shows that the line through the origin and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ has the same slope as the line through the origin and $3\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and hence is the same line.

§6. Lines, Planes, and Hyperplanes

The principal geometric objects here are lines and their higher dimensional analogs.

Scalar multiplication was seen to have a simple geometric interpretation: if v is a vector and c is a number then the vectors 0, v, and cv lie on a straight line. This line can be identified as the line through the origin in the direction determined by v.

Exercise 6–1. Why does this description of the line make sense?

More concretely, the **line through the origin in the direction** v is $\{cv : c \in \mathbf{R}\}$.

Exercise 6–2. Is the line through the origin in the direction v also $\{3cv : c \in \mathbf{R}\}$?

Exercise 6–3. What is the relationship between the line through the origin in the direction $\begin{pmatrix}1\\1\end{pmatrix}$ and the line through the origin in the direction $\begin{pmatrix}2\\2\end{pmatrix}$?

A plane is uniquely determined by 3 non-collinear points. The **plane through the origin with direction vectors** u **and** v is $\{cu + dv : c \in \mathbf{R}, d \in \mathbf{R}\}$. What algebraic criterion can be used to see if two different descriptions are actually descriptions of the same plane?

Example 6–1. Is the plane through the origin with direction vectors $\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\2\\3\end{pmatrix}$ the same as the plane through the origin with direction vectors $\begin{pmatrix}2\\2\\2\end{pmatrix}$ and $\begin{pmatrix}2\\3\\4\end{pmatrix}$? A point on the first plane can be written as $c\begin{pmatrix}1\\1\\1\end{pmatrix} + d\begin{pmatrix}1\\2\\3\end{pmatrix}$. Now $\begin{pmatrix}2\\3\\4\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}1\\2\\3\end{pmatrix}$, so that $\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}2\\3\\4\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix}$ and making this substitution gives $c\begin{pmatrix}1\\1\\1\end{pmatrix} + d\begin{pmatrix}1\\2\\3\end{pmatrix} = c\begin{pmatrix}1\\1\\1\end{pmatrix} + d(\begin{pmatrix}2\\3\\4\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix}) = (c-d)\begin{pmatrix}1\\1\\1\end{pmatrix} + d\begin{pmatrix}2\\3\\4\end{pmatrix} = ((c-d)/2)\begin{pmatrix}2\\2\\2\end{pmatrix} + d\begin{pmatrix}2\\3\\4\end{pmatrix}$. Thus a point on the first plane lies on the second plane.

Exercise 6–4. Show that a point on the second plane lies on the first plane.

A **linear combination** of the vectors $v_1, \ldots, v_d$ is an expression of the form $c_1v_1 + \ldots + c_dv_d$ where $c_1, \ldots, c_d$ are numbers.

Example 6–2. One linear combination of $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ is $3\begin{pmatrix}1\\0\end{pmatrix}+5\begin{pmatrix}0\\1\end{pmatrix}$. Another linear combination of the same vectors is $-2\begin{pmatrix}1\\0\end{pmatrix}+7\begin{pmatrix}0\\1\end{pmatrix}$.

Example 6–3. Is $\begin{pmatrix}3\\4\end{pmatrix}$ a linear combination of $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$? Is $\begin{pmatrix}3\\4\end{pmatrix}$ a linear combination of $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}2\\0\end{pmatrix}$?

Each point in a plane through the origin is a linear combination of the direction vectors. A plane is therefore the set of *all possible* linear combinations of the direction vectors. The set of points consisting of all possible linear combinations of the vectors $v_1, \ldots, v_d$ is called the **space spanned by** $v_1, \ldots, v_d$.

Example 6–4. The space spanned by $\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right\}$ is the plane $\mathbf{R}^2$.

Not all lines and planes pass through the origin. If u and v are vectors, the **line through** u **in the direction** v is $\{u + cv : c \in \mathbf{R}\}$.

Exercise 6–5. Is u on the line through u in the direction v?

Example 6–5. The line through $\begin{pmatrix}1\\2\end{pmatrix}$ in the direction $\begin{pmatrix}1\\1\end{pmatrix}$ is

$$\left\{\begin{pmatrix}1\\2\end{pmatrix} + c\begin{pmatrix}1\\1\end{pmatrix} : c \in \mathbf{R}\right\}.$$

More conventionally this line has the equation $y = x + 1$.

Exercise 6–6. Is the line of the previous example the same as the line through $\begin{pmatrix}0\\1\end{pmatrix}$ in the direction $\begin{pmatrix}3\\3\end{pmatrix}$?

Lines can be given a simple physical interpretation too.

Example 6–6. If u and v are vectors, define the function f with domain $\mathbf{R}$ by the formula $f(t) = u + tv$. As t varies, the values of $f(t)$ trace out the line through u in the direction v. Physically, the value $f(t)$ can be interpreted as the position of a particle at time t. Thus f is describing the motion of a particle that travels in a straight line, and is at location u at time zero.

Exercise 6–7. What is the velocity of such a particle?

The algebraic description of a general plane is quite similar, except that two direction vectors are required. The **plane through the vector** u **with direction vectors** v **and** w is the set $\{u + cv + dw : c \in \mathbf{R}, d \in \mathbf{R}\}$.

Exercise 6–8. What is the plane through $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with the direction vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$?

Exercise 6–9. What is the plane through $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with the direction vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$?

More generally, the **hyperplane through** u **in the directions** $\{v_1, \ldots, v_d\}$ is the set $\{u + c_1 v_1 + \ldots + c_d v_d : c_1, \ldots, c_d$ are real numbers$\}$. The dimension of this hyperplane is the dimension of the space spanned by the vectors $v_1, \ldots, v_d$.

As in the case of lines and planes through the origin, a general hyperplane has many different algebraic descriptions. Two descriptions will describe the same geometric plane provided that the spaces spanned by the two sets of direction vectors are the same, and the difference of the 'through' vectors lies in the space spanned by the direction vectors.

Problems

Problem 6–1. What geometric object is the line through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in the direction $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$?

Problem 6–2. Is the line through the origin in the direction v the same as the line through the origin in the direction $-v$?

Problem 6–3. Is the line through $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in the direction $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ the same as the line through $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ in the direction $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$?

Problem 6–4. Suppose f is a function with domain $\mathbf{R}$ and range $\mathbf{R}^2$ given by $f(t) = \begin{pmatrix} t+5 \\ 2t-3 \end{pmatrix}$. Use the definition of derivative to compute $f'(t)$.

Solutions to Problems

Problem 6–1. The single point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Problem 6–2. Yes, since the line consists of all multiples of the given vector.

Problem 6–3. Yes, the direction vectors are non-zero multiples of each other and $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Problem 6–4. Here $(f(t+h) - f(t))/h = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ so $f'(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Solutions to Exercises

Exercise 6–1. The direction determined by v is the direction one would travel when walking in a straight line from the origin to the point v.

Exercise 6–2. Yes, this is the same set as before, because both sets describe all possible multiples of the vector v.

Exercise 6–3. These two lines are the same line.

Exercise 6–4. $c\begin{pmatrix}2\\2\\2\end{pmatrix}+d\begin{pmatrix}2\\3\\4\end{pmatrix}=2c\begin{pmatrix}1\\1\\1\end{pmatrix}+d(\begin{pmatrix}1\\1\\1\end{pmatrix}+\begin{pmatrix}1\\2\\3\end{pmatrix})=(2c+d)\begin{pmatrix}1\\1\\1\end{pmatrix}+d\begin{pmatrix}1\\2\\3\end{pmatrix}$.

Exercise 6–4. Yes, since $\begin{pmatrix}3\\4\end{pmatrix}=3\begin{pmatrix}1\\0\end{pmatrix}+4\begin{pmatrix}0\\1\end{pmatrix}$. There are no numbers c and d so that $\begin{pmatrix}3\\4\end{pmatrix}=c\begin{pmatrix}1\\0\end{pmatrix}+d\begin{pmatrix}2\\0\end{pmatrix}$, so $\begin{pmatrix}3\\4\end{pmatrix}$ is not a linear combination of $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}2\\0\end{pmatrix}$.

Exercise 6–5. Yes, simply take $c=0$ to see that u is on the line.

Exercise 6–6. Yes, this is a second description of the same line.

Exercise 6–7. The velocity at time t is $f'(t)$. The velocity is a vector.

Exercise 6–8. This is the entire plane $\mathbf{R}^2$.

Exercise 6–9. This is the line through $\begin{pmatrix}1\\1\end{pmatrix}$ in the direction $\begin{pmatrix}1\\0\end{pmatrix}$.

§7. Linear Functions

In the one dimensional case, functions whose graphs are lines played a central role. The generalization of these functions to higher dimensions will play a key role in the study of calculus in higher dimensions.

Example 7–1. The function $f(x) = ax$ has a graph which is a straight line passing through the origin. What is the higher dimensional analog of this function?

To answer the question raised in the example, look at the one dimensional case in a slightly different way. If $f : \mathbf{R} \to \mathbf{R}$ is a function, the graph of f is the set $\left\{ \begin{pmatrix} x \\ f(x) \end{pmatrix} : x \in \mathbf{R} \right\}$. A line through the origin is the set of all multiples of a fixed vector. Now $\begin{pmatrix} x \\ f(x) \end{pmatrix} = x \begin{pmatrix} 1 \\ f(x)/x \end{pmatrix}$ and the vector $\begin{pmatrix} 1 \\ f(x)/x \end{pmatrix}$ does *not* depend on x only if $f(x)/x$ is a number, say a. Thus $f(x) = ax$ is required for the graph of f to be a line through the origin.

Suppose now $g : \mathbf{R}^2 \to \mathbf{R}$ is a function whose graph is a plane through the origin. What is the formula for g? Following the reasoning in the one dimensional case, the graph of g is $\left\{ \begin{pmatrix} x \\ y \\ g\begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix} : x \in \mathbf{R}, y \in \mathbf{R} \right\}$. Now if $\begin{pmatrix} x \\ y \\ g\begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix}$ is to be of the form $xv + yw$ where v and w are vectors that do not depend on either x or y, then v must be of the form $\begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix}$ for some number a, and w must be of the form $\begin{pmatrix} 0 \\ 1 \\ b \end{pmatrix}$ for some number b. But this means the formula for g is $g\begin{pmatrix} x \\ y \end{pmatrix} = ax + by$.

This line of reasoning has uncovered the needed generalizations of the one dimensional case to the case in which the domain is higher dimensional. A function f with domain $\mathbf{R}^2$ and range $\mathbf{R}$ is **linear** if there are numbers a and b for which $f\begin{pmatrix} x \\ y \end{pmatrix} = ax + by$ for all x and y. Similarly, a real valued function h with domain $\mathbf{R}^3$ is linear if there are numbers a, b, and c so that $h\begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz$ for all x, y, and z. The graphs of these linear functions are planes passing through the origin.

Exercise 7–1. Show that the graph of $f\begin{pmatrix} x \\ y \end{pmatrix} = ax + by$ is a plane passing through the origin.

Expressions of the form $ax + by$ and $ax + by + cz$ arise quite frequently. Notice that there are as many constants as variables in such expressions. This suggests the idea of collecting the constants and variables together into vectors, one vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ containing the constants and the other vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ containing the variables. The dot product of these two vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz$ then reproduces the original expression. This use of the dot product allows a second way of saying that a function is linear. A function h is linear if there is a vector u of constants so that $h(v) = u \bullet v$ for all vectors v.

Example 7–2. The function $g\begin{pmatrix} x \\ y \end{pmatrix} = 3y = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix}$ is linear, as was seen in the earlier example.

Exercise 7–2. Is the function $h\begin{pmatrix} x \\ y \end{pmatrix} = 3x^2$ linear?

The simple vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in $\mathbf{R}^2$, and the corresponding vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ in $\mathbf{R}^3$, will play an important role in the following discussion. These vectors are called the **standard basis vectors** and are denoted by e_1 and e_2 in $\mathbf{R}^2$, and by e_1, e_2, and e_3 in $\mathbf{R}^3$. More concretely, in $\mathbf{R}^2$, $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, while in $\mathbf{R}^3$, $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

An important fact about linear functions is that **a linear function is completely determined by its values on the standard basis vectors**.

Example 7–3. Consider the linear function $f\begin{pmatrix} x \\ y \end{pmatrix} = ax + by$. Then $f(e_1) = a$ and $f(e_2) = b$ so that $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(e_1) \\ f(e_2) \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix}$. Once the values of f are known on the basis vectors, the values of f can be computed anywhere!

Linear functions with higher dimensional range are those functions each of whose component functions is linear.

Example 7–4. The function $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - 3y \\ 5x + 2y \end{pmatrix}$ is linear, since each component function $g_1\begin{pmatrix} x \\ y \end{pmatrix} = 2x - 3y = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix}$ and $g_2\begin{pmatrix} x \\ y \end{pmatrix} = 5x + 2y = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix}$ is linear.

As an extension of the idea above used for real valued functions, the numbers appearing in formula of the linear function will be collected into a rectangular grid called a matrix. A **matrix** is simply a rectangular array of numbers. Notationally, the numbers in the matrix, called the **entries**, are blocked off using parentheses.

Example 7–5. The grid $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ is a matrix. The grid $\begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix}$ is also a matrix. The matrix $\begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix}$ has 2 **rows** and 3 **columns**. The size of a matrix is always specified by giving the number of rows first. So this matrix is a 2×3 matrix (read: '2-by-3 matrix'). The entry in the first row and second column is 3.

Exercise 7–3. What is the entry in the second row and first column of the matrix $\begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix}$?

A matrix can be associated with a linear function in a simple way.

Example 7–6. The matrix associated with the linear function g of the previous example is $\begin{pmatrix} 2 & -3 \\ 5 & 2 \end{pmatrix}$. Notice that the first *row* of the matrix is the vector of constants used to define the first component function of g; the second *row* of the matrix is the vector of constants used to define the second component function of g.

More generally, the matrix associated with a given linear transformation has as its ith row the vector used to write the ith component function of the linear transformation as a dot product.

Exercise 7–4. What is the matrix of $g\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y \\ 2y + 3z \\ 5z \end{pmatrix}$?

To carry this idea a bit further, the product of a matrix and a vector can be defined. If A is an $r \times c$ matrix and x is a c dimensional vector, the product Ax is the r dimensional vector whose ith coordinate is the dot product of the ith row of A with the vector x.

Example 7–7. The product $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix} \\ \begin{pmatrix} 3 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix} = \begin{pmatrix} x + 2y \\ 3x + 4y \end{pmatrix}$.

The definition of multiplication was chosen precisely so that any linear function can be written in the form of a product of the matrix of the linear function and the vector of variables.

Example 7–8. The exercise above shows that the matrix of the linear transformation

$g\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y \\ 2y+3z \\ 5z \end{pmatrix}$ is $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{pmatrix}$. So $g\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Example 7–9. Continuing the previous example,

$$g\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

which is the first column of the matrix of g. Also $g\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ is the second column of the matrix of g. Finally $g\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix}$ which is the third column of the matrix of g.

The last example illustrates a basic point. The *columns* of the matrix of a linear function can be found by computing the value of the linear function on the standard basis vectors in the domain of the function. Once the matrix of the function is known, the value on *any* vector can be computed. So a linear function is completely determined by its value on the standard basis vectors.

Example 7–10. This property of linear functions can be useful in finding the formula for a linear function with desired properties. What is the formula for the linear function which rotates the point in the plane through an angle of θ in the counterclockwise direction with the origin as pivot? To find the formula for this linear function R, only $R(e_1)$ and $R(e_2)$ must be computed. Simple geometry gives $R(e_1) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and $R(e_2) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$. Thus $R\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

A final important property of linear functions is that lines in the domain of a linear function are mapped to lines in the range of the linear function.

Example 7–11. Suppose $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x-y \\ x-3y \end{pmatrix}$. The line $t\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in the domain of f is mapped to the line $f\begin{pmatrix} t \\ 2t \end{pmatrix} = \begin{pmatrix} 0 \\ -5t \end{pmatrix} = t\begin{pmatrix} 0 \\ -5 \end{pmatrix}$ in the range of f.

Exercise 7–5. Where does f map the box $\left\{\begin{pmatrix} x \\ y \end{pmatrix} : 0 \le x \le 1, 0 \le y \le 1\right\}$ in its domain?

This geometric fact is expressed algebraically in the requirement that in order for a function f to be linear, the equality $f(\alpha u + \beta v) = \alpha f(u) + \beta f(v)$ must hold for any two vectors u and v and any two numbers α and β. This fact is computationally very useful.

The most important facts about linear functions are

(1) A function f is linear if and only if for any two vectors u and v and any two numbers α and β, $f(\alpha u + \beta v) = \alpha f(u) + \beta f(v)$.

(2) A function f is linear if and only if there is a matrix A of constants so that $f(u) = Au$ for all vectors u.

(3) A linear function f is completely determined by the values of f on the standard basis vectors in the domain of f. In fact, the columns of the matrix A are the values of f on the standard basis vectors.

(4) A linear function f maps lines in the domain of f into lines in the range of f. Also parallel lines in the domain of f are mapped into parallel lines in the range of f.

Problems

Problem 7–1. If the graph of the function $m : \mathbf{R} \to \mathbf{R}^2$ is $\left\{ \begin{pmatrix} x \\ 2x \\ 3x \end{pmatrix} : x \in \mathbf{R} \right\}$, is the function m a linear function? Briefly justify your answer.

Problem 7–2. True or False: $\begin{pmatrix} x \\ y \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix} = \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2$.

Problem 7–3. True or False: It is possible for a linear function f with domain $\mathbf{R}^2$ to have $f(e_1) = 3$, $f(e_2) = 5$, and $f\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 9$.

Problem 7–4. True or False: If $g : \mathbf{R}^2 \to \mathbf{R}^3$ is a linear function then the matrix of g has 2 rows and 3 columns.

Problem 7–5. If $k : \mathbf{R}^2 \to \mathbf{R}^3$ is a linear function and $k\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$, what is $k\begin{pmatrix} 5 \\ -10 \end{pmatrix}$?

Problem 7–6. Is the function $g\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2x + 3y - 9z$ linear? If so, write g as a dot product.

Problem 7–7. What is the matrix of the linear function $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - 3y + 6z \\ y - 6z \\ x + 7y \end{pmatrix}$?

Problem 7–8. Suppose $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - y \\ x - 3y \end{pmatrix}$. Compute $g\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Compute $g\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Write the linear function g in matrix form by filling in the blanks: $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Problem 7–9. True or False: If f is a linear function then $f(0) = 0$.

Problem 7–10. Is $\left(2\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right) \bullet \begin{pmatrix} 1 \\ 7 \end{pmatrix} = 2\left(\begin{pmatrix} 3 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 7 \end{pmatrix}\right)$? Is there anything special about the number 2 here? Is there anything special about the two vectors?

Problem 7–11. For what value of θ is $\begin{pmatrix} a \\ b \end{pmatrix} \bullet \left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}\right)$ a maximum? A minimum? Zero?

Solutions to Problems

Problem 7–1. The function m is linear, since the graph of m is a line through the origin. Also the formula for m is $m(x) = \begin{pmatrix} 2x \\ 3x \end{pmatrix}$.

Problem 7–2. True, by using the definition of norm.

Problem 7–3. From the first two conditions, $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix}$ so $f\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \cdot 3 + 2 \cdot 5 = 13 \neq 9$. False.

Problem 7–4. False. The matrix has 3 rows and 2 columns.

Problem 7–5. $k\begin{pmatrix} 5 \\ -10 \end{pmatrix} = k\left(5\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right) = 5k\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 20 \\ 25 \\ 30 \end{pmatrix}$.

Problem 7–6. Here $g\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -9 \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, so g is linear.

Problem 7–7. The matrix of f is $\begin{pmatrix} 2 & -3 & 6 \\ 0 & 1 & -6 \\ 1 & 7 & 0 \end{pmatrix}$.

Problem 7–8. $g\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \times 1 - 0 \\ 1 - 3 \times 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. $g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \times 0 - 1 \\ 0 - 3 \times 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$. $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Problem 7–9. True. Notice that the zeros appearing here may well be the zero vector in different dimensional spaces.

Problem 7–10. Yes, a general fact is that if a is a number and u and v are vectors then $(au) \bullet v = a(u \bullet v)$. In this sense, the dot product behaves like multiplication.

Problem 7–11. Use the calculus of one variable to show that the maximum occurs when $\theta = 0$, the minimum occurs when $\theta = \pi$ and the value is zero when $\theta = \pi/2$.

Solutions to Exercises

Exercise 7–1. A general point on the graph of f has the form $\begin{pmatrix} x \\ y \\ f\begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix} = \begin{pmatrix} x \\ y \\ ax+by \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ b \end{pmatrix}$ where x and y are arbitrary real numbers.

Exercise 7–2. No, since there is no vector $\begin{pmatrix} a \\ b \end{pmatrix}$ of constants so that $h\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix}$.

Exercise 7–3. The entry in row 2 and column 1 is 5.

Exercise 7–4. The matrix is $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{pmatrix}$.

Exercise 7–5. The linear function f maps the box to the parallelogram with vetices at the origin, $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, and $\begin{pmatrix} -1 \\ -3 \end{pmatrix}$.

§8. Derivatives

The derivative of a real valued function f of a single variable is $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$. Re-writing this expression after dropping the limit shows that for small numbers h the equality

$$f(a+h) - f(a) = f'(a)h$$

is approximately true. Since $f'(a)$ is a number, the right side of this approximate equality defines a linear function of h. The definition of derivative is therefore the formal statement that the function $f(a+h) - f(a)$ is approximately a linear function of h when h is near 0. Viewed in this way, the definition of derivative in higher dimensions is easy to grasp. A function f has a derivative at the point a if for vectors h that are near the zero vector the function $f(a+h) - f(a)$ is approximately a linear function of h. This approximating linear function is called the **derivative** of f and the matrix of this linear function is denoted $df(a)$. Thus

$$f(a+h) - f(a) = df(a)h$$

is approximately true for small vectors h. How is this approximating linear function $df(a)$ determined?

Example 8–1. Suppose $g\begin{pmatrix} x \\ y \end{pmatrix} = x+y$. For a small vector $\begin{pmatrix} h \\ k \end{pmatrix}$, $g\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h \\ k \end{pmatrix}\right) - g\begin{pmatrix} x \\ y \end{pmatrix} = h+k = (1 \quad 1)\begin{pmatrix} h \\ k \end{pmatrix}$. So the matrix of the approximating linear function is $dg\begin{pmatrix} x \\ y \end{pmatrix} = (1 \quad 1)$ in this case.

Notice that the approximating linear function must have the same domain and range space as the original function. Since in the example $g : \mathbf{R}^2 \to \mathbf{R}$, the matrix of the linear function must have 1 row and 2 columns.

Example 8–2. Suppose $j\begin{pmatrix} x \\ y \end{pmatrix} = x^2+y^2$. Then $j\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h \\ k \end{pmatrix}\right) - j\begin{pmatrix} x \\ y \end{pmatrix} = (x+h)^2 + (y+k)^2 - (x^2+y^2) = 2xh + 2yk + h^2 + k^2$. So the linear function of $\begin{pmatrix} h \\ k \end{pmatrix}$ which best approximates this difference is $dj\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} = 2xh + 2yk = (2x \quad 2y)\begin{pmatrix} h \\ k \end{pmatrix}$. Thus $dj\begin{pmatrix} x \\ y \end{pmatrix} = (2x \quad 2y)$. In this case the linear approximation is not exact. The term $h^2 + k^2$ represents the error in the linear approximation. The important point is that this error goes to zero faster than h and k go to zero. Computationally, the entries in the matrix of the approximating linear function are obtained as follows. The first entry is obtained by differentiating j with respect to the variable x, treating y as a number; the second entry is obtained by differentiating j with respect to y treating x as a number.

The methodology of the example is completely general. A key observation of the last section is that a linear function is determined by its values on the standard basis vectors. Suppose the small vector is $he_1 = \begin{pmatrix} h \\ 0 \end{pmatrix}$, where the number h is small. Then $f\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} h \\ 0 \end{pmatrix}\right) - f\begin{pmatrix} x \\ y \end{pmatrix} = df\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} h \\ 0 \end{pmatrix}$ approximately, so $df(e_1) = \left(f\begin{pmatrix} x+h \\ y \end{pmatrix} - f\begin{pmatrix} x \\ y \end{pmatrix}\right)/h$, approximately. The approximation becomes exact as $h \to 0$. Thus $df(e_1) = \lim_{h\to 0}\left(f\begin{pmatrix} x+h \\ y \end{pmatrix} - f\begin{pmatrix} x \\ y \end{pmatrix}\right)/h$. This is nothing more than the formula for computing the derivative of f treating x as the variable and y as though it were a number. A similar computation holds in the other coordinates.

Example 8–3. The computations of the preceding example and exercise show that the linear function which approximates $f\begin{pmatrix} x \\ y \end{pmatrix} = x^2y$ near the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is $df\begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

The example has illustrated the general methodology used to identify the approximating linear function. The computational steps involve the familiar rules of calculus applied while treating all but one of the variables as though they were numbers.

To formalize the results of this discussion, define the **partial derivative** of the function f in the x direction at the point a, denoted $D_1f(a)$, by the formula

$$D_1f(a) = \lim_{m\to 0} \frac{f(a + me_1) - f(a)}{m}.$$

Similarly, the partial derivative of f in the y direction at the point a, denoted $D_2f(a)$ is

$$D_2f(a) = \lim_{m\to 0} \frac{f(a + me_2) - f(a)}{m}.$$

Notice that there are as many partial derivatives of f at the point a as the dimension of the domain of f. The ith partial derivative of f is defined by using the ith standard basis vector. The derivative of $f : \mathbf{R}^2 \to \mathbf{R}$ is then the matrix

$$df\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} D_1f\begin{pmatrix} x \\ y \end{pmatrix} & D_2f\begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix}.$$

A similar formula defines df in the higher dimensional cases.

Example 8–4. Suppose $g\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + 3xy - y^3$. Then $D_1g\begin{pmatrix} x \\ y \end{pmatrix} = 2x + 3y$ and

$D_2 g\begin{pmatrix} x \\ y \end{pmatrix} = 3x - 3y^2$. Also $D_1 g\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 18$ and $D_2 g\begin{pmatrix} 3 \\ 4 \end{pmatrix} = -39$. So $dg\begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (18 \quad -39)\begin{pmatrix} x \\ y \end{pmatrix}$.

Example 8–5. Suppose $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x + 2xz - y^2 + yz$. Then $D_1 f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 + 2z$, $D_2 f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2y + z$ and $D_3 f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2x + y$. So $df\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (7 \quad -1 \quad 4)$ and $df\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 7x - y + 4z$.

Partial derivatives can be given a geometric interpretation. As m varies the vectors $a + me_1$ form the line through a in the direction e_1. Thus $D_1 f(a)$ represents the rate at which the values of f are changing as one moves near a in the e_1 direction. A similar interpretation attaches to the other partial derivatives.

The geometric interpretation of the preceding paragraph suggests other possibilities. The **directional derivative** of f in the direction u at the point a, denoted $D_u f(a)$, is the number defined by the formula $D_u f(a) = \lim_{m \to 0} \frac{f(a + mu) - f(a))}{\| mu \|}$. Not surprisingly, $D_u f(a) = df(a) \bullet u / \| u \|$.

There is some alternate notation for partial derivatives that is common. The multitude of notational options is due to different developments of this theory in the past 150 years. Of course, once notation is introduced, the notation develops a fan club which never quite lets its favorite notation die! One sometimes writes $\frac{\partial}{\partial x} f\begin{pmatrix} x \\ y \end{pmatrix} = D_1 f\begin{pmatrix} x \\ y \end{pmatrix}$ and $\frac{\partial}{\partial y} f\begin{pmatrix} x \\ y \end{pmatrix} = D_2 f\begin{pmatrix} x \\ y \end{pmatrix}$. Similar expressions are used in higher dimensions. When $f : \mathbf{R}^2 \to \mathbf{R}$, the **gradient vector** of f, defined by $\nabla f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} D_1 f\begin{pmatrix} x \\ y \end{pmatrix} \\ D_2 f\begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix}$ is often used instead of the derivative. The connection is that $df\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \nabla f\begin{pmatrix} x \\ y \end{pmatrix} \bullet \begin{pmatrix} a \\ b \end{pmatrix}$.

Example 8–6. In what direction is the directional derivative the largest? The gradient vector gives an expression for the directional derivative as a dot product: $D_u f\begin{pmatrix} x \\ y \end{pmatrix} = df\begin{pmatrix} x \\ y \end{pmatrix} u / \| u \| = \nabla f\begin{pmatrix} x \\ y \end{pmatrix} \bullet u / \| u \|$. Given the expression for the directional derivative as a dot product, the directional derivative $D_u f(a)$ is the largest when the direction u is the same as the gradient $\nabla f(a)$. Thus the gradient vector $\nabla f(a)$ points in the direction in which the values of f are increasing the fastest.

Exercise 8–1. In what direction are the values of f decreasing the fastest?

The possibility exists that the partial derivatives exist but that there still is no approximating linear function. This arises rarely in practice, and will not be examined in further detail here.

The derivative of a function f whose range is a higher dimensional space is found by using the derivatives of the component functions. Naturally, this means that df is a matrix with as many rows as the dimension of the range space of f.

Example 8–7. Suppose $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - 2xy \\ y^3 - x \end{pmatrix}$. Then for the component functions $df_1\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2a - 2b \\ -2a \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix}$ and $df_2\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 3b^2 \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix}$. So the derivative of f itself is $df\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2a - 2b & -2a \\ -1 & 3b^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

In summary, the derivative of a function f at the point a is the linear function $df(a)$ which makes the equality $f(a + h) - f(a) = df(a)h$ approximately true for all vectors h close to the zero vector.

Problems

Problem 8–1. True or False: If $f : \mathbf{R}^3 \to \mathbf{R}^2$ is differentiable at the point p, the matrix of $df(p)$ has 2 rows and 3 columns.

Problem 8–2. If $h\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xe^y - ye^z$ compute $dh\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\nabla h\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Problem 8–3. Suppose $f\begin{pmatrix} x \\ y \end{pmatrix} = xe^y - y\sin x$. Compute $D_1 f\begin{pmatrix} x \\ y \end{pmatrix}$. Compute $D_2 f\begin{pmatrix} x \\ y \end{pmatrix}$. Compute $df\begin{pmatrix} x \\ y \end{pmatrix}$. Compute $df\begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$.

Problem 8–4. Show that

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^2}{x^2+y^2}\right) \neq \lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^2}{x^2+y^2}\right).$$

Problem 8–5. Suppose $f\begin{pmatrix} x \\ y \end{pmatrix} = 7$. What is $df\begin{pmatrix} x \\ y \end{pmatrix}$?

Problem 8–6. For the function $g\begin{pmatrix} x \\ y \end{pmatrix} = 4x - x^2 - y^2$ compute $dg\begin{pmatrix} x \\ y \end{pmatrix}$.

Problem 8–7. Suppose $g\begin{pmatrix} x \\ y \end{pmatrix}$ is a linear function. What is $dg\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$?

Problem 8–8. Suppose $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} xe^z \\ y - z\cos x \end{pmatrix}$. What is $df\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$? Find an approximate value for $f\begin{pmatrix} 0.1 \\ 0.2 \\ 0.3 \end{pmatrix}$.

Solutions to Problems

Problem 8–1. True. Like f itself, $df(p)$ requires 3 dimensional input and produces 2 dimensional output.

Problem 8–2. Here $D_1 h\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^y$, $D_2 h\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xe^y - e^z$ and $D_3 h\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -ye^z$ so $dh\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} e^y & xe^y - e^z & -ye^z \end{pmatrix}$ while $\nabla h\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} e^y \\ xe^y - e^z \\ -ye^z \end{pmatrix}$.

Problem 8–3. $D_1 f\begin{pmatrix} x \\ y \end{pmatrix} = e^y - y\cos x$. $D_2 f\begin{pmatrix} x \\ y \end{pmatrix} = xe^y - \sin x$. $df\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^y - y\cos x & xe^y - \sin x \end{pmatrix}$. $df\begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = a$.

Problem 8–4. The limit on the left is 1, while the limit on the right is 0. This illustrates some of the difficulties that can arise in higher dimensional spaces.

Problem 8–5. $df\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$.

Problem 8–6. $dg\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 - 2x & -2y \end{pmatrix}$.

Problem 8–7. A good linear approximation to a linear function must be the linear function itself. Thus $dg\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = g\begin{pmatrix} x \\ y \end{pmatrix}$.

Problem 8–8. The matrix of

$$df\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} e^z & 0 & xe^z \\ z\sin x & 1 & -\cos x \end{pmatrix},$$

so $df\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y - z \end{pmatrix}$. Use $a = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $h = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.3 \end{pmatrix}$ in the approximate equality for the derivative to obtain $f\begin{pmatrix} 0.1 \\ 0.2 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.1 \\ -0.1 \end{pmatrix}$, approximately.

Solutions to Exercises

Exercise 8–1. The direction $-\nabla f(a)$.

§9. Some Applications

The major applications of multivariate calculus parallel the applications of univariate calculus.

Example 9–1. What is the maximum value of $f(x) = 6x - x^2 - 2$ on the interval $[0, 8]$? The candidates for the location of a maximum (or minimum) value of f consist of those points where the derivative of f is zero, those points where f' is not defined, and the endpoints of the interval. Here $f'(x) = 6 - 2x$, which is zero at $x = 3$; there are no points where the derivative is undefined. So the candidate points are 0, 3, and 8. Computation gives $f(0) = -2$, $f(3) = 7$, and $f(8) = -18$. So the maximum of f occurs at $x = 3$ and the maximum value of f on the interval is 7. Notice that the minimum value of f on this interval occurs at the right hand endpoint of the interval.

Example 9–2. What is the maximum value of $f\begin{pmatrix} x \\ y \end{pmatrix} = 6xy - x^2 - x - y - y^2 - 5$ on the region $R = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : 0 \le x \le 5, 0 \le y \le 9 \right\}$? Analogously with the univariate case, the candidate points for the location of a maximum (or minimum) are the points at which $df\begin{pmatrix} x \\ y \end{pmatrix}$ is the zero linear transformation, the points where the derivative is not defined, and the boundary (edges) of the region. In this case the matrix of $df\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6y - 2x - 1 & 6x - 2y - 1 \end{pmatrix}$, which is the zero matrix only at the points $\begin{pmatrix} x \\ y \end{pmatrix}$ for which *both* of the equations $6y - 2x - 1 = 0$ and $6x - 2y - 1 = 0$ are satisfied. The only point at which both of these equalities hold is $\begin{pmatrix} 2/10 \\ 1/10 \end{pmatrix}$. The other candidate points may lie on the boundary of R. This boundary consists of 4 line segments, and along each segment the variables are determined in some way. Along the lower segment, $y = 0$ and $f\begin{pmatrix} x \\ 0 \end{pmatrix} = -x^2 - 5$ for $0 \le x \le 5$. This function of one variable has candidate points at $x = 0$ and $x = 5$. So $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$ become candidate points for the location of the maximum of f on R. Along the right hand segment, $x = 5$ and $f\begin{pmatrix} 5 \\ y \end{pmatrix} = 29y - y^2 - 10$ for $0 \le y \le 9$. So $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 9 \end{pmatrix}$ are added to the candidate list. Along the top segment $y = 9$ and $f\begin{pmatrix} x \\ 9 \end{pmatrix} = 53x - x^2 - 95$ for $0 \le x \le 5$, so $\begin{pmatrix} 5 \\ 9 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 9 \end{pmatrix}$ are added to the list. Finally, analysis of the left hand segment adds $\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$ to the candidate list. Summarizing, the candidates for location of the maximum are $\begin{pmatrix} 2/10 \\ 1/10 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 5 \\ 9 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 9 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$.

The corresponding values of f are $f\begin{pmatrix} 2/10 \\ 1/10 \end{pmatrix} = -5.23$, $f\begin{pmatrix} 0 \\ 0 \end{pmatrix} = -5$, $f\begin{pmatrix} 5 \\ 0 \end{pmatrix} = -35$, $f\begin{pmatrix} 5 \\ 9 \end{pmatrix} = 145$, $f\begin{pmatrix} 0 \\ 9 \end{pmatrix} = -95$ and $f\begin{pmatrix} 0 \\ 1/2 \end{pmatrix} = -5.75$. So the maximum of f occurs at $\begin{pmatrix} 5 \\ 9 \end{pmatrix}$ and the maximum value is 145.

Some additional complications arise in the case in which the values of the variables are not restricted.

Example 9–3. What is the maximum value of the function $g\begin{pmatrix} x \\ y \end{pmatrix} = 10 - x^2 - y^2$? In this case inspection of the function shows that the maximum value is 10 and is attained at the origin. Mechanical computation alone would give $dg\begin{pmatrix} x \\ y \end{pmatrix} = (-2x \quad -2y)$, and identify the origin as a candidate point for the location of a maximum or minimum. But is the origin the location of a maximum or minimum, or a false alarm?

A general method of answering the question raised in the example is as follows. If the values of the function $f\begin{pmatrix} x \\ y \end{pmatrix}$ must become arbitrarily large and positive as $\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|$ becomes large, then the function can not have an absolute maximum and must have an absolute minimum. Thus the candidate points must be the location of either an absolute minimum, a local minimum, or a local maximum. If the values of the function $f\begin{pmatrix} x \\ y \end{pmatrix}$ must become arbitrarily large and negative as $\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|$ becomes large, then the function can not have an absolute minimum and must have an absolute maximum. The candidate points must be the location of either an absolute maximum, a local maximum, or a local minimum.

Example 9–4. In the previous example, the function values must become arbitrarily large and negative as $\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|$ becomes large. So g must have an absolute maximum and the origin is the location of either an absolute maximum, a local maximum, or a local minimum. Since there are no other candidate points and g must have an absolute maximum, the origin is the location of the absolute maximum.

Because of the complicated nature of the equations for finding the location of candidate points, purely numerical methods are also used. One of the most common of these methods is called the **gradient search** method. The method is based on the fact from the last section that the gradient points in the direction of the rate of greatest increase in the values of the function. If the maximum value of a function f is to be found, choose a starting point p and compute $\nabla f(p)$. Move from p to the point $p_1 = p + m\nabla f(p)$ a short distance from p in the direction of the gradient vector. Now move from p_1 a short distance in the direction of $\nabla f(p_1)$ to the point

p_2. Continue in this way until the gradient vector is approximately the zero vector. This point is (approximately) the location of a local maximum. A more detailed discussion of this method can be found elsewhere.

Multivariate functions are often used in physics. One of the most basic applications is as the position of a particle in space.

Suppose $p(t)$ is the position of a particle in space at time t. The physical interpretation of $p(0)$ is as the position of the particle at the time $t = 0$ at which observation begins. The physical interpretation of $p'(t)$ is as the **velocity** of the particle at time t; $p''(t)$ is the **acceleration** of the particle at time t. Notice that both the velocity and acceleration are vectors. The norm $\| p'(t) \|$ of the velocity vector is the **speed** of the particle at time t. Physically, the velocity is the rate at which the position of the particle is changing, the acceleration is the rate at which the velocity is changing, and the speed is the rate at which the distance travelled by the particle is changing.

Example 9–5. Suppose $p(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$ is the position of a particle at time t. Such a particle is moving on a spiral path in the positive z direction. Then $p'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}$ and $p''(t) = \begin{pmatrix} -\cos t \\ -\sin t \\ 0 \end{pmatrix}$. The vector $p'(t)$ is tangent to the path of motion $p(t)$ at time t. The speed of this particle at time t is $\| p'(t) \| = \sqrt{2}$. The particle moves with constant speed, but non-constant velocity.

Sometimes the domain of the function to be maximized (or minimized) is a lower dimensional surface instead of a region.

Example 9–6. A rectangular box is to be designed which must have a volume of 8 cubic meters. What should the dimensions of the box be in order to use the least amount of material? The amount $M\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ of material used to build a box with length x, width y, and height z is $M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2xy + 2yz + 2xz$. The function M is to be minimized when $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ takes values in the surface $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : xyz = 8 \right\}$.

In order to attack this problem, the nature of surfaces is examined in a bit more detail. The key observation is this. A surface is completely covered by *all* of the one dimensional curves which lie in the surface. This fact allows the surface to be studied by looking at all of these one dimensional curves. Since one dimensional objects are so simple (at least compared to their higher dimensional counterparts) many computational difficulties can be eliminated by using this fact.

A simple application of this fact allows a geometric interpretation of the gradient of a function used to define a surface. Suppose $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \right\}$. If c is a function with domain **R** and range contained in S then $f(c(t)) = 0$ for all t. For small values of h, $f(c(t+h)) - f(c(t)) = df(c(t))(c(t+h) - c(t))$, so $\frac{d}{dt} f(c(t)) = df(c(t))c'(t) = 0$ for all t. In vector form $\nabla f(c(t)) \bullet c'(t) = 0$. So the gradient of f is perpendicular to the tangent of every curve which lies in the surface defined by the equation $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$.

Example 9–7. Returning to the previous example, define the function f by the formula $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xyz - 8$. Then the surface $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \right\}$. The previous discussion shows that $\nabla f\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is perpendicular to the tangent vector to every curve which takes values in S. Now suppose that p is a point in S at which M achieves a maximum or a minimum, and that c is a curve taking values in S with $c(0) = p$. Then $(M \circ c)'(0) = dM(c(0))c'(0) = 0$, so $\nabla M(p)$ is also perpendicular to every tangent vector to a curve lying in the surface S near p. This means that the two gradient vectors $\nabla f(p)$ and $\nabla M(p)$ must be multiples of each other at the critical point p. The point p must also lie on the surface. In the particular case of this example, $\nabla M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2y + 2z \\ 2x + 2z \\ 2x + 2y \end{pmatrix}$ and $\nabla f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}$. These two vectors are multiples of each other only when $x = y = z$. The point with equal coordinates that lies on the surface is the point $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$, which is the point that minimizes M.

The technique developed in the last example is called the **Lagrange multiplier method**. In summary, to find the critical points of a function M subject to a restriction specified by the requirement $f = 0$, find those points p for which

(1) $f(p) = 0$ and

(2) there is a number m so that $\nabla M(p) = m\nabla f(p)$.

Keep in mind that such points p can correspond to a maximum or a minimum of M, or neither.

These types of problems can also be solved by using the restriction to eliminate one of the variables.

Example 9–8. In the preceding example the restriction $xyz = 8$ can be used to write z (say) in terms of x and y. Thus $z = 8/xy$. Then $M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M\begin{pmatrix} x \\ y \\ 8/xy \end{pmatrix} = 2xy + 16/y + 16/x$ with the variables x and y no longer restricted except that each must be positive. The derivative of this unrestricted function of two variables is $(\,2y - 16/x^2 \quad 2x - 16/y^2\,)$ which is the zero matrix only when $x = y = 2$. This is the location of an absolute minimum. The value of $z = 8/xy = 8/2\times 2 = 2$, as before.

The advantage of the Lagrange multiplier method is that the restriction does not have to be used to eliminate one of the variables. The disadvantage is that the equations to be solved tend to be more complicated. In practice, the simpler of these two approaches should be followed.

Partial derivatives often arise in constructing a mathematical model of a physical situation. This usually leads to an equation among partial derivatives of the unknown function. The resulting partial differential equation must then be solved for the function of interest. This type of application and the methods of solving such equations will not be discussed here.

Problems

Problem 9–1. What is the maximum value of $f\begin{pmatrix} x \\ y \end{pmatrix} = xy$ on the triangular region $T = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : 0 \le y \le x, 0 \le x \le 4 \right\}$?

Problem 9–2. What is the maximum value of $g\begin{pmatrix} x \\ y \end{pmatrix} = e^{-x-y}$ on $\mathbf{R}^2$?

Problem 9–3. What is the maximum and minimum value of $g\begin{pmatrix} x \\ y \end{pmatrix} = xy$?

Problem 9–4. Find the maximum and minimum of the function $f\begin{pmatrix} x \\ y \end{pmatrix} = 10 - x^2 - y^2$ if $\begin{pmatrix} x \\ y \end{pmatrix}$ must lie on the line $y = 3x + 5$.

Problem 9–5. What is the maximum value of $f\begin{pmatrix} x \\ y \end{pmatrix} = 2xy + 28x - x^2 - x^4 + 4y - y^2$?

Problem 9–6. At what point(s) does the function $g\begin{pmatrix} x \\ y \end{pmatrix} = 4x - x^2 - y^2$ achieve its maximum value? Its minimum value?.

Problem 9–7. Suppose $p(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$ is the position of a particle at time t. How far does the particle travel between times $t = 0$ and $t = 1$?

Problem 9–8. Suppose $p(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ is the position of a particle in the plane at time t. How far does the particle travel between times $t = 0$ and $t = 6$? At time $t = 6$ how far is the particle from where it was at time $t = 0$?

Problem 9–9. A box is to be constructed that holds 8 cubic meters. The bottom of the box requires 4 times the amount of material per unit area as the sides or the top. What are the dimensions of the box that uses the least amount of material?

Problem 9–10. Suppose $f\begin{pmatrix} x \\ y \end{pmatrix} = xy - y^2$, and that $g(t) = \begin{pmatrix} t \sin t \\ e^t \end{pmatrix}$. What is $(f \circ g)'(t)$? Hint: What does the chain rule look like here?

Solutions to Problems

Problem 9–1. Here $df\begin{pmatrix} x \\ y \end{pmatrix} = (y \quad x)$, which is only the zero matrix at the origin. So one candidate point is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Analysis of the boundaries shows that the only candidate points are at the vertices of the triangle. Now $f\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$, $f\begin{pmatrix} 4 \\ 0 \end{pmatrix} = 0$, and $f\begin{pmatrix} 4 \\ 4 \end{pmatrix} = 16$, so the maximum is at $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$ and the maximum value is 16.

Problem 9–2. The function g has no maximum value on this region. The values of g become arbitrarily large as $x + y$ becomes large and negative.

Problem 9–3. Although the origin is a candidate point, the origin is neither the location of a maximum or a minimum. The function has neither a maximum or a minimum.

Problem 9–4. Eliminating y as a variable in the original function gives the function of one variable as $10 - x^2 - (3x + 5)^2 = -15 - 10x^2 - 30x$ which has a candidate point at $x = -3/2$. This is the location of an absolute maximum. Thus $\begin{pmatrix} -3/2 \\ 1/2 \end{pmatrix}$ is the location of the absolute maximum of the original function and the maximum value is $f\begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix} = 9.5$. The function has no minimum value along this line.

Problem 9–5. Here $df\begin{pmatrix} x \\ y \end{pmatrix} = (2y + 28 - 2x - 4x^3 \quad 2x + 4 - 2y)$. Solving shows that $df\begin{pmatrix} x \\ y \end{pmatrix} = (0 \quad 0)$ only at the point $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$. Since the highest power terms appear with negative coefficients, the values of f become large and negative whenever $\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|$ is large. So $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ is the location of an absolute maximum, and the maximum value is $f\begin{pmatrix} 2 \\ 4 \end{pmatrix} = 36$.

Problem 9–6. The candidate points for the location of a maximum or minimum are those points at which $dg\begin{pmatrix} x \\ y \end{pmatrix}$ is the zero matrix. Here $dg\begin{pmatrix} x \\ y \end{pmatrix} = (4 - 2x \quad -2y)$ and there is only one such point, namely, $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Since as either $x \to \infty$ or $y \to \infty$ the values of g become more and more negative, g does not have a minimum value. So $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ is the location of the absolute maximum. As an alternate approach, $g\begin{pmatrix} x \\ y \end{pmatrix} = 4 - (x - 2)^2 - y^2$, so $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ is the location of an absolute maximum.

Problem 9–7. The distance travelled depends on the speed. If $D(t)$ is the distance travelled by time t, $D(t + h) = D(t) + h\left\| p'(t) \right\|$ for small h, so that $D'(t) = \left\| p'(t) \right\|$. Thus the distance travelled between times $t = 0$ and $t = 1$ is

$$\int_0^1 \| p'(t) \| \, dt = \int_0^1 \sqrt{2}\, dt = \sqrt{2}.$$

Problem 9–8. The particle has travelled $\int_0^6 \| p'(t) \| \, dt = 6$ distance units in the 6 time units between $t = 0$ and $t = 6$. At the end of this time, the particle is $\| p(6) - p(0) \| = 0.2822$ distance units from where it began. Note that the particle is travelling counterclockwise around the circumference of the unit circle.

Problem 9–9. Here the amount $M\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ of material required to make the box is $M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5xy + 2xz + 2yz$, and the requirement is $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xyz - 8 = 0$. The multiple requirement then shows that $x = y$ and $z = 5x/2 = 5y/2$, from which $x = (16/5)^{1/3} = y = 1.473$, with the corresponding value for $z = 3.684$. This can also be solved by eliminating one of the variables and finding the unrestricted minimum of the resulting function of two variables.

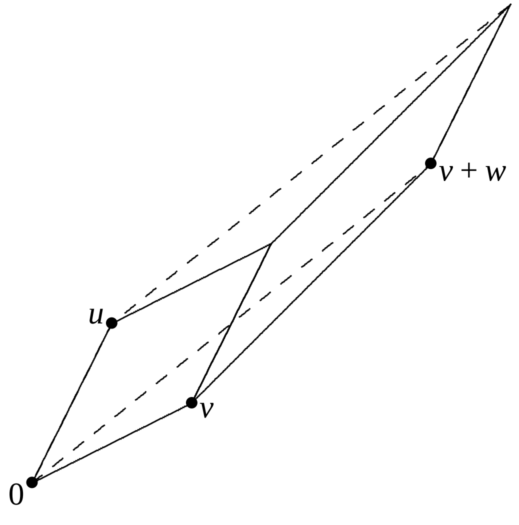
Problem 9–10. For small values of h, $f(g(t+h)) - f(g(t)) = df(g(t))(g(t+h) - g(t))$, approximately. Dividing by h and taking limits gives $(f \circ g)'(t) = df(g(t))g'(t)$.

§10. Determinants and Volume

Suppose you want to find the volume of a box whose opposite sides are parallel, but which is not necessarily a rectangular box. For the moment, consider the case of a two dimensional box, for which volume means area and the box is simply a parallelogram. If two of the sides are determined by the vectors u and v, denote the area by $A(u, v)$. What properties does this area function have that would assist in computing its value?

First note that $A(2u, v) = 2A(u, v)$, and a similar property holds when 2 is replace by any positive constant. A similar property holds for multiples of the second argument as well.

Second, geometric considerations show that if w is any vector then $A(u, v+w) = A(u, v) + A(u, w)$. The illustration is given in the picture below. The area of the parallelogram with two dashed sides is equal to the sum of the areas of the two parallelograms with solid sides, since the side-angle-side congruence theorem shows that the two triangles have the same area. A similar property holds for addition in the first argument.



Third, geometric considerations also show that $A(u, u) = 0$.

Combining these properties shows that the function A behaves almost like a linear transformation with respect to each argument separately. The only difficulty is that negative multiples can't pull out properly like positive multiples can.

Example 10–1. Unfortunately, these properties and the requirement that the area always be non-negative are not compatible. To see this, $0 = A(0, v) = A(u -$

$u, v) = A(u, v) + A(-u, v)$. So $A(-u, v) = -A(u, v)$ follows as a consequence of these properties. Negative values for the area function are unavoidable!

The unavoidability of negative values can be turned to our advantage. To do this, *define* a function which does have the full linearity property. The **determinant** function on $\mathbf{R}^2$, denoted by $\det(u, v)$ is the function with the properties

(1) $\det(\alpha u + \beta w, v) = \alpha \det(u, v) + \beta \det(w, v)$ for all vectors u, v, and w and all numbers α and β.

(2) $\det(u, \alpha v + \beta w) = \alpha \det(u, v) + \beta \det(u, w)$ for all vectors u, v, and w and all numbers α and β.

(3) $\det(u, u) = 0$ for any vector u.

(4) $\det(u, v) = -\det(v, u)$ for any vectors u and v.

(5) $\det(e_1, e_2) = 1$.

These properties are sufficient to allow computation of the determinant.

Example 10–2. Compute $\det(\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix})$. Since $\begin{pmatrix}1\\2\end{pmatrix} = e_1 + 2e_2$, and $\begin{pmatrix}3\\4\end{pmatrix} = 3e_1 + 4e_2$ using the properties above gives $\det(\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}) = \det(e_1 + 2e_2, 3e_1 + 4e_2) = \det(e_1, 3e_1 + 4e_2) + 2\det(e_2, 3e_1 + 4e_2) = 3\det(e_1, e_1) + 4\det(e_1, e_2) + 6\det(e_2, e_1) + 8\det(e_2, e_2) = 4\det(e_1, e_2) + 6\det(e_2, e_1) = -2\det(e_1, e_2) = -2$, since $\det(e_1, e_2) = 1$.

Exercise 10–1. Show that $\det(\begin{pmatrix}a\\c\end{pmatrix}, \begin{pmatrix}b\\d\end{pmatrix}) = ad - bc$.

The only remaining problem is discover the relationship between the determinant function and the area function.

Consider the two values $\det(-u, v) = -\det(u, v)$. Geometric considerations show that the parallelograms with sides $-u$ and v and with sides u and v have equal area. Notice that starting from the origin and traversing the perimeter of the parallelogram beginning in the direction of the first vector argument, the interior of the parallelogram will lie to the right in one case and to the left in the other. Since when traversing the perimeter of the parallelogram with sides e_1 and e_2 (in that order) the interior of the parallelogram lies to the left, such a paralellogram has **positive orientation**. If the interior lies to the right the parallelogram has **negative orientation**. Thus the value of the determinant function has a geometric interpretation: a positive value of the determinant is equal to the area of the parallelogram, and the parallelogram has a positive orientation; a negative value of the determinant is equal to the negative of the area of the parallelogram, and the parallelogram has a negative orientation. Thus $A(u, v) = |\det(u, v)|$ and the sign of the determinant indicates only the orientation of the parallelogram.

Another point of view will help make the interpretation of orientation more visible. Suppose B is a linear function with domain and range space $\mathbf{R}^2$. The columns of B are the values of Be_1 and Be_2, and these two vectors determine the parallelogram that results when B is applied to all points inside the unit square (the parallelogram in the domain space of B with sides e_1 and e_2). Thus $\det(Be_1, Be_2)$ measures how B changes area and orientation. For simplicity of notation, write $\det(B)$ instead of the bulkier $\det(Be_1, Be_2)$.

The exercise above shows that if $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det(B) = ad - bc$.

The geometric interpretation of determinant as the area of the output of the linear function applied to the unit square shows that the linear function B has an inverse function if and only if $\det(B) \neq 0$.

This discussion extends to spaces of any dimension. Notice that the determinant function always requires as many arguments as the dimension of the space, so the determinant function can only be applied to square matrices, that is, matrices with the same number of rows as columns. The property of determinant that changes with dimension is the last one. In $\mathbf{R}^3$, that requirement becomes $\det(e_1, e_2, e_3) = 1$; and generally $\det(e_1, e_2, \ldots, e_d) = 1$ in d dimensional space.

Problems

Problem 10–1. What is $\det \begin{pmatrix} 3 & -4 \\ 2 & -1 \end{pmatrix}$?

Problem 10–2. What is $\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 6 \\ 1 & 0 & 1 \end{pmatrix}$?

Problem 10–3. Suppose B is the linear function with matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. How does B change area? Does B preserve orientation?

Problem 10–4. Suppose $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y \\ xy \end{pmatrix}$. How does f change area near the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$? Near the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$? Does f preserve orientation?

Solutions to Problems

Problem 10–1. The determinant is 5.

Problem 10–2. The determinant is 4. This is computed by writing each column in terms of the basis vectors and expanding using linearity, as in the example of this section.

Problem 10–3. The determinant of the matrix of B is -2, so B magnifies areas by a factor of 2, and also changes orientation.

Problem 10–4. The matrix of $df\begin{pmatrix} x \\ y \end{pmatrix}$ is $\begin{pmatrix} 2x & -1 \\ y & x \end{pmatrix}$ which has determinant $2x^2 + y$. At $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ this determinant is zero, so f is collapsing space to a lower dimension near this point. Near $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ the determinant is 10, so f is magnifying area by a factor of 10 near this point, and is also preserving orientation there.

Solutions to Exercises

Exercise 10–1. Simply expand as in the example.

§11. Changing Variables in Multiple Integrals

Changing variables in multiple integrals is more complicated than in the case of single integrals. The method does have a simple geometric interpretation. Over a small region A containing a point p, the value of f on the image set $g(A)$ is approximately $f(g(p))$, so for small regions A, $\int_{g(A)} f\begin{pmatrix} x \\ y \end{pmatrix} dx{\times}dy = f(g(p)){\times}\text{volume of } g(A)$, approximately. But the volume of $g(A)$ is approxmately $|\det dg(p)| \times$ volume of A, so

$$\begin{aligned}\int_{g(A)} f\begin{pmatrix} x \\ y \end{pmatrix} dx \times dy &= f(g(p)) \times \text{volume of } g(A) \\ &= f(g(p))|\det dg(p)| \times \text{volume of } A \\ &= \int_A (f \circ g)\begin{pmatrix} x \\ y \end{pmatrix} |\det dg\begin{pmatrix} x \\ y \end{pmatrix}| \, dx \times dy\end{aligned}$$

approximately. This argument can be made precise by taking limits as the region shrinks to zero, and adding together the integrals over the resulting small pieces.

Change of Variable Theorem. *If g is a function with domain A in $\mathbf{R}^2$ and range in $\mathbf{R}^2$ which has an inverse function and if f is function whose domain includes $g(A)$ then*

$$\int_{g(A)} f\begin{pmatrix} x \\ y \end{pmatrix} dx \times dy = \int_A (f \circ g)\begin{pmatrix} x \\ y \end{pmatrix} |\det dg\begin{pmatrix} x \\ y \end{pmatrix}| \, dx \times dy.$$

A similar statement holds when the domain and range of g lie in $\mathbf{R}^3$, or a space of any other dimension.

Here $|\det dg\begin{pmatrix} x \\ y \end{pmatrix}|$ is the absolute value of the determinant of the *matrix* of the linear function $dg\begin{pmatrix} x \\ y \end{pmatrix}$. The function $|\det dg|$ is called the **Jacobian** of g.

The typical case is that in which the function g is chosen so that A is a rectangle.

Example 11–1. Suppose $B = \left\{\begin{pmatrix} x \\ y \end{pmatrix} : 0 \le x+y \le 2, -2 \le x-y \le 0\right\}$. What is $\int_B e^x \, dx{\times}dy$? A box is a set defined by inequalities in which the variable(s) appear between fixed numerical limits. The form of the set B suggests that using $x{+}y$ and $x{-}y$ as the new variables will produce a box as the new region of integration. For notational convenience, write $u = x{+}y$ and $v = x{-}y$. Then simple algebra gives $x = (u{+}v)/2$ and $y = (u{-}v)/2$. Define $g\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (u+v)/2 \\ (u-v)/2 \end{pmatrix}$ and $A = \left\{\begin{pmatrix} u \\ v \end{pmatrix} : 0 \le u \le 2, -2 \le v \le 0\right\}$. Then $g(A)$ is the original region of integration, and $dg\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$, so $|\det dg\begin{pmatrix} u \\ v \end{pmatrix}| = 1/2$. Thus $\int_B e^x \, dx \times dy = \int_A e^{(u+v)/2}(1/2) \, du \times dv$, which is far more easily evaluated.

Another common case is that in which either the original integrand or the region of integration exhibit circular symmetry. This situation arises in applications since gravitational and electrical fields exhibit such symmetry. In this situation the use of **polar coordinates** produces a rectangular region of integration in polar coordinate space.

Example 11–2. As an illustration, consider the quarter circle region

$$Q = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x^2 + y^2 \le 1, 0 \le x \le 1, 0 \le y \le 1 \right\}$$

in the first quadrant. What is $\int_Q x^2 + y^2 \, dx \times dy$? This integral can be computed by simply writing down an iterated integral, but the computations are messy and error prone. Recall that polar coordinates in $\mathbf{R}^2$ are defined in terms of the usual coordinates x and y by the formulas $x = r\cos\theta$ and $y = r\sin\theta$. The requirements on r and θ are that $r \ge 0$ and $0 \le \theta < 2\pi$. The function $g\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$ maps from polar coordinate space to regular space. Now Q is the image under g of the rectangular region $R = \left\{ \begin{pmatrix} r \\ \theta \end{pmatrix} : 0 \le r \le 1, 0 \le \theta \le \pi/2 \right\}$. So $\int_Q x^2 + y^2 \, dx \times dy = \int_{g(R)} x^2 + y^2 \, dx \times dy = \int_R r^2 |\det dg| \, dr \times d\theta$. Here $dg\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$, so $|\det dg| = r$. So $\int_R r^2 |\det dg| \, dr \times d\theta = \int_0^1 \int_0^{\pi/2} r^3 \, d\theta dr = \pi/8$.

Keep in mind that changing variables can simplify the computation of multiple integrals in many cases by changing the region of integration to a rectangle. The trade off is that the integrand becomes more complicated.

Problems

Problem 11–1. What is the value of the integral in the first example of this section?

Problem 11–2. Sketch the region of integration for the integral $\int_0^1 \int_{x^2}^{2x^2} 1\, dy\, dx$, write an equivalent iterated integral with the order of integration reversed, and evaluate the double integral.

Problem 11–3. Compute the volume of the region in $\mathbf{R}^3$ defined by the inequalities $x \geq 0$, $y \geq 0$, $z \geq 0$, $x + y + z \leq 1$.

Problem 11–4. Let R be the region bounded by $y = x^2$, $y = 2x^2$, and $x = 1$ in $\mathbf{R}^2$. Compute $\int_R xy^2\, dx \times dy$.

Problem 11–5. Suppose $A = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 0 \leq y \leq 2, 0 \leq y \leq x, x \leq z \leq 2x \right\}$. Compute $\int_A e^{-x}\, dx \times dy \times dz$.

Problem 11–6. Let R be the region in $\mathbf{R}^2$ in which $1 \leq x^2 + y^2 \leq 2$ and $y \geq |x|$. Sketch the region and compute $\int_R x^2 + y^2\, dx \times dy$.

Problem 11–7. Suppose $C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : 1 \leq xy \leq 3, x \leq y \leq 4x \right\}$. Compute $\int_C dx \times dy$.

Problem 11–8. Compute $\int_0^\infty \int_0^\infty e^{-x^2-y^2}\, dx\, dy$.

Solutions to Problems

Problem 11–1. $\int_A e^{(u+v)/2}(1/2)\,du \times dv = \int_0^2 \int_{-2}^0 e^{(u+v)/2}\,dv\,du = 4e - 8 + 4e^{-1}.$

Problem 11–2. $\int_0^1 \int_{\sqrt{y/2}}^{\sqrt{y}} 1\,dx\,dy + \int_1^2 \int_{\sqrt{y/2}}^{1} 1\,dx\,dy = 1/3.$

Problem 11–3. The volume is 1/6.

Problem 11–4. One method is to change variables. The definition of R suggests $u = y/x^2$ and $v = x$ as the new variables, from which $x = v$ and $y = uv^2$. Define $g(u, v) = (v, uv^2)$ as a mapping from $\mathbf{R}^2$ in u-v coordinates to $\mathbf{R}^2$ with x-y coordinates. Let $A = \{(u, v) : 1 \le u \le 2, 0 \le v \le 1\}$. Then $g(A) = R$, $\det dg\begin{pmatrix} u \\ v \end{pmatrix} = -v^2$ and the change of variables formula gives

$$\begin{aligned}\int_R xy^2 &= \int_A u^2v^5 \times v^2\,du \times dv \\ &= \int_1^2 \int_0^1 u^2v^7\,dv\,du \\ &= 7/24.\end{aligned}$$

Problem 11–5. From the definition of A a reasonable choice of new variables would be $u = y$, $v = y/x$, and $w = z/x$, since with this choice the new region of integration is the box $B = \left\{\begin{pmatrix} u \\ v \\ w \end{pmatrix} : 0 \le u \le 2, 0 \le v \le 1, 1 \le w \le 2\right\}$. Solving gives $x = u/v$, $y = u$, and $z = uw/v$ and the Jacobian matrix is $\begin{pmatrix} 1/v & -u/v^2 & 0 \\ 1 & 0 & 0 \\ w/v & -uw/v^2 & u/v \end{pmatrix}$ which has determinant u^2/v^3. The desired integral is equal to $\int_1^2 \int_0^1 \int_0^2 e^{-u/v} u^2/v^3\,du\,dv\,dw = 2 - 4e^{-2}$. This integral can also be computed by simply setting up an iterated integral without a change of variables.

Problem 11–6. Using polar coordinates gives the equivalent integral as $\int_{\pi/4}^{3\pi/4} \int_1^{\sqrt{2}} r^3\,dr\,d\theta = 3\pi/8.$

Problem 11–7. Use the new variables $u = xy$ and $v = y/x$ for which the region of integration becomes $\left\{\begin{pmatrix} u \\ v \end{pmatrix} : 1 \le u \le 3, 1 \le v \le 4\right\}$. Then $x = \sqrt{u/v}$ and $y = \sqrt{uv}$ and the Jacobian is $1/2v$. The value of the integral is $2\ln(2)$.

Problem 11–8. Here the region in polar space is

$$R = \left\{\begin{pmatrix} r \\ \theta \end{pmatrix} : 0 \le r < \infty, 0 \le \theta \le \pi/2\right\}.$$

Computation gives the value of the integral as $\pi/4$.

§12. Forces, Work, Fluid Flow

Most of the remainder of the discussion here deals with mathematics that was constructed to deal with physical problems. Most of these physical problems have simple interpretations.

A **force** is a physical quantity that has both a magnitude and direction. For this reason, forces are often represented mathematically by vectors. In some applications, the force varies depending on the location in space.

Example 12–1. The force of gravitational attraction of a point mass depends on the distance of the attracted particle from the point mass. If the point mass is located at the origin, the inverse square law states that the gravitational force exerted by this particle on a point at $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is proportional to $\dfrac{1}{(x^2+y^2+z^2)^{3/2}} \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$.

A reasonable way of representing such a force field would use a vector with its 'tail' at $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and its 'head' in the direction of the origin. Vectors do not have this sort of property since the 'tail' of a vector is always located at the origin. To overcome this difficulty suppose $p \in \mathbf{R}^3$. The collection of pairs (p, v) for $v \in \mathbf{R}^3$ is called the **tangent space** of $\mathbf{R}^3$ at p and is denoted $\mathbf{R}^3_p$. The elements of the tangent space are called **tangent vectors**. Addition and scalar multiplication of tangent vectors are defined by the formulas $(p, v) + (p, w) = (p, v + w)$ and $c(p, v) = (p, cv)$. A similar approach is taken in higher dimensional spaces.

There are two intuitive interpretations of the tangent space. First, $(p, v) \in \mathbf{R}^3_p$ represents an arrow with tail at p and head at $p + v$. The second interpretation is as follows. Suppose $C : \mathbf{R} \to \mathbf{R}^3$ is a differentiable curve with $C(0) = p$. Then $C'(0)$ is the velocity of C at 0, and this vector is tangent to the curve C at the point $C(0) = p$. Thus $(p, C'(0))$ is the velocity vector with its tail located at p.

A similar definition and properties holds in spaces of other dimensions.

Example 12–2. In the previous example, the gravitational field would be represented by the collection of tangent vectors $\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \frac{1}{(x^2+y^2+z^2)^{3/2}} \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix} \right)$ in $\mathbf{R}^3_{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}$.

A function $f : \mathbf{R}^2 \to \mathbf{R}^2$ (or $f : \mathbf{R}^3 \to \mathbf{R}^3$) is often called a **vector field**. Typical pictures of vector fields show the vector $f(p)$ with its tail at p. By an abuse of notation f can be reinterpreted so that $f(p)$ takes the value $(p, f(p))$ in $\mathbf{R}^2_p$. This makes the pictures typically drawn more mathematically accurate.

The standard basis of $\mathbf{R}^3_p$ is of course (p, e_1), (p, e_2), and (p, e_3).

Exercise 12–1. What is the standard basis of $\mathbf{R}^2_p$?

One important phyiscal idea is the notion of work. Suppose a force field has value F at all points in space, and a particle is moved in this field from the origin to the point d. Notice that both F and d are vectors. The **work** done in moving from the origin to the point d is defined to be $F \bullet d$.

Example 12–3. If the force is $F = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $d = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$ the work done in moving from the origin to d is $F \bullet d = 13$.

Exercise 12–2. What work is done in moving from the origin to $\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$ in this same force field?

A positive value for the work done implies that the force field is doing the work; a negative value implies that work is being done against the field.

One of the problems to be addressed is how work should be calculated when the force field has varying values and the path of motion is not a straight line. The ideas of calculus enter here because any path of motion is approximately a straight line over short enough distances.

Example 12–4. Consider computing the amount of work in the general force field $F\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ when a particle moves from the origin to a point d along some path. Suppose the path is specified by a given function $p(t)$ which represents the position of the particle at time t. Assume the particle is at the origin at time $t = 0$ and is at d at time $t = 1$. If h is a small number, during the time interval from t to $t+h$ the particle moves approximately in a straight line along the tangent vector $p(t+h) - p(t) = p'(t)h$. The force applied to the particle is approximately $F(p(t))$. So the work done in this time interval is approximately $F(p(t)) \bullet p'(t)h$. This argument shows that if $W(t)$ is the work done between time 0 and t, then

$$\begin{aligned} W'(t) &= \lim_{h \to 0} \frac{W(t+h) - W(t)}{h} \\ &= F(p(t)) \bullet p'(t). \end{aligned}$$

So the total work done is $W(1) - W(0) = \int_0^1 F(p(t)) \bullet p'(t)\,dt$ by the Fundamental Theorem of Calculus.

Exercise 12–3. Compute this integral when $F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $p(t) = t\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$ for $0 \le t \le 1$.

This methodology depends on giving the path through the field a parametric representation. Physically, the amount of work done should depend only on the path, not on the way the path is traversed.

Exercise 12–4. Compute the integral if the *same* path is parameterized by $q(t) = t^2\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$, for $0 \le t \le 1$. Is the integral the same?

The change of variable formula can be used to show that the value of the integral depends only on the path, not on the parameterization of the path. This will be done in a later section.

Another important physical problem deals with fluid flow. In this connection, the velocity of a particle of fluid at each point in space is specified. The **velocity field** is then a vector field in the same way that the force field considered above is a vector field.

Example 12–5. A straight river flowing from left to right with river banks along the x-axis and the line $y = 1$ might have velocity field $v\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y(1-y) \\ 0 \end{pmatrix}$ for $0 \le y \le 1$.

Exercise 12–5. What is the value of the velocity field at $\begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$?

For a 2 dimensional flow, as in the example, what amount of fluid crosses a given line segment per unit time? This is a difficult question to answer for a flow with varying velocity. Suppose the velocity of the flow was the same at each point in space, say v, and that the line segment extended from the origin to the point d. A particle of fluid crossed the line segment in a unit of time that just ended precisely if the particle currently is in the parallelogram with vertices at the origin, v, $v + d$, and d. So the amount of fluid flow is nothing more than the area of this parallelogram, which is $\det(v, d)$, the determinant of the matrix with v and d as columns. The sign of this determinant has a physical interpretation here. The sign is positive if the flow is from left to right as viewed by a person standing at the origin and looking toward the point d; the sign is negative if the person would see the flow moving from right to left.

Exercise 12–6. If $v = e_1$ and $d = e_2$ is the flow moving from left to right or from right to left? What if $v = e_2$ and $d = e_1$?

Once again, the analysis of the simple case of a constant velocity flow across a straight line segment leads to the solution of the general problem. Suppose a path in the plane is parameterized by a function $p(t)$, for $0 \le t \le 1$. Then over a short time interval starting at t, the path is approximately a straight line segment $p'(t)$, and the velocity of particles near this segment is approximately $V(p(t))$, where $V\begin{pmatrix} x \\ y \end{pmatrix}$ is the velocity of a fluid particle at $\begin{pmatrix} x \\ y \end{pmatrix}$. The amount of fluid flowing across this short segment per unit of time is then $\det(\, V(p(t)) \quad p'(t)\,)$. The total amount of fluid flow across the whole path is then $\int_0^1 \det(\, V(p(t)) \quad p'(t)\,)\, dt$.

Example 12–6. Suppose the velocity field for the flow is $V\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$ and the path is the straight line segment from the origin to $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. One parameterization of this path is $p(t) = t\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, for $0 \le t \le 1$. The net flow of fluid across this path per unit time is then $\int_0^1 \det(\, V(p(t)) \quad p'(t)\,)\, dt = \int_0^1 \det\begin{pmatrix} -3t & 2 \\ 2t & 3 \end{pmatrix} dt = \int_0^1 -13t\, dt = -13/2$. From the point of view of an observer standing at the origin looking toward $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ the flow is from right to left.

Exercise 12–7. What if the path in the example is a circle of radius 1 centered at the origin?

For a 3 dimensional flow, the amount of fluid crossing a two dimensional surface could be measured. Again, the general case is difficult to analyze. Suppose the velocity field takes the same value v at each point, and the two dimensional surface is the parallelogram with vertices at the origin, a, $a + b$, and b. The same geometric intuition shows that the fluid flux across this parallelogram in one time unit is $\det(v, a, b)$, the determinant of the matrix with v, a, and b as columns. What is the significance of the sign of this determinant? Imagine that the parallelogram with vertices at 0, a, $a + b$, and b is painted on a piece of glass. The side of the glass with the property that when traversing the vertices in this order the inside of the parallelogram lies to the left is the ‘outside’ and the other side is the ‘inside’. This determinant is positive if the flow is from the inside to the outside, and negative if the flow is from the outside to the inside.

The general flow can be analyzed using the same methodology as in the other cases. The primary challenge is to find a reasonable parameterization of the path or surface involved. The core observation is that there should be as many parameters as the dimension of the path or surface being parameterized.

Example 12–7. If a velocity vector field is given, what is the net outflow of the fluid through the surface of a sphere of radius 3 centered at the origin? The surface of the sphere is a 2 dimensional object, and so should be parameterized using two variables. A natural approach is to use latitude and longitude coordinates. Define a mapping $g\begin{pmatrix}\theta\\ \phi\end{pmatrix} = \begin{pmatrix}3\cos\theta\cos\phi\\ 3\sin\theta\cos\phi\\ 3\sin\phi\end{pmatrix}$, and take the domain of g to be the rectangle $Q = \left\{\begin{pmatrix}\theta\\ \phi\end{pmatrix} : 0 \le \theta < 2\pi, -\pi/2 \le \phi \le \pi/2\right\}$. The function g maps Q onto the sphere of radius 3 centered at the origin. If the velocity field is $v\begin{pmatrix}x\\ y\\ z\end{pmatrix}$, then the net fluid flow is $\int_Q \det\left(v(g\begin{pmatrix}\theta\\ \phi\end{pmatrix}) \quad dg\begin{pmatrix}\theta\\ \phi\end{pmatrix}\right) d\theta \times d\phi$.

The remainder of the discussion here develops a more systematic approach to these types of computations. As was seen in the fluid flow examples, that parameterization is physically irrelevant, but was central for the discussion as developed here. The first step will be to develop tools that will push the parameterization further into the background and allow the physical intuition to be used more directly. This systemized approach will also expose the analog of the Fundamental Theorem of Calculus in this more complicated setting.

Problems

Problem 12–1. A particle moves in a force field $R\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$ along a path $p(t) = \begin{pmatrix} t-1 \\ 2t-3 \end{pmatrix}$ for $0 \le t \le 1$. How much work is done?

Problem 12–2. Suppose a and b are two vectors in $\mathbf{R}^3$. Find a parameterization of the line segment joining a and b.

Problem 12–3. Suppose the two dimensional fluid flow has a velocity field given by $v\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y(1-y) \\ 0 \end{pmatrix}$ for $0 \le y \le 1$ and all x. What is the rate at which fluid crosses the line segment from the origin to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ per unit time?

Problem 12–4. Suppose the velocity field of a fluid flow is $V\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 0 \end{pmatrix}$. What is the net flow from the inside to the outside of the box with corners at $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$?

Problem 12–5. The velocity of a particle of fluid at the point $\begin{pmatrix} x \\ y \end{pmatrix}$ is $V\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x^2 \end{pmatrix}$. A curve is given parametrically by $p(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$ for $0 \le t \le 1$. Compute that net rate at which fluid crosses the curve per unit time. From the point of view of an observer standing at the origin and looking along the curve, is the net fluid flow crossing the curve from left to right, or from right to left?

Problem 12–6. The force acting on a particle at the location $\begin{pmatrix} x \\ y \end{pmatrix}$ is given by $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$. Give a parameterization of the straight line path connecting the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ to the point $\begin{pmatrix} 4 \\ 7 \end{pmatrix}$. Find the work done when a particle moves along the straight line path from $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ to $\begin{pmatrix} 4 \\ 7 \end{pmatrix}$. Is the work done by the force field, or against the force field?

Problem 12–7. How much work is done by the gravitational field $F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \dfrac{-1}{(x^2+y^2+z^2)^{3/2}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ on a particle moving from $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$?

Solutions to Problems

Problem 12–1. The work done is $\int_0^1 \begin{pmatrix} 3-2t \\ t-1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \end{pmatrix} dt = 1.$

Problem 12–2. To move from a to b in a straight line, follow the line through a in the direction $b - a$. One path that follows this line is $p(t) = a + t(b - a)$, for $0 \le t \le 1$.

Problem 12–3. One parameterization of the line segment is $p(t) = \begin{pmatrix} 0 \\ t \end{pmatrix}$ for $0 \le t \le 1$. The rate of flow across this segment per unit time is then $\int_0^1 \det \begin{pmatrix} t(1-t) & 0 \\ 0 & t \end{pmatrix} dt = 1/6.$

Problem 12–4. Parameterize each of the four sides of the box in such a way that a positive value for the net flow indicates flow from inside the box to the outside of the box. One parameterization for the left side of the box is $p(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1-t \end{pmatrix}$. The net outflow through this side of the box is then $\int_0^1 \det \begin{pmatrix} -t & 0 \\ 0 & -1 \end{pmatrix} dt = \int_0^1 t\,dt = 1/2$. Applying similar methods for the other 3 sides and adding the results gives a net outflow of 4.

Problem 12–5. Here $p'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$, and the net flow is $\int_0^1 \det \begin{pmatrix} -t^2 & 1 \\ t^2 & 2t \end{pmatrix} dt = \int_0^1 -2t^3 - t^2\,dt = -1/2 - 1/3 = -5/6$. Since the net flow is negative, the net flow is from right to left.

Problem 12–6. One parameterization is $p(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\left(\begin{pmatrix} 4 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 1+3t \\ 2+5t \end{pmatrix}$ for $0 \le t \le 1$. There are many others. Using this parameterization, $p'(t) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and the work is $\int_0^1 \begin{pmatrix} -1-3t \\ -2-5t \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 5 \end{pmatrix} dt = \int_0^1 -13 - 34t\,dt = -13 - 17 = -30$. Since the result is negative, the work is done against the field.

Problem 12–7. One path is parameterized as $p(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, for $0 \le t \le 1$. The work done is $\int_0^1 \frac{-1}{(3(t+1)^2)^{3/2}} 3t\,dt = 4 - 3\sqrt{2} = -0.2426.$

Solutions to Exercises

Exercise 12–1. The standard basis of $\mathbf{R}^2_p$ is (p, e_1) and (p, e_2).

Exercise 12–2. The work done is $\begin{pmatrix}1\\2\\3\end{pmatrix} \bullet \begin{pmatrix}-4\\0\\1\end{pmatrix} = -1$.

Exercise 12–3. Here $p'(t) = \begin{pmatrix}-4\\0\\1\end{pmatrix}$ and the integral is -1, as before.

Exercise 12–4. In this case $q'(t) = 2t\begin{pmatrix}-4\\0\\1\end{pmatrix}$ and the integral is

$$\int_0^1 \begin{pmatrix}1\\2\\3\end{pmatrix} \bullet 2t\begin{pmatrix}-4\\0\\1\end{pmatrix} dt = -\int_0^1 2t\,dt = -1,$$

as before.

Exercise 12–5. The value is the tangent vector $\left(\begin{pmatrix}1\\1/2\end{pmatrix}, \begin{pmatrix}1/4\\0\end{pmatrix}\right)$. The river flows fastest at its center, and has velocity near zero near its banks.

Exercise 12–6. In the first case, the flow moves from left to right. In the second case, the flow moves from right to left.

Exercise 12–7. One parameterization of the path is $p(t) = \begin{pmatrix}\cos t\\ \sin t\end{pmatrix}$ for $0 \le t \le 2\pi$. The net flow is then $\int_0^{2\pi} \det\begin{pmatrix}-\sin t & -\sin t\\ \cos t & \cos t\end{pmatrix} dt = 0$.

§13. Tensors and Differential Forms

The physical examples of the previous section involve integrands which have a nice form. A real valued function of k vector arguments which is linear in each argument separately is called a **tensor** of order k.

Example 13–1. The usual dot product is a tensor of order 2 since if v is fixed $v \bullet u$ is linear in the u argument.

Example 13–2. If v_1, v_2, and v_3 are vectors in $\mathbf{R}^3$ the function defined by the formula $T(v_1, v_2, v_3) = \det\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ is a tensor of order 3.

Example 13–3. A linear function with domain $\mathbf{R}^2$ and range $\mathbf{R}$ is a tensor of order 1.

There are some simple linear functions which will be particularly useful. The notation is rather poor, but has been in use for more than a century. Denote by x the function with domain $\mathbf{R}^2$ and range $\mathbf{R}$ specified by the formula $x\begin{pmatrix} a \\ b \end{pmatrix} = a$. Similarly, y is the function defined by the formula $y\begin{pmatrix} a \\ b \end{pmatrix} = b$. The functions x and y are nothing more than projections onto the coordinate directions. These same names are also used for functions with domain $\mathbf{R}^3$. Thus $x\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a$ and $y\begin{pmatrix} a \\ b \\ c \end{pmatrix} = b$, too.

Exercise 13–1. What is the rule defining the function z, whose domain is $\mathbf{R}^3$?

Any tensor of order 1 can be written in terms of these basic functions.

Example 13–4. Suppose $f\begin{pmatrix} a \\ b \end{pmatrix} = 3a + 4b$. Then $f\begin{pmatrix} a \\ b \end{pmatrix} = 3x\begin{pmatrix} a \\ b \end{pmatrix} + 4y\begin{pmatrix} a \\ b \end{pmatrix}$. One would usually write more simply that $f = 3x + 4y$. Note that in this expression x and y are functions and not variables! Also $df = 3\,dx + 4dy$ in terms of the derivatives dx and dy of the basic tensors x and y.

As the determinant example above demonstrates, some tensors have the property that an interchange in the order of the arguments changes the sign of the value of the tensor. Such a tensor is **alternating**.

There is a simple method of building up alternating tensors of higher orders from the simple tensors x and y (and z). The **wedge product** (or **exterior product**) $x \wedge y$ is the second order alternating tensor given by the formula $x \wedge y(u, v) = \det\begin{pmatrix} x(u) & x(v) \\ y(u) & y(v) \end{pmatrix}$.

Exercise 13–2. Show that the wedge product thus defined is an alternating tensor of order 2.

Exercise 13–3. Compute $x \wedge y(\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix})$.

Exercise 13–4. What is $x \wedge x$?

$$\text{Similarly, } x \wedge y \wedge z(u, v, w) = \det \begin{pmatrix} x(u) & x(v) & x(w) \\ y(u) & y(v) & y(w) \\ z(u) & z(v) & z(w) \end{pmatrix}.$$

The wedge product of two general alternating tensors is computed by expanding each tensor in terms of the basic wedge products $x \wedge y$ above and then adding the wedge products of the corresponding basis elements, taking care to maintain the proper order of the factors.

Example 13–5. What is $(2x-7y)\wedge(x+6y)$? Taking care with the order of the factors and expanding gives $(2x-7y)\wedge(x+6y) = (2x)\wedge x+(2x)\wedge(6y)-(7y)\wedge x-(7y)\wedge(6y) = 2x \wedge x + 12x \wedge y - 7y \wedge x - 42y \wedge y = 19x \wedge y$, after using the facts that $x \wedge x = 0$, $y \wedge y = 0$ and $x \wedge y = -y \wedge x$.

In the phyiscal examples of the last section, some of the objects appearing in the computations were tangent vectors (such as $p'(t)$). This suggests that the main interest will be with tensors whose inputs are tangent vectors. A **differential** k **form** (or simply a k **form**) on $\mathbf{R}^2$ is a function ω with domain $\mathbf{R}^2$ for which $\omega(p)$ is an alternating tensor of order k whose domain is the tangent space $\mathbf{R}^2_p$. Note that there is no obvious 'differential' in the definition of differential forms! The intuition should be that the 'differential' comes from the tangent vectors which are the arguments of the form. A similar definition applies in spaces of other dimension.

A differential form must have, for each p, an expression in terms of the wedge products of basic first order tensors on $\mathbf{R}^2_p$. What are these basic first order tensors? The basic first order tensors of $\mathbf{R}^2$ are the functions x and y. Since each of these basic tensors is linear, $dx(p) = x$ for all p. Hence the basic first order tensors on $\mathbf{R}^2_p$ are $dx(p)$ and $dy(p)$. Every differential form ω can therefore be written as a sum of terms each of which is a real valued function of 2 variables times a wedge product of these basic tensors.

Exercise 13–5. What are the basic differential forms on $\mathbf{R}^3$?

To gain physical intuition, consider the simple differential form dx. What is the value $dx\begin{pmatrix}a\\b\end{pmatrix}\left(\begin{pmatrix}a\\b\end{pmatrix}, \begin{pmatrix}c\\d\end{pmatrix}\right)$? Since x is the function with formula $x\begin{pmatrix}a\\b\end{pmatrix} = a$, $dx\begin{pmatrix}a\\b\end{pmatrix} = (1 \quad 0)$. Thus $dx\begin{pmatrix}a\\b\end{pmatrix}\left(\begin{pmatrix}a\\b\end{pmatrix}, \begin{pmatrix}c\\d\end{pmatrix}\right) = (1 \quad 0)\begin{pmatrix}c\\d\end{pmatrix} = c$. Here c is the *change* in the x component of the tangent vector with tail at $\begin{pmatrix} a \\ b \end{pmatrix}$ and head at $\begin{pmatrix} a+c \\ b+d \end{pmatrix}$. So dx represents an incremental change in the x coordinate. A similar interpretation applies to dy and dz.

With this physical intuition, the language of differential forms makes the description of the physical problems of the previous section simple.

Example 13–6. Suppose $F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -z \\ y \end{pmatrix}$ is a force field on $\mathbf{R}^3$. The work form for this force field is $W = x\,dx - z\,dy + y\,dz$. The form expresses at an intuitive level the earlier argument that for small changes in position the force field is essentially constant and that in this case the work done is the dot product of the field with the displacement vector. Notice that, formally, the work form is nothing more than $F\begin{pmatrix} x \\ y \\ z \end{pmatrix} \bullet \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$.

Exercise 13–6. What is the work form associated with the force field $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix}$ on $\mathbf{R}^2$?

Example 13–7. What form would be associated with the fluid flow with velocity field $V\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$ in $\mathbf{R}^2$? Using the same physical intuition, the fluid flow form for this flow would by $\det\begin{pmatrix} y & dx \\ x & dy \end{pmatrix} = y\,dy - x\,dx$.

There is also physical intuition associated with the wedge product of differential forms. In $\mathbf{R}^2$, the form $dx \wedge dy$ is the area of a parallelogram with sides parallel to the x and y axes. This interpretation comes from the computation $dx \wedge dy\begin{pmatrix} a \\ b \end{pmatrix}\left(\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right), \left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} e \\ f \end{pmatrix}\right)\right) = \det\begin{pmatrix} c & e \\ d & f \end{pmatrix}$. In $\mathbf{R}^3$, the same wedge product is the area of the projection (shadow) of a 3 dimensional parallelogram on the x-y plane.

Exercise 13–7. Verify this physical interpretation by an appropriate computation.

Example 13–8. Suppose a fluid flow in $\mathbf{R}^3$ has the same velocity $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ at all points. Then as was seen earlier, the fluid flow through a parallelogram spanned by $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$ is $\det\begin{pmatrix} 2 & a & d \\ 3 & b & e \\ 4 & c & f \end{pmatrix}$. Expanding this determinant gives the value $2(bf - ce) - 3(af - cd) + 4(ae - bd)$. This is the same value as the 2 form $\omega = 2dy \wedge dz + 3dz \wedge dx + 4dx \wedge dy$ applied to the tangent vectors $\left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}a\\b\\c\end{pmatrix}\right)$ and $\left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}d\\e\\f\end{pmatrix}\right)$. The wedge products in this form are giving the areas of the projection of the parallelogram onto the respective coordinate planes. The ordering in the factors in the wedge products maintains the earlier interpretation of the sign of the determinant in this case.

Exercise 13–8. What form would be associated with a general fluid flow in $\mathbf{R}^3$?

As the problems in the earlier section indicate, forms exist to be integrated, as it is the integral of the form that is of physical significance. How should the integral of a form be defined?

Problems

Problem 13–1. What is $(3x\,dx + 5y\,dy)\begin{pmatrix}1\\2\end{pmatrix}\left(\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}\right)$?

Problem 13–2. Simplify $(3x\,dx + 5y\,dy) \wedge dx$.

Problem 13–3. What is the interpretation of a 2 form in $\mathbf{R}^2$?

Problem 13–4. What is the interpretation of a 3 form in $\mathbf{R}^3$?

Solutions to Problems

Problem 13–1. $(3xdx + 5ydy)\begin{pmatrix}1\\2\end{pmatrix}(\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}) = 9 + 40 = 49.$

Problem 13–2. $(3xdx + 5ydy) \wedge dx = 3xdx \wedge dx + 5ydy \wedge dx = -5ydx \wedge dy.$

Problem 13–3. This must be an oriented area.

Problem 13–4. This must be an oriented volume.

Solutions to Exercises

Exercise 13–1. The rule is $z\begin{pmatrix} a \\ b \\ c \end{pmatrix} = c$.

Exercise 13–2. Interchanging two columns of a matrix changes the sign of its determinant.

Exercise 13–3. $x \wedge y(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}) = \det\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = -2$.

Exercise 13–4. $x \wedge x = 0$, since a matrix with two equal columns has determinant 0.

Exercise 13–5. Here the basic forms are dx, dy, and dz.

Exercise 13–6. The work form is $-y\,dx - x\,dy$.

Exercise 13–7.

$$dx \wedge dy\begin{pmatrix} a \\ b \\ c \end{pmatrix}\left(\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} d \\ e \\ f \end{pmatrix}\right), \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} g \\ h \\ i \end{pmatrix}\right)\right)$$
$$= \det\begin{pmatrix} d & g \\ e & h \end{pmatrix}.$$

Exercise 13–8. Suppose the velocity vector of the flow is $V\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ where each of v_1, v_2, and v_3 may be functions of x, y, and z. The 2 form associated with this flow is then $v_1 dy \wedge dz + v_2 dz \wedge dx + v_3 dx \wedge dy$.

§14. Pulling Back and Integrating Forms

A 1-form can only be reasonably integrated over a set of dimension 1, even though the form is usually defined over a set of larger dimension. The earlier physical examples illustrated that the 1 dimensional set will be parameterized using 1 parameter. This parameterization will be done using a function of a single variable which takes values in a higher dimensional space. If the form can be pulled back to a form defined on the domain of this function, the integral of the pulled back form can be computed in the usual way. The only question is how the form can be pulled back.

Forms are defined on tangent spaces. To understand how a form should be pulled back, consider the basic form dx on $\mathbf{R}^2$. Suppose a 1 dimensional path in $\mathbf{R}^2$ is parameterized by a function $p(t)$ whose domain is the interval $[0, 1]$ in $\mathbf{R}$. Grab a tangent vector (a, b) in $\mathbf{R}^1_a$. Under the mapping p, this tangent vector is mapped to a tangent vector in $\mathbf{R}^2_{p(a)}$, namely the tangent vector $(p(a), p(a + b) - p(a)) = (p(a), p'(a)\, b)$. This tangent vector can be used as input for the form dx at the point $p(a)$. Direct computation gives $dx(p(a))(p(a), p'(a)\, b) = (1 \quad 0\,)p'(a)b$. This is defined to be the value of the **pull back** p^*dx of the form dx by p at the tangent vector (a, b).

Example 14–1. To gain some additional insight, consider a specific case. Suppose $p(t) = \begin{pmatrix} 2t^2 \\ 3t \end{pmatrix}$, for $0 \le t \le 1$ is a path in $\mathbf{R}^2$. What is p^*dx? Here $p'(t) = \begin{pmatrix} 4t \\ 3 \end{pmatrix}$ so by the formula just developed, $(p^*dx)(a)(a, b) = 4ab$. A key insight is that this is the same as the value of the form $4t\, dt(a)(a, b)$.

Exercise 14–1. Compute $4t\, dt(a)(a, b)$.

The example illustrates a simple method of pulling back a form. The parameterization expresses each of the variables in the range space in terms of the variables in the domain space. The values for the basic forms dx and dy in the range space can then be obtained by mechanical computation, similar to that used when making a simple change of variable in an integral.

Example 14–2. Using the parameterization of the previous problem gives the expressions $x = 2t^2$ and $y = 3t$ in terms of the variable t, so that $dx = 4t\, dt$ and $dy = 3\, dt$. So, $p^*(5x\, dx - 7y^2\, dy) = 5(2t^2)\, 4t\, dt - 7(3t)^2\, 3\, dt = (40t^3 - 189t^2)\, dt$ is the pull back of the form $5x\, dx - 7y^2\, dy$ under p.

The integral of a form over a curve (or surface) is defined to be the integral of the pull back of a form under a parameterization of the curve (or surface) over the domain of the parameterization.

Observe that a 1-form can only be integrated over a 1 dimensional object; a 2-form can only be integrated over a 2 dimensional object. For this reason, a parameterization of a 1 dimensional object is sometimes called a **1 cube**, and a parameterization of a 2 dimensional object a **2 cube**.

Example 14–3. Consider the form $\omega = 3\,dx + 4\,dy$ on $\mathbf{R}^2$ and the 1 dimensional curve parameterized by $f(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$ for $0 \le t \le 1$. In order to compute $\int_f \omega$, first compute $f^*\omega = (8t+3)\,dt$. Then $\int_f \omega = \int_0^1 (8t+3)\,dt = 7$. This integral represents the work done by moving a particle through the constant force field $(3,4)$ along the arc of the parabola parameterized by f.

Example 14–4. The **arc length form** on $\mathbf{R}^2$ is $ds = \sqrt{(dx)^2 + (dy)^2}$. The physical intuition for the name of the form comes from the fact that ds is the distance travelled when moving through a small change of position $\begin{pmatrix} dx \\ dy \end{pmatrix}$ along a straight line path. The pull back of the arc length form under a parameterized path $p(t)$ gives the distance travelled under a small displacement along the curve parameterized by p. Thus $\int_p ds$ is the length of the curve parameterized by p.

Exercise 14–2. What is the arc length form on $\mathbf{R}^3$?

Example 14–5. In $\mathbf{R}^3$ there is a surface area form which represents the area of a small surface element of a 2 dimensional surface in 3 dimensional space. To uncover this form, consider first the formula for the volume of a 3 dimensional parallelogram spanned by the vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$, and $\begin{pmatrix} g \\ h \\ i \end{pmatrix}$. The volume is $\det\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} = a(ei-fh)-b(di-fg)+c(dh-eg)$ as was computed earlier. If the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is chosen to be perpendicular to the other 2 vectors and of unit length, this volume will be numerically equal to the 2 dimensional area of the parallelogram spanned by the other 2 vectors. The perpendicularity requirements on a, b, and c are then $ad+be+cf = 0$ and $ag+bh+ci = 0$. Multiply the first of these by g and the second by d and subtract to get $b(eg-dh)+c(fg-di) = 0$. So one choice for the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is $\begin{pmatrix} ei-fh \\ fg-di \\ dh-eg \end{pmatrix}$, divided by its length. The area of the parallelogram spanned by $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$ and $\begin{pmatrix} g \\ h \\ i \end{pmatrix}$ is then $\sqrt{(ei-fh)^2 + (di-fg)^2 + (dh-eg)^2}$. The surface area form is therefore $dS = \sqrt{(dy \wedge dz)^2 + (dx \wedge dz)^2 + (dx \wedge dy)^2}$, since this form has the same value on these tangent vectors. The integral $\int_p dS$ is the surface area of the 2 dimensional surface parameterized by p. The sign of the integral discloses the orientation of the surface.

Problems

Problem 14–1. For the 1 form $\omega = xy^2\,dx + y\,dy$ and $f(t) = \begin{pmatrix} t \\ t \end{pmatrix}$, for $0 \le t \le 1$, compute $f^*\omega$ and $\displaystyle\int_f \omega$.

Problem 14–2. For the 1 form $\omega = xy^2\,dx + y\,dy$ and $f(t) = (t, t^2)$, for $0 \le t \le 1$, compute $f^*\omega$ and $\displaystyle\int_f \omega$.

Problem 14–3. For the 1 form $\omega = xy^2\,dx + y\,dy$ find a parameterization f whose image (range) is the line segment from $(1, 1)$ to $(3, 7)$ and compute $f^*\omega$ and $\displaystyle\int_f \omega$.

Problem 14–4. Find the length of the curve with parameterization $p(t) = \begin{pmatrix} 4t \\ 2t^2 \end{pmatrix}$ for $0 \le t \le 1$.

Problem 14–5. Argue that the gravitational field generated by a point of mass M at the origin is $\dfrac{-GM}{(x^2 + y^2 + z^2)^{3/2}}(x\,dx + y\,dy + z\,dz)$. **Spherical coordinates** $\begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}$ on $\mathbf{R}^3$ are defined by $x = r\cos\theta\cos\phi$, $y = r\sin\theta\cos\phi$, and $z = r\sin\phi$. Find the equivalent representation of this field in spherical coordinates $r = \sqrt{x^2 + y^2 + z^2}$, θ, and ϕ. Hint: Use an appropriate pullback.

Problem 14–6. Use spherical coordinates on $\mathbf{R}^3$ to compute the volume of a sphere of radius 5 centered at the origin.

Problem 14–7. Find the surface area of a sphere of radius 5 centered at the origin.

Solutions to Problems

Problem 14–1. $f^*\omega = (t^3 + t)\,dt$ and $\displaystyle\int_f \omega = \int_0^1 (t^3 + t)\,dt = 1/4 + 1/2 = 3/4$.

Problem 14–2. $f^*\omega = (t^5 + 2t^3)\,dt$ so $\int_f \omega = \int_0^1 (t^5 + 2t^3)\,dt = 3/2$.

Problem 14–3. One choice is $f(t) = \begin{pmatrix}1\\1\end{pmatrix} + t\left(\begin{pmatrix}3\\7\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix}\right)$. This gives $f^*\omega = (2t+1)(6t+1)^2 2\,dt + (6t+1)6\,dt$.

Problem 14–4. Pull back the arc length form to get $p^*(ds) = \sqrt{16\,dt^2 + 16t^2\,dt} = 4\sqrt{1+t^2}\,dt$. The length of the curve is $\displaystyle\int_0^1 4\sqrt{1+t^2}\,dt = 2\sqrt{2} - 2\ln(\sqrt{2} - 1)$.

Problem 14–5. Pull back the 1 form using the mapping

$$f\begin{pmatrix}r\\\theta\\\phi\end{pmatrix} = \begin{pmatrix}r\cos\theta\cos\phi\\ r\sin\theta\cos\phi\\ r\sin\phi\end{pmatrix},$$

noting that $x^2 + y^2 + z^2 = r^2$.

Problem 14–6. The volume is $500\pi/3$.

Problem 14–7. Spherical coordinates can be used again, but this time the radius is not a variable, but a known constant. The surface area is 100π.

Solutions to Exercises

Exercise 14–1. $4t\,dt(a)(a,b) = 4a\,dt(a)(a,b) = 4ab$.

Exercise 14–2. On $\mathbf{R}^3$ the arc length form is $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$.

§15. Orientation

The physical interpretation of the integral of a 1 form as work suggests that the direction in which the path of integration is traversed will affect the sign of the integral. This is in fact the case. In computing the integral of a 2 form on a 2 cube the parameterization of the cube is important.

Example 15–1. The surface area form $dS = \sqrt{(dx \wedge dy)^2 + (dx \wedge dz)^2 + (dy \wedge dz)^2}$ of the last section can be pulled back to compute surface area. What is the surface area of the rectangle in $\mathbf{R}^3$ with vertices at the origin, $\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}1\\2\\0\end{pmatrix}$, and $\begin{pmatrix}0\\2\\0\end{pmatrix}$?

The surface area must be 2, the area of the rectangle. To compute using the methodology of forms and pull backs, parameterize the rectangle using the function $p\begin{pmatrix}s\\t\end{pmatrix} = \begin{pmatrix}s\\t\\0\end{pmatrix}$, for $0 \le s \le 1$, $0 \le t \le 2$. Since this parameterization gives $x = s$, $y = t$ and $z = 0$, then $dx = ds$, $dy = dt$, and $dz = 0$. So $p^*(dS) = \sqrt{(ds \wedge dt)^2}$. Does this simplify to $ds \wedge dt$ or $dt \wedge ds$? The answer to this question affects the sign of the integral of the pull back.

In the discussion of volumes and determinants, the orientation of the volume was seen to affect the sign of the determinant. A similar situation exists here, with 2 dimensional surfaces inside 3 dimensional space. Such a surface has both a top and a bottom (or an inside and outside). A parameterization of the surface will be orientation preserving if any triangle with the standard orientation in the domain of the parameterization is mapped to a curve which traces out a counterclockwise path when viewed from the top (or outside) in the range of the parameterization.

Example 15–2. For the parameterization p of the previous example, if a triangle in the domain of p is traced out in a counterclockwise direction, the image triangle will also be traced out in a counterclockwise direction, as viewed from a point on the positive z axis.

An orientation preserving parameterization should always be used to pull back a form to the natural ordering of the wedge products before integration.

Example 15–3. The parameterization $q\begin{pmatrix}s\\t\end{pmatrix} = \begin{pmatrix}1-s\\t\\0\end{pmatrix}$ is orientation reversing. Tracing the border of a triangle in the domain of q in a counterclockwise direction will trace the border of the image triangle in the clockwise direction, as viewed from the positive z axis. If q pulls back a form to the natural order of wedge products, the integral will have the wrong sign.

The upshot of the previous discussion is that unless care is taken, the integral of the pull back may be computed with the incorrect sign.

The following gives the technical details of the previous discussion. Suppose f is a 2 cube in $\mathbf{R}^3$. There may well be another 2 cube g with the same image set in $\mathbf{R}^3$. How are $\int_f \omega$ and $\int_g \omega$ related? Write $g^*\omega = h\, ds \wedge dt$. Then

$$\begin{aligned} f^*\omega &= (g \circ g^{-1} \circ f)^*\omega \\ &= (g^{-1} \circ f)^*(g^*\omega) \\ &= h \circ (g^{-1} \circ f) \det(d(g^{-1} \circ f))\, ds \wedge dt. \end{aligned}$$

This computation and the change of variables formula then gives

$$\begin{aligned} \int_f \omega &= \int f^*\omega \\ &= \int (g \circ g^{-1} \circ f)^*\omega \\ &= \int (g^{-1} \circ f)^*(g^*\omega) \\ &= \int h \circ (g^{-1} \circ f)\ \det(d(g^{-1} \circ f))\, ds \times dt \\ &= \int h \circ (g^{-1} \circ f)\, |\det(d(g^{-1} \circ f))|\, ds \times dt \\ &= \int h\, ds \times dt \\ &= \int g^*\omega \\ &= \int_g \omega \end{aligned}$$

under the sole proviso that $\det(d(g^{-1} \circ f)) > 0$. The two 2 cubes f and g have the **same orientation** if the condition $\det(d(g^{-1} \circ f)) > 0$ holds. A similar computation can be made in other cases.

§16. The Fundamental Theorem of Calculus

Since the integral of a form has been seen to have physical significance, the question of differentiation of forms naturally arises. Once meaning is attached to the differentiation of forms, the analog of the Fundamental Theorem of Calculus for forms can be developed. As in the case of functions, this theorem is of great value in computing integrals of forms.

One of the basic interpretations of a derivative is as a rate of change. Thus the derivative of a form should represent the rate at which the form is changing.

Example 16–1. To gain some insight into what the rate of change of a form is, consider the fluid flow form associated with a velocity field that is the same at all points in $\mathbf{R}^2$. For concreteness, suppose the velocity field is $V\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ at all points $\begin{pmatrix} x \\ y \end{pmatrix}$. The associated form is then $F = 2\,dy - 3\,dx$. What is dF, the derivative of this form? Since the velocity field is the same everywhere, two observers measuring the flow across a line segment of a given length in a given direction will get the same value, no matter where the two observers are standing. So this flow does not change at all with the position of the observer. Hence $dF = 0$.

In particular, the observation of the example implies that $d(dx) = 0$, $d(dy) = 0$ and $d(dz) = 0$, since all of these basic forms are associated with constant velocity fluid flows.

Consider now the case of a general flow in $\mathbf{R}^2$, with $V\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ where v_1 and v_2 are functions of x and y representing the components of the velocity in the x and y directions. How can the flow change when moving from $\begin{pmatrix} x \\ y \end{pmatrix}$ to a nearby point? The only way the flow can change is if the change in the velocity of the flow causes a net buildup or net depletion of fluid in the region between the measurement locations. Using earlier intuition, suppose the nearby point is at $\begin{pmatrix} x+dx \\ y+dy \end{pmatrix}$. Looking at the horizontal component of the fluid motion, the net buildup is $v_1\begin{pmatrix} x+dx \\ y \end{pmatrix} dy - v_1\begin{pmatrix} x \\ y \end{pmatrix} dy \approx \left(v_1\begin{pmatrix} x \\ y \end{pmatrix} + \frac{\partial}{\partial x}v_1\begin{pmatrix} x \\ y \end{pmatrix} dx\right) dy - v_1\begin{pmatrix} x \\ y \end{pmatrix} dy = \frac{\partial}{\partial x}v_1\begin{pmatrix} x \\ y \end{pmatrix} dx\,dy$. Similarly, for the vertical component of the velocity the net buildup is $v_2\begin{pmatrix} x \\ y+dy \end{pmatrix} dx - v_2\begin{pmatrix} x \\ y \end{pmatrix} dx \approx \left(v_2\begin{pmatrix} x \\ y \end{pmatrix} + \frac{\partial}{\partial y}v_2\begin{pmatrix} x \\ y \end{pmatrix} dy\right) dx - v_2\begin{pmatrix} x \\ y \end{pmatrix} dx = \frac{\partial}{\partial y}v_2\begin{pmatrix} x \\ y \end{pmatrix} dx\,dy$. The total net buildup is therefore $\left(\frac{\partial}{\partial x}v_1\begin{pmatrix} x \\ y \end{pmatrix} + \frac{\partial}{\partial y}v_2\begin{pmatrix} x \\ y \end{pmatrix}\right) dx\,dy$. Here the product $dx\,dy$ represents the area of the small rectangle, which is also true of the form $dx \wedge dy$. This physical argument suggests that $dF = \left(\frac{\partial}{\partial x} v_1\binom{x}{y} + \frac{\partial}{\partial y} v_2\binom{x}{y}\right) dx \wedge dy$ in this case.

Physically, a positive value of $\frac{\partial}{\partial x} v_1\binom{x}{y} + \frac{\partial}{\partial y} v_2\binom{x}{y}$ would mean that there was a net depletion of the fluid in this small rectangle, since the fluid was speeding up as it moved from left to right and bottom to top. For this reason the sum $\frac{\partial}{\partial x} v_1\binom{x}{y} + \frac{\partial}{\partial y} v_2\binom{x}{y}$ is called the **divergence** of the vector field $V\binom{x}{y} = \binom{v_1}{v_2}$ and is denoted $\mathrm{div}V$.

Exercise 16–1. What is the divergence of a vector field $V\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ in $\mathbf{R}^3$?

As in the calculus of functions, computing in this way is tedious and error prone. Mechanical methods of computation are necessary to make the theory useful. Fortunately, such mechanical methods are near at hand.

The form for the flow $V\binom{x}{y} = \begin{pmatrix} v_1\binom{x}{y} \\ v_2\binom{x}{y} \end{pmatrix}$ is $F = v_1\binom{x}{y}\, dy - v_2\binom{x}{y}\, dx$. Now both v_1 and v_2 are real valued functions, and $dv_1\binom{x}{y} = \frac{\partial}{\partial x} v_1\binom{x}{y}\, dx + \frac{\partial}{\partial y} v_1\binom{x}{y}\, dy$. Also $dv_2\binom{x}{y} = \frac{\partial}{\partial x} v_2\binom{x}{y}\, dx + \frac{\partial}{\partial y} v_2\binom{x}{y}\, dy$. Substituting these expressions into the formula for F and replacing the usual product with the differentials by the wedge product gives $dF = \left(\frac{\partial}{\partial x} v_1\binom{x}{y}\, dx + \frac{\partial}{\partial y} v_1\binom{x}{y}\, dy\right) \wedge dy - \left(\frac{\partial}{\partial x} v_2\binom{x}{y}\, dx + \frac{\partial}{\partial y} v_2\binom{x}{y}\, dy\right) \wedge dx = \mathrm{div}V\binom{x}{y}\, dx \wedge dy$, as above. The computational scheme is now complete.

Example 16–2. Using this technique, $d(xy\,dx + 7y^2\,dy) = (y\,dx + x\,dy) \wedge dx + (14y\,dy) \wedge dy = -x\,dx \wedge dy$.

Exercise 16–2. What is $d(e^{xy}\,dx - \sin(xy)\,dy)$?

This computational procedure shows that for any form F, $d(dF) = 0$. So the first fact above about $d(dx) = 0$ was not peculiar to this simple form alone.

Having developed the notion of differentiation of forms paves the way for the development of the Fundamental Theorem of Calculus for forms.

To gain some insight into how the fundamental theorem for forms should work, consider again the simple work form $W = 2\,dx + 3\,dy$. Suppose the objective is to find the amount of work done when moving along a line segment from the origin to the point $\begin{pmatrix}5\\7\end{pmatrix}$. Instead of parameterizing the path and computing as was done earlier, first find an antiderivative of W. A moments reflection shows that the function $P\begin{pmatrix}x\\y\end{pmatrix} = 2x+3y$ has the property that $dP = W$. In this context, the function P is a **potential** for the form W. This simply means that P is a function for which $dP = W$.

Exercise 16–3. Show that $d(2x + 3y) = W$. Are there any other functions whose derivative is W?

Having found an antiderivative for W, the fundamental theorem should allow the work done to be computed as a difference in the values of the antiderivative at the endpoints of the path. In this case, the work done when traversing the path would be $P\begin{pmatrix}5\\7\end{pmatrix} - P\begin{pmatrix}0\\0\end{pmatrix} = 2\cdot 5 + 3\cdot 7 - (2\cdot 0 + 3\cdot 0) = 31$.

Exercise 16–4. Show that this value is correct by computing the work done by pulling back the form.

Example 16–3. Not every form is a derivative. *If* the form F is the derivative of another form, say $F = dW$, then from the general fact above, $dF = d(dW) = 0$. The form $F = xy\,dx + y\,dy$ is not a derivative, since $dF = (x\,dy + y\,dx) \wedge dx + dy \wedge dy = -x\,dx \wedge dy$ is not zero.

There is an additional technicality which even shows up in the usual function setting.

Example 16–4. The function $F(x)$ which takes the value 0 for $x < 2$ and the value 1 for $x > 2$ has derivative equal to 0 everywhere except at $x = 2$. Because of this single point of difficulty, the Fundamental Theorem fails to hold for this function F. As a particular case, $\int_0^4 F'(t)\,dt = 0 \neq F(4) - F(0) = 1$. The interval on which F' is defined has a hole at $x = 2$ and this single hole is sufficient to cause the Fundamental Theorem to fail.

To distinguish the cases caused by this technicality, some terminology is introduced. A form F is **closed** if $dF = 0$. A form F is **exact** if F is a derivative. An exact form *must* be closed. This is just the general fact above about the second derivative of a form. An important fact called the **Poincaré Lemma** says that in a region which lacks holes a closed form is exact.

Example 16–5. The form $F = x^2\,dx + y^2\,dy$ is closed, since $dF = 0$. Since this form makes sense anywhere, in any region lacking holes F is exact, too. If $P\begin{pmatrix}x\\y\end{pmatrix}$

is a potential for this form, $dP = \frac{\partial}{\partial x}P\begin{pmatrix}x\\y\end{pmatrix}dx + \frac{\partial}{\partial y}P\begin{pmatrix}x\\y\end{pmatrix}dy$. Thus $\frac{\partial}{\partial x}P\begin{pmatrix}x\\y\end{pmatrix} = x^2$, so $P\begin{pmatrix}x\\y\end{pmatrix} = x^3/3 + g(y)$ for any function g of y alone. Using this expression gives $\frac{\partial}{\partial y}P\begin{pmatrix}x\\y\end{pmatrix} = g'(y)$ which should be equal to y^2. Thus $g(y) = y^3/3$ is one choice for g. Thus one choice for P is $P\begin{pmatrix}x\\y\end{pmatrix} = x^3/3 + y^3/3$. This gives a general method for finding potentials.

Example 16–6. The gravitational force field $F\begin{pmatrix}x\\y\end{pmatrix} = (x^2+y^2)^{-3/2}\begin{pmatrix}-x\\-y\end{pmatrix}$ on $\mathbf{R}^2$ has associated work form $W = \frac{-x\,dx - y\,dy}{(x^2+y^2)^{3/2}}$. Simple computation shows that $dW = 0$. A potential for W is $g\begin{pmatrix}x\\y\end{pmatrix} = (x^2+y^2)^{-1/2}$. The potential is defined everywhere except at the origin, and $dg = W$ at all points at which W is defined. So W is exact.

The gravitational field example has some interesting consequences. How much work is done by a particle moving once around a circle of radius 1 centered at the origin? Since the work form for the gravitational field has a potential, and the starting and ending point of the path are the same, no work is done.

Looking more carefully at this question brings another idea into play. The circular path is the boundary (border) of the solid circular region. The relationship between the boundary of the circle and the solid circle is the same as the relationship between the endpoints of an interval and the whole interval. Denote by ∂C the boundary of the solid circular region C. By analogy, $\int_{\partial C} W = \int_C dW$. While true for the gravitational field, this result fails generally unless the solid region has no holes.

Example 16–7. The form $B = \frac{-y\,dx + x\,dy}{x^2+y^2}$ is very similar to the work form for the gravitational field. Direct computation shows that $dB = 0$ so that B is closed. In fact, $d(\arctan(y/x)) = B$ too where this function is defined. But since $\arctan(y/x)$ is not defined on the x-axis, it is *not* a potential for the form B. So B is not exact.

Exercise 16–5. For the form B compute the work done when traversing a semicircular path from $\begin{pmatrix}-1\\0\end{pmatrix}$ to $\begin{pmatrix}1\\0\end{pmatrix}$ which passes above the origin and for a semicircular path that passes below the origin.

This equality between the integral of a form on the boundary of a region and the integral of the derivative of the form over the region itself constitutes the Fundamental Theorem of Calculus for differential forms.

Stoke's Theorem. *If F is a form and R is a set without holes then*

$$\int_R dF = \int_{\partial R} F.$$

When computing using Stoke's Theorem, the boundary ∂R is given the orientation inherited from the region R.

Example 16–8. For the fluid flow form $F = y\,dx - x\,dy$, what is the net outflow from the square with vertices at the origin, $\begin{pmatrix}1\\0\end{pmatrix}$, $\begin{pmatrix}1\\1\end{pmatrix}$, and $\begin{pmatrix}0\\1\end{pmatrix}$ per unit time? Here $dF = -2\,dx \wedge dy$, so by Stoke's Theorem the net outflow per unit time is $\int_0^1 \int_0^1 -2\,dx\,dy = -2$. Since the net outflow is negative, there is in fact an inflow of 2 per unit time.

Exercise 16–6. Compute the net outflow by pulling back the form.

Example 16–9. In $\mathbf{R}^2$ suppose the two dimensional vector field $v = \begin{pmatrix}v_1\\v_2\end{pmatrix}$ specfies the velocity vector of a fluid flow at each point in the plane via the associated 2 form $\omega = -v_2\,dx + v_1\,dy$. If R is a region in the plane then $\int_{\partial R} \omega$ represents the rate at which fluid is leaving the region R. By Stoke's Theorem

$$\int_{\partial R} \omega = \int_R d\omega = \int_R (D_1 v_1 + D_2 v_2)\,dx \wedge dy.$$

Recall that $D_1 v_1 + D_2 v_2$ is the divergence of the vector field v. This identity restates the earlier interpretation of the divergence as the net rate at which fluid is leaving a region. This special case of Stoke's Theorem is often called the **Divergence Theorem**.

The symbol ∇ is introduced to formally represent the 'vector' $\begin{pmatrix}D_1\\D_2\end{pmatrix}$ in $\mathbf{R}^2$ or $\begin{pmatrix}D_1\\D_2\\D_3\end{pmatrix}$ in $\mathbf{R}^3$. The divergence of a vector field v is then formally written as $\nabla \cdot v$.

Exercise 16–7. A vector field is **incompressible** if its divergence is zero. Explain the use of this terminology.

Example 16–10. In $\mathbf{R}^3$ suppose $F = \begin{pmatrix}F_1\\F_2\\F_3\end{pmatrix}$ is a three dimensional vector field. The **curl** of F is defined to be the formal cross product $\nabla \times F = \begin{pmatrix}D_2F_3 - D_3F_2\\D_3F_1 - D_1F_3\\D_1F_2 - D_2F_1\end{pmatrix}$. If S

is a two dimensional surface in $\mathbf{R}^3$ then Stoke's Theorem gives

$$\int_{\partial S} F_1\,dx + F_2\,dy + F_3\,dz =$$
$$\int_S (D_2F_3 - D_3F_2)\,dy \wedge dz + (D_3F_1 - D_1F_3)\,dz \wedge dx + (D_1F_2 - D_2F_1)\,dx \wedge dy.$$

(This last integral is often written as $= \int_S \nabla \times F \cdot n\,d\sigma$ where n is the unit normal to the surface and $d\sigma$ is the surface area element.) Hence if the curl of v is zero, the work done in moving a particle along any closed curve is zero. This is easily seen to imply that the 1 form representing the work done by F is closed (and so, exact in a reasonable geometric region). Thus zero curl for F implies that the 1 form is the derivative of a potential.

Problems

Problem 16–1. If $F = e^y\,dx + e^x\,dy$ what is dF?

Problem 16–2. If $F = x\,dy \wedge dz + zx\,dz \wedge dx + zy\,dx \wedge dy$, what is dF?

Problem 16–3. The work form associated with a force field is $W = (6x + 5y)\,dx + (5x + 4y)\,dy$. What is dW? If possible, find a potential for W. If it is not possible to find a potential for W explain why not.

Problem 16–4. A force field on $\mathbf{R}^3$ is $F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ x+z \\ y \end{pmatrix}$. What work form W is associated with this force field? What is dW? Is there a potential? Find the work done by a particle moving along the line segment connecting the origin to the point $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Problem 16–5. For the work form $W = y^3\,dx - x^3\,dy$ find the work done by a particle moving counterclockwise once around the circle of radius 4 centered at the origin.

Problem 16–6. From the form $F = xy\,dx + yz\,dy + xy\,dz$ a fluid flow is obtained as dF. What is the net rate of flow per unit time through the part of the paraboloid $z = 9 - x^2 - y^2$ above the x-y plane?

Problem 16–7. For the work form $W = (x + y^2)\,dx + (y + z^2)\,dy + (z + x^2)\,dz$ find the work done by a particle traversing the edge of the rectangle with vertices at $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ in the counterclockwise direction.

Problem 16–8. Find the net outflow of fluid per unit time for a flow with velocity field $V\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3y^2z^3 \\ 9x^2yz^2 \\ -4xy^2 \end{pmatrix}$ from the cube with vertices $\begin{pmatrix} \pm 2 \\ \pm 2 \\ \pm 2 \end{pmatrix}$.

Problem 16–9. Suppose f is a vector field on $\mathbf{R}^3$ with component functions f_1, f_2, and f_3. Define three associated differential forms by the equations

$$\omega_f^1 = f_1\,dx + f_2\,dy + f_3\,dz$$
$$\omega_f^2 = f_1\,dy \wedge dz + f_2\,dz \wedge dx + f_3\,dx \wedge dy$$
$$\omega_f^3 = (f_1 + f_2 + f_3)\,dx \wedge dy \wedge dz.$$

Give a physical interpretation of each of these three forms.

Problem 16–10. If $f : \mathbf{R}^3 \to \mathbf{R}$ show that $df = \omega^1_{\nabla f}$.

Problem 16–11. Show that if $F : \mathbf{R}^3 \to \mathbf{R}^3$ then $d(\omega^1_F) = \omega^2_{\text{curl } F}$ and that $d(\omega^2_F) = \omega^3_{\text{div } F}$.

Problem 16–12. Use the previous problem to show that curl $\nabla f = 0$ and that div curl $F = 0$.

Problem 16–13. Consider the force field $\omega = (2x + y)\,dx + x\,dy$ in $\mathbf{R}^2$. Find the amount of work done in moving a particle from $(1, -2)$ to $(2, 1)$.

Problem 16–14. Is the graviational field d $F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \dfrac{-1}{(x^2 + y^2 + z^2)^{3/2}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ the gradient of some function?

Problem 16–15. Argue that the work done by the gravitational field depends only on the norm of the endpoints of the path traveled, not on the path itself.

Problem 16–16. In $\mathbf{R}^2$, $n = n(x)$ is a unit vector which is normal to the boundary of a region S at the point x and ds is the arc length element. Let $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ be a two dimensional vector field on $\mathbf{R}^2$. Show that $\int_C F \cdot n\, ds = \int_C -F_2\, dx + F_1\, dy$.

Problem 16–17. In $\mathbf{R}^3$, $n = n(x)$ is a unit vector which is normal to the surface S at the point x and $d\sigma$ is the surface area element. Let $F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$ be a three dimensional vector field on $\mathbf{R}^3$. Show that $\int_S F \cdot n\, d\sigma = \int_S F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy$.

Solutions to Problems

Problem 16–1. $dF = (e^x - e^y)\, dx \wedge dy$.

Problem 16–2. $dF = dx \wedge dy \wedge dz + y\, dz \wedge dx \wedge dy = (1 + y)\, dx \wedge dy \wedge dz$.

Problem 16–3. Computing gives $dW = (6\, dx + 5\, dy) \wedge dx + (5\, dx + 4\, dy) \wedge dy = 0$, so W is closed. Since W is defined everywhere, W is exact. One potential for W is $P\begin{pmatrix} x \\ y \end{pmatrix} = 3x^2 + 5xy + 2y^2$.

Problem 16–4. The associated work form is $W = y\, dx + (x + z)\, dy + y\, dz$, for which $dW = dy \wedge dx + dx \wedge dy + dz \wedge dy + dy \wedge dz = 0$. One potential is $P\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xy + yz$. The work done along this path is $P\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - P\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 8$.

Problem 16–5. The form W is not exact, but Stoke's Theorem can still be applied. Here $dW = -3(x^2 + y^2)\, dx \wedge dy$. If C is the circular disk, the work done is $\int_C -3(x^2 + y^2)\, dx \times dy = \int_0^{2\pi} \int_0^4 -3r^3\, dr\, d\theta = -384\pi$.

Problem 16–6. If P denotes the paraboloid surface, $\displaystyle\int_P dF = \int_{\partial P} F$ by Stoke's Theorem. Now the boundary of P is a circle of radius 3 in the x-y plane which can be parameterized as $p(t) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}$ for $0 \le t \le 2\pi$. So $\displaystyle\int p^* F = \int_0^{2\pi} -\cos t \sin^2 t\, dt = 0$.

Problem 16–7. Stoke's Theorem says that the work done is the same as the integral of dW over the solid rectangle, which is $\displaystyle\int_0^1 \int_0^1 -2y\, dx\, dy = -1$.

Problem 16–8. The associated fluid flow form is $F = 9x^2y^2\, dy \wedge dz + 9x^2yz^2\, dz \wedge dx - 4xy^2\, dx \wedge dy$. Stoke's Theorem gives the flow as the integral of dF over the solid cube. Here $dF = 9x^2z^2\, dx \wedge dy \wedge dz$, so this integral is $\displaystyle\int_{-2}^2 \int_{-2}^2 \int_{-2}^2 9x^2z^2\, dx\, dy\, dz = 1024$.

Problem 16–9. The form ω_f^1 represents the work done by the force field f, the form ω_f^2 represents the rate a which a fluid with velocity vector f is passing through a surface, and ω_f^3 represents the rate at which a fluid with velocity vector f is leaving a 3 dimensional region.

Problem 16–13. Since $d\omega = 0$ the form is closed and hence exact here. It is easy to see that if $\eta = x^2 + xy$ then $d\eta = \omega$. The work done is therefore $\eta(2, 1) - \eta(1, -2) = 6 - (-1) = 7$.

Problem 16–14. Recall that if $f\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is a real valued function, the gradient of f

is the vector $\begin{pmatrix} D_1 f\begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ D_2 f\begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ D_3 f\begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{pmatrix}$. What is the gradient of the function $(x^2+y^2+z^2)^{-1/2}$?

Problem 16–15. If $p(t)$ is a parameterization of the path, then $\frac{d}{dt}\|p(t)\|^{-1} = F(p(t))\bullet p'(t)$, by the chain rule. So the Fundamental Theorem of Calculus applies.

Problem 16–16. If c is a 1 cube parameterizing S then $ds = \left\| c'(t) \right\| dt$. Find a formula for n and use an appropriate pull back to the x-y plane.

Problem 16–17. If $c = (c_1, c_2, c_3)$ is a 2 cube parameterizing S then n is proportional to $\begin{pmatrix} D_1 c_1 \\ D_1 c_2 \\ D_1 c_3 \end{pmatrix} \times \begin{pmatrix} D_2 c_1 \\ D_2 c_2 \\ D_2 c_3 \end{pmatrix}$. What is $d\sigma$ in terms of c?

Solutions to Exercises

Exercise 16–1. The divergence of the vector field $V\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ is $\frac{\partial}{\partial x}v_1 + \frac{\partial}{\partial y}v_2 + \frac{\partial}{\partial z}v_3$.

Exercise 16–2. $d(e^{xy}\,dx - \sin(xy)\,dy) = (ye^{xy}\,dx + xe^{xy}\,dy) \wedge dx - (y\cos(xy)\,dx + x\cos(xy)\,dy) \wedge dy = xe^{xy}\,dy \wedge dx - y\cos(xy)\,dx \wedge dy = (-xe^{xy} - y\cos(xy))\,dx \wedge dy$.

Exercise 16–3. Here $d(2x + 3y) = 2\,dx + 3\,dy = W$. The same is true for the function $2x + 3y + 17$. There are many other choices.

Exercise 16–4. A parameterization of the path is $p(t) = t\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ for $0 \le t \le 1$, so that $p^*W = 2(5\,dt) + 3(7\,dt) = 31\,dt$. Hence $\int p^*W = \int_0^1 31\,dt = 31$.

Exercise 16–5. One parameterization of a path above the origin is $p(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$ for $-\pi \le t \le 0$. This gives the work done as $\int_{-\pi}^{0} -1\,dt = \pi$. One parameterization of a path below the origin is $p(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ for $-\pi \le t \le 0$, which gives the work done as $\int_{-\pi}^{0} 1\,dt = -\pi$.

Exercise 16–6. Here it is convenient to use a different parameterization for each side of the square. The boundary of the square should be traversed in a counterclockwise direction to measure outflow. All parameterizations here are for $0 \le t \le 1$. For the bottom edge: $x = t$ and $y = 0$ so $dx = dt$ and $dy = 0$ giving both the pull back and integral as 0. For the right edge: $x = 1$ and $y = t$ so that $dx = 0$ and $dy = 1$ giving the pull back as $-dt$ and the integral as -1. For the top: $x = (1 - t)$ and $y = 1$ so that $dx = -dt$ and $dy = 0$ giving the pull back as $-dt$ and the integral as -1. For the left edge: $x = 0$ and $y = 1 - t$ so that $dx = 0$ and $dy = -dt$ giving the pull back and integral as 0. The total integral is -2.

Exercise 16–7. If the divergence is zero, the previous example shows that the net flow across any boundary must be zero. So the fluid can not be compressable.