ΣΧΕΔΙΑΣΜΟΣ ΚΑΙ ΠΡΟΓΡΑΜ-ΜΑΤΙΣΜΟΣ ΠΑΡΑΓΩΓΗΣ
Έλεγχος Αποθεμάτων Υπό Αβέβαιη Ζήτηση

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PRODUCTION PLANNING AND SCHEDULING

Inventory Control Under Uncertain Demand

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The Newsvendor model

- **Assumptions/notation**
  - Single-period horizon
  - Uncertain demand in the period: D (parts) assume continuous random variable
    Density function and cumulative distribution function of D: $f(x)$ and $F(x)$

$$F(a) = P(D \leq a) = \int_{x=0}^{a} f(x)dx \qquad f(a) = \left.\frac{dF(x)}{dx}\right|_{x=a}$$

  - Infinite production/replenishment rate (instantaneous replenishment)
  - Zero lead time
  - Overage cost rate: cost per unit of positive inventory remaining at the end of the period: c_o (€ per left-over part)
  - Underage cost rate: cost per unit of unsatisfied demand (negative ending inventory) : c_u (€ per missing part or unsatisfied demand)
  - No fixed setup production/order cost
- **Decision**
  - Order quantity at the beginning of the period: Q (parts)

The Newsvendor model

- **Definitions**
  - (Positive) inventory remaining at the end of the period: I^+
  - Unsatisfied demand (negative inventory) at the at the end of the period: I^-

$$I^+ = (Q - D)^+ \equiv \max(Q - D, 0)$$

$$I^- = (D - Q)^+ \equiv \max(D - Q, 0)$$

  - Total overage and underage cost at the end of the period: $G(Q, D)$

$$G(Q,D) = c_o I^+ + c_u I^- = c_o (Q - D)^+ + c_u (D - Q)^+$$

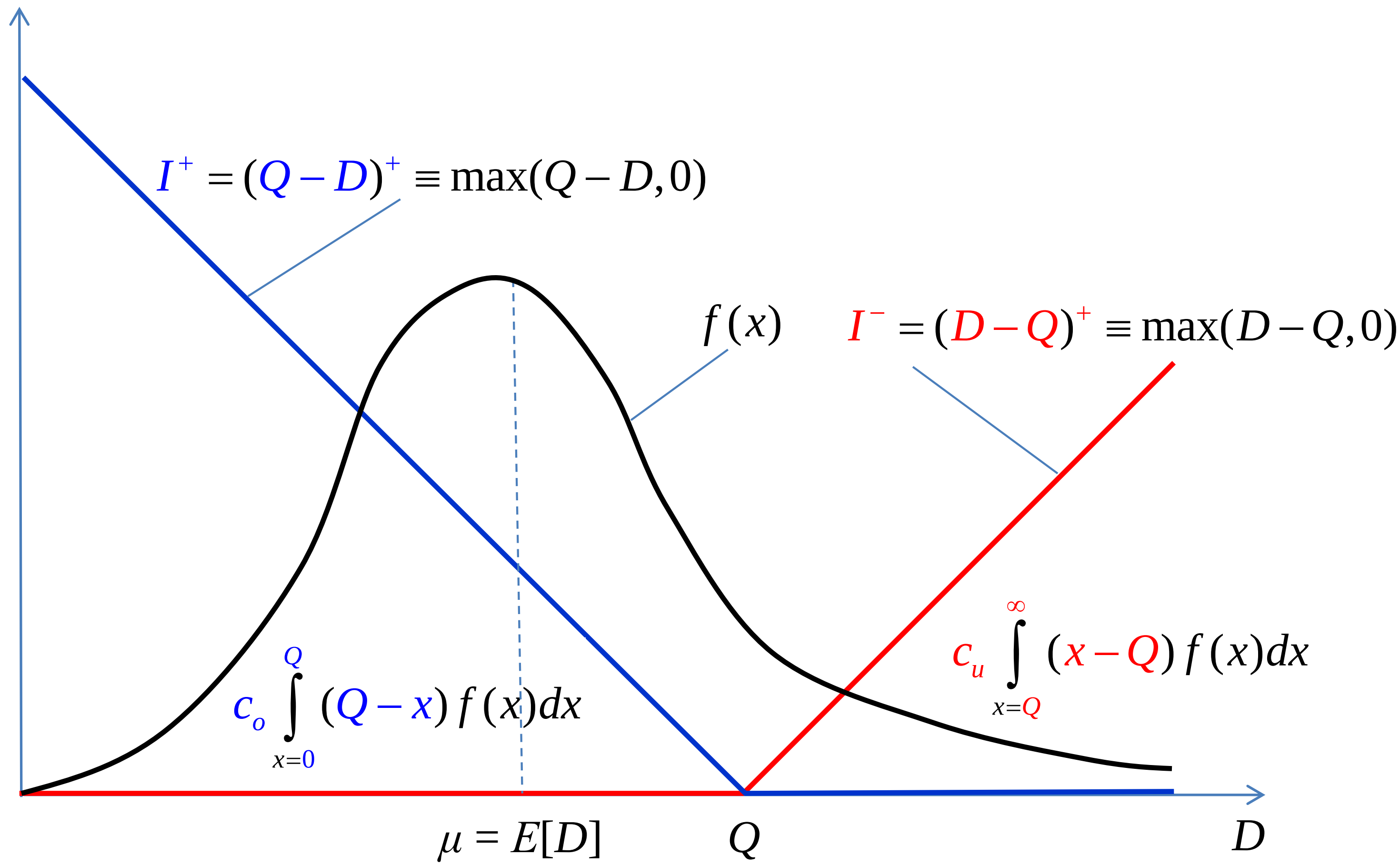
  - Expected cost: $G(Q)$

$$G(Q) = \underset{D}{E}[G(Q,D)] = \int_{x=0}^{\infty} G(Q,x) f(x) dx$$

$$= c_o \int_{x=0}^{\infty} (Q - x)^+ f(x) dx + c_u \int_{x=0}^{\infty} (x - Q)^+ f(x) dx$$

$$= c_o \int_{x=0}^{Q} (Q - x) f(x) dx + c_u \int_{x=Q}^{\infty} (x - Q) f(x) dx$$

The Newsvendor model



The Newsvendor model

- **Problem**

$$\underset{Q}{\text{Minimize}} \; G(Q)$$

- **First derivative of cost function**

$$\frac{dG(Q)}{dQ} = \frac{d}{dQ}\left[c_o \int_{x=0}^{Q} (Q-x)\, f(x)dx + c_u \int_{x=Q}^{\infty} (x-Q)\, f(x)dx \right]$$

$$= c_o \int_{x=0}^{Q} \frac{d}{dQ}(Q-x)\, f(x)dx + 1(Q-x)\, f(x)\Big|_{x=Q} - 0(Q-x)\, f(x)\Big|_{x=0} +$$

$$c_u \int_{x=Q}^{\infty} \frac{d}{dQ}(x-Q)\, f(x)dx + 0(x-Q)\, f(x)\Big|_{x=\infty} - 1(x-Q)\, f(x)\Big|_{x=Q}$$

$$= c_o \int_{x=0}^{Q} 1\, f(x)dx + c_u \int_{x=Q}^{\infty} (-1)\, f(x)dx$$

$$= c_o F(Q) - c_u [1 - F(Q)]$$

The Newsvendor model

- **Second derivative**

$$\frac{dG^2(Q)}{dQ^2} = \frac{d}{dQ}\left[c_o F(Q) - c_u[1 - F(Q)]\right] = (c_o + c_u) f(Q) \geq 0$$

- **First-order condition for minimization**

$$\left.\frac{dG(Q)}{dQ}\right|_{Q=0} = 0 \quad \Rightarrow \quad c_o F(Q) - c_u[1 - F(Q)] = 0 \quad \Rightarrow \quad (c_o + c_u) F(Q) = c_u$$

$$\Rightarrow \quad \boxed{Q^* : F(Q^*) = \frac{c_u}{c_o + c_u}} \quad \Rightarrow \quad \boxed{Q^* = F^{-1}\left(\frac{c_u}{c_o + c_u}\right)}$$

Note:

– $F(Q)$ is the fill rate, i.e. the probability that a demand will be satisfied!

– Recall: EOQ model with backorders: $F^* = \dfrac{b}{h + b}$

The Newsvendor model

- **Special case:** $D \sim$ Normal(μ, σ)

$$F(Q) = P(D \le Q) = P\left(\underbrace{\frac{D-\mu}{\sigma}}_{\text{Normal}(0,1)} \le \frac{Q-\mu}{\sigma} \right) = \underbrace{\Phi\left(\frac{Q-\mu}{\sigma}\right)}_{\substack{\text{standardized Normal} \\ \text{cumulative distribution} \\ \text{function}}}$$

$$\Rightarrow \quad \boxed{F(Q) = \Phi(z), \quad z = \frac{Q-\mu}{\sigma}}$$

$\Rightarrow$ $\Phi(z)$ and hence $F(Q)$ can be evaluated from standardized Normal tables

z	$\Phi(z)$	$\Phi(z)-0.5$
0	0.5000	0.0000
0.85	0.8023	0.3023
1.19	0.9015	0.4015
1.65	0.9505	0.4505
2.33	0.9901	0.4901
3.09	0.9990	0.4990

The Newsvendor model

- **Special case:** $D \sim$ Normal(μ, σ) **cont'd**

$$Q^*:\ F(Q^*) = \frac{c_u}{c_o + c_u} \quad \Rightarrow \quad \Phi\left(\frac{Q^* - \mu}{\sigma}\right) = \frac{c_u}{c_o + c_u} \quad \Rightarrow \quad \frac{Q^* - \mu}{\sigma} = \underbrace{\Phi^{-1}\left(\frac{c_u}{c_o + c_u}\right)}_{z_{c_u/(c_o+c_u)}}$$

$$\Rightarrow \quad \boxed{Q^* = \mu + \sigma z_{c_u/(c_o+c_u)}}$$

- **Example**

$D \sim$ Normal(120, 45)

buying price c = 30

selling price S = 110

salvage price s = 10

$$\Rightarrow \begin{cases} c_u = S - c = 110 - 30 = 80 \\ c_o = c - s = 30 - 10 = 20 \end{cases}$$

$$\Rightarrow \quad \frac{c_u}{c_o + c_u} = \frac{80}{20 + 80} = 0.80 \quad \Rightarrow \quad z_{0.80} = 0.85$$

$$\Rightarrow \quad Q^* = \mu + \sigma z_{0.80} = 120 + 45 \cdot 0.85 = 120 + 38.25 = 158.85 \approx 159$$

The Newsvendor model

- **Extension: Discrete demand**
  - Uncertain demand in the period: D (parts) assume discrete random variable

    Probability mass function and cumulative distribution function of D: $p(x)$ and $F(x)$

    $$F(a) = P(D \le a) = \sum_{x \le a} p(x) \qquad p(a) = F(a) - F(a-1)$$
  - Expected cost: $G(Q)$

    $$G(Q) = \underset{D}{E}[G(Q,D)] = \sum_{x} G(Q,x)\,p(x) = c_o \sum_{x=0}^{Q-1} (Q-x)\,p(x) + c_u \sum_{x=Q}^{\infty} (x-Q)\,p(x)$$
  - Problem: $\underset{Q}{\text{Minimize}}\ G(Q)$
  - First-order difference

    $$G(Q+1) - G(Q) = c_o \sum_{x=0}^{Q} p(x) - c_u \sum_{x=Q+1}^{\infty} p(x) = c_o F(Q) - c_u[1 - F(Q)]$$
  - First-order condition for minimization

    $$Q^* : \text{smallest } Q \text{ such that } G(Q+1) - G(Q) \ge 0 \iff c_o F(Q) - c_u[1 - F(Q)] \ge 0$$

    $$\Rightarrow \boxed{Q^* : \text{smallest } Q \text{ such that } F(Q^*) \ge \frac{c_u}{c_o + c_u}}$$

The Newsvendor model

- **Extension: Staring inventory** $y > 0$
  - Still want to be at Q^* after ordering, because Q^* is the minimizer of $G(Q)$
  - Order quantity: U
  - Optimal policy now depends on starting inventory:

$$U^*(y) = \begin{cases} Q^* - y, & \text{if } y < Q^* \\ 0, & \text{if } y \geq Q^* \end{cases}$$

Note:

- $U^* \equiv$ optimal order quantity
- $Q^* \equiv$ optimal "order-up-to" point $\equiv$ inventory target level $\equiv$ base stock level

The Newsvendor model

- **Interpretation of c_o and c_u for the single-period model**
  - S = selling price (€ per part)
  - c = variable cost (€ per part)
  - h = holding cost (€ per part per period)
  - p = loss-of-goodwill cost (€ per part short per period)

$$G(Q,D) = \underbrace{cQ}_{\text{order cost}} + \underbrace{h(Q-D)^+}_{\text{leftover inventory cost}} + \underbrace{p(D-Q)^+}_{\text{shortage cost}} \underbrace{- S\underbrace{\min(Q,D)}_{\text{sales}}}_{\text{sales revenue}}$$

$$G(Q) = \underset{D}{E}[G(Q,D)] = cQ + h\int_0^Q (Q-x)f(x)dx + p\int_Q^\infty (x-Q)f(x)dx - S\underbrace{[\underbrace{\int_0^Q xf(x)dx}_{\mu-\int_Q^\infty xf(x)dx} + \int_Q^\infty Qf(x)dx]}_{\mu-\int_Q^\infty (x-Q)f(x)dx}$$

$$= cQ + h\int_0^Q (Q-x)f(x)dx + (p+S)\int_Q^\infty (x-Q)f(x)dx - S\mu$$

$$\frac{dG(Q)}{dQ} = 0 \quad \Rightarrow \quad c + hF(Q) - (p+S)(1-F(Q)) = 0$$

$$\Rightarrow \quad \boxed{Q^*: F(Q) = \frac{p+S-c}{p+S+h}} \quad \Rightarrow \quad \boxed{c_u = p+S-c}, \quad \boxed{c_o = h+c}$$

The Newsvendor model

- **Extension: infinite periods (infinite horizon) with backorders**

  Same assumptions as single-period model except that:

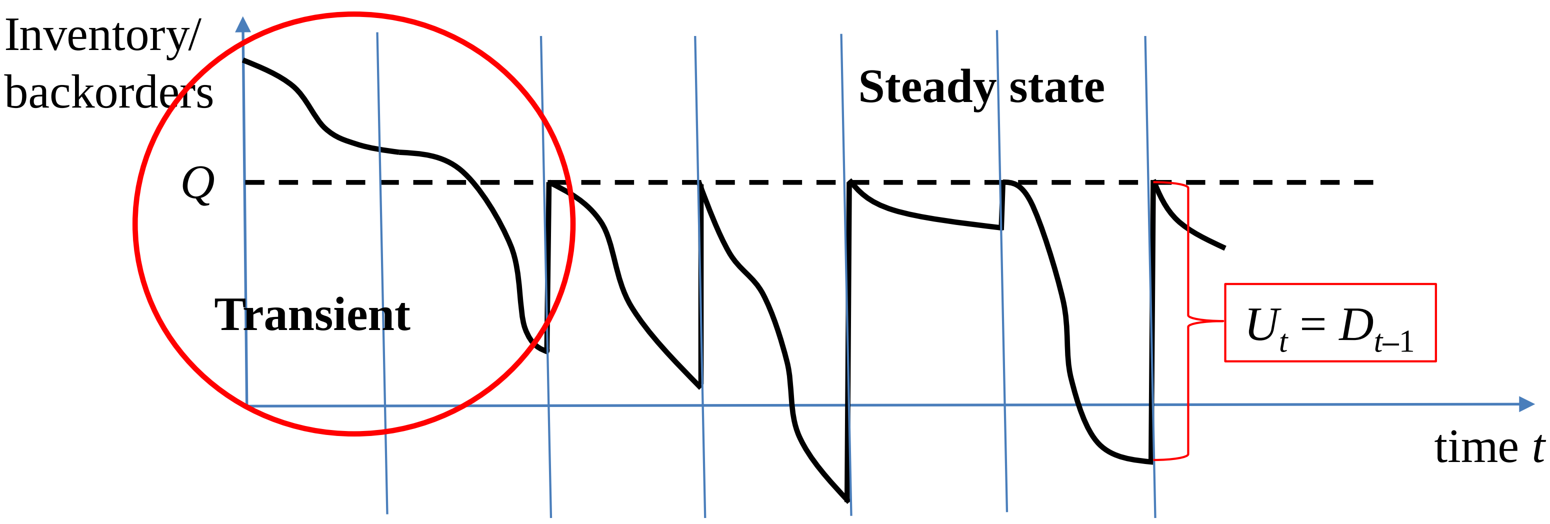
  – D_t = demand in period t; D_1, D_2, D_3, … are i.i.d. with distribution $f(x)$, $F(x)$

  – U_t = amount ordered in period t

  – Optimal policy in each period is "order-up-to" Q

  – In the long-run, the inventory can never be higher than Q

  $\Rightarrow$ In steady-state (long run): $U_t = D_{t-1}$



The Newsvendor model

- **Extension: infinite periods (infinite horizon) with backorders (cont'd)**
  - Total cost in a period with demand D

$$G(Q,D) = \underbrace{(c-S)D}_{\text{order cost } - \text{ sales revenue}} + \underbrace{h(Q-D)^+}_{\text{inventory holding cost}} + \underbrace{p(D-Q)^+}_{\text{backorder cost}}$$

  - Expected average cost per period

$$G(Q) = \underset{D}{E}[G(Q,D)] = (c-S)\mu + hE[(Q-D)^+] + pE[(D-Q)^+]$$

  - First-order condition for minimizing $G(Q)$

$$\frac{dG(Q)}{dQ} = 0 \quad \Rightarrow \quad hF(Q) - p(1-F(Q)) = 0$$

$$\Rightarrow \quad \boxed{Q^* : F(Q^*) = \frac{p}{p+h}} \quad \Rightarrow \quad \text{Newsventor formula: } \boxed{c_u = p}, \quad \boxed{c_o = h}$$

Note:

- c and S play no role in determining Q^*, because in the long run, all demands are satisfied regardless of Q; therefore, the expected average ordering cost minus revenue per period is $(c - S)\mu$ regardless of Q.

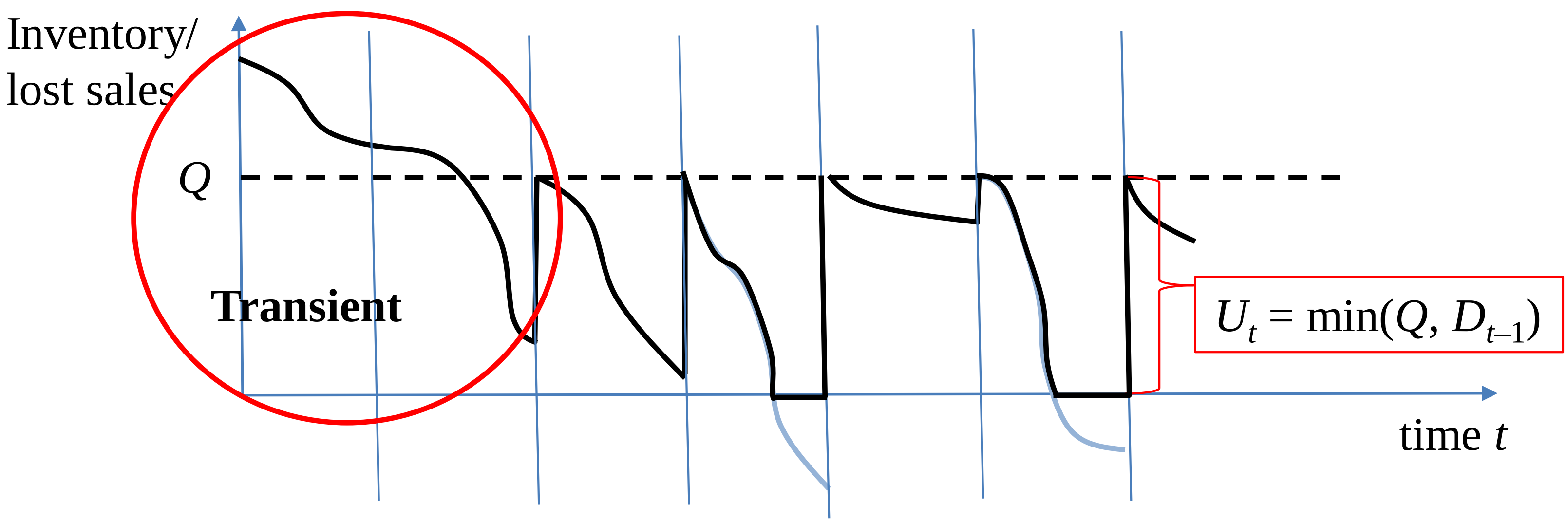
The Newsvendor model

- **Extension: infinite periods (infinite horizon) with lost sales**

  Same assumptions as infinite horizon with backorders except that:

  – Unmet demand is not backordered but is lost

  – In steady-state (long run): $U_t = \min(Q, D_{t-1})$



The Newsvendor model

- **Extension: infinite periods (infinite horizon) with lost sales (cont'd)**
  - Total cost in a period with demand D

$$G(Q,D) = (c-S)\min(Q,D) + h(Q-D)^+ + p(D-Q)^+$$

  - Expected average cost per period

$$G(Q) = \underset{D}{E}[G(Q,D)] = (c-S)\left[\mu - E[(D-Q)^+]\right] + hE[(Q-D)^+] + pE[(D-Q)^+]$$

  - First-order condition for minimizing $G(Q)$

$$\frac{dG(Q)}{dQ} = 0 \quad \Rightarrow \quad hF(Q) - (p+S-c)(1-F(Q)) = 0$$

$$\Rightarrow \quad \boxed{Q^* : F(Q^*) = \frac{p+S-c}{p+S-c+h}} \quad \Rightarrow$$ Newsvendor formula: $\boxed{c_u = p+S-c}$, $\boxed{c_o = h}$

Note:

- c and S now play a role in determining Q^*, because in the long run, the demand satisfied and the orders are min(Q, D), so they depend on Q; therefore, the expected average ordering cost minus revenue per period is $(c - S)E[\min(Q, D)]$.

Lot size – Reorder point (Q, R) model

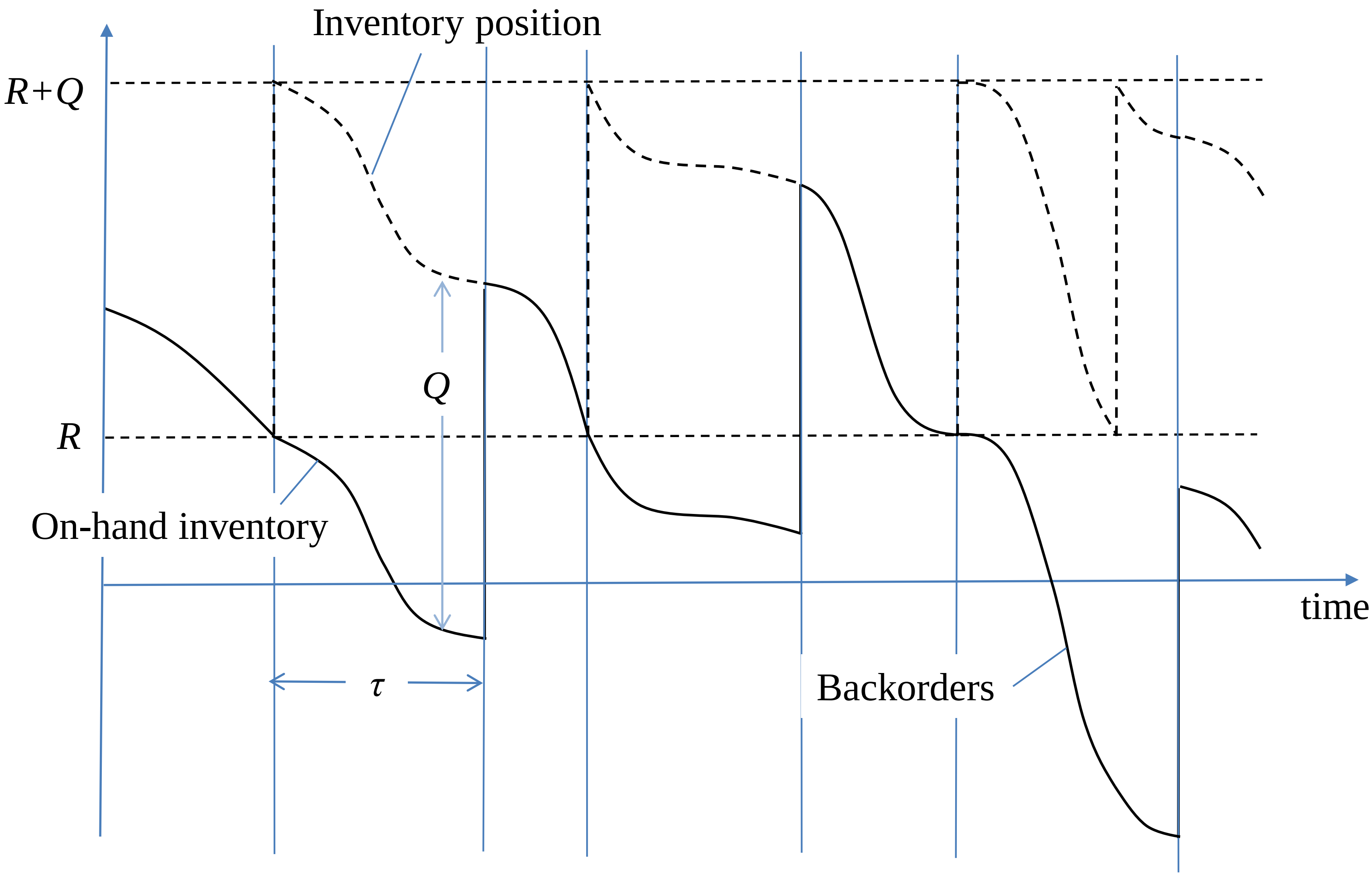
- **Assumptions**
  - Infinite horizon
  - Continuous review (as opposed to periodic review)
  - D_t: random stationary demand per unit time (e.g., daily demand)
    mean $\lambda \equiv E[D_t]$, variance $\sigma_t^2 = E[(D_t - \lambda)^2]$
  - Unmet demand is either backordered or lost
  - τ: Fixed replenishment order lead time
  - Costs:
    - Variable unit production/order cost: c (€ per part)
    - Fixed setup production/order cost: K (€ per production run/order)
    - Inventory holding cost rate: h (€ per part per unit time)
    - Stock-out (shortage/penalty) cost rate (2 cases / 4 situations: see next)
- **Order policy**
  - (Q, R) policy: order Q when *inventory position* falls below R
- **Decision variables**
  - Q: lot size (reorder quantity)
  - R: reorder point

(Q, R) model

- **Assumptions on stock-outs and stock-out cost rate**
  - **Case 1: Backordered demand**
    - p_1 (€ per stock-out occasion)
    - p_2 (€ per part short)
    - p_3 (€ per part short per unit time)
  - **Case 2: Lost sales**
    - p_L (€ per lost sale)

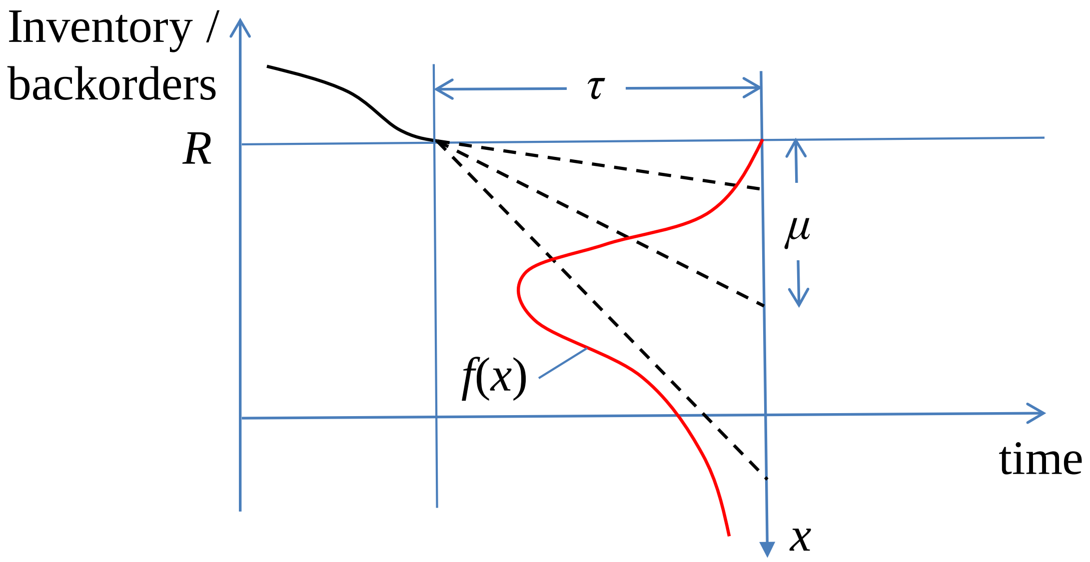
In this course, we only deal with p_2

(*Q*, *R*) model: Backordered demand



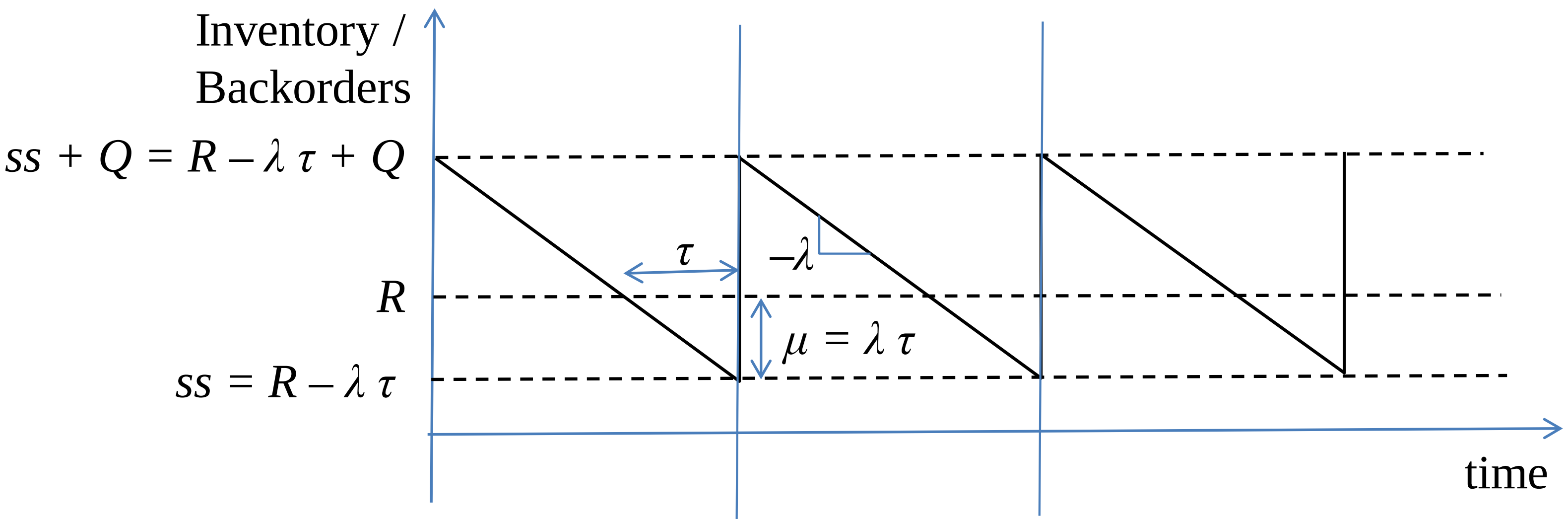
(Q, R) model: Backordered demand

- **Analysis**
  - D: demand during lead time τ
    - Density function and cumulative distribution function of D: $f(x)$ and $F(x)$
    - $D = D_1 + D_2 + \ldots + D_\tau$
    - Mean: $\mu \equiv E[D] = E[D_1 + D_2 + \ldots + D_\tau] = \tau\, E[D_t] = \tau\, \lambda$
    - Variance: $\sigma^2 \equiv Var[D] = E[(D - \mu)^2] = Var[D_1 + D_2 + \ldots + D_\tau] = \tau\, Var[D_t] = \tau\, \sigma_t^2$



(Q, R) model: Backordered demand

- **Inventory holding cost**
  - Safety stock $ss \equiv R - \mu = R - \lambda\,\tau$
  - Expected average inventory approximation $\bar{I} \approx ss + Q/2 = R - \lambda\,\tau + Q/2$ (underestimates true value)
  - Expected average inventory hold cost $= h\,\bar{I} = h(R - \lambda\,\tau + Q/2)$



(Q, R) model: Backordered demand

- **Setup cost**
  - Expected average order frequency = λ/Q
  - Expected average setup cost per unit time = $K\,\lambda/Q$
- **Stock-out (penalty) cost**
  - Assumption: $\tau << Q/\lambda \;\Rightarrow$ stock-out per cycle depends only on R
  - $B(R)$: Expected stock-out cost per cycle (depends on definition of stock-out cost rate)
  - Expected average stock-out cost per unit time = $B(R)\,\lambda/Q$

- **Total expected average cost per unit time**

$$G(Q,R) = h\left(\frac{Q}{2} + R - \lambda\tau\right) + K\frac{\lambda}{Q} + B(R)\frac{\lambda}{Q}$$

(Q, R) model: Backordered demand

- **Optimization problem**

$$\underset{Q,R}{\text{Minimize}}\ G(Q,R) = h\left(\frac{Q}{2} + R - \lambda\tau\right) + K\frac{\lambda}{Q} + \frac{\lambda}{Q}B(R)$$

- **Optimality conditions**

$$\frac{\partial G(Q,R)}{\partial Q} = \frac{h}{2} - \frac{K\lambda}{Q^2} - \frac{\lambda B(R)}{Q^2} = 0 \quad \Rightarrow \quad Q^2 = \frac{2\lambda[K + B(R)]}{h}$$

$$\Rightarrow \quad \boxed{Q = \sqrt{\frac{2\lambda[K + B(R)]}{h}}} \ (1)$$

$$\frac{\partial G(Q,R)}{\partial R} = h + \frac{\lambda}{Q}\frac{dB(R)}{dR} = 0$$

$$\Rightarrow \quad \boxed{\frac{dB(R)}{dR} = -\frac{hQ}{\lambda}} \ (2)$$

(Q, R) model: Backordered demand

- **Optimality condition (2): Case 2**: Stock-out cost p_2 € per part short

$$B(R) = p_2 \underbrace{E[(D-R)^+]}_{\substack{n(R)\equiv\text{expected number}\\ \text{of stock-outs per cycle}}} = p_2 \underbrace{\int_{x=R}^{\infty} (x-R) f(x) dx}_{n(R)} \Rightarrow \frac{dB(R)}{dR} = -p_2[1-F(R)]$$

$$\text{Condition (2)}: \quad -p_2[1-F(R)] = -\frac{hQ}{\lambda} \Rightarrow \boxed{F(R) = 1 - \frac{hQ}{p_2\lambda}}$$

(*Q*, *R*) model: Backordered demand

- **Simultaneous solution of conditions (1) and (2)**

  Solve by fixed-point iteration:

  1. Given *R*, solve (1) to find *Q*
  2. Given *Q*, solve (2) to find *R*
  3. Repeat until convergence

(Q, R) model: Backordered demand

Illustration for case 2: Stock-out cost p_2 € per part short

- **Optimality conditions for case 2**

$$\boxed{Q = \sqrt{\frac{2\lambda[K + p_2 n(R)]}{h}}} \quad (1) \qquad \boxed{F(R) = 1 - \frac{hQ}{p_2 \lambda}} \quad (2)$$

- **Assumption:** D~Normal(μ, σ)

  Use standardized cumulative distribution function (cdf) $\Phi(z)$ to compute $F(R)$

$$F(R) = P(D \leq R) = P\left(\underbrace{\frac{D-\mu}{\sigma}}_{\text{Normal}(0,1)} \leq \frac{R-\mu}{\sigma} \right) = \Phi\left(\frac{R-\mu}{\sigma}\right)$$

$$\Rightarrow \quad \boxed{F(R) = \Phi(z), \quad z = \frac{R-\mu}{\sigma}}$$

- $\Phi(z)$ and hence $F(R)$ can be evaluated from standardized Normal cdf tables

(Q, R) model: Backordered demand

Use standardized loss function $L(z)$ to compute $n(R)$

$$Y \sim \text{Normal}(0,1) \quad \Rightarrow \quad L(z) \equiv E[(Y-z)^+] = \int_{y=z}^{\infty} (y-z) \underbrace{\varphi(y)}_{\substack{\text{Normal}(0,1) \\ \text{density function}}} dy$$

$$n(R) = E[(D-R)^+] = E\left[\sigma\left(\underbrace{\frac{D-\mu}{\sigma}}_{\text{Normal}(0,1)} - \frac{R-\mu}{\sigma}\right)^+\right] = \sigma L\left(\frac{R-\mu}{\sigma}\right)$$

$$\Rightarrow \quad \boxed{n(R) = \sigma L(z), \quad z = \frac{R-\mu}{\sigma}}$$

$L(z)$ and hence $n(R)$ can be evaluated from standardized loss function tables

It can be shown that

$$L(z) = \varphi(z) - z[1-\Phi(z)]$$

$$\Rightarrow n(R) = \sigma L(z) = \sigma\varphi(z) + (\mu - R)[1-\Phi(z)], \quad z = \frac{R-\mu}{\sigma}$$

(Q, R) model: Backordered demand

Under the assumption D~Normal(μ, σ) , the optimality conditions become

$$Q = \sqrt{\frac{2\lambda[K + p_2 \sigma L(z)]}{h}} \quad (1)$$

$$\Phi(z) = 1 - \frac{Qh}{p_2 \lambda} \quad (2)$$

$$z = \frac{R - \mu}{\sigma} \quad (3)$$

(Q, R) model: Backordered demand

Fixed point iteration algorithm for case 2 under the assumption D~Normal(μ, σ)

$$Q_0 = \sqrt{\frac{2\lambda K}{h}}, \quad z_0 = \Phi^{-1}\left(1 - \frac{Q_0 h}{p\lambda}\right), \quad R_0 = \mu + \sigma z_0, \quad n = 1$$

Step 1: $Q_n = \sqrt{\frac{2\lambda[K + p_2 \sigma L(z_{n-1})]}{h}}$

Step 2: $z_n = \Phi^{-1}\left(1 - \frac{Q_n h}{p_2 \lambda}\right)$

Step 3: $R_n = \mu + \sigma z_n$

Step 4: $|Q_n - Q_{n-1}| \geq \varepsilon$ OR $|R_n - R_{n-1}| \geq \varepsilon \Rightarrow n \leftarrow n+1$, GOTO Step 1

(Q, R) model: Service Levels

- **Service levels in (Q, R) systems**
  - **Type 1 Service (**replaces stock-out cost p_1 € per stock-out occasion)

    $S_1 \equiv$ Probability of not stocking out during the lead time

    $S_1 = P(D \le R) = F(R)$
  - **Optimization problem**

$$\underset{Q,R}{\text{Minimize}}\, G(Q,R) = h\left(\frac{Q}{2} + R - \lambda\tau\right) + K\frac{\lambda}{Q}$$

$$\text{subject to } F(R) \ge \alpha \quad \text{(i.e., subject to } S_1 \ge \alpha\text{)}$$

  - Solution

$$\boxed{Q^* = \sqrt{\frac{2K\lambda}{h}} = \text{EOQ}}$$

$$R^* = \text{minimum } R \text{ such that } F(R) \ge \alpha$$

$$D \text{ continuous r.v.} \quad \Rightarrow \quad \boxed{R^* = F^{-1}(\alpha)}$$

(Q, R) model: Service Levels

- **Type 2 Service (**replaces stock-out cost p_2 € per part short)
  $S_2 \equiv$ Proportion of demands met from stock
  $S_2 = 1 - n(R)/Q$

- **Optimization problem**

$$\underset{Q,R}{\text{Minimize}}\ G(Q,R) = h\left(\frac{Q}{2} + R - \lambda\tau\right) + K\frac{\lambda}{Q}$$

$$\text{subject to}\ \ 1 - \frac{n(R)}{Q} \geq \beta \quad (\text{i.e., subject to } S_2 \geq \beta)$$

- **Note:** Now the constraint depends on both R and Q

(Q, R) model: Service Levels

– **Type 2 Service (cont'd)**

Approximate solution

$$\boxed{Q^* \approx \sqrt{\frac{2K\lambda}{h}} = \text{EOQ}}$$

$$R^* = \text{minimum } R \text{ such that } n(R) \le Q^*(1-\beta)$$

$$D \text{ continuous r.v.} \Rightarrow \boxed{n(R^*) = Q^*(1-\beta)}$$

$$D \sim \text{Normal}(\mu, \sigma) \Rightarrow n(R^*) \equiv \sigma L(z^*) = Q^*(1-\beta)$$

$$\Rightarrow \boxed{R^* = \mu + \sigma z^*, \quad z^* = L^{-1}\left(\frac{Q^*(1-\beta)}{\sigma}\right)}$$

(Q, R) model: Service Levels

– **Type 2 Service (cont'd)**

More accurate solution

Consider first-order conditions (1) and (2) for case 2

$$Q = \sqrt{\frac{2\lambda[K + p_2 n(R)]}{h}} \quad (1), \qquad F(R) = 1 - \frac{Qh}{p_2 \lambda} \quad (2)$$

$$(2) \Rightarrow p_2 = \frac{Qh}{[1 - F(R)]\lambda} \equiv \text{imputed stock-out cost}$$

$$(1) \Rightarrow Q = \sqrt{\frac{2\lambda\{K + Qhn(R)/[1 - F(R)]\lambda\}}{h}} \equiv \text{quandratic function in } Q$$

positive root:

$$\boxed{Q = \frac{n(R)}{1 - F(R)} + \sqrt{\frac{2K\lambda}{h} + \left(\frac{n(R)}{1 - F(R)}\right)^2}} \quad (3)$$

$$\boxed{n(R) = (1 - \beta)Q} \quad (4) \Rightarrow \qquad L(z) = \frac{(1 - \beta)Q}{\sigma}$$

(Q, R) model: Random Lead Time

- **Extension: Random lead-time**
  - L: random lead time
  - Mean: $\tau \equiv E[L]$, variance $\sigma_L{}^2 \equiv E[(L - \tau)^2]$
  - D: demand during lead time L
    - Density function and cumulative distribution function of D: $f(x)$ and $F(x)$
    - $D = D_1 + D_2 + \ldots + D_L$, where L is a random variable
    - It can be shown (see next page) that:

      Mean: $\mu \equiv E[D] = \tau\,\lambda$

      Variance: $\sigma^2 \equiv Var[D] = E[(D - \mu)^2] = \tau\,\sigma_t{}^2 + \lambda^2\sigma_L{}^2$

Everything else holds!!

(Q, R) model: Random Lead Time

- Derivation of μ and σ^2

Mean: $\mu \equiv E[D] = \underset{L}{E}[\underset{D|L}{E}[D \mid L]] = \underset{L}{E}[L\lambda] = \tau\lambda$

Variance: $\sigma^2 \equiv Var[D] = E[(D-\mu)^2] = E[D^2 - 2\mu D + \mu^2] = E[D^2] - 2\mu E[D] + E[\mu^2]$

$= \underset{L}{E}[\underset{D|L}{E}[D^2 \mid L]] - 2\mu^2 + \mu^2 = \tau\sigma_t^2 + \lambda^2\sigma_L^2 + \lambda^2\tau^2 - \mu^2 = \tau\sigma_t^2 + \lambda^2\sigma_L^2 + \lambda^2\tau^2 - \tau^2\lambda^2$

$= \tau\sigma_t^2 + \lambda^2\sigma_L^2$

where we used:

$\underset{D|L}{E}[D^2 \mid L] = \underset{D|L}{E}[Var[D \mid L] + E[D \mid L]^2] = L\sigma_t^2 + L^2\lambda^2$

$\underset{L}{E}[\underset{D|L}{E}[D^2 \mid L]] = \underset{L}{E}[L\sigma_t^2 + L^2\lambda^2] = \tau\sigma_t^2 + \lambda^2(Var[L] + \tau^2) = \tau\sigma_t^2 + \lambda^2(\sigma_L^2 + \tau^2) = \tau\sigma_t^2 + \lambda^2\sigma_L^2 + \lambda^2\tau^2$