MATH 335: Applied Analysis I
Formulas

First Order Eqs

Separable Eqs

$$\boxed{\frac{dy}{dx} = -\frac{M(x)}{N(y)}}, \qquad y(x_0) = y_0$$

$$\boxed{\int N(y)\,dy + \int M(x)\,dx = C}$$

$$\int_{y_0}^{y} N(s)\,ds + \int_{x_0}^{x} M(u)\,du = 0$$

Linear Eqs

$$\boxed{\frac{dy}{dx} + p(x)y = g(x)}, \qquad y(x_0) = y_0$$

Integrating factor

$$\boxed{\mu(x) = \exp\left(\int p(x)\,dx\right)}$$

$$\boxed{y(x) = \frac{1}{\mu(x)}\left\{\int \mu(x)g(x)\,dx + C\right\}}$$

$$\mu(x) = \exp\left(\int_{x_0}^{x} p(s)\,ds\right)$$

$$y(x) = \frac{1}{\mu(x)}\left\{\int_{x_0}^{x} \mu(t)g(t)\,dt + y_0\right\}$$

Bernoulli Eq

$$\frac{dy}{dx} + p(x)y = q(x)y^n, \qquad n \neq 0, 1$$

Substitution: $y = v^{\frac{1}{1-n}} \Rightarrow$ linear eq.

$$\frac{dv}{dx} + (1-n)p(x)v = (1-n)q(x)$$

Riccati Eq

$$\frac{dy}{dx} = f(x) + g(x)y + h(x)y^2$$

If $y_1(x)$ is a solution, then substitution $y = y_1 + \frac{1}{v}$ $\Rightarrow$ linear eq.

$$\frac{dv}{dx} + [g(x) + 2h(x)y_1(x)]v = -h(x)$$

Homogeneous Eqs

$$\boxed{\frac{dy}{dx} = F\left(\frac{y}{x}\right)}, \qquad y(x_0) = y_0$$

Substitution: $\boxed{y = xv}$ $\Rightarrow$ separable eq.

$$x\frac{dv}{dx} = F(v) - v$$

$$\int \frac{dv}{F(v) - v} = \int \frac{dx}{x}$$

Orthogonal trajectories

$$f(x, y) = C$$

$$\frac{dy}{dx} = \frac{\partial_y f}{\partial_x f}$$

Exact Eqs

$$\boxed{M(x, y)\,dx + N(x, y)\,dy = 0}$$

Check:

$$\boxed{\partial_y M(x,y) = \partial_x N(x,y)}$$

Solution:

$$\boxed{\Psi(x,y) = C},$$

where

$$\partial_x \Psi(x,y) = M(x,y), \qquad \partial_y \Psi(x,y) = N(x,y)$$

$$\Psi = \int M\,dx + \int N\,dy - \int \left(\int \partial_y M\,dx \right) dy$$

Integrating factor:

$$\partial_y(\mu M) = \partial_x(\mu N)$$

If $(M_y - N_x)/N = f(x)$, then $\mu = \mu(x)$

$$\int \frac{d\mu}{\mu} = \int \frac{M_y - N_x}{N}\,dx$$

If $(N_x - M_y)/M = h(y)$, then $\mu = \mu(y)$

$$\int \frac{d\mu}{\mu} = \int \frac{N_x - M_y}{M}\,dy$$

Autonomous Eqs

$$\boxed{\frac{dy}{dx} = f(y)}$$

Equilibrium solutions (Critical points)

$$\boxed{f(y) = 0}$$

Second Order Eqs

Eqs with Dependent Variable Missing

$$y'' = f(t, y')$$

Substitution $v(t) = y'(t)$

$$v' = f(t, v)$$

$$y(t) = \int v(t)\,dt + C$$

Eqs with Independent Variable Missing

$$y'' = f(y, y')$$

Substitution $v(y) = y'$

$$v\frac{dv}{dy} = f(y, v)$$

$$t = \int \frac{dy}{v(y)} + C$$

Second Order Linear Eqs

Homogeneous Eq

$$\boxed{Ly = y'' + p(t)y' + q(t)y = 0}$$

Fundamental Solutions $y_1,\ y_2$
Wronskian

$$\boxed{W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0}$$

General Solution

$$\boxed{y(t) = c_1 y_1(t) + c_2 y_2(t)}$$

Homogeneous Eqs with Constant Coeff

$$\boxed{ay'' + by' + cy = 0}$$

Characteristic Eq ($y = e^{rt}$)

$$\boxed{ar^2 + br + c = 0}$$

$$\boxed{r_{1,2} = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right)}$$

Fundamental Solutions
I. Real Distinct Roots ($r_1 \neq r_2$)

$$\boxed{y_1 = e^{r_1 t}, \qquad y_2 = e^{r_2 t}}$$

II. Complex Distinct Roots ($r_{1,2} = \lambda \pm i\mu,\ \mu \neq 0$)

$$\boxed{y_1 = e^{\lambda t}\cos(\mu t), \qquad y_2 = e^{\lambda t}\sin(\mu t)}$$

III. Repeated Roots ($r_1 = r_2 = r$)

$$\boxed{y_1 = e^{rt}, \qquad y_2 = te^{rt}}$$

Euler Eq

$$t^2 y'' + \alpha t y' + \beta y = 0$$

Substitution $x = \ln t$.
Characteristic Eq. ($y = t^r$)

$$r(r-1) + \alpha r + \beta = 0$$

Real Distinct Roots ($r_1 \neq r_2$)

$$y_1 = t^{r_1}, \qquad y_2 = t^{r_2}$$

Complex Distinct Roots ($r_{1,2} = \lambda \pm i\mu$)

$$y_1 = t^{\lambda}\cos(\mu \ln t), \qquad y_2 = t^{\lambda}\sin(\mu \ln t)$$

Repeated Roots ($r_1 = r_2 = r$)

$$y_1 = t^r, \qquad y_2 = t^r \ln t$$

Reduction of Order
If y_1 is solution of

$$y'' + p(t)y' + q(t) = 0$$

then $\boxed{y = vy_1} \Longrightarrow$

$$v'' + \left(p + 2\frac{y_1'}{y_1}\right)v' = 0$$

Non-homogeneous Eq

$$\boxed{y'' + p(t)y' + q(t)y = g(t)}$$

Particular Solution $Y(t)$
General Solution:

$$\boxed{y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)}$$

$$\boxed{Y(t) = -y_1 \int \frac{y_2\, g}{W(y_1, y_2)} dt + y_2(t) \int \frac{y_1\, g}{W(y_1, y_2)} dt}$$

$$\begin{aligned} g = P_n(t) &\Longrightarrow Y = t^s Q_n(t) \\ g = P_n(t)e^{\alpha t} &\Longrightarrow Y = t^s Q_n(t) e^{\alpha t} \\ g = P_n(t)e^{(\alpha \pm i\beta)t} &\Longrightarrow Y = t^s Q_n(t) e^{\alpha t}\cos(\beta t) \\ & \qquad\quad + t^s R_n(t) e^{\alpha t}\sin(\beta t) \end{aligned}$$

Higher Order Linear Eqs

Homogeneous Eq

$$\boxed{Ly = y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_0(t)y = 0}$$

Fundamental Solutions $y_1, \dots,\ y_n$
Wronskian

$$\boxed{W(y_1, \dots, y_n) = \begin{vmatrix} y_1 & \dots & y_n \\ y_1' & \dots & y_n' \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} \neq 0}$$

General Solution

$$\boxed{y(t) = c_1 y_1(t) + \cdots + c_n y_n(t)}$$

Homogeneous Eqs with Constant Coeff

$$\boxed{a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0}$$

Characteristic Eq ($y = e^{rt}$)

$$\boxed{a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0}$$

Fundamental Solutions
I. Real Distinct Roots ($r_1 \neq r_2$)

$$\boxed{y_1 = e^{r_1 t}, \dots, y_n = e^{r_n t}}$$

II. Complex Distinct Roots ($r_{1,2} = \lambda \pm i\mu,\ \mu \neq 0$)

$$\boxed{y_1 = e^{\lambda t}\cos(\mu t), \qquad y_2 = e^{\lambda t}\sin(\mu t)}$$

III. Repeated Roots (r with multiplicity s)

$$\boxed{y_1 = e^{rt},\ y_2 = te^{rt}, \dots, y_s = t^{s-1}e^{rt}}$$

Repeated Roots ($\lambda \pm i\mu$ with multiplicity s)

$$\boxed{y_1 = e^{rt}\cos(\mu t), \dots, \quad y_s = t^{s-1}e^{rt}\cos(\mu t)}$$

$$\boxed{y_{s+1} = e^{rt}\sin(\mu t), \dots, y_{2s} = t^{s-1}e^{rt}\sin(\mu t)}$$

Non-homogeneous Eq

$$\boxed{Ly = y^{(n)} + p_{n-1}(t)y^{(n-1)} + \dots + p_0(t)y = g(t)}$$

Particular Solution $Y(t)$
General Solution:

$$\boxed{y(t) = c_1 y_1(t) + \dots + c_n y_n(t) + Y(t)}$$

$$\boxed{Y(t) = \sum_{k=1}^{n} y_k \int \frac{W_k}{W}\, g\, dt}$$

(W_k is obtained from W by replacing the kth column by $(0, \dots, 0, 1)$)

$$\begin{aligned}
g = P_m(t) &\implies Y = t^s Q_m(t)\\
g = P_m(t)e^{\alpha t} &\implies Y = t^s Q_m(t)e^{\alpha t}\\
g = P_m(t)e^{(\alpha\pm i\beta)t} &\implies Y = t^s Q_m(t)e^{\alpha t}\cos(\beta t)\\
&\qquad\ \ + t^s R_m(t)e^{\alpha t}\sin(\beta t)
\end{aligned}$$

Series Solutions of Linear Eqs

Convention $a_{-k} = 0$ $(k = 1, 2, \dots)$.
Power series

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum a_n (x - x_0)^n$$

Radius of convergence

$$\boxed{R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|}$$

Multiplication of series

$$\sum a_n (x - x_0)^n \sum b_k (x - x_0)^k = \sum c_n (x - x_0)^n$$

$$\boxed{c_n = \sum_{k=0}^{n} a_k b_{n-k}}$$

Taylor series

$$y(x) = \sum_{k=0}^{\infty} \frac{y^{(k)}(x_0)}{k!}(x - x_0)^k$$

f is analytic at x_0 if Taylor series converges to $f(x)$ for $|x - x_0| < R$
Shift of summation index

$$\boxed{\sum a_n (x - x_0)^{n-N} = \sum a_{n+N} (x - x_0)^n}$$

2nd Order Linear Eq

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

Ordinary Point

$P(x_0) \neq 0$, P, Q, R analytic at x_0

$$\boxed{y = \sum_{n=0}^{\infty} a_n (x - x_0)^n}$$

$$a_0 = y(x_0), \qquad a_1 = y'(x_0)$$

Regular Singular Point $x_0 = 0$

$P(0) = 0$,

$$x\frac{Q(x)}{P(x)} \quad \text{and} \quad x^2\frac{R(x)}{P(x)} \quad \text{analytic at } 0$$

$$p_0 = \lim_{x\to 0} x\frac{Q(x)}{P(x)}, \qquad q_0 = \lim_{x\to 0} x^2\frac{R(x)}{P(x)}$$

Indicial Equation

$$r(r-1) + p_0 r + q_0 = 0$$

Exponents at Singularity

$$r_{1,2} = \frac{1}{2}\left[-(p_0-1)^2 \pm \sqrt{(p_0-1)^2 - 4q_0}\right]$$

$$r_1 - r_2 = \sqrt{(p_0-1)^2 - 4q_0}$$

Fundamental Solutions:

Case I. r_1, r_2 real, $r_1 - r_2$ not integer.

$$\boxed{y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r_1}}$$

$$\boxed{y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2}}$$

$\boxed{a_0 = b_0 = 1}$

General Solution

$$\boxed{y(x) = \sum_{n=0}^{\infty} \left[d_n x^{r_1} + e_n x^{r_2}\right] x^n}$$

d_0, e_0 arbitrary.

Case II. r_1, r_2 complex.

$$\boxed{y_1(x) = \mathrm{Re} \sum_{n=0}^{\infty} a_n x^{n+r_1}}$$

$$\boxed{y_2(x) = \mathrm{Im} \sum_{n=0}^{\infty} a_n x^{n+r_1}}$$

$\boxed{a_0 = 1}$

General Solution for $r_{1,2} = \alpha \pm i\beta$

$$\boxed{y(x) = \sum_{n=0}^{\infty} \left[d_n \cos(\beta \ln x) + e_n \sin(\beta \ln x)\right] x^{n+\alpha}}$$

d_0, e_0 arbitrary, real

Case III. $r_1 = r_2$

$$\boxed{y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r_1}}$$

$$\boxed{y_2(x) = y_1(x) \ln x + \sum_{n=1}^{\infty} b_n x^{n+r_1}}$$

$$b_n = \frac{\partial}{\partial r} a_n(r)\Big|_{r=r_1}$$

$\boxed{a_0 = 1, b_0 = 0}$

General Solution

$$\boxed{y(x) = \sum_{n=0}^{\infty} (d_n \ln x + e_n) x^{n+r_1}}$$

d_0, e_0 arbitrary

Case IV. $r_1 - r_2 = N$ positive integer

$$\boxed{y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r_1}}$$

$$\boxed{y_2(x) = a y_1(x) \ln x + \sum_{n=0}^{\infty} c_n x^{n+r_2}}$$

$$c_n = \frac{\partial}{\partial r}(r - r_2) a_n(r)\Big|_{r=r_2}$$

$$a = \lim_{r \to r_2} (r - r_2) a_N(r)$$

$\boxed{a_0 = c_0 = 1}$

General Solution

$$\boxed{y(x) = \sum_{n=0}^{N-1} d_n x^{n+r_2} + \sum_{n=0}^{\infty} \left[e_n \ln x + f_n\right] x^{n+N+r_2}}$$

d_0, e_0 arbitrary

Laplace Transform

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

$$\mathcal{L}\{cf(t)\} = cF(s)$$
$$\mathcal{L}^{-1}\{cF(s)\} = cf(t)$$

$$\mathcal{L}\{f_1(t) + f_2(t)\} = F_1(s) + F_2(s)$$
$$\mathcal{L}^{-1}\{F_1(s) + F_2(s)\} = f_1(t) + f_2(t)$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1}(0)$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$
$$\mathcal{L}^{-1}\{F^{(n)}(s)\} = (-t)^n f(t)$$

$$\mathcal{L}\{e^{ct} f(t)\} = F(s-c)$$
$$\mathcal{L}^{-1}\{F(s-c)\} = e^{ct} f(t)$$

$$\mathcal{L}\{f(ct)\} = \frac{1}{c} F\left(\frac{s}{c}\right)$$
$$\mathcal{L}^{-1}\{F(ks)\} = \frac{1}{k} f\left(\frac{t}{k}\right)$$

Step (Heaviside) Function

$$u_c(t) = \theta(t-c) = \begin{cases} 0, & t < c \\ 1, & t \ge c \end{cases}$$

$$\mathcal{L}\{\theta(t-c)\} = \frac{e^{-cs}}{s}$$

$$\mathcal{L}\{\theta(t-c) f(t-c)\} = e^{-cs} F(s)$$

$$\mathcal{L}^{-1}\{e^{-cs} F(s)\} = \theta(t-c) f(t-c)$$

$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2 + a^2}$$

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2 + a^2}$$

Impulse (Dirac) Function

$$\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1$$

$$\int_{-\infty}^{\infty} \delta(t - t_0) f(t)\, dt = f(t_0)$$

$$\delta(t - t_0) = \frac{d}{dt} \theta(t - t_0)$$

$$\mathcal{L}\{\delta(t - t_0)\} = \theta(t_0) e^{-st_0}$$

Convolution

$$h(t) = (f * g)(t) = \int_0^t f(t-\tau) g(\tau)\, d\tau$$

$$f * g = g * f$$

$$f * (g_1 + g_2) = f * g_1 + f * g_2$$

$$(f * g) * h = f * (g * h)$$

$$f * 0 = 0$$

$$\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{f(t)\} \mathcal{L}\{g(t)\}$$
$$H(s) = F(s) G(s)$$