ΣΧΕΔΙΑΣΜΟΣ ΚΑΙ ΠΡΟΓΡΑΜ-ΜΑΤΙΣΜΟΣ ΠΑΡΑΓΩΓΗΣ
Έλεγχος Αποθεμάτων Υπό Σταθερή Ζήτηση

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Πανεπιστήμιο Θεσσαλίας

Τμήμα Μηχανολόγων Μηχανικών

PRODUCTION PLANNING AND SCHEDULING
Inventory Control Under Constant Demand

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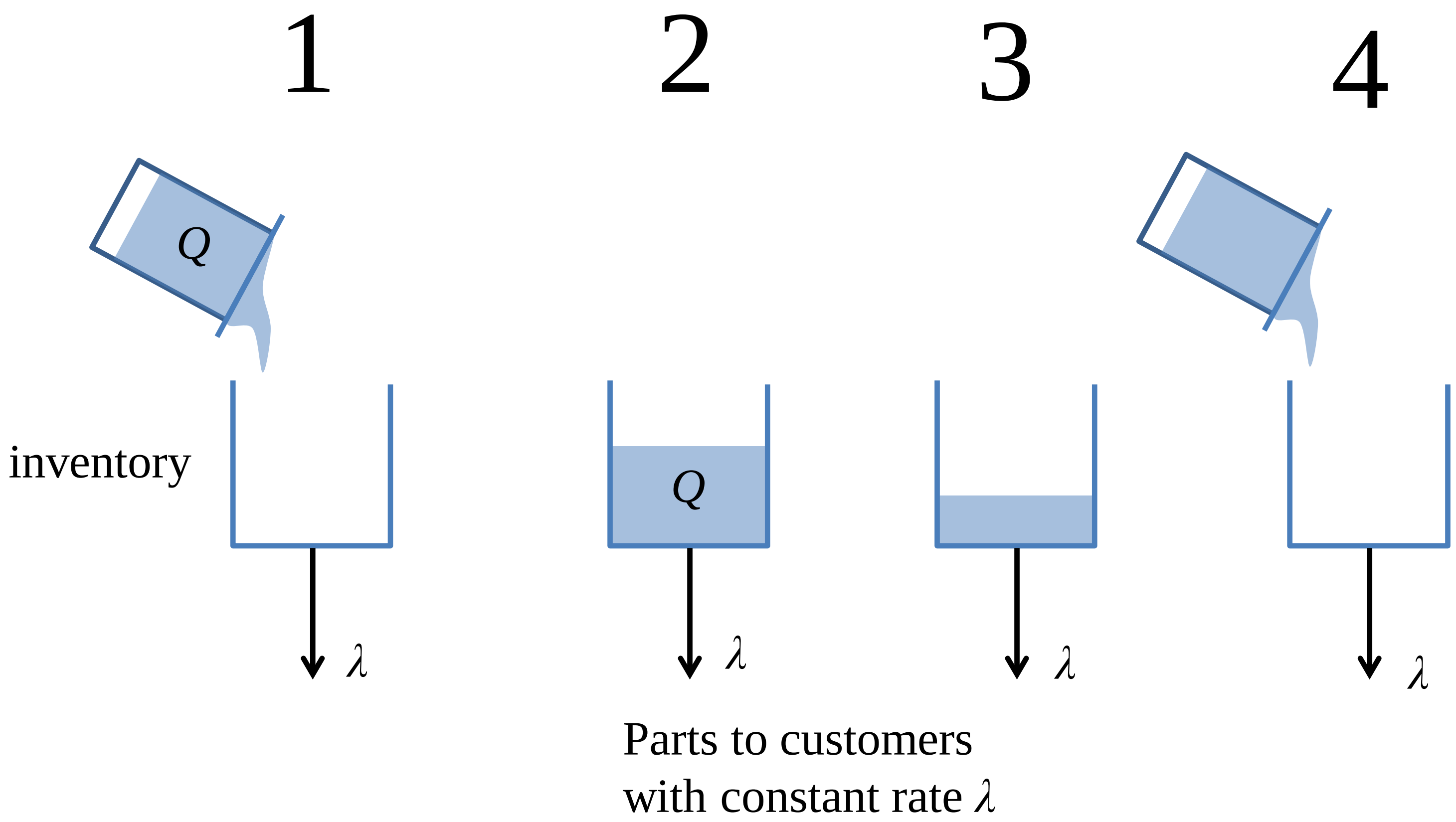
Economic Order Quantity (EOQ): basic model



EOQ: basic model

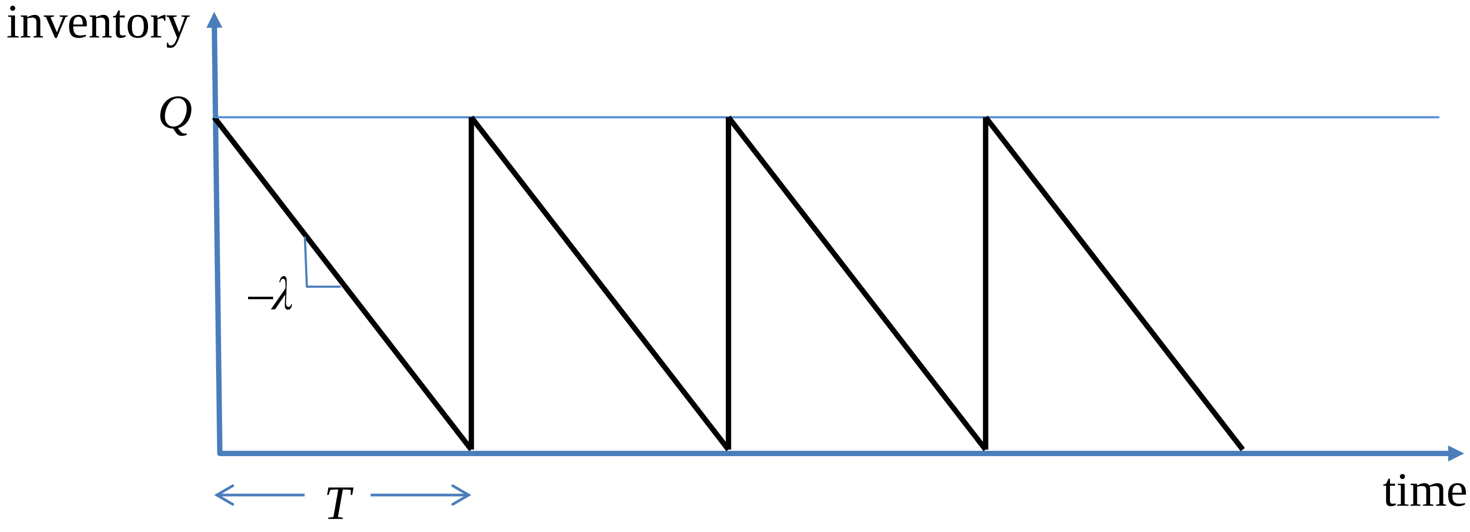
- **Assumptions/notation**
  - Constant demand rate: λ (parts per unit time)
  - Shortages not permitted
  - Infinite production/replenishment rate (instantaneous replenishment)
  - Zero lead time
  - Variable unit production/order cost: c (€ per part)
  - Fixed setup production/order cost: K (€ per production run/order)
  - Interest rate: I (€ per € invested per unit time)

- **Computation**
  - Inventory holding cost rate: $h = Ic$ (€ per part per unit time)

- **Decision**
  - Reorder quantity: Q (parts per production run/order)

- **Reference**
  - Harris, F. W. 1990 (reprint from 1913). How many parts to make at once. *Operations Research* 38 (6) 947–950.

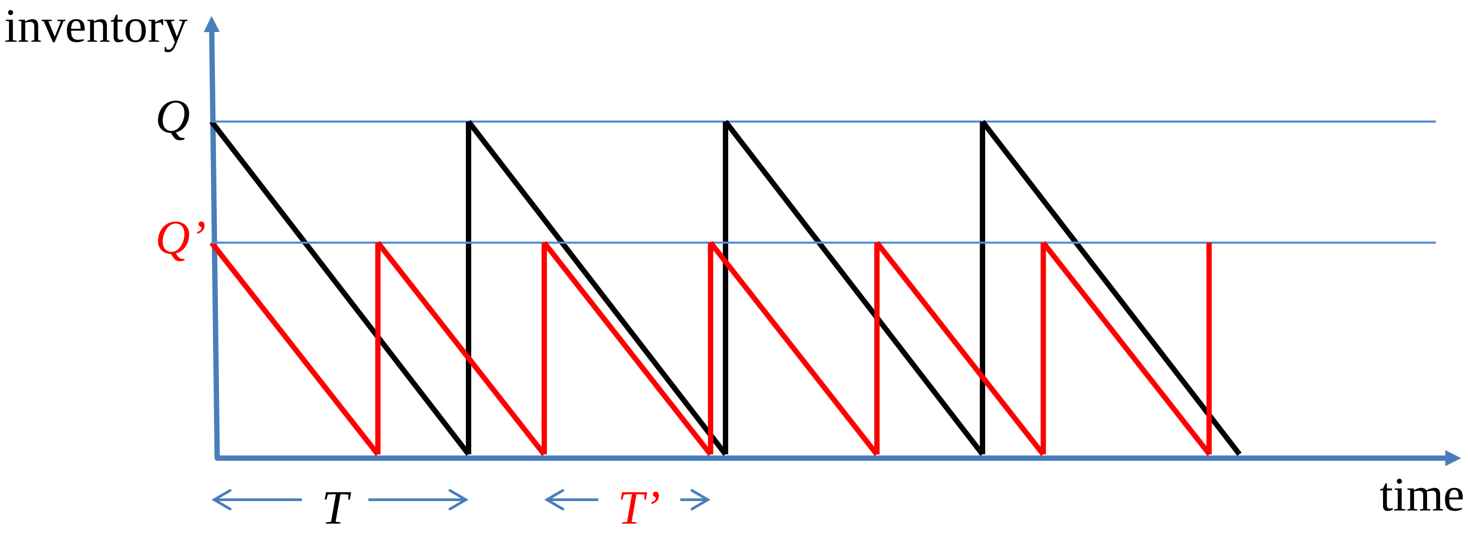
EOQ: basic model



- **Computation**
  - Reorder period (cycle length): $T = Q/\lambda$ (time per cycle)
  - Reorder frequency: $N = 1/T = \lambda/Q$ (orders/cycles per unit time)

EOQ: basic model

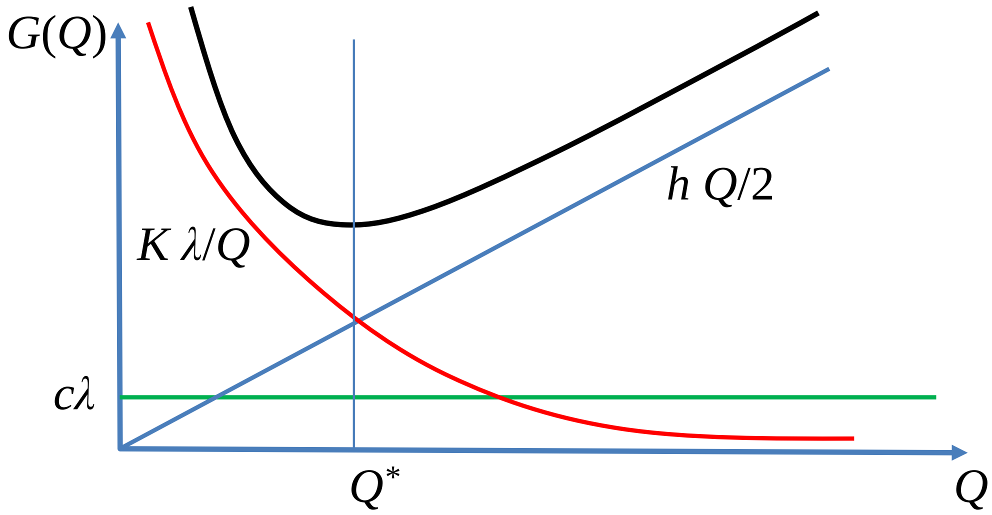
- **Main issue**
  - **Tradeoff** between **fixed setup cost** and **inventory holding cost**



EOQ: basic model

- **(Unconstrained) optimization problem**

$$\text{Minimize}_{Q} \quad \underbrace{G(Q)}_{\text{total average cost}} = \underbrace{K\frac{\lambda}{Q}}_{\text{av. fixed order cost}} + \underbrace{c\lambda}_{\text{av. variable order cost}} + \underbrace{h\frac{Q}{2}}_{\text{av. inventory holding cost}}$$



EOQ: basic model

- **Solution**

  Solve optimality condition

$$Q^*: \quad \frac{dG(Q)}{dQ} = 0 \quad \Rightarrow \quad -\frac{K\lambda}{Q^2} + \frac{h}{2} = 0$$

$$\Rightarrow \quad \boxed{Q^* = \sqrt{\frac{2K\lambda}{h}}} \quad \Rightarrow \quad \boxed{T^* = \frac{Q^*}{\lambda} = \sqrt{\frac{2K}{h\lambda}}}$$

$$\Rightarrow \quad \boxed{G^* \equiv G(Q^*) = \sqrt{2K\lambda h} + c\lambda}$$

- **Insight**: $K\uparrow \Rightarrow Q^*\uparrow, \quad h\uparrow \Rightarrow Q^*\downarrow$

EOQ: basic model

- **Sensitivity**

$$\underbrace{G'(Q)}_{\text{partial average cost}} = K\frac{\lambda}{Q} + h\frac{Q}{2} \quad \Rightarrow \quad \boxed{G'(Q^*) = \sqrt{2K\lambda h}}$$

  - Suppose an arbitrary order quantity Q is chosen

$$\Rightarrow \quad \frac{G'(Q)}{G'(Q^*)} = \cdots = \frac{1}{2}\left(\frac{Q^*}{Q} + \frac{Q}{Q^*}\right)$$

  - Example:

$$Q = 2Q^* \quad \Rightarrow \quad \frac{G'(Q)}{G'(Q^*)} = \frac{1}{2}\left(\frac{Q^*}{2Q^*} + \frac{2Q^*}{Q^*}\right) = \frac{1}{2}\left(\frac{1}{2} + 2\right) = 1.25$$

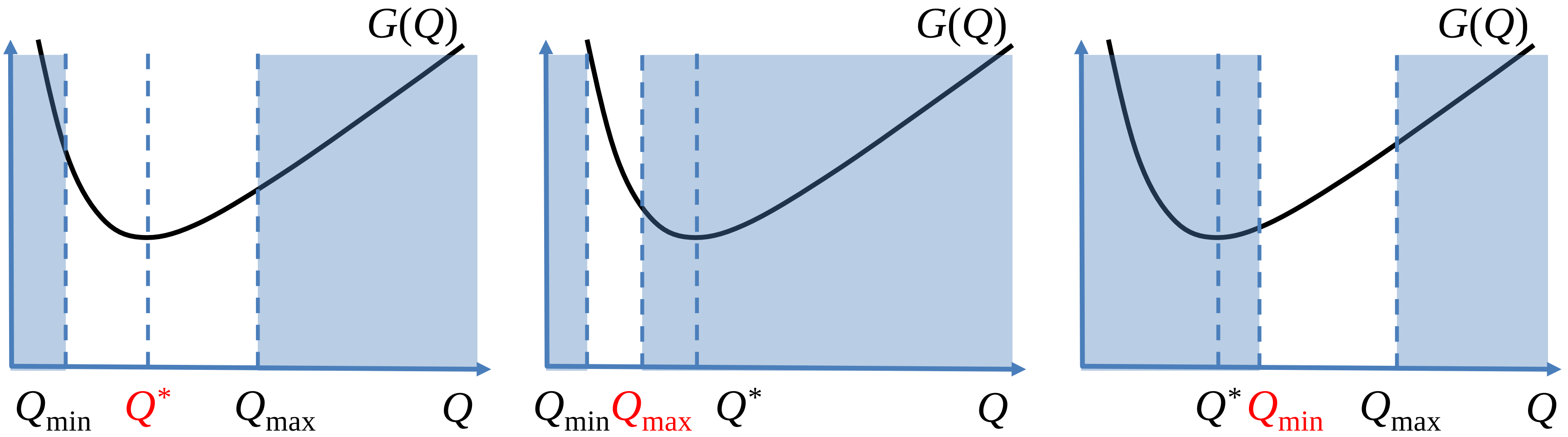
    In words: 100% error in choosing Q $\Rightarrow$ 25% increase in cost

  - **Conclusion**: Performance is not very sensitive to errors in the decision variable Q

EOQ: basic model

- **Constrained optimization problem**
  - Suppose that $Q_{\min} \le Q \le Q_{\max}$

$$\Rightarrow \quad Q^*_{\text{constr}} = \max\left[\min\left(Q^*, Q_{\max}\right), Q_{\min}\right]$$



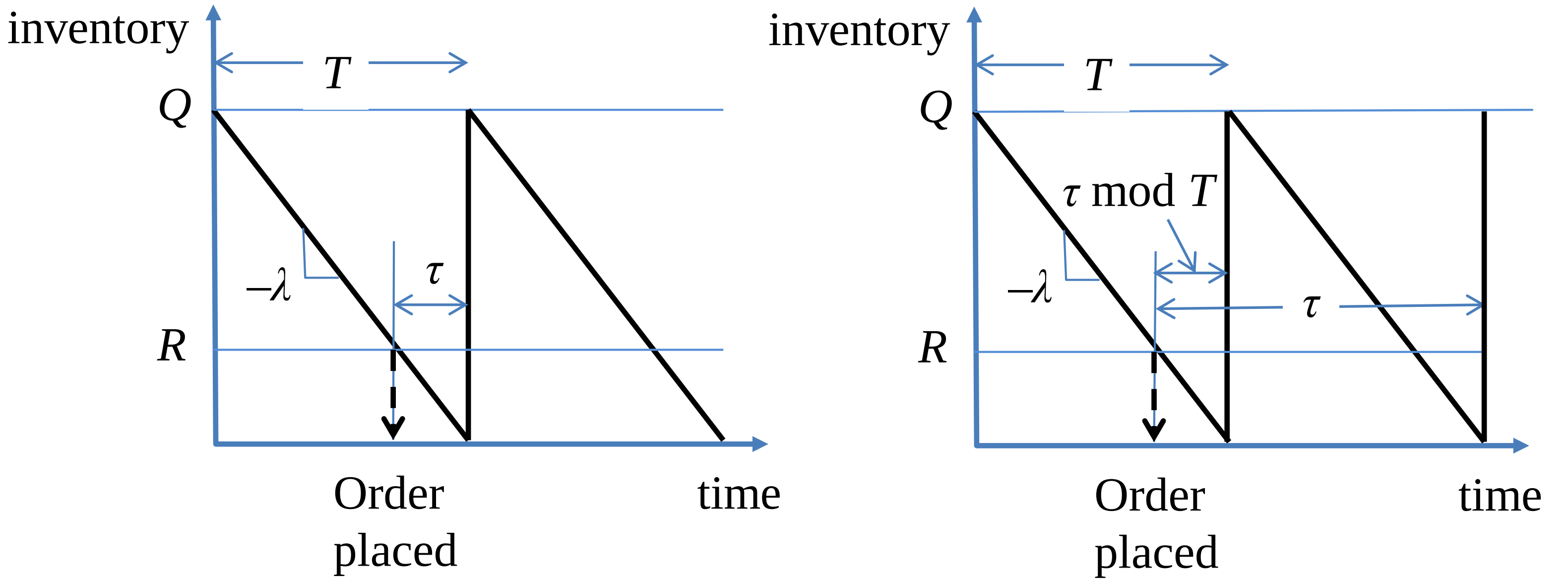
  - Alternatively, suppose that $T_{\min} \le T \le T_{\max}$

$$\Rightarrow \quad T_{\min}\lambda \le Q \le T_{\max}\lambda \quad \Rightarrow \quad Q^*_{\text{constr}} = \max\left[\min\left(Q^*, T_{\max}\lambda\right), T_{\min}\lambda\right]$$

EOQ: basic model

- **Non-zero order lead time τ**
  - Same as EOQ with zero lead time except that order is placed when inventory reaches reorder point R, where

$$R = \begin{cases} \lambda\tau, & \text{if } \tau < \mathrm{T} \\ \lambda(\tau \bmod T), & \text{if } \tau \geq T \end{cases}$$

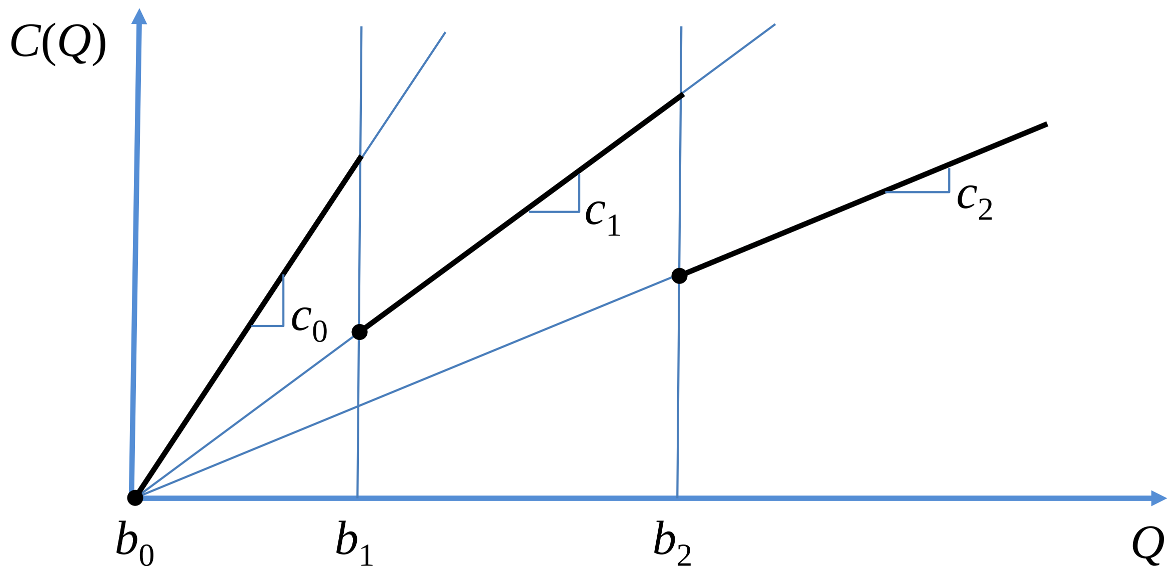


EOQ: model quantity discounts

- **Case1: Same discount for all units**

Total cost for buying Q units, $C(Q) = cQ$, where

$$c = \begin{cases} c_0 & \text{for } b_0 \le Q < b_1 \\ c_1 & \text{for } b_1 \le Q < b_2 \\ c_2 & \text{for } b_2 \le Q \end{cases} \quad \text{where } c_0 > c_1 > c_2$$



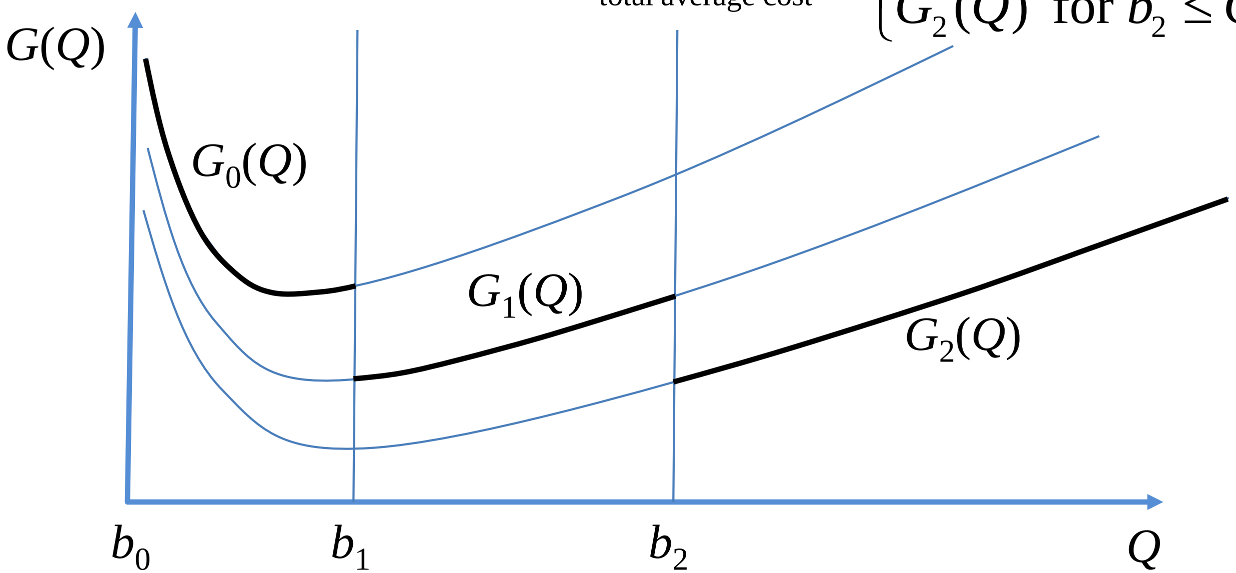
EOQ: model quantity discounts

- **Total average cost function for discount level *j***

$$G_j(Q) = K\frac{\lambda}{Q} + \lambda c_j + \underbrace{Ic_j}_{h_j}\frac{Q}{2}, \quad j = 0,1,2$$

- **Constrained optimization problem**

$$\underset{Q}{\text{Minimize}} \quad \underbrace{G(Q)}_{\text{total average cost}} = \begin{cases} G_0(Q) & \text{for } b_0 \le Q < b_1 \\ G_1(Q) & \text{for } b_1 \le Q < b_2 \\ G_2(Q) & \text{for } b_2 \le Q \end{cases}$$



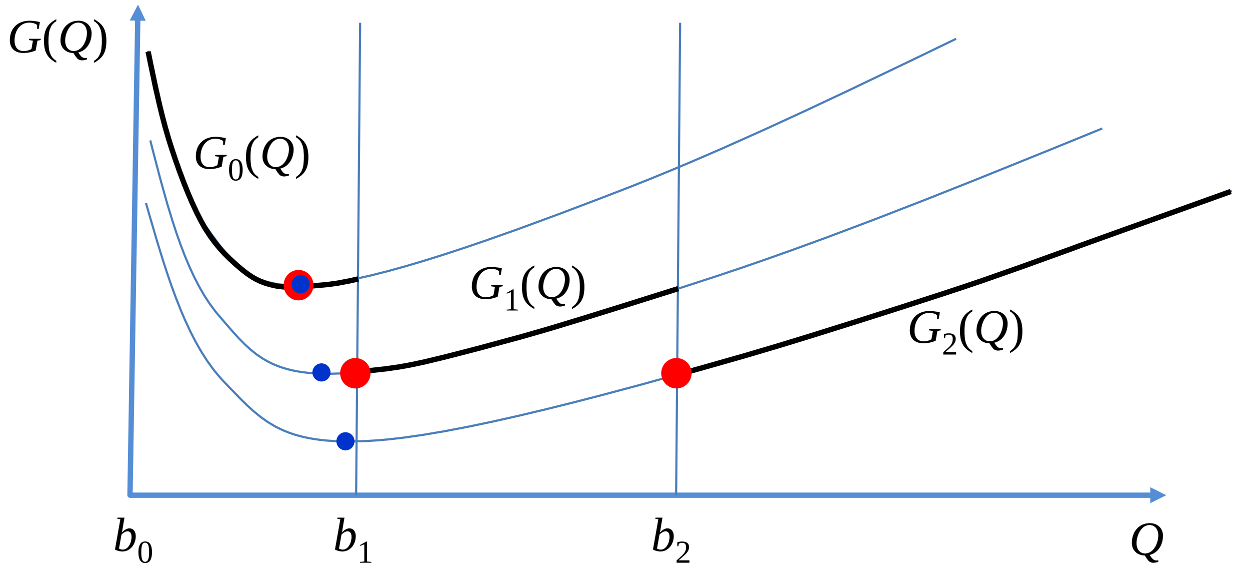
EOQ: model quantity discounts

- **Solution**

- Unconstrained EOQ for discount level j: $Q_j^* = \sqrt{\frac{2K\lambda}{Ic_j}}, \quad j = 0,1,2$
- Constrained optimal order quantity: $Q_{j,\text{constr}}^* = \max\left[\min\left(Q_j^*, b_{j+1}\right), b_j\right], \quad j = 0,1,2$

Note: $b_3 = \infty$

$\Rightarrow$ $j^* = \arg\min_j \left\{G_j(Q_{j,\text{constr}}^*)\right\}$ $\Rightarrow$ $Q^* = Q_{j^*,\text{constr}}^*$ $\Rightarrow$ $G^* = G(Q^*) = G_{j^*}(Q_{j^*,\text{constr}}^*)$



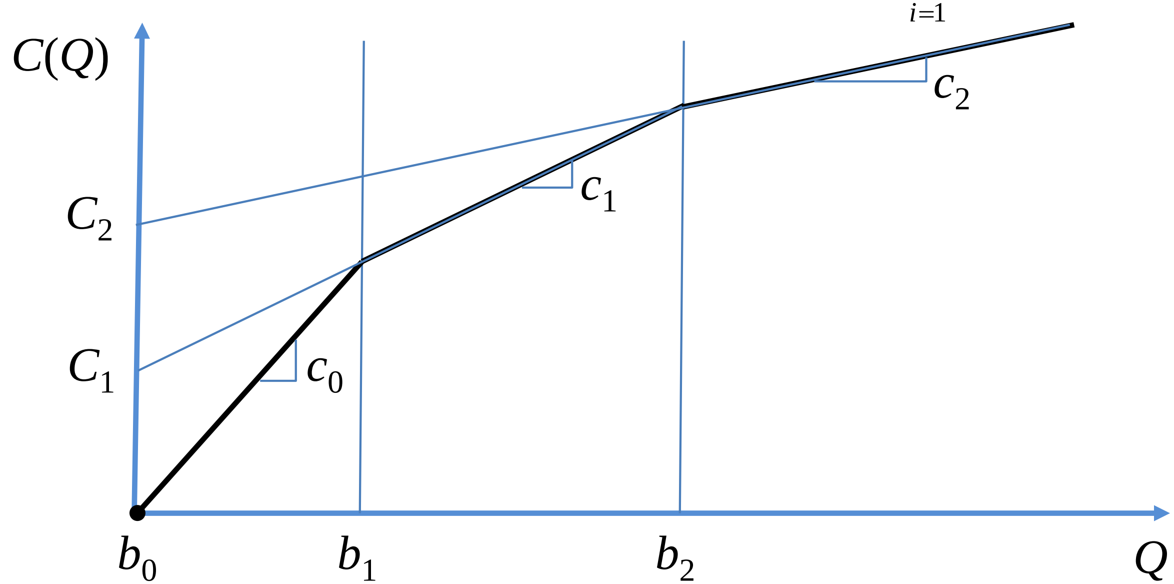
EOQ: model quantity discounts

- **Case2: Incremental quantity discounts**

Total cost for buying Q units, $C(Q)$, where

$$C(Q)=\begin{cases} c_0 Q & \text{for } b_0 \le Q < b_1 \\ c_0 b_1 + c_1(Q-b_1) = (c_0-c_1)b_1 + c_1 Q = C_1 + c_1 Q & \text{for } b_1 \le Q < b_2 \\ c_0 b_1 + c_1(b_2-b_1) + c_2(Q-b_2) = (c_0-c_1)b_1 + (c_1-c_2)b_2 + c_2 Q = C_2 + c_2 Q & \text{for } b_2 \le Q \end{cases}$$

where, for simplification, we have used the notation: $C_j = \sum_{i=1}^{j}(c_{i-1}-c_i)b_i$. Note: $C_0 = 0$



EOQ: model quantity discounts

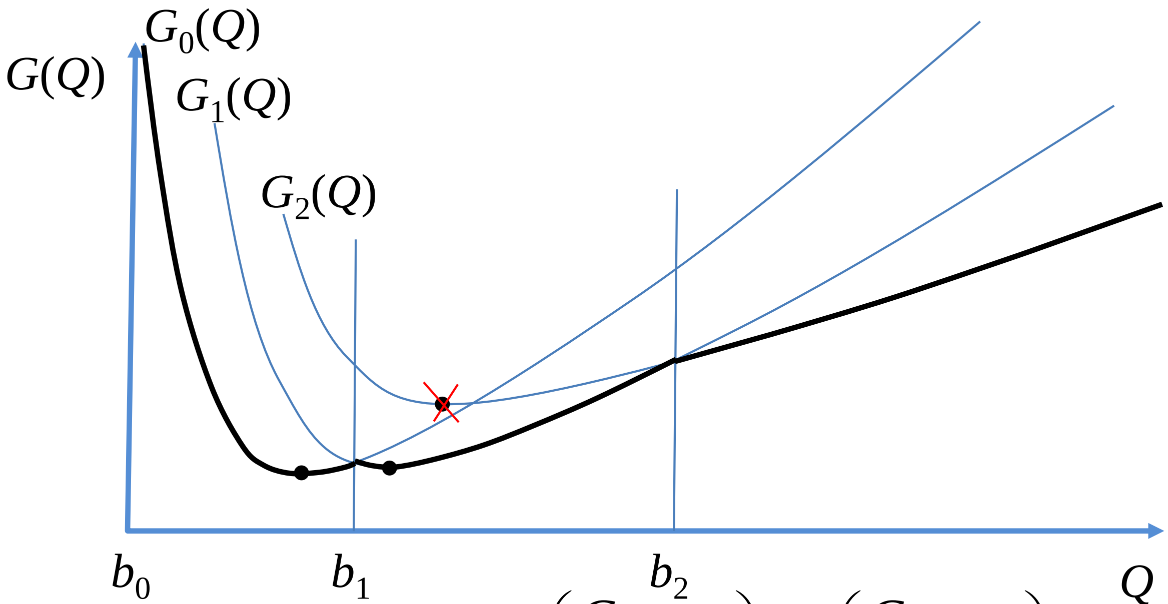
Total average cost per part when ordering Q units, $C(Q)/Q$, where

$$\frac{C(Q)}{Q}=\begin{cases} c_0 & \text{for } b_0 \le Q < b_1 \\ \dfrac{\overbrace{(c_0-c_1)b_1}^{C_1}}{Q}+c_1=\dfrac{C_1}{Q}+c_1 & \text{for } b_1 \le Q < b_2 \\ \dfrac{\overbrace{(c_0-c_1)b_1+(c_1-c_2)b_2}^{C_2}}{Q}+c_2=\dfrac{C_2}{Q}+c_2 & \text{for } b_2 \le Q \end{cases}$$

Total average cost per unit time

$$G(Q)=K\frac{\lambda}{Q}+\lambda\underbrace{\frac{C(Q)}{Q}}_{\text{equivalent to } c}+I\underbrace{\frac{C(Q)}{Q}}_{\text{equivalent to } c}\frac{Q}{2}$$

EOQ: model quantity discounts



$$G_j(Q) = K\frac{\lambda}{Q} + \lambda\left(\frac{C_j}{Q} + c_j\right) + I\left(\frac{C_j}{Q} + c_j\right)\frac{Q}{2}$$

$$= (K + C_j)\frac{\lambda}{Q} + \lambda c_j + Ic_j\frac{Q}{2} + \frac{IC_j}{2}$$

$$\Rightarrow \quad Q_j^* = \sqrt{\frac{2(K + C_j)\lambda}{Ic_j}}$$

EOQ: model quantity discounts

- **Final solution**

$$\Rightarrow \quad \boxed{j^* = \arg\min_j \left\{ G_j(Q_j^*) : b_j \le Q_j^* < b_{j+1} \right\}}$$

$$\Rightarrow \quad \boxed{Q^* = Q_{j^*}^*} \quad \Rightarrow \quad \boxed{G^* = G(Q^*) = G_{j^*}(Q_{j^*}^*)}$$

EOQ: Resource-constrained multi-product systems

- **Assumptions**
  - n products
  - λ_i, K_i, c_i, h_i : parameters for product i
  - Budget or space or other constraint
  - Average cost per unit time for product i

$$G_i(Q_i) = K_i \frac{\lambda_i}{Q_i} + c_i \lambda_i + h_i \frac{Q_i}{2} \quad \Rightarrow \quad Q_i^* = \sqrt{\frac{2K_i\lambda_i}{h_i}}, \quad i = 1, 2, \ldots, n$$

  - Total average cost per unit time

$$G(Q_1, Q_{2,} \ldots, Q_n) = \sum_{i=1}^{n} G_i(Q_i)$$

EOQ: Resource-constrained multi-product systems

- **Constrained minimization problem**

$$\underset{Q_1,Q_2,\ldots,Q_n}{\text{Minimize}}\; G(Q_1,Q_2,\ldots,Q_n) \;\text{ subject to }\; \sum_{i=1}^{n} c_i Q_i \le C$$

  - E.g. C is budget/space cap (upper limit)

- **Solution**

case 1) If $\sum_{i=1}^{n} c_i Q_i^* \le C \;\Rightarrow\;$ constraint is not active $\;\Rightarrow\; \boxed{Q_{i,\text{constr}}^* = Q_i^*}$

case 2) If $\sum_{i=1}^{n} c_i Q_i^* > C \;\Rightarrow\;$ constraint is active $\;\Rightarrow\; Q_i^*$ not feasible

In this case, we know that the constraint is binding at the optimal solution

$\Rightarrow$ Problem to solve

$$\underset{Q_1,Q_2,\ldots,Q_n}{\text{Minimize}}\; G(Q_1,Q_2,\ldots,Q_n) \;\text{ subject to }\; \sum_{i=1}^{n} c_i Q_i = C$$

EOQ: Resource-constrained multiple product systems

- **Solution for case 2**
  - Introduce Lagrange multiplier θ

$$\underset{Q_1,Q_2,\ldots,Q_n,\theta}{\text{Minimize}}\; G(Q_1,Q_2,\ldots,Q_n,\theta)=\sum_{i=1}^{n}\left(\frac{K_i\lambda_i}{Q_i}+\frac{h_iQ_i}{2}\right)+\theta\sum_{i=1}^{n}\left(c_iQ_i-C\right)$$

  - Necessary conditions for optimality

$$\frac{\partial G}{\partial Q_i}=0, \quad \Rightarrow \quad -\frac{K_i\lambda_i}{Q_i^2}+\frac{h_i}{2}+\theta c_i=0 \quad \Rightarrow \quad \boxed{Q^*_{i,\text{constr}}=\sqrt{\frac{2K_i\lambda_i}{h_i+2\theta^* c_i}}, \quad i=1,2,\ldots,n \quad (1)}$$

$$\frac{\partial G}{\partial \theta}=0 \quad \Rightarrow \quad \boxed{\sum_{i=1}^{n}c_iQ^*_{i,\text{constr}}=C \quad (2)}$$

  - Solve numerically: Try different values of θ until optimality conditions (1) and (2) hold

EOQ: Resource-constrained multiple product systems

– Special Case:

$$\frac{c_1}{h_1} = \frac{c_2}{h_2} = \cdots = \frac{c_n}{h_n} = \frac{c}{h}$$

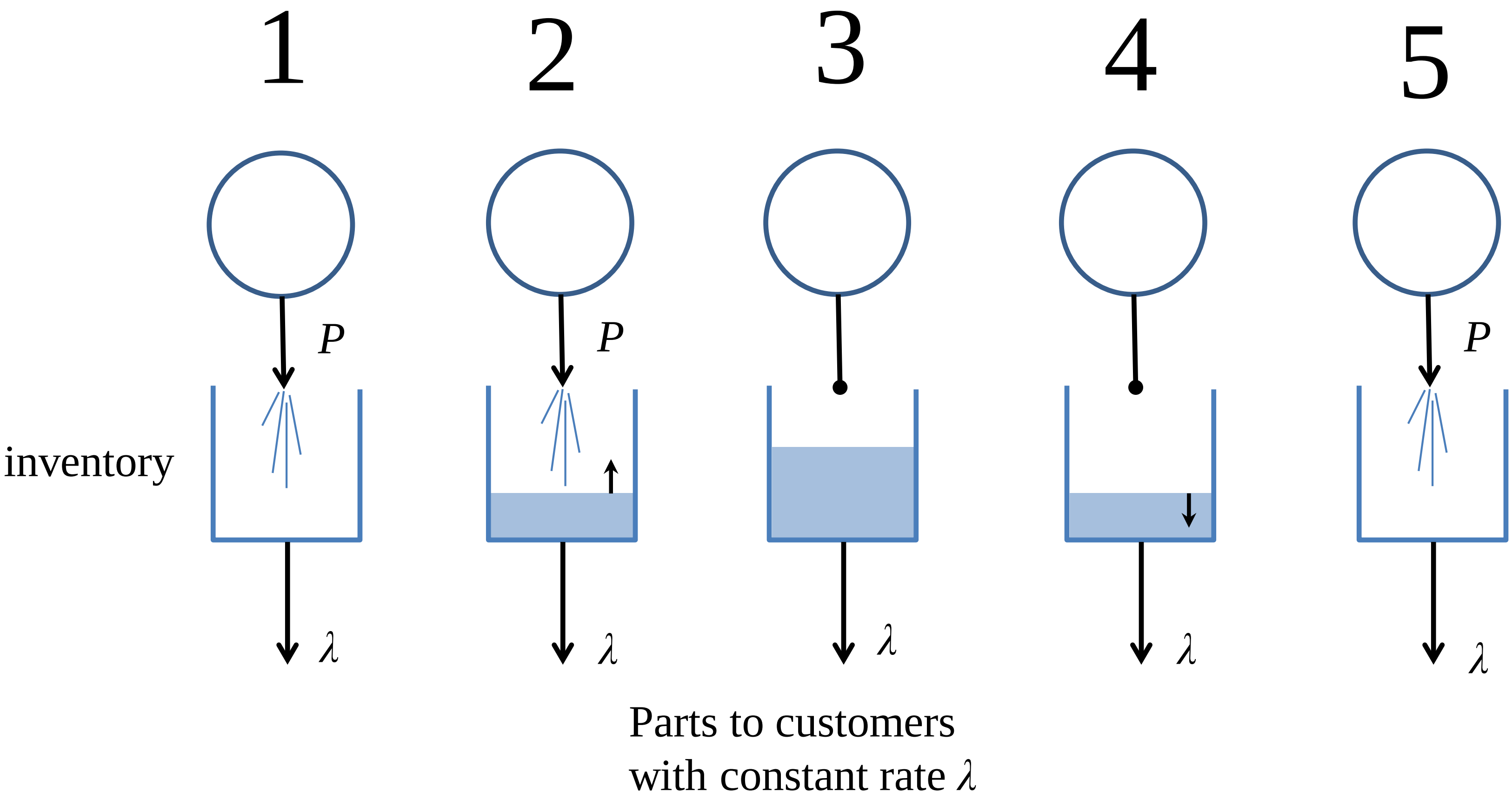
– In this case:

$$(1) \quad \Rightarrow \quad Q^*_{i,\text{constr}} = \sqrt{\frac{2K_i\lambda_i}{h_i + 2\theta^* c_i}} = \sqrt{\frac{2K_i\lambda_i}{h_i}}\sqrt{\frac{1}{1+2\theta^* c/h}} = Q^*_i\sqrt{\frac{1}{1+2\theta^* c/h}}$$

$$\Rightarrow \quad \boxed{Q^*_{i,\text{constr}} = Q^*_i m, \quad i = 1, 2, \ldots, n} \quad \text{where} \quad m = \sqrt{\frac{1}{1+2\theta^* c/h}}$$

$$(2) \quad \Rightarrow \quad \sum_{i=1}^{n} c_i Q^*_{i,\text{constr}} = C \quad \Rightarrow \quad \sum_{i=1}^{n} c_i Q^*_i m = C \quad \Rightarrow \quad \boxed{m = \frac{C}{\sum_{i=1}^{n} c_i Q^*_i}}$$

Economic Production Lot (EPL): EOQ with finite production rate



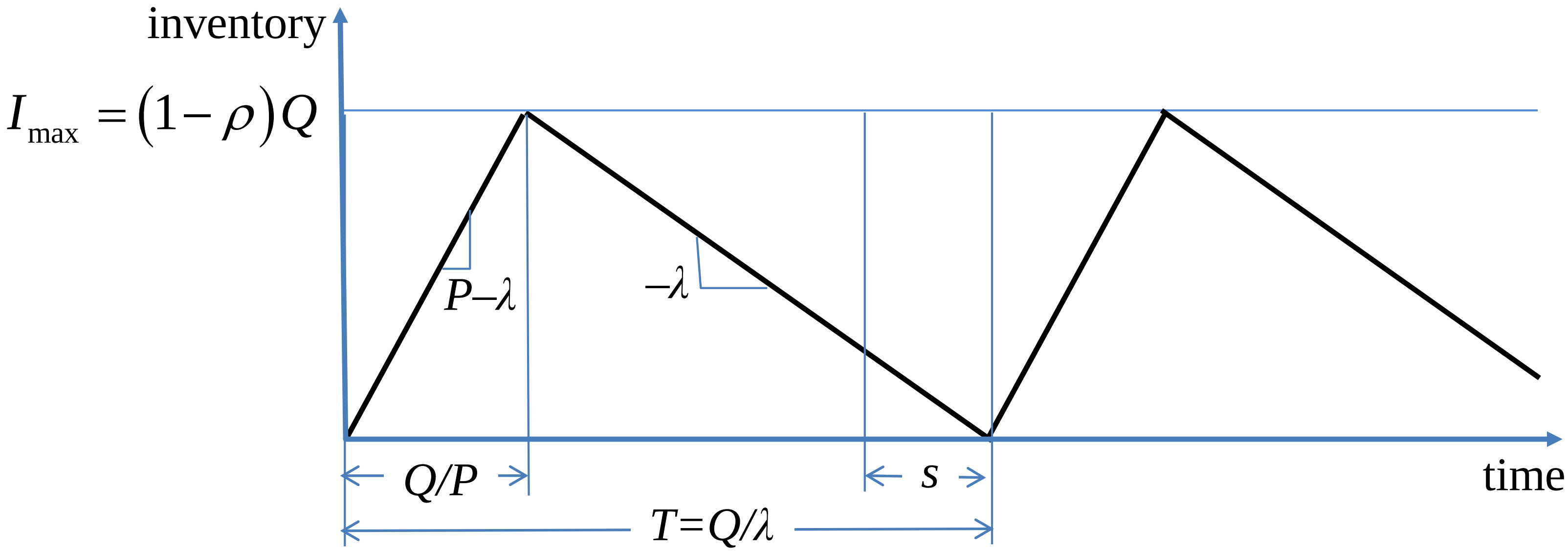
EPL

- **Assumptions/notation**

  Same as basic EOQ, except that:

  – Finite production/replenishment rate P (parts per unit time) with $P > \lambda$

  – Setup time to produce a new production lot s



Maximum inventory: $I_{\max} = (P - \lambda)Q/P = (1 - \lambda/P)Q = (1 - \rho)Q$, where $\rho = \lambda/P \equiv$ utilization factor, $1–\rho \equiv$ fraction of time machine is not producing

EPL

- **Unconstrained optimization problem**

$$\underset{Q}{\text{Minimize}} \quad G(Q) = K\frac{\lambda}{Q} + c\lambda + h\frac{(1-\rho)Q}{2}$$

$$Q^*: \quad \frac{dG(Q)}{dQ} = 0 \quad \Rightarrow \quad -\frac{K\lambda}{Q^2} + \frac{h(1-\rho)}{2} = 0$$

$$\Rightarrow \quad \boxed{Q^* = \sqrt{\frac{2K\lambda}{h(1-\rho)}}} \quad \Rightarrow \quad \boxed{T^* = \frac{Q^*}{\lambda} = \sqrt{\frac{2K}{\lambda h(1-\rho)}}}$$

$$\Rightarrow \quad \boxed{G^* \equiv G(Q^*) = \sqrt{2K\lambda h(1-\rho)} + c\lambda}$$

- **Limiting case:**

$$\boxed{\lim_{P\to\infty} Q^* = \lim_{P\to\infty}\sqrt{\frac{2K\lambda}{h\left(1-\lambda/P\right)}} = \sqrt{\frac{2K\lambda}{h}} = \text{EOQ!}}$$

EPL

- **What about the setup time** s?
  - Cycle time T must be large enough to accommodate s

$$\underbrace{T}_{\text{cycle time}} \geq \underbrace{\frac{Q}{P}}_{\text{production time}} + \underbrace{s}_{\text{setup time}} \Rightarrow \frac{Q}{\lambda} \geq \frac{Q}{P} + s \Rightarrow \boxed{Q \geq \frac{\lambda s}{1 - \lambda/P} = \frac{\lambda s}{1-\rho} \equiv Q_{\min}}$$

$$\Rightarrow \boxed{Q^*_{\text{constr}} = \max(Q^*, Q_{\min})}$$

  - Alternatively

$$T \geq \frac{Q}{P} + s \Rightarrow T \geq \frac{T\lambda}{P} + s = T\rho + s$$

$$\Rightarrow T(1-\rho) \geq s \Rightarrow T \geq \frac{s}{1-\rho} \equiv T_{\min}$$

$$\Rightarrow \boxed{T^*_{\text{constr}} = \max(T^*, T_{\min})}$$

EPL

- **What if there is a maximum storage capacity $I_{\max}$?**

  $\Rightarrow Q_{\min} \leq Q \leq I_{\max}/(1-\rho) \equiv Q_{\max}$

$$\Rightarrow \quad Q^*_{\text{constr}} = \max\left[\min\left(Q^*, Q_{\max}\right), Q_{\min}\right]$$

- **Sensitivity analysis**

  Same as EOQ model: Cost not very sensitive to errors in Q

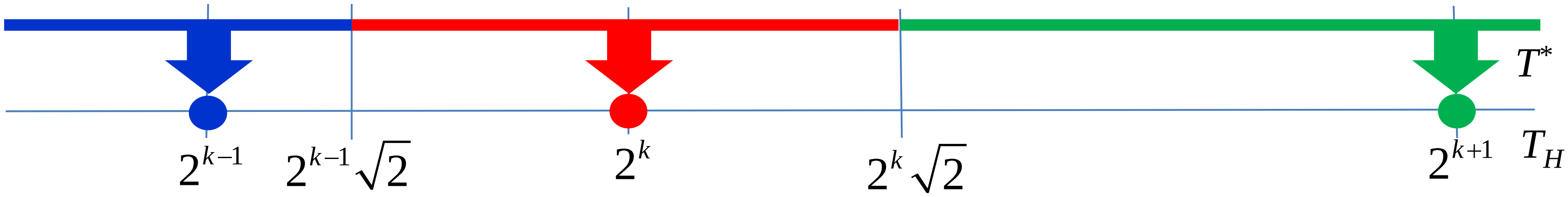
$$\Rightarrow \quad \frac{G'(Q)}{G'(Q^*)} = \cdots = \frac{1}{2}\left(\frac{Q^*}{Q} + \frac{Q}{Q^*}\right) \qquad \Rightarrow \quad \frac{G'(T)}{G'(T^*)} = \cdots = \frac{1}{2}\left(\frac{T}{T^*} + \frac{T^*}{T}\right)$$

EPL

- **"Power-of-2" heuristic for choosing *T***

  Suppose that period (cycle) length T is restricted to be a "power-of-2" multiple of the base time unit, i.e., $T_H = 2^k$, for some $k = 0, 1, 2,\ldots$

- Which k to choose? Rule: $k: \quad 2^{k-1}\sqrt{2} \le T^* < 2^k\sqrt{2} \quad \Rightarrow \quad T_H = 2^k$



- **How bad is the cost increase?**

  Worst case:

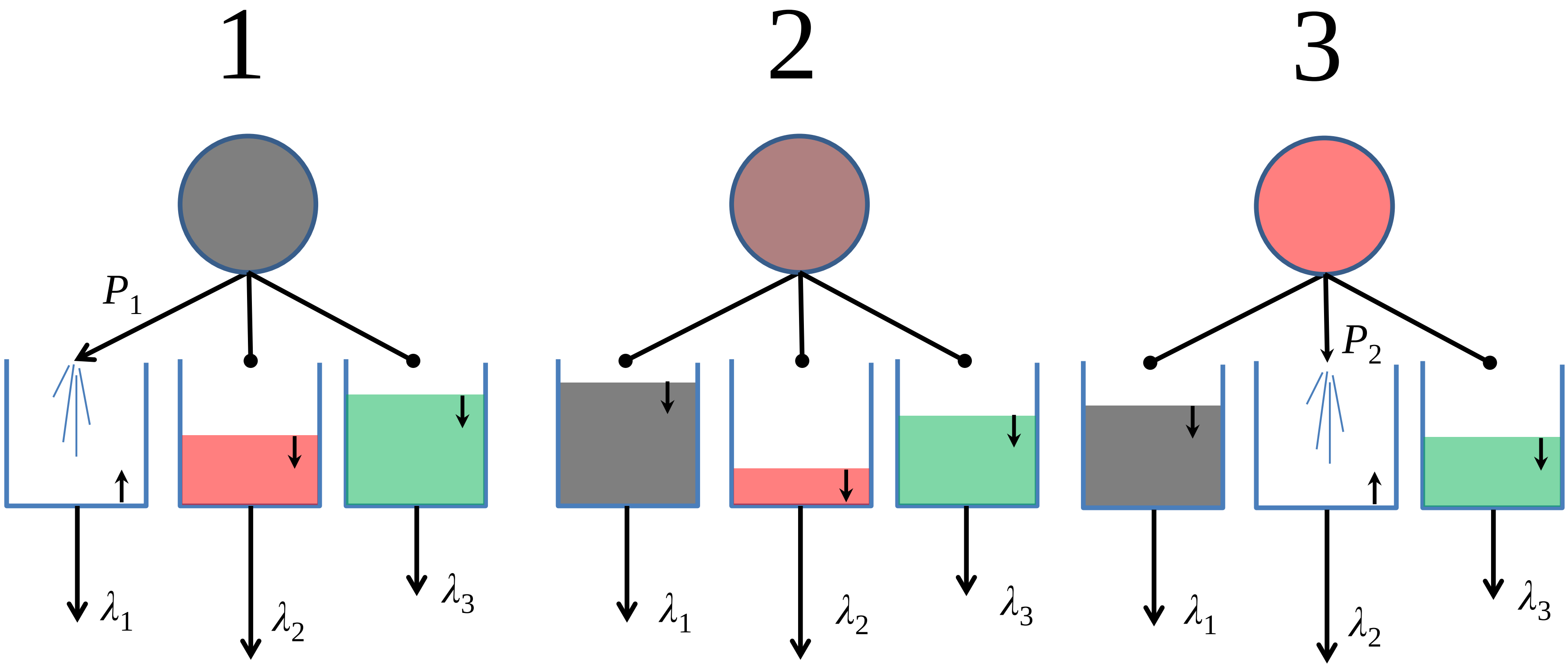
$$\text{If } T^* = 2^{k-1}\sqrt{2} \quad \Rightarrow \quad \frac{G'(T_H)}{G'(T^*)} = \frac{1}{2}\left(\frac{2^k}{2^{k-1}\sqrt{2}} + \frac{2^{k-1}\sqrt{2}}{2^k}\right) = \frac{1}{2}\left(\frac{2}{\sqrt{2}} + \frac{\sqrt{2}}{2}\right) = 1{,}06$$

$$\text{If } T^* = 2^{k}\sqrt{2} \quad \Rightarrow \quad \frac{G'(T_H)}{G'(T^*)} = \frac{1}{2}\left(\frac{2^k}{2^{k}\sqrt{2}} + \frac{2^{k}\sqrt{2}}{2^k}\right) = \frac{1}{2}\left(\frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{1}\right) = 1{,}06$$

- **Conclusion:**

  Using the best T_H will result in an increase in G' of at most 6% with respect to using T^*!

Economic Lot Scheduling Problem (ELSP)



Parts to customers
with constant rate λ

Economic Lot Scheduling Problem (ELSP)

- **Assumptions**

  Same as EPL, except that

  – n products

  – λ_i, K_i, c_i, h_i, s_i: parameters for product i

  – Cyclic scheduling: All products must be produced by the same machine in a cyclic fashion

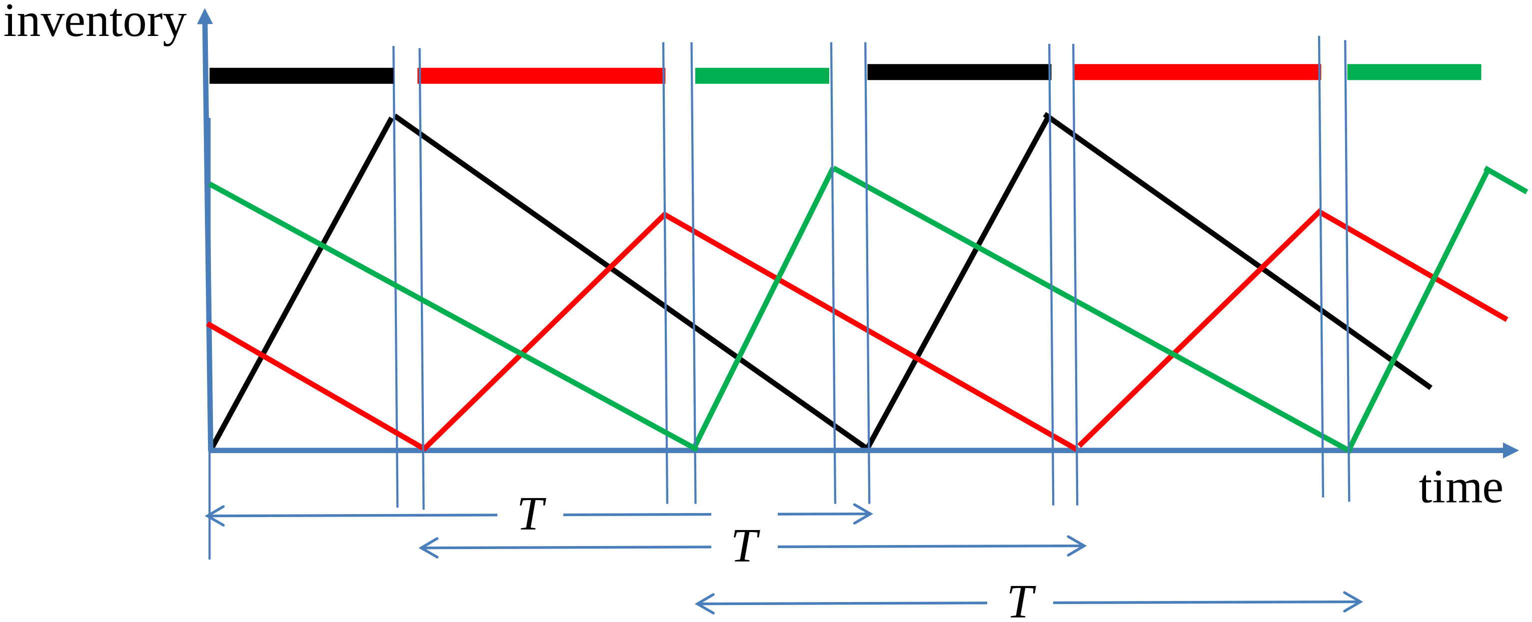
  – Simple cycle: Each product is produced **only once** in each cycle

  – Cycle pattern: $(1 - 2 - 3 - \ldots - n \mid 1 - 2 - 3 - \ldots - n \mid 1 - 2 - 3 - \ldots)$

- **Computation**

  – Utilization factor for product i: $\rho_i = \lambda_i / P_i$

ELSP



- Strong dependency among products: They all have the same cycle time T
- Once T is determined then the production lot sizes can be computed:

$$\boxed{Q_i = \lambda_i T}$$

ELSP

- **Average cost per unit time for product *i***

$$G_i(Q_i) = K_i \frac{\lambda_i}{Q_i} + c_i \lambda_i + h_i \frac{(1-\rho_i)Q_i}{2}$$

- **Total average cost per unit time**

$$G(Q_1, Q_2, \ldots, Q_n) = \sum_{i=1}^{n} G_i(Q_i)$$

- **Problem**

$$\underset{Q_1, Q_2, \ldots, Q_n}{\text{Minimize}}\, G(Q_1, Q_2, \ldots, Q_n) \quad \text{subject to} \quad Q_i = \lambda_i T, \quad i = 1, 2, \ldots, n$$

- **Solution**

Replace Q_i by $\lambda_i T$, and formulate a minimization problem with respect to T

ELSP

- **New Total average cost per unit time**

$$\underset{T}{\text{Minimize}}\ G(T) = \sum_{i=1}^{n} G_i(T) = \sum_{i=1}^{n} K_i \frac{\lambda_i}{\lambda_i T} + c_i\lambda_i + h_i \frac{(1-\rho_i)\lambda_i T}{2}$$

$$= \frac{1}{T}\sum_{i=1}^{n} K_i + T\sum_{i=1}^{n} h_i \frac{(1-\rho_i)\lambda_i}{2} + \sum_{i=1}^{n} c_i\lambda_i$$

$$= \frac{A}{T} + BT + C \quad \text{(same form as EOQ model)}$$

- **Optimal solution**

$$\boxed{T^* = \sqrt{\frac{2\sum_{i=1}^{n} K_i}{\sum_{i=1}^{n} \lambda_i h_i (1-\rho_i)}}} \Rightarrow \boxed{Q_i^* = \lambda_i T^*, \quad i = 1,2,\ldots,n}$$

ELSP

- **What about setup times s_i?**
  - Common cycle time T must be large enough to accommodate all s_i

$$T \geq \sum_{i=1}^{n} \frac{Q_i}{P_i} + s_i \quad \Rightarrow \quad T \geq \sum_{i=1}^{n} \frac{\lambda_i T}{P_i} + s_i = \sum_{i=1}^{n} \rho_i T + s_i = T \sum_{i=1}^{n} \rho_i + \sum_{i=1}^{n} s_i$$

$$\Rightarrow \quad \boxed{T \geq \frac{\sum_{i=1}^{n} s_i}{1 - \sum_{i=1}^{n} \rho_i} = T_{\min}}$$

$$\Rightarrow \quad \boxed{T^*_{\text{constr}} = \max(T^*, T_{\min})} \quad \Rightarrow \quad \boxed{Q^*_{i,\text{constr}} = \lambda_i T^*_{\text{constr}}}$$

ELSP

- **More complicated cycles**

  Assumption

  – Each product i is produced m_i times in each cycle

$$\Rightarrow \quad m_i Q_i = \lambda_i T \quad \Rightarrow \quad \boxed{Q_i = \frac{\lambda_i T}{m_i}}$$

- **Same approach as with simple cycle**

  Replace Q_i by $\lambda_i T/m_i$, and formulate a minimization problem with respect to T

ELSP

- **Average cost per unit time**

$$\underset{T}{\text{Minimize}}\; G(T) = \sum_{i=1}^{n} G_i(T) = \sum_{i=1}^{n} K_i \frac{\lambda_i m_i}{\lambda_i T} + c_i\lambda_i + h_i \frac{(1-\rho_i)\lambda_i T}{2m_i}$$

$$= \frac{1}{T}\sum_{i=1}^{n} K_i m_i + T\sum_{i=1}^{n} h_i \frac{(1-\rho_i)\lambda_i}{2m_i} + \sum_{i=1}^{n} c_i\lambda_i$$

- **Optimal solution**

$$\boxed{T^* = \sqrt{\frac{2\sum_{i=1}^{n} m_i K_i}{\sum_{i=1}^{n} h_i \frac{\lambda_i(1-\rho_i)}{m_i}}}} \Rightarrow \boxed{Q_i^* = \frac{\lambda_i T^*}{m_i}, \quad i = 1,2,\ldots,n}$$

ELSP

- **What about setup times s_i?**
  - Common cycle time T must be large enough to accommodate all s_i

$$T \geq \sum_{i=1}^{n} m_i \left(\frac{Q_i}{P_i} + s_i \right) \quad \Rightarrow \quad T \geq \sum_{i=1}^{n} m_i \left(\frac{\lambda_i T}{m_i P_i} + s_i \right) = \sum_{i=1}^{n} m_i \left(\frac{\rho_i T}{m_i} + s_i \right) = T \sum_{i=1}^{n} \rho_i + \sum_{i=1}^{n} m_i s_i$$

$$\Rightarrow \quad \boxed{T \geq \frac{\sum_{i=1}^{n} m_i s_i}{1 - \sum_{i=1}^{n} \rho_i} = T_{\min}}$$

$$\Rightarrow \quad \boxed{T^*_{\text{constr}} = \max(T^*, T_{\min})} \quad \Rightarrow \quad \boxed{Q^*_{i,\text{constr}} = \frac{\lambda_i T^*_{\text{constr}}}{m_i}, \quad i = 1, 2, \ldots, n}$$

ELSP

- **How to choose good values for m_i**

  Use "powers-of-2" method, i.e. set $m_i = 2^{k_i}$ for some $k_i \in \{0,1,2,3,\ldots\}$ for i = 1,2,…,n

  Algorithm for computing k_i

  – **Step 1**: Compute unconstrained optimal cycle time of each product in isolation and find the minimum of these times

$$T_i^* = \sqrt{\frac{2K_i}{h_i \lambda_i (1-\rho_i)}}, \quad i = 1, 2, \ldots, n$$

$$T_{\min}^* = \min_i \left(T_i^* \right)$$

  – **Step 2**: Compute relative production frequency of each product in isolation

$$N_i^* = \frac{T_i^*}{T_{\min}^*}, \quad i = 1, 2, \ldots, n$$

ELSP

- **Step 3**: "Round" N_i to the nearest "power-of-2" using the rule

$$2^{k_i-1}\sqrt{2} \le N_i < 2^{k_i}\sqrt{2} \quad \Rightarrow \quad N_i^{\text{round}} = 2^{k_i}$$

- Example:

$$k_i = 0: \quad 2^{-1}\sqrt{2} = 0.707 \le N_i < 1.414 = 2^0\sqrt{2} \quad \Rightarrow \quad N_i^{\text{round}} = 2^0 = 1$$

$$k_i = 1: \quad 2^0\sqrt{2} = 1.414 \le N_i < 2.828 = 2^1\sqrt{2} \quad \Rightarrow \quad N_i^{\text{round}} = 2^1 = 2$$

$$k_i = 2: \quad 2^1\sqrt{2} = 2.828 \le N_i < 5.656 = 2^2\sqrt{2} \quad \Rightarrow \quad N_i^{\text{round}} = 2^2 = 4$$

$$\vdots$$

- **Step 4**: Find the largest rounded frequency and call it $N_{\max}^{\text{round}}$
- **Step 5**: Compute multiple m_i: $m_i = \dfrac{N_{\max}^{\text{round}}}{N_i^{\text{round}}}$
- **Step 6**: Compute T^* with these multiples. T^* will be $\approx N_{\max}^{\text{round}} T_{\min}^*$
- **Step 7**: Compute $T_{\text{constr}}^* = \max(T^*, T_{\min})$

ELSP

– **Note**: To compute N_i^{round} in step 3, think as follows:

$N_i^{\text{round}} = 2^{k_i^*}$, where k_i^* is the smallest integer k_i such that $N_i < 2^{k_i}\sqrt{2}$

The above inequality can be written as:

$$N_i < 2^{k_i}\sqrt{2} \quad \Rightarrow \quad N_i \big/ \sqrt{2} < 2^{k_i} \quad \Rightarrow \quad \ln\left(N_i \big/ \sqrt{2}\right) < \ln\left(2^{k_i}\right)$$

$$\Rightarrow \quad \ln\left(N_i \big/ \sqrt{2}\right) < k_i \ln(2) \quad \Rightarrow \quad \ln\left(N_i / \sqrt{2}\right) \Big/ \ln(2) < k_i$$

$$\Rightarrow \quad \boxed{k_i^* = \left\lfloor 1 + \ln\left(N_i / \sqrt{2}\right) \Big/ \ln(2) \right\rfloor}$$

where $\lfloor x \rfloor \equiv$ floor of $x \equiv$ largest integer $\leq x$

e.g., $\lfloor 4.9 \rfloor = 4, \quad \lfloor 4.2 \rfloor = 4, \quad \lfloor 4.0 \rfloor = 4$

Example

$$N_i = 5.2 \quad \Rightarrow \quad k_i^* = \left\lfloor 1 + \ln\left(5.2 / \sqrt{2}\right) \Big/ \ln(2) \right\rfloor = \lfloor 2.878 \rfloor = 2$$