Random Vibrations - Reliability
Exercises
Due: February 20, 2015

Set 1: Gaussian Distributions

1. The sum $Z = X + Y$ of two independent Gaussian random variables $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ is Gaussian with mean $\mu_Z = \mu_X + \mu_Y$ and variance $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2$, i.e. $Z \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$.

Hint: Estimate marginal distribution $f(z) = \int f(z, x)\, dx = \int f(z \mid x) f(x)\, dx$ and use the fact that $X \sim N(\mu_X, \sigma_X^2)$ and $Z \mid X \sim N(X + \mu_Y, \sigma_Y^2)$.

2. The sum $Z = X + Y$ of two Gaussian random variables $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ is Gaussian with mean $\mu_Z = \mu_X + \mu_Y$ and variance $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y$, where ρ is the correlation coefficient given by $\rho = E[XY]/(\sigma_X \sigma_Y)$.

3. The mixture distribution is defined by

$$f(\underline{x}) = \sum_{i=1}^{n} w_i f_i(\underline{x})$$

Where $f_i(\underline{x})$, $i = 1, \ldots, n$, are the mixture components and w_i are mixture weights which are non-negative $w_i \geq 0$ and satisfy $\sum_{i=1}^{n} w_i = 1$. The mixture components $f_i(\underline{x})$ are probability distributions. Show that $f(\underline{x})$ is a probability distribution. Estimate the first and second moment of the mixture distribution in terms of the first and second moment of the mixture components. Estimate the variance of the mixture distribution.

4. The mixture of Gaussian distributions is defined by

$$f(\underline{x}) = \sum_{i=1}^{n} w_i f_i(\underline{x})$$

where the mixture components $f_i(\underline{x})$, $i = 1, \ldots, n$, are Gaussian, i.e. $f_i(\underline{x}) = N(\underline{x}; \underline{\mu}_i, \Sigma_i)$ and w_i are mixture weights which are non-negative $w_i \geq 0$ and satisfy $\sum_{i=1}^{n} w_i = 1$. Estimate the mean and the variance of the mixture distribution. Find the marginal distribution of a parameter x_j in $\underline{x}$.

Set 2: Prior System Analysis

5. Consider the mathematical model of a system represented by the equation

$$Y = aX_1 + bX_2 + E \tag{1}$$

where X_1 and X_2 are uncertain parameters of the mathematical model of the system, $Y \in R$ is the output quantity of interest (QoI), and $E \in R$ represents the model error which is quantified by a Gaussian distribution $E \sim N(0,S)$, where $S \in R$. The parameters X_1 and X_2 are assumed to be independent with mean μ_1 and μ_2, respectively. Also the standard deviation of the parameters X_1 and X_2 are σ_1 and σ_2, respectively. The uncertainty in the output QoI is quantified by the simplified measures of uncertainty such as the mean μ_Y and standard deviation σ_Y. The variables a and b are known constants. Given the uncertainty in the parameters X_1 and X_2, find the uncertainty in the output QoI Y, i.e. find μ_Y and standard deviation σ_Y.

6. Consider the mathematical model of a physical process represented by the equation

$$Y = a\cos(X_1 - 1) + E \tag{2}$$

where X_1 is the uncertain parameter of the mathematical model of the system, $Y \in R$ is the output quantity of interest (QoI), and $E \in R$ represents the model error which is quantified by a Gaussian distribution $E \sim N(0,S)$, where $S \in R$. The variable a is known constant. The uncertainty in the output QoI is quantified by the simplified measures of uncertainty such as the mean $\underline{\mu}_Y$ and standard deviation σ_Y. Given the uncertainty in the parameter X_1, find the uncertainty in the output QoI Y in the following cases.

   a. The parameter X_1 has mean $\mu_1 = 1$ and standard deviation σ.

   b. The parameter X_1 is Gaussian with mean $\mu_1 = 1$ standard deviation σ. Find the result for any other distribution of the uncertain variable X_1 with mean $\mu_1 = 1$ and standard deviation σ.

   c. The parameter X_1 is uniform with upper and lower bounds $1-b$ and $1+b$. Use the analytical approximations based on Taylor series expansion up to the quadratic term. Also find the exact estimate of the mean $\underline{\mu}_Y$ and standard deviation σ_Y and investigate the effect of level of the uncertainty in X_1 on the accuracy of the Taylor series expansion estimate by plotting the errors

$$|\mu_Y^{approx} - \mu_Y^{exact}| / \mu_Y^{exact}$$

$$|\sigma_Y^{approx} - \sigma_Y^{exact}| / \sigma_Y^{exact}$$

   as a function of b ranging from 0 to 1. Comment on the results.

Set 3: Bayesian Inference and Posterior System Analysis

7. The posterior distribution of the parameters of a model is given by

$$p(\theta_1, \theta_2 \mid D, I) \propto \exp\left[-\frac{1}{2}\left(\theta_1^2 + \theta_2^2 + 2\mu\theta_1\theta_2 - 2\mu\theta_1 - 2\theta_2 + 1\right)\right]$$

Find the uncertainty region and plot it in the two-dimensional parameter space (θ_1, θ_2).

Hint: Need to find the most probable point, the Hessian, the covariance matrix and then clearly plot the contour plots of the posterior distribution in the two-dimensional parameter space, indicate the principal direction of the ellipsoid, as well as the length of the uncertainty along the principal axes of the ellipsoid.

8. Consider the mathematical model of a physical process/system represented by the equation

$$Y = aX_1 + E \tag{3}$$

where X_1 is the uncertain parameter of the mathematical model of the system, $Y \in R$ is the output quantity of interest (QoI), and $E \in R$ represents the model error which is quantified by a Gaussian distribution $E \sim N(0, S)$, where $S \in R$ is known. Given the single measurement $\hat{Y} = y_0$,

a. find the posterior uncertainty in the model parameter X_1. The prior uncertainty in X_1 is quantified by
   i. a uniform distribution with very large bounds
   ii. a Gaussian distribution with mean μ and standard deviation σ

b. For the case (i), find the uncertainty in the output quantity of interest

$$Z = bY + \eta \tag{4}$$

where the error term η is a Gaussian distribution with mean zero and variance S_0.

9. Consider the mathematical model of a physical process/system represented by the equation

$$Y = aX^2 + E \tag{5}$$

where X_1 is the uncertain parameter of the mathematical model of the system, $Y \in R$ is the output quantity of interest (QoI), and $E \in R$ represents the model error which is quantified by a Gaussian distribution $E \sim N(0, S)$, where $S \in R$ is known. Given the single measurement $\hat{Y} = y_0$,

a. find the posterior uncertainty in the model parameter X_1 using Bayesian central limit theorem. The prior uncertainty in X_1 is quantified by a Gaussian distribution with mean μ and standard deviation σ.

b. Approximate the uncertainty in the output quantity of interest

$$Z = bY + \eta \tag{6}$$

where the error term η is a Gaussian distribution with mean zero and variance S_0.

10. Consider a mathematical model of a system represented by the difference equation

$$Y_k = g(Y_{k-1}, \mu) + E$$

where E is a Gaussian distribution, i.e. $E \sim N(0, \sigma^2)$. Given the observations $D \equiv (\hat{Y}_0, \hat{Y}_1, \hat{Y}_2, \ldots, \hat{Y}_N) \equiv \{\hat{Y}_k\}_{0\to N}$ covering all time instances, we are interesting in updating the uncertainty in the variables μ and σ^2. Find the likelihood of the model parameters μ and σ^2.

11. The posterior uncertainty in two parameters x_1 and x_2 is found to be Gaussian with mean $\underline{\hat{x}} = (3,3)^T$ and covariance matrix $C = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$, where $-1 < \rho < 1$

   a. Plot the spread of uncertainty around the best estimate $\underline{\hat{x}}$ (that is, plot a contour plot corresponding to $Q(\underline{x}) = 1$), for values of $\rho = 0, 0.1, 0.5, 0.9$

      Hint: Solve the eigenvalue problem $C\underline{\upsilon} = \mu\underline{\upsilon}$ and use the results in class to draw the contour plots. Note that $H^{-1} = C$ and that $\underline{u}$ and λ obtained from the eigenvalue problem $H\underline{u} = \lambda\underline{u}$ developed in class are related to $\underline{\upsilon}$ and μ as follows:

$$\underline{\upsilon} = \underline{u}$$

$$\mu = \frac{1}{\lambda}$$

   b. Also, estimate the uncertainty in the marginal distribution of x_1 or x_2. Can the uncertainty in the marginal distribution of x_1 or x_2 describe the spread of uncertainty in the two dimensional space (x_1, x_2) of the two parameters?

12. <u>Inference of Acceleration of Gravity and Air Resistance Coefficient for a Falling Object</u>

Consider the mathematical model of a falling object with mass m, acceleration of gravity g and air resistance force $F_{res} = -m\beta\upsilon^2$, where β is the air resistance coefficient. Using Newton's law, the equation of motion of the falling object is

$$m\frac{d\upsilon(t)}{dt} = mg - m\beta\upsilon^2(t) \tag{7}$$

or equivalently

$$a(t) = g - \beta\upsilon^2(t)$$

Measurements for the acceleration and the velocity of the falling object are obtained at regular time intervals $k\Delta t$. The acceleration measurements are denoted by $(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_N) \equiv \{\hat{a}_k\}_{1\to N}$ and the corresponding velocity measurements are denoted by $(\hat{\upsilon}_1, \hat{\upsilon}_2, \ldots, \hat{\upsilon}_N) \equiv \{\hat{\upsilon}_k\}_{1\to N}$. Given the observation data $D \equiv (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_N, \hat{\upsilon}_1, \hat{\upsilon}_2, \ldots, \hat{\upsilon}_N)$ of the acceleration and velocity of the falling object at time instances $t = \Delta t, 2\Delta t, \ldots, N\Delta t$, respectively, we are interesting in estimating the uncertainty of the parameters g and β of the system. Note that the measurements and the model predictions satisfy the model error equation

$$\hat{a}_k = g - \beta \hat{\upsilon}_k^2 + E_k \tag{8}$$

$k = 1, \ldots, N$, where the measurement error terms E_k are independent identically distributed (iid) and follow a zero-mean Gaussian distribution $E_k \sim N(0, \sigma^2)$. The value of the variance σ^2 is given.

Assume a uniform prior for the parameter set (g, β) and derive the expressions for the

1. Posterior PDF $p(g, \beta \mid D, \sigma, I)$.
2. The function $L(g, \beta) = -\ln p(g, \beta \mid D, \sigma, I)$
3. The MPV (or best estimate) $(\hat{g}, \hat{\beta})$ of (g, β)
4. The uncertainty in the parameter space (g, β)
5. Derive the Gaussian asymptotic approximation for the posterior PDF of $p(g, \beta \mid D, \sigma, I)$. Is the Gaussian representation of the posterior uncertainty exact or approximate for this case?
6. Find the marginal distribution of the parameter β. Specifically,
    a. Give the uncertainty in β in terms of the mean and the standard deviation of the marginal distribution of β.
    b. Find the minimum number of data points required so that the uncertainty in β is less that a given value λ.
7. Find the uncertainty in the resistance force $F_{res} = -m\beta\upsilon^2$ given the uncertainties in the parameters (g, β):
    a. Compute the mean of F_{res}
    b. Compute the standard deviation of F_{res}
    c. Find the probability density function that describes the uncertainty in F_{res}

Set 4. Information Entropy and Principal of Maximum Information Entropy

13. Estimate the information entropy for the exponential distribution $p(x) = \lambda \exp(-\lambda x)$, $x \geq 0$.
14. Show that the maximum entropy distribution defined within the interval $[a,b]$ is the uniform distribution.
15. Show that the maximum entropy distribution given only the mean is the exponential distribution.

Set 5: Markov Chain Monte Carlo Algorithms

16. The posterior probability density function of a set of two parameters $\underline{\theta} = (\theta_1, \theta_2)^T$ is Gaussian with mean $\underline{0}$ and diagonal covariance matrix

$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$$

Let $\underline{\theta}^{(j)}$ be the current sample in the Markov Chain Monte Carlo algorithm generated using a Metropolis-Hasting algorithm. Following Metropolis-Hasting algorithm, let $\underline{\xi}$ be the candidate sample drawn from a uniform distribution centered at the current sample $\underline{\theta}^{(j)}$. Let $\underline{\theta}^{(j)} = (1,0)^T$. If $\underline{\xi} \sim U([0,3],[0,1])$, is drawn from a uniform distribution with bounds $[0,3]$ for the first component ξ_1 and $[0,1]$ for the second component ξ_2

1. find the probability that the next sample in the chain will be $\underline{\theta}^{(j+1)} = \underline{\xi} = (0,1)^T$
2. find the probability that the next sample in the chain will be $\underline{\theta}^{(j+1)} = \underline{\xi} = (0,3)^T$
3. find the probability that the next sample in the chain will be $\underline{\theta}^{(j+1)} = \underline{\xi} = (3,0)^T$