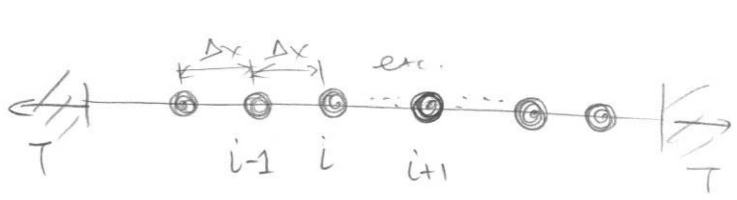
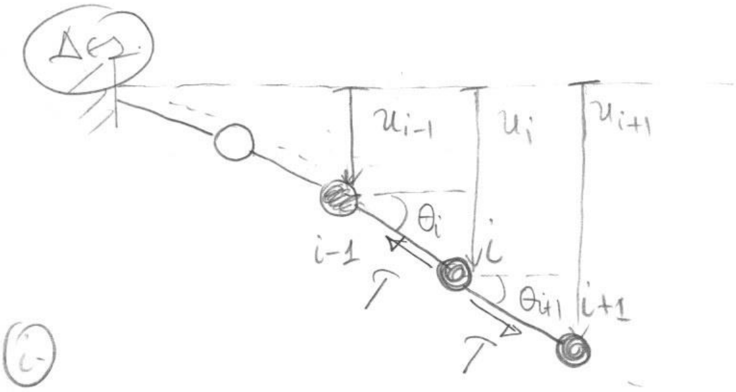




$$\tan\theta_i = \frac{u_i - u_{i-1}}{\Delta x}$$

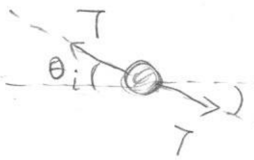
$$\tan\theta_{i+1} = \frac{u_{i+1} - u_i}{\Delta x}$$

(i) Στον κόμβο i :

$$m_i u_i = T \sin\theta_{i+1} - T \sin\theta_i$$

για μικρές γωνίες $\longrightarrow$ $\sin\theta_{i+1} \simeq \theta_{i+1}$, $\sin\theta_i \simeq \theta_i$

κόμβος i



$$\rightarrow m_i \ddot{u}_i = T\theta_{i+1} - T\theta_i \Rightarrow$$

$$\Delta m \ddot{u}_i = T\left(\theta_{i+1} - \theta_i\right) \Rightarrow$$

$$\boxed{\Delta m \ddot{u}_i = T\left(\frac{u_{i+1} - u_i}{\Delta x} - \frac{u_i - u_{i-1}}{\Delta x}\right)} \quad (1)$$

(ii) Για $\Delta x \to dx$ και $\Delta m \to dm$ ~> Νδο $T\dfrac{\partial^2 u}{\partial x^2} = \rho A \dfrac{\partial^2 u}{\partial t^2}$

$$\partial m = \partial(\rho A x) = \rho A \partial x$$

$$(1) \rightsquigarrow \partial m \frac{\partial^2 u}{\partial t^2} = T\left(\frac{u(x+\partial x, t) - u(x,t)}{\partial x} - \frac{u(x,t) - u(x-\partial x, t)}{\partial x}\right)$$

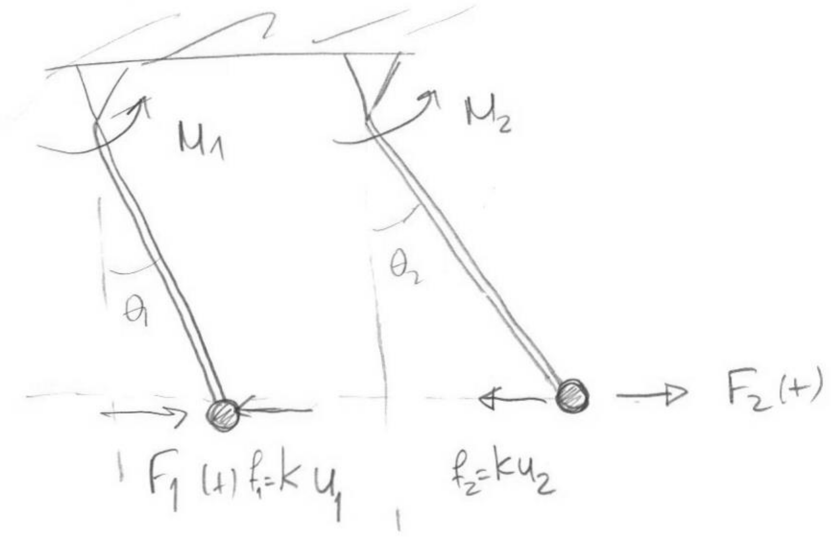
$$\rightsquigarrow \rho A \partial x \frac{\partial^2 u}{\partial t^2} = T\left(\frac{u(x+\partial x, t) \overbrace{- u(x,t) - u(x,t)} + u(x-\partial x, t)}{\partial x}\right)$$

$$\rho A \frac{\partial^2 u}{\partial t^2} = T \overbrace{\frac{u(x+\partial x, t) - 2u(x,t) + u(x-\partial x, t)}{\partial x^2}}^{/\!/ \; u_{xx}}$$

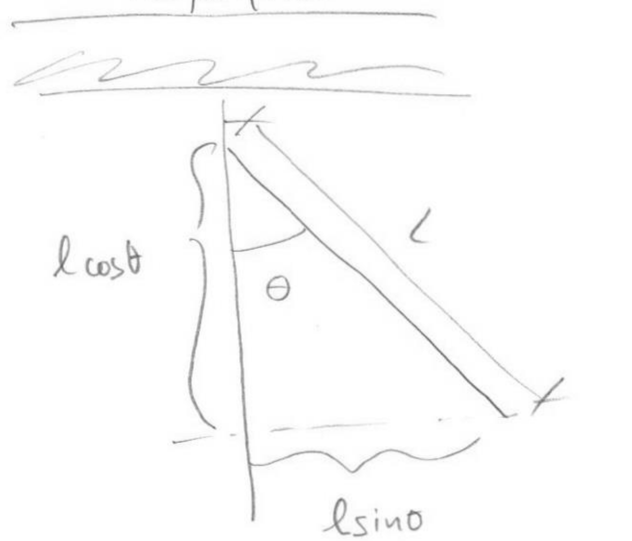
: κεντρώα έκφραση πεπερασμένων διαφορών 2ης τάξης και 2ης παραγώγου.

$$\rightarrow \rho A \frac{\partial^2 u(x,t)}{\partial t^2} = T \frac{\partial^2 u(x,t)}{\partial x^2}$$

ΑΣΚΗΣΗ Α2



Γεωμετρία.



άρα $f_1 = kl(\theta_1-\theta_2)$ $\quad (l_1 = l_2 = l)$

$f_2 = kl(\theta_2-\theta_1)$ $\quad (m_1 = m_2 = m)$

άρα $u_1 = l(\sin\theta_1 - l\sin\theta_2)$

λόγω $\theta_1, \theta_2 << 1$

$\sin\theta_1, \sin\theta_2 \simeq \theta_1, \theta_2$

αντίστοιχα.

1] Εξισώσεις κίνησης :

▷ κάθε ράβδο ξεχωριστά

ΔΕΣ$_1$ $\quad \sum M = I_1 \ddot{\theta}_1 \Rightarrow$

Euler για συμπαγή μάζα:

$I_1 = m_1 \cdot l_1^2 = m_1 l^2 = m l^2$

$$M_1 + F_1 - \underbrace{f_1}_{u_1}(l\cos\theta_1) = I_1 \ddot{\theta}_1 \Longrightarrow$$

$$M_1 + F_1 - kl(\theta_1-\theta_2)\, l\cos\theta_1 = m l^2 \ddot{\theta}_1 \Rightarrow$$

$$\boxed{M_1 + F_1 - kl^2(\theta_1-\theta_2) = m l^2 \ddot{\theta}_1} \quad (1)$$

$\boxed{\cos\theta_1 \simeq 1 \quad \cos\theta_2 \simeq 1}$

ΔΕΣ$_2$ $\quad \sum M = I_2 \ddot{\theta}_2 \Rightarrow$

$$M_2 + F_2 - f_2 = I_2 \ddot{\theta} \Rightarrow \boxed{M_2 + F_2 - kl^2(\theta_2-\theta_1) = m l^2 \ddot{\theta}_2} \quad (2)$$

Θα μπορούσα να διαιρέσω τις (1) και (2) με το l^2

$$(1),(2) \Rightarrow \begin{cases} \dfrac{M_1}{l^2} + \dfrac{F_1}{l^2} - k(\theta_1-\theta_2) = m\ddot{\theta}_1 \\ \dfrac{M_2}{l^2} + \dfrac{F_2}{l^2} - k(\theta_2-\theta_1) = m\ddot{\theta}_2 \end{cases} \Rightarrow$$

2] Γραφή σε μητρωϊκή μορφή.

$$m\ddot{\theta}_1 + k(\theta_1 - \theta_2) = \frac{F_1}{\ell^2} + \frac{M_1}{\ell^2}$$

$$m\ddot{\theta}_2 + k(\theta_2 - \theta_1) = \frac{F_2}{\ell^2} + \frac{M_2}{\ell^2}$$

$$\boxed{\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \frac{1}{\ell^2} \begin{bmatrix} F_1 + M_1 \\ F_2 + M_2 \end{bmatrix}}$$

Όπου $M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$ και $k = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$

Μητρώο Δυνάμεων - Ροπών : $\underset{\sim}{F} = \frac{1}{\ell^2} \begin{bmatrix} F_1 + M_1 \\ F_2 + M_2 \end{bmatrix}$

3] Εύρεση Ιδιοσυχνοτήτων και Ιδιομορφών

Αρχικά να υπολογίσω την ορίζουσα $k - \omega^2 M$

$$-\omega^2 M = \begin{bmatrix} -\omega^2 m & 0 \\ 0 & -\omega^2 m \end{bmatrix} \quad \text{άρα} \quad \boxed{k - \omega^2 M = \begin{bmatrix} k-\omega^2 m & -k \\ -k & k-\omega^2 m \end{bmatrix}}$$

θα πρέπει να ισχύει:

$$\det\left(k - \omega^2 M\right) = 0 \Rightarrow (k-\omega^2 m)^2 - (-k)^2 = 0 \Rightarrow$$

$$(k-\omega^2 m)^2 - k^2 = 0 \Rightarrow (k - \omega^2 m + k)(\cancel{k} - \omega^2 m - \cancel{k}) = 0$$

i] $2k - \omega_1^2 m = 0 \Rightarrow \omega_1^2 = \frac{2k}{m} \Rightarrow \omega_1 = \pm\sqrt{\frac{2k}{m}}$ (με ενδιαφέρει "+")

ii] $-\omega_2^2 m = 0 \Rightarrow \omega_2 = 0$ Συμβαίνει για $T \to \infty$

από σχέση i) με αντικατάσταση στην Χαρακτηριστική εξίσωση

έχουμε :

$$\left(k - \omega_1^2 M\right) \hat{u}_1 = 0 \Rightarrow$$

$$\begin{bmatrix} k-\omega_1^2 m & -k \\ -k & k-\omega_1^2 m \end{bmatrix} \begin{Bmatrix} \hat{u}_{11} \\ \hat{u}_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$(k-\omega^2 m)\hat{u}_{11} + (-k)\hat{u}_{21} = 0$$ → κρατώ την μια ειτων 2 εξισώσεων

$$\cancel{(-k)\hat{u}_{21} + (k-\omega^2 m)\hat{u}_{11}} = 0$$

$$\rightsquigarrow (k-\omega^2 m)\hat{u}_{11} - k\hat{u}_{21} = 0 \Rightarrow$$

$$\hat{u}_{11} = \frac{k}{k-\omega^2 m}\hat{u}_{21} \xrightarrow{\omega_1^2 = \frac{2k}{m}} \boxed{\hat{u}_{11} = \frac{k}{k-\frac{2k}{\cancel{m}}\cancel{m}}\hat{u}_{21} = \frac{k}{-k}\hat{u}_{21} = -\hat{u}_{21}}$$

θέτω $\hat{u}_{11} = 1$

$$\boxed{\underset{\sim}{\hat{u}_1} = \begin{Bmatrix} \hat{u}_{11} \\ \hat{u}_{21} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}}$$

από σχέση ii] $\omega_2 = 0$

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} \hat{u}_{11} \\ \hat{u}_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$k\,\hat{u}_{11} - k\,\hat{u}_{21} = 0 \Rightarrow \hat{u}_{11} = \hat{u}_{21}$$

Εάν θεωρήσω $\hat{u}_{11} = \hat{u}_{21} = 1$

$$\hat{\underline{u}}_2 = \begin{Bmatrix} \hat{\underline{u}}_{11} \\ \hat{\underline{u}}_{21} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Για την κανονικοποίηση ενδεικτικά

$$\hat{m}_1 = u_1^T M u_1 = \underbrace{\begin{Bmatrix} 1 & -1 \end{Bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}}_{[m \; -m]} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = m + m = 2m$$

άρα $\boxed{\underline{\phi}_1 = \dfrac{1}{\sqrt{2m}} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}}$

αντίστοιχα $\boxed{\underline{\phi}_2 = \dfrac{1}{\sqrt{2m}} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}}$

$$\ddot{\xi}_1 + 2\xi\omega_1 + \omega_1^2 \xi_1 = \phi_1^T \underset{\sim}{F} = \frac{1}{\sqrt{2m}} [1 \; -1] \; \underset{\sim}{F}$$

$$\ddot{\xi}_1 + \frac{2k}{m}\xi_1 = \frac{1}{\sqrt{2m}}[1 \; -1]\underset{\sim}{F} \longrightarrow$$

για $F = f_0 \cos(\Omega t) \rightsquigarrow$

$\underset{\sim}{F}(t) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} f_0 \cos(\Omega t)$

$$\ddot{\xi}_1 + \frac{2k}{m}\xi_1 = \frac{1}{\sqrt{2m}}[1 \; -1] \; F_0 \cos(\Omega t) \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \Rightarrow$$

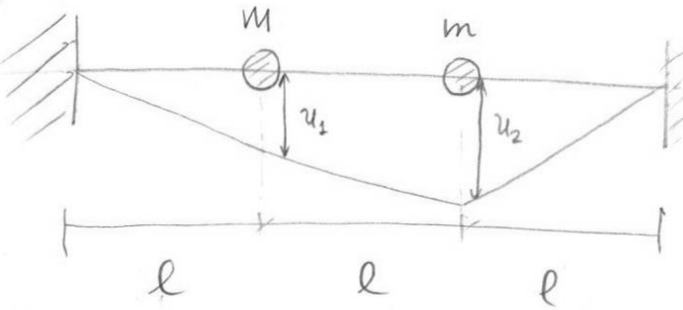
$$\ddot{\xi}_1 + \frac{2k}{m}\xi_1 = \left[\frac{F_0\cos(\Omega t)}{\sqrt{2m}} \; \frac{-F_0\cos(\Omega t)}{\sqrt{2m}}\right] \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \frac{F_0\cos(\Omega t)}{\sqrt{2m}}$$

$\eta_1 = \dfrac{\Omega}{\sqrt{\frac{2k}{m}}}$, $\tan\phi_1 = \dfrac{2\xi_1 \eta_1}{1-\eta_1^2}$

$$\xi_1 = X(\eta_1, \xi) \frac{f_0}{\sqrt{2m}\;\frac{2k}{m}} \cos(\Omega t - \varphi) = X(\eta_1, \xi) \frac{m F_0}{2k\sqrt{2m}} \cos(\Omega t - \varphi)$$

$\omega_2 = 0 \Rightarrow \ddot{\xi}_2 = \dfrac{1}{\sqrt{2m}}[1 \; 1] f_0 \cos(\Omega t) \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \rightsquigarrow \boxed{\xi_2 = \dfrac{-f_0}{\Omega^2} \dfrac{1}{\sqrt{2m}} \cos(\Omega t)}$

ΑΣΚΗΣΗ Β2 Δίνεται $l_1 = l_2 = l_3 = l$ και $m_1 = m_2 = m$



$\sin\theta_1 \simeq \theta_1$

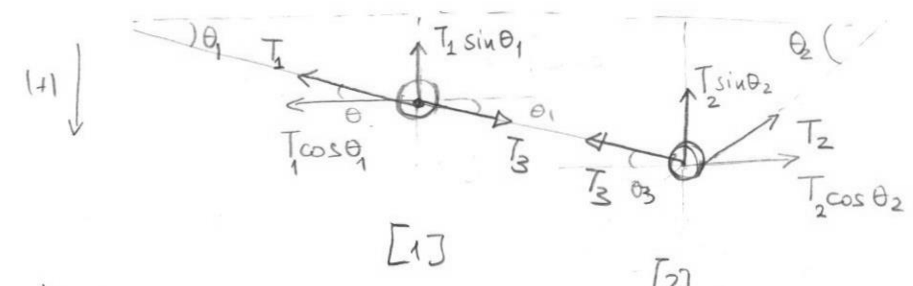
$\sin\theta_2 \simeq \theta_2$

εάν $\theta_1 \ll 1$, $\theta_2 \ll 1$

1] Εξίσωση κίνησης

Μελετώ την μετατοπισμένη κατάσταση :

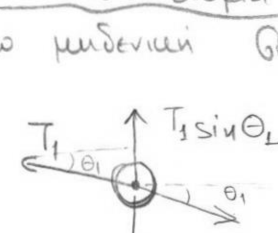
ΔΕΣ (συνολικά)



Απομονώνω κάθε σώμα για ευκολία

θεωρώ μηδενική βαρύτητα ($g=0$)

[1] :



Εξίσωση στον y-άξονα

$$\Sigma F_y = m\ddot{u}_1 \Rightarrow$$

$$-T_1 \sin\theta_1 + T_3 \sin\theta_3 = m\ddot{u}_1 \Rightarrow$$

$$\boxed{-T_1\theta_1 + T_3\theta_3 = m\ddot{u}_1} \quad (\mathrm{I})$$

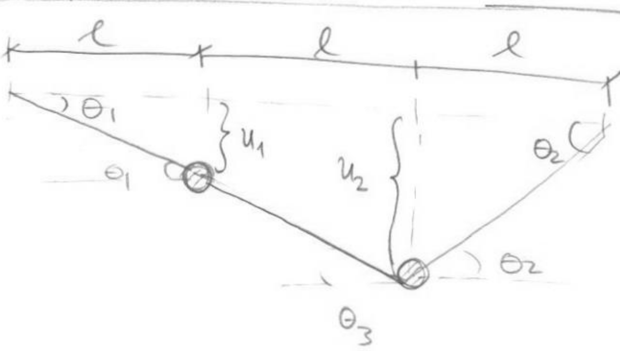
[2] :

$$\Sigma F_y = m\ddot{u}_2 \Rightarrow$$

$$-T_2 \cancel{\sin\theta_2}^{\theta_2} - T_3 \cancel{\sin\theta_3}^{\theta_3} = m\ddot{u}_2 \Rightarrow$$

$$\boxed{-T_2\theta_2 - T_3\theta_3 = m\ddot{u}_2} \quad (\mathrm{II})$$

Γεωμετρικά στοιχεία προβλήματος



$$\left.\begin{aligned} \tan\theta_1 &= \frac{u_1}{l} \simeq \theta_1 \\ \tan\theta_2 &= \frac{u_2}{l} \simeq \theta_2 \\ \tan\theta_3 &= \frac{u_2-u_1}{l} \simeq \theta_3 \end{aligned}\right\} (*)$$

$$(\mathrm{I}) \xrightarrow{(*)} -T_1\frac{u_1}{l} + T_3\frac{u_2-u_1}{l} = m\ddot{u}_1 \Rightarrow$$

$$\boxed{m\ddot{u}_1 + T_1\frac{u_1}{l} - T_3\frac{u_2-u_1}{l} = 0} \quad (\mathrm{III})$$

$$(\mathrm{II}) \xrightarrow{(*)} -T_2\frac{u_2}{l} - T_3\frac{u_2-u_1}{l} = m\ddot{u}_2 \Rightarrow$$

$$\boxed{m\ddot{u}_2 + T_2\frac{u_2}{l} + T_3\frac{u_2-u_1}{l} = 0} \quad (\mathrm{IV})$$

από (III), (IV) φτιάχνω τα μητρώα

2.) Μητρωϊκή Μορφή

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} \frac{T_1+T_3}{l} & -\frac{T_3}{l} \\ -\frac{T_3}{l} & \frac{T_3}{l}+\frac{T_2}{l} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Εάν υποθέσω ότι $T_1 = T_3 = T_2 = T$

$$\underbrace{\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}}_{M} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \underbrace{\begin{bmatrix} \frac{2T}{l} & -\frac{T}{l} \\ -\frac{T}{l} & \frac{2T}{l} \end{bmatrix}}_{K} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

3] Εύρεση ιδιοσυχνοτήτων και ιδιομορφών

Λύνω το ιδιοπρόβλημα

$$\left(\underline{k} - \omega^2 \underline{M}\right)\underline{\hat{u}} = \underline{0}$$

$$\det\left[\underline{k} - \omega^2 \underline{M}\right] = 0$$

$$-\omega^2 \underline{M} = \begin{bmatrix} -\omega^2 m & 0 \\ 0 & -\omega^2 m \end{bmatrix}$$

$$\underline{k} - \omega^2 \underline{M} = \begin{bmatrix} \frac{2T}{\ell} & -\frac{T}{\ell} \\ -\frac{T}{\ell} & \frac{2T}{\ell} \end{bmatrix} + \begin{bmatrix} -\omega^2 m & 0 \\ 0 & -\omega^2 m \end{bmatrix} \Rightarrow$$

$$\underline{k} - \omega^2 \underline{M} = \begin{bmatrix} \frac{2T}{\ell} - \omega^2 m & -\frac{T}{\ell} \\ -\frac{T}{\ell} & \frac{2T}{\ell} - \omega^2 m \end{bmatrix}$$

$$\det\left(\underline{k} - \omega^2 \underline{M}\right) = \left(\frac{2T}{\ell} - \omega^2 m\right)^2 - \left(\frac{T}{\ell}\right)^2 = 0 \Rightarrow$$

$$\left(\frac{2T}{\ell} - \omega^2 m + \frac{T}{\ell}\right)\left(\frac{2T}{\ell} - \omega^2 m - \frac{T}{\ell}\right) = 0 \Rightarrow$$

(κρατάω το "+")

$$\left(\frac{3T}{\ell} - \omega^2 m\right)\left(\frac{T}{\ell} - \omega^2 m\right) = 0$$

(i) $\omega_1^2 = \dfrac{3T}{m\ell} \Rightarrow \omega_1 = \pm\sqrt{\dfrac{3T}{m\ell}}$

(ii) $\omega_2^2 = \dfrac{T}{m\ell} \Rightarrow \omega_2 = \pm\sqrt{\dfrac{T}{m\ell}}$

(κρατάω το "+")

για $w_1^2 = \frac{3T}{ml}$ (περίπτωση (i))

$$(K - \omega_1^2 M)\,\hat{\underline{u}}_1 = \underline{0}$$

$$\begin{bmatrix} \frac{2T}{l} - \omega_1^2 m & -\frac{T}{l} \\ -\frac{T}{l} & \frac{2T}{l} - \omega_1^2 m \end{bmatrix} \begin{Bmatrix} \hat{u}_{11} \\ \hat{u}_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \rightsquigarrow$$

$$\left(\frac{2T}{l} - \omega_1^2 m\right)\hat{u}_{11} - \frac{T}{l}\hat{u}_{21} = 0 \xrightarrow{w_1^2 = \frac{3T}{ml}}$$

$$\left(\frac{2T}{l} - \frac{3T}{\cancel{m}l}\cancel{m}\right)\hat{u}_{11} - \frac{T}{l}\hat{u}_{21} = 0 \rightsquigarrow$$

$$-\frac{T}{l}\hat{u}_{11} - \frac{T}{l}\hat{u}_{21} = 0 \Rightarrow \boxed{\hat{u}_{11} = -\hat{u}_{21}}$$

και άρα για $\hat{u}_{21} = 1$

$$\hat{\underline{u}}_1 = \begin{Bmatrix} \hat{u}_{11} \\ \hat{u}_{21} \end{Bmatrix} = \begin{Bmatrix} -u_{21} \\ \hat{u}_{21} \end{Bmatrix} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

για $\omega_2^2 = \frac{T}{ml}$ (περίπτωση (ii)) $\rightsquigarrow$ $(K - \omega_2^2 M)\,\hat{\underline{u}}_2 = \underline{0}$

$$\left(\frac{2T}{l} - \omega_2^2 m\right)\hat{u}_{11} - \frac{T}{l}\hat{u}_{21} = 0 \xrightarrow{\omega_2^2 = \frac{T}{ml}}$$

$$\left(\frac{2T}{l} - \frac{T}{\cancel{m}l}\cancel{m}\right)\hat{u}_{11} - \frac{T}{l}\hat{u}_{21} = 0 \Rightarrow \frac{T}{l}\hat{u}_{11} - \frac{T}{l}\hat{u}_{21} = 0 \Rightarrow$$

$$\boxed{\hat{u}_{11} = \hat{u}_{21}}$$

Επομένως $\hat{\underline{u}}_2 = \begin{Bmatrix} \hat{u}_{11} \\ \hat{u}_{21} \end{Bmatrix}$

Εάν θέσω $\hat{u}_{11} = \hat{u}_{21} = 1$

$$\hat{\underline{u}}_2 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

<u>κανονικοποίηση</u> $\hat{m}_1 = u_1^T M u_1 = \begin{Bmatrix} -1 & 1 \end{Bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = m + m = 2m$

$\underline{\phi}_1 = \frac{1}{\sqrt{2m}} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$ και $\hat{m}_2 = u_2^T M u_2 = \underbrace{\begin{Bmatrix} 1 & 1 \end{Bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}}_{[m \ m]} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = 2m$

$$\underset{\sim}{\phi}_2 = \frac{1}{\sqrt{2m}} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

για $F = F_0 \cos(\Omega t)$, $\underset{\sim}{F} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} F_0 \cos(\Omega t)$

$$\ddot{\xi}_1 + \cancel{2\zeta\omega_1} + \omega_1^2 \xi_1 = \phi_1^T \underset{\sim}{F} = \frac{1}{\sqrt{2m}} \begin{bmatrix} -1 & 1 \end{bmatrix} F_0 \cos(\Omega t) \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \Rightarrow$$

$$\ddot{\xi}_1 + \frac{3T}{ml} \xi_1 = \begin{bmatrix} \frac{-F_0 \cos(\Omega t)}{\sqrt{2m}} & \frac{F_0 \cos(\Omega t)}{\sqrt{2m}} \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \Rightarrow$$

$$\ddot{\xi}_1 + \frac{3T}{ml} \xi_1 = - \frac{F_0 \cos(\Omega t)}{\sqrt{2m}}$$

$$\xi_1 = X(n_1, \zeta) \frac{-\frac{F_0}{\sqrt{2m}}}{\frac{3T}{ml}} \cos(\Omega t - \varphi) = X(n_1, \zeta) \frac{-F_0 ml}{\sqrt{2m}\, 3T} \cos(\Omega t - \varphi)$$

$$n_1 = \frac{\Omega}{\omega_1} = \frac{\Omega}{\sqrt{\frac{3T}{ml}}} \quad , \quad \tan\phi_1 = \frac{2\zeta_1 n_1}{1 - n_1^2}$$

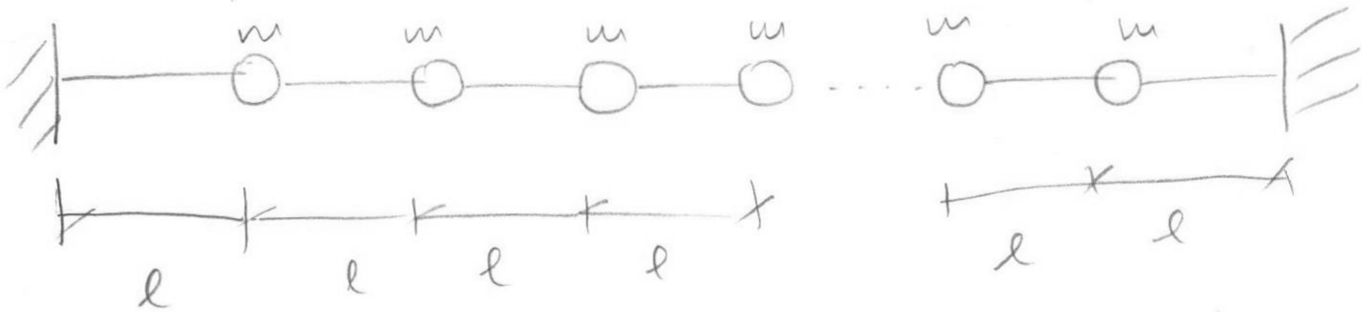
$$\ddot{\xi}_2 + \frac{T}{ml}\xi_2 = \frac{1}{\sqrt{2m}}\begin{bmatrix}1 & 1\end{bmatrix} F_0 \cos(\Omega t)\begin{Bmatrix}1\\0\end{Bmatrix} \Rightarrow$$

$$\ddot{\xi}_2 + \frac{T}{ml}\xi_2 = \frac{F_0\cos(\Omega t)}{\sqrt{2m}} \Rightarrow$$

$$\xi_2 = X(\eta_2, \zeta)\frac{\frac{F_0}{\sqrt{2m}}}{\frac{T}{ml}}\cos(\Omega t - \phi) = X(\eta_2, \zeta)\frac{F_0\, ml}{T\sqrt{2m}}\cos(\Omega t - \phi)$$

$$\eta_1 = \frac{\Omega}{\sqrt{\frac{T}{ml}}} \quad , \quad \tan\phi_2 = \frac{2\zeta_2\eta}{1-\eta_2^2}$$

Σύστημα Bn

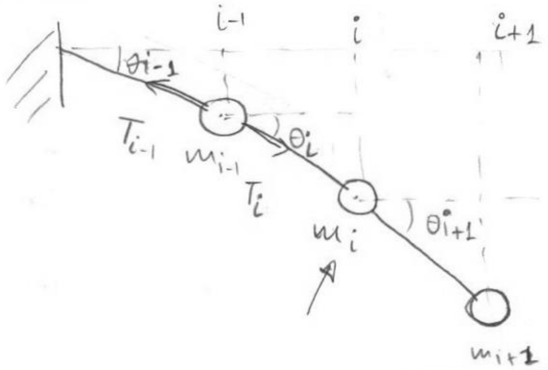


Έχω n μάζες άρα $l_{i-1} \dots l_n$

$\theta_{i-1} \dots \theta_n$

$T_{i-1} \dots T_n$

ΔΕΣ



$\sin\theta_i \simeq \theta_i$

θ_{i+1} $\qquad\qquad$ θ_i

π.χ για σώμα (i) $\boxed{m_i \ddot{u}_i = T_{i+1} \sin\theta_{i+1} - T_i \sin\theta_i}$ (1)

από γεωμετρία

$$\left.\begin{aligned} \tan\theta_{i+1} &= \frac{u_{i+1} - u_i}{l} \\ \tan\theta_i &= \frac{u_i - u_{i-1}}{l} \\ \tan\theta_n &= \frac{u_n}{l} \end{aligned}\right\} \quad (2)$$

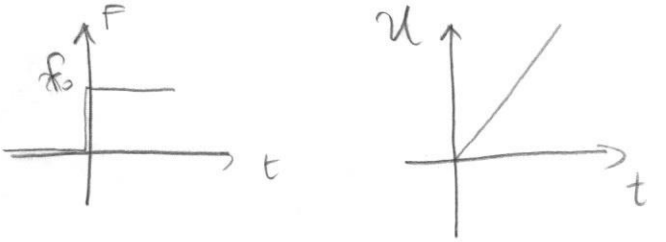
$$(1) \xrightarrow{(2)} m_i \ddot{u}_i = T_{i+1} \frac{u_{i+1} - u_i}{l} - T_i \frac{u_i - u_{i-1}}{l}$$

Εν 17

Ασκ 1

Χιωτέλλη Μαρία
Δήμητρα

Step response – Βημ. Απόκριση ⟶ γραμμική για μηδ. αρχικές συνθήκες. εάν η διέγερση είναι <u>βηματική</u>



$$u = \frac{f_0}{k}$$

Λύση ΣΔΕ για αποτελ. με απόσβεση:

$(m\ddot{u}(t) + c\dot{u} + ku = F(t))$

$$u_h(t) = e^{-\zeta\omega_0 t}\left(\alpha \sin(\omega t) + b\cos(\omega t)\right)$$

$$\boxed{u = e^{-\zeta\omega_0 t}\left[\alpha\cos(\omega_\delta t) + b\sin(\omega_\delta t)\right] + \frac{f_0}{k}}$$

συνεισφορά απο βηματική διεγ.

$$\text{Α.Σ} \begin{cases} u(0) = u_0 & [1] \\ \dot{u}(0) = v_0 & [2] \end{cases}$$

$$[1] \Rightarrow \alpha + \frac{f_0}{k} = u_0 \Rightarrow \boxed{\alpha = u_0 - \frac{f_0}{k}}$$

$$\dot{u} = -\zeta\omega_0 e^{-\zeta\omega_0 t}\left(\alpha\cos(\omega_\delta t) + b\sin(\omega_\delta t)\right) + e^{-\zeta\omega_0 t}\left(-\alpha\omega_\delta\sin(\omega_\delta t) + b\omega_\delta\cos(\omega_\delta t)\right) \Rightarrow$$

Θα πρέπει $\dot{u}(t_{max}) = 0$

$$\underbrace{e^{-\zeta\omega_0 t}}_{\neq 0}\underbrace{\left(\cos(\omega_\delta t)\left(-\zeta\omega_0\alpha + b\omega_\delta\right) - \sin(\omega_\delta t)\left(\zeta\omega_0 b + \alpha\omega_\delta\right)\right)}_{=0} = 0$$

$\omega_\delta = \omega_0\sqrt{1-\zeta^2}$

$$\Rightarrow \boxed{\frac{-\zeta\omega_0\alpha + b\omega_\delta}{\zeta\omega_0 b + \alpha\omega_\delta} = \tan(\omega_\delta t)}$$

$$\tan(\omega_\delta t) = \frac{\cancel{\omega_0}\left(\sqrt{1-\zeta^2}\, b - \zeta a\right)}{\cancel{\omega_0}\left(\zeta b + a\sqrt{1-\zeta^2}\right)}$$

Εύρεση του β $\quad \dot{x}(0) = V_0$

$$\cancelto{1}{e^{-\zeta\omega_0 0}} \cancelto{1}{\cos 0} \left(-\zeta\omega_0 \alpha + \beta u_\delta\right) = V_0 \Rightarrow$$

$$\zeta\omega_0\left(u_0 - \frac{f}{k}\right) + V_0 = \beta u_\delta \Rightarrow \beta = \frac{V_0 + \zeta\omega_0\left(u - \frac{f}{k}\right)}{u_\delta}$$

Λυση των α, β

$$\frac{\left(\frac{V_0 + \zeta\omega_0\left(u_0 - \frac{f_0}{k}\right)}{\omega_0}\right) - \zeta\left(u_0 - \frac{f_0}{k}\right)}{\zeta\left(\frac{V_0 + \zeta\omega_0\left(u_0 - \frac{f_0}{k}\right)}{\omega_0\sqrt{1-\zeta^2}}\right) + \left(u_0 - \frac{f_0}{k}\right)\sqrt{1-\zeta^2}} = \tan(\omega_\delta t) \Rightarrow$$

$$\omega_\delta = \omega_0\sqrt{1-\zeta^2}$$

$$\frac{\left(\cdots\right) - \zeta\left(\cdots\right)}{\zeta\left(V_0 + \zeta\omega_0\left(u_0 - \frac{f_0}{k}\right)\right) + \omega_0\left(1-\zeta^2\right)\left(u_0 - \frac{f_0}{k}\right)} = \tan(\omega_\delta t)$$

$$\Rightarrow \frac{\sqrt{1-\zeta^2}\left(V_0 + \cancel{\zeta\omega_0\left(u_0 - \frac{f_0}{k}\right)} - \cancel{\zeta\omega_0\left(u_0 - \frac{f_0}{k}\right)}\right)}{\zeta\left(V_0 + \zeta\omega_0\left(u_0 - \frac{f_0}{k}\right)\right) + \omega_0\left(1-\zeta^2\right)\left(u_0 - \frac{f_0}{k}\right)} = \tan(\omega_\delta t)$$

Εάν θεωρήσω $V_0 = u_0 = 0$ ως αρχ. συνθήκη

$$\frac{V_0\sqrt{1-\zeta^2}}{\zeta\left(\cancelto{0}{V_0} + \zeta\omega_0\left(\cancelto{0}{u_0} - \frac{f_0}{k}\right)\right) + \omega_0\left(1-\zeta^2\right)\left(\cancelto{0}{u_0} - \frac{f_0}{k}\right)} = \tan(\omega_\delta t)$$

$$\frac{\cancelto{0}{V_0}\sqrt{1-\zeta^2}}{\zeta\left(-\zeta\omega_0\frac{f_0}{k}\right) + \omega_0\left(1-\zeta^2\right)\left(-\frac{f_0}{k}\right)} = \tan(\omega_\delta t) \Rightarrow \tan(\omega_\delta t) = 0 \Rightarrow \omega_\delta t = n\pi$$

για

για $\nu = 1 \Rightarrow t = \frac{\pi}{\omega_\delta}$: $\omega_\delta = \omega_0 \sqrt{1-\zeta^2}$

$$u = e^{-\zeta \omega_0 \frac{\pi}{\omega_\delta}} \left(\alpha \cos\left(\cancel{\omega_\delta} \frac{\pi}{\cancel{\omega_\delta}}\right) + \cancel{b \sin\left(\omega_\delta \frac{\pi}{\omega_\delta}\right)} \right) + \frac{f_0}{k}$$

$$\rightarrow u = e^{-\zeta \omega_0 \frac{\pi}{\omega_\delta}} \left(- \alpha \right) + \frac{f_0}{k} \Longrightarrow$$

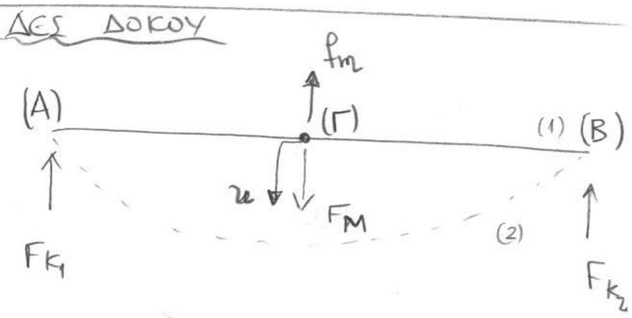
το θέσαμε μηδ.

$$\boxed{u = - e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} \left(\cancel{u_0} - \frac{f_0}{k} \right) + \frac{f_0}{k}}$$

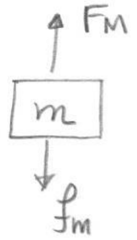
$$\rightsquigarrow \boxed{u_{max} = \frac{f_0}{k} \left(1 + e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} \right)}$$

ΑΣΚ 2   ΔΕΣ ΔΟΚΟΥ

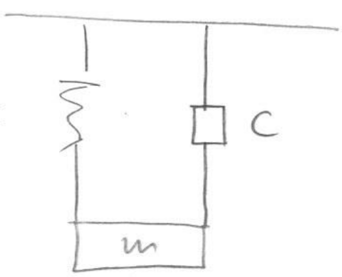
1.



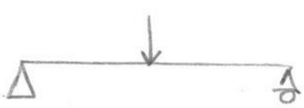
ΔΕΣ Μάζας m



Μοντελοποιείται ως σύστημα ταλαντωτή - αποσβεστήρα.

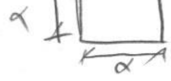


Στην περ. μας είναι παρόμοιο



$$\boxed{f_m = \frac{48EI}{L^3} u} \rightsquigarrow \text{δηλ. } \boxed{k = \frac{48EI}{L^3}}$$

Διατομή δοκαριού



$$I = \frac{1}{12} b h^3 \overset{b=h}{=} \frac{1}{12} b^4 = \frac{1}{12} a^4 \rightsquigarrow \boxed{I = \frac{a^4}{12}}$$

2] Ιδιοσυχν. συνάρτηση μηχ. στοιχείων? $\left(\omega_\delta = \omega_0 \sqrt{1-\zeta^2} \right)$

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \omega_\delta = \sqrt{\frac{48EI}{L^3 m}} \sqrt{1-\zeta^2} \qquad I = \frac{\alpha^4}{12}$$

$$\Rightarrow \boxed{\omega_\delta = \underbrace{\sqrt{\frac{48E\alpha^4}{12L^3 m}}}_{\omega_0} \sqrt{1-\zeta^2}}$$: αποσβεννύμενη ιδιοσυχνότητα.

$$\Rightarrow \boxed{\omega_0 = \sqrt{\frac{48E\alpha^4}{12L^3 m}}}$$: ιδιοσυχνότητα.

3] $x_r(t) = ?$ (αλλιώς $u_r(t) = ?$)

Λόγω της ύπαρξης της μάζας m έχω <u>βηματική διέγερση</u>

$$\frac{F}{k} = \frac{mg}{k}$$

άρα $$u(t) = e^{-\zeta\omega_0 t}\left(\alpha \cos(\omega_\delta t) + \beta \sin(\omega_\delta t)\right) + \frac{mg}{k}$$

$$u(0) = u_0 \Rightarrow \alpha + \frac{mg}{k} = u_0 \Rightarrow \alpha = u_0 - \frac{mg}{k}$$

Όντας τεντωμένο το ελατήριο (αρχική θέση) η δύν. του ελατηρίου εξαρτάται από την M, m : $F_{\varepsilon\lambda} = k u_0 \Rightarrow M_{total}\, g = k u_0 \Rightarrow$

$$(M+m)g = k u_0 \Rightarrow \boxed{u_0 = \frac{(M+m)g}{k}}$$

$$\alpha = \frac{Mg + \cancel{mg} - \cancel{mg}}{k} \Rightarrow \boxed{\alpha = \frac{Mg}{k}}$$

$$\dot{u}(t) = -\zeta\omega_0 e^{-\zeta\omega_0 t}\left(\alpha \cos(\omega_\delta t) + \beta \sin(\omega_\delta t)\right) + e^{-\zeta\omega_0 t}\left(-\alpha\omega_\delta \sin(\omega_\delta t) + \beta\omega_\delta \cos(\omega_\delta t)\right)$$

$$\dot{u}(0) = -\zeta\omega_0 \alpha + \beta\omega_\delta = 0 \Rightarrow -\zeta\omega_0 \frac{Mg}{k} = -\beta\omega_\delta \Rightarrow \boxed{\beta = \frac{\zeta\omega_0}{\omega_\delta}\frac{Mg}{k}}$$

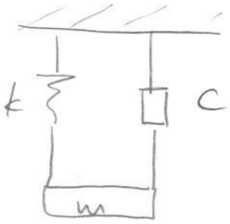
$$\boxed{u_r(t) = e^{-\zeta\omega_0 t}\left(\frac{Mg}{k}\cos(\omega_\delta t) + \frac{\zeta\omega_0}{\omega_\delta}\frac{Mg}{k}\sin(\omega_\delta t)\right) + \frac{mg}{k}}$$

ή $$\boxed{u_r(t) = e^{-\zeta\omega_0 t}\,\frac{Mg}{k}\left(\cos(\omega_\delta t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_\delta t)\right) + \frac{mg}{k}}$$

$$\boxed{u_r(t) = \frac{Mg}{k}e^{-\zeta\omega_0 t}\left[\frac{1}{\sqrt{1-\zeta^2}}\cos(\omega_\delta t - \phi)\right] + \frac{mg}{k}}$$

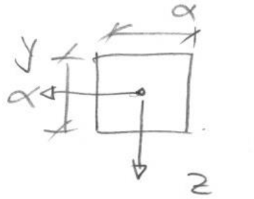
όπου $\tan\phi = \dfrac{\zeta}{\sqrt{1-\zeta^2}}$

4] $\varepsilon_r(t) = ?$



$\sigma = \varepsilon E \longrightarrow \varepsilon = \dfrac{\sigma}{E}$ → είναι η max δύναμη που προκύπτει με ανάλυση καθαρής κάμψης μέσω επίλυσης τάσης αντοχής.

$$\sigma_r = \frac{M_r\, z_{max}}{I} = \frac{F\ell\, a\, \cancel{12}}{a^4\, \cancel{4} * 2} \Rightarrow$$



$z_{max} = \dfrac{a}{2}$

$$\boxed{\sigma_r = \frac{3F\ell}{2a^3}} \quad (1)$$

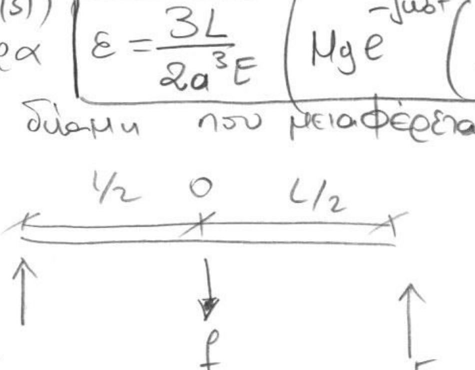
$$\boxed{\varepsilon = \frac{3F\ell}{E\,2a^3}} \quad (2)$$

$M_r = \dfrac{FL}{4}$

εφόσον $F = Mg\,e^{-\zeta\omega_0 t}\left(\dfrac{1}{\sqrt{1-\zeta^2}}\cos(\omega_\delta t-\phi) + mg\right)$ με $\tan\phi = \dfrac{\zeta}{\sqrt{1-\zeta^2}}$ (3)

(από (2) + (3)) άρα $$\boxed{\varepsilon = \frac{3L}{2a^3E}\left(Mg\,e^{-\zeta\omega_0 t}\left(\frac{1}{\sqrt{1-\zeta^2}}\cos(\omega_\delta t - \phi) + mg\right)\right) \text{ με } \tan\phi = \frac{\zeta}{\sqrt{1-\zeta^2}}}$$

5] δύναμη που μεταφέρεται στην κολ. K_2 → ΔΕΞ ΔΟΚΟΥ
Χωρίς την μάζα m



$F_{K_1} + F_{K_2} - f = 0 \Rightarrow 2F_{K_2} = f \Rightarrow \boxed{F_{K_2} = \frac{f}{2}}$

$\Sigma M_o = 0 \Rightarrow F_{K_1}\cancel{\tfrac{L}{2}} - F_{K_2}\cancel{\tfrac{L}{2}} = 0$ (5)

$$F_{k_2} = \frac{F}{2} = \frac{1}{2}\left(M_g e^{-\zeta\omega_0 t}\left(\frac{1}{\sqrt{1-\zeta^2}}\cos(\omega_\delta t - \phi)\right) + mg\right) \quad \text{με } \tan\phi = \frac{\zeta}{\sqrt{1-\zeta^2}}$$

δύναμη στην κολώνα k_2

6] Επιτίμηση C , γνωστή η $\varepsilon(t)$

$$\sigma(t) = E\varepsilon(t) \Rightarrow \frac{3F(t)L}{2a^3} = E\,\varepsilon(t)$$

$$u = \frac{F(t)}{k}$$

$$\frac{3kuL}{2a^3} = E\,\varepsilon(t) \Rightarrow$$

$$\frac{3\cdot \cancel{48}^{24} \cancel{E}I\,u\,\cancel{L}}{\cancel{2}a^3 L^{3}} = \cancel{E}\,\varepsilon(t) \Rightarrow$$

$$\frac{\cancel{3}^{6}\cdot\cancel{24}^{2}\,\cancel{a^4}^{a}\,u}{a^3\,\cancel{12}\,L^2} = \varepsilon(t) \Rightarrow \boxed{u = \frac{L^2\varepsilon(t)}{6a}}$$

Για λογ. Μείωση θέλω μέγιστο!! δηλ. θα πρέπει $\dot{u}(t) = 0 \Rightarrow \varepsilon(t) = 0$

$\Rightarrow \dot{\varepsilon}(t) = 0$

Λογ. Μείωση

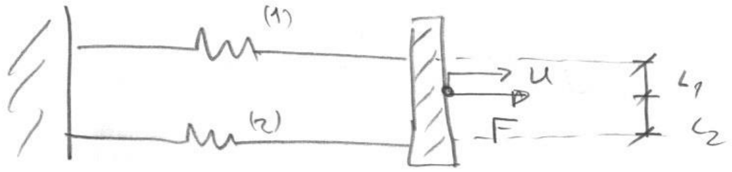
$$\frac{u_i}{u_{i+1}} = \frac{u(t_i)}{u(t_i+T_\delta)} = e^{\zeta\omega_0\frac{2\pi}{\omega_0\sqrt{1-\zeta^2}}} = e^{\zeta\frac{2\pi}{\sqrt{1-\zeta^2}}}$$

$$\underbrace{\ln\frac{u(t_i)}{u(t_i+T_d)}}_{\delta} = \zeta\frac{2\pi}{\sqrt{1-\zeta^2}} \Rightarrow (1-\zeta^2)\delta^2 = \zeta^2 2\pi \Rightarrow \delta^2 - \zeta^2\delta^2 = \zeta^2 4\pi^2 \Rightarrow$$

$$\zeta^2(4\pi^2+\delta^2) = \delta^2 \Rightarrow$$

Επιτίμηση $\longrightarrow$ $\boxed{\zeta = \sqrt{\frac{\delta^2}{4\pi^2+\delta^2}}}$

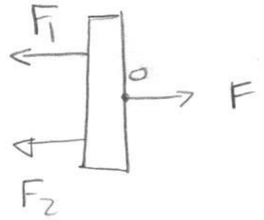
Ασκ3 $k_{eq} = ?$



① Κ.Ε $F_1 = k_1 u_1$

$F_2 = k_2 u_2$

Η πλάκα

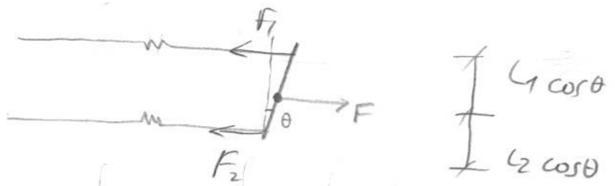


② Ε.Ι $F_1 + F_2 = F$

④ κινημ.

③ ΕΞΙΣΟΡΡΟΠΙΑ

$\Sigma M_O = 0 \Rightarrow F_1 l_1 = F_2 l_2$



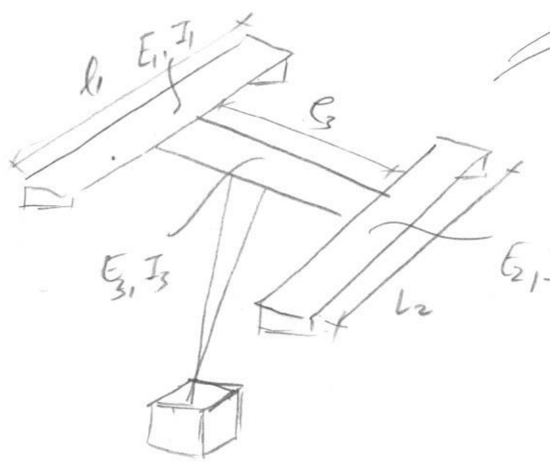
$$\boxed{\frac{l_1}{l_1+l_2} u_2 + \frac{l_2}{l_1+l_2} u_1 = u} \quad (4)$$

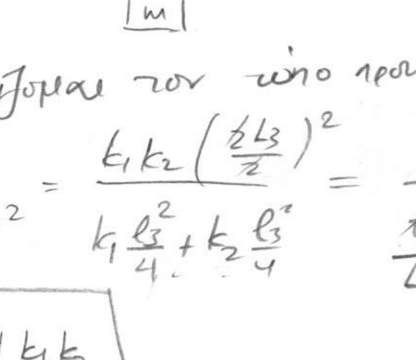
από (3),(2) $\Rightarrow F = F_2\left(1 + \frac{l_2}{l_1}\right)$

(4),(3) $\Rightarrow \; ... \; \Rightarrow F\left(\frac{k_1 l_1^2 + k_2 l_2^2}{k_1 k_2 (l_1+l_2)}\right) = (l_1+l_2) u$

$$\rightsquigarrow \boxed{k_{eq} = \frac{k_1 k_2 (l_1+l_2)^2}{k_1 l_1^2 + k_2 l_2^2}} \quad (I)$$

Ασκ4 $\omega_0 = ?$





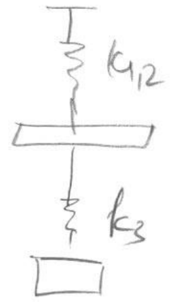
Δανειζόμαστε τον από πριν αποτ.

$$(I) \Rightarrow k_{1,2} = \frac{k_1 k_2 \left(\frac{2 l_3}{2}\right)^2}{k_1 \frac{l_3^2}{4} + k_2 \frac{l_3^2}{4}} = \frac{k_1 k_2 \cancel{l_3^2}}{\frac{\cancel{l_3^2}}{4}(k_1+k_2)}$$

$$\rightarrow \boxed{k_{1,2} = \frac{4 k_1 k_2}{k_1 + k_2}}$$

αποδεχόμαστε λόγω κάμψης + εφαρμογής δυν. στοιχείου $\xrightarrow{} k_i$

$$f_i = \frac{48 E I_i}{l_i^3} u_i$$



Σε σειρα τα 2 ελατηρια

$$k_{eq} = \frac{1}{\frac{1}{k_{1,2}} + \frac{1}{k_3}} = \frac{1}{\frac{k_3 + k_{1,2}}{k_{1,2} k_3}} = \frac{k_3 k_{12}}{k_3 + k_{1,2}}$$

$$k_{eq} = \frac{\frac{4 k_1 k_2}{k_1 + k_2} k_3}{k_3 + \frac{4 k_1 k_2}{k_1 + k_2}} \quad \xrightarrow{k_i = \frac{48 E I_i}{l_i^3}} \quad k_{eq} = \frac{\frac{4 k_1 k_2 k_3}{k_1 + k_2}}{\frac{k_3(k_1 + k_2) + 4 k_1 k_2}{k_1 + k_2}}$$

$$\rightsquigarrow k_{eq} = \frac{4 k_1 k_2 k_3}{k_3 k_1 + k_3 k_2 + 4 k_1 k_2} = \frac{4 \cdot 48^3 \left(\frac{E I_1}{L_1^3} \frac{E I_2}{L_2^3} \frac{E I_3}{L_3^3} \right)}{48^2 \left(\frac{E I_3 E I_1}{L_3^3 L_1^3} + 48^2 \frac{E I_3 E I_2}{L_3^3 L_2^3} + 48^2 \frac{4 E I_1 E I_2}{L_1^3 L_2^3} \right)}$$

$$\rightsquigarrow k_{eq} = \frac{4 \cdot 48 \frac{E^3 I_1 I_2 I_3}{L_1^3 L_2^3 L_3^3}}{\frac{(L_2^3 E I_3 E I_1) + (L_1^3 E I_3 E I_2) + (L_3^3 4 E I_1 E I_2)}{L_1^3 L_2^3 L_3^3}} \Rightarrow$$

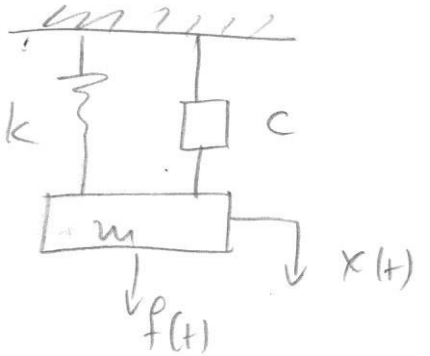
$$\rightsquigarrow k_{eq} = \frac{192 E^3 I_1 I_2 I_3}{E^2 \left(L_2^3 I_3 I_1 + L_1^3 I_3 I_2 + 4 L_3^3 I_1 I_2 \right)}$$

$$\boxed{k_{eq} = 192 \frac{E (I_1 I_2 I_3)}{\left(4 L_3^3 I_1 I_2 + I_3 \left(L_2^3 I_1 + L_1^3 I_2 \right) \right)}}$$

ιδιοσυχνοτητα συστηματος

Ιδιοσυχν. $\omega_0 = \sqrt{\frac{k_{eq}}{m}} \Rightarrow \boxed{\omega_0^2 = \frac{192 E (I_1 I_2 I_3)}{m \left(4 L_3^2 I_1 I_2 + I_3 \left(L_2^3 I_1 + L_1^3 I_2 \right) \right)}}$

Άσκ. 5 :



(i.) Ν.δ.ο $\omega_n = \sqrt{\frac{g}{\delta_{st}}}$

$$\omega_n = \sqrt{\frac{k}{m}} \quad (1)$$

$$f = k\delta_{st} \Rightarrow mg = k\delta_{st} \Rightarrow$$

$$k = \frac{mg}{\delta_{st}} \quad (2)$$

$$(1)+(2) \Rightarrow \boxed{\omega_n = \sqrt{\frac{\cancel{m}g}{\delta_{st}\cancel{m}}} = \sqrt{\frac{g}{\delta_{st}}}} \quad \checkmark$$

(ii.) αποσυν. → 4 βισκοαποσβ. + 4 ελατ.

για $\underline{\zeta = 1}$: σταθερά απόσβεσης κρισίμως αποσβεσμένου.

Εφόσον είναι παράλληλη διάταξη : $\begin{cases} C_{eq} = 4C \\ k_{eq} = 4k \end{cases}$

Για όλο το σύστημα :

$$2\zeta\omega_0 = \frac{C_{eq}}{m} \overset{\zeta=1}{\Rightarrow} \quad 2\omega_0 = \frac{C_{eq}}{m} \Rightarrow C_{eq} = 2m\omega_0 \Rightarrow$$

$$C_{eq} = 2m\sqrt{\frac{k}{m}} = 2m\sqrt{\frac{g}{\delta_{st}}}$$

όμως για κάθε ξεχωριστά $4\,C = 2m\sqrt{\frac{g}{\delta_{st}}} \Rightarrow$

$$\boxed{C = \frac{1}{2}m\sqrt{\frac{g}{\delta_{st}}}}$$

Άσκ 6

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = f(t)$$

ή $\ddot{x} + \alpha\dot{x} + \beta x = f(t)$

(i.) Ν.δ.ο ο ταλαντωτής ασταθής $(x(t) \to \infty, t \to \infty)$ όταν $\zeta < 0$

(Χ.Ε.) $\lambda^2 + 2\zeta\omega_0\lambda + \omega_0^2 = 0 \Rightarrow \lambda^2 + \alpha\lambda + \beta = 0$

$$\Delta = 4\zeta^2\omega_0^2 - 4\omega_0^2 = 4\omega_0^2\underbrace{(\zeta^2 - 1)} \longrightarrow 3\ \text{περ.} \begin{cases} \zeta = 1 \\ \zeta > 1 \\ \zeta < 1 \Rightarrow \zeta < 0 \end{cases}$$

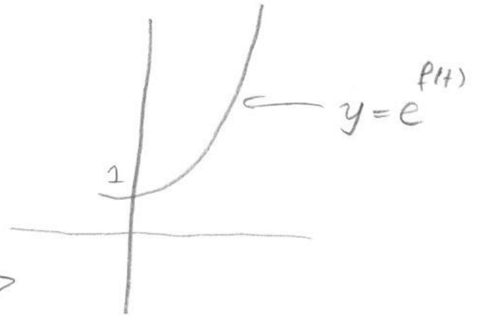
για $\zeta < 1$ που είναι αυτή η περ. μας

$$u(t) = e^{-\zeta\omega_0 t}\sqrt{a^2+b^2}\cos(\omega_d t - \phi) \quad \text{με} \quad \tan\phi = \frac{b}{a}$$

αστάθεια εάν $x(t) \to \infty$ για $t \to \infty$

$t \to \infty$

Εάν τότε $e^{-\zeta\omega_0 t} \to \infty$



άρα θα πρέπει $-\zeta\omega_0 t > 0 \Rightarrow$

όπου $\underline{\zeta < 0}$ αυτό σημαίνει ότι $\lim_{t\to\infty}\left(e^{-\zeta\omega_0 t}\right) = \infty$

(ii) $x(t) \to \infty$, $t \to \infty$ $\quad b < 0$: ΝΔΟ αστάθεια

$b < 0 \Rightarrow \Delta > 0 \Rightarrow$ 2 πρ. ρίζες $\to$ Χ.Ε: $\lambda^2 + a\lambda + b = 0$

$\Delta = \alpha^2 - 4b \qquad \lambda_{1,2} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4b}}{2}$

$\frac{-\alpha + \sqrt{\alpha^2-4b}}{2} = \lambda_1$

$\frac{-\alpha - \sqrt{\alpha^2-4b}}{2} = \lambda_2$

Λύση $u(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$

$$= Ae^{\left(\frac{-\alpha+\sqrt{\alpha^2-4b}}{2}\right)t} + Be^{\left(\frac{-\alpha-\sqrt{\alpha^2-4b}}{2}\right)t}$$

Εχ. $\frac{-\alpha + \sqrt{\alpha^2-4b}}{2} > 0 \Rightarrow -\alpha + \sqrt{\alpha^2-4b} > 0 \Rightarrow$

$\sqrt{\alpha^2-4b} > \alpha$

Επομένως $e^{\left(-\frac{\alpha+\sqrt{\alpha^2-4b}}{2}\right)t} \to \infty$

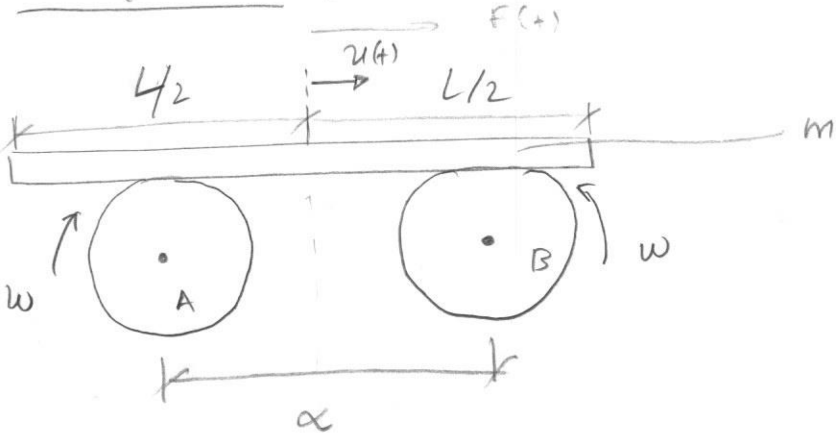
ενώ $e^{\left(\frac{-\alpha-\sqrt{\alpha^2-4b}}{2}\right)t} \to 0$

$\lim_{t\to\infty} u(t) = \infty$

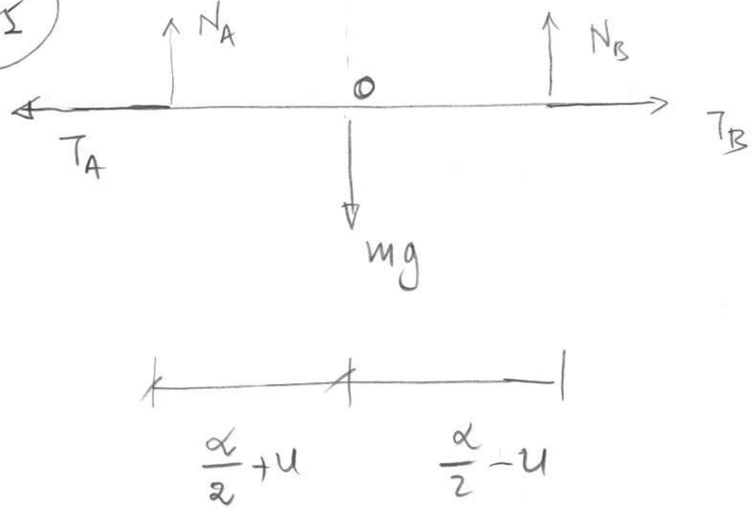
Askisi_11_kai_12_.pdf

11] Διατηρεί ως / κάθε κύλινδρος ασκεί αντίδραση περ.

▷ θέλω εξ. κίνησης στην οριζόντια κατεύθ.



ΔΕΣ



<u>ΕΞ.ΙΣ</u> : $N_B + N_A = mg$ (1)

<u>ΕΞ.ΡΟΠ</u> :

$$\Sigma M_o = 0 \Rightarrow N_A\left(\frac{\alpha}{2}+u\right) = N_B\left(\frac{\alpha}{2}-u\right)$$

$$N_A + N_A\,\frac{\left(\frac{\alpha}{2}+u\right)}{\left(\frac{\alpha}{2}-u\right)} = mg$$

$$N_A\left(1+\frac{\frac{\alpha}{2}+u}{\frac{\alpha}{2}-u}\right) = mg$$

$$1+\frac{\frac{\alpha}{2}+u}{\frac{\alpha}{2}-u} = \frac{\frac{\alpha}{2}\cancel{-u}+\frac{\alpha}{2}\cancel{+u}}{\frac{\alpha}{2}-u} = \frac{\alpha}{\frac{\alpha}{2}-u}$$

άρα $N_A\left(\frac{\alpha}{\frac{\alpha}{2}-u}\right) = mg$

$$F(t) = T_B - T_A = \mu N_B - \mu N_A = \mu\left(N_B - N_A\right)$$

$$= \mu\left(mg - 2N_A\right) = \mu\left(mg - 2mg\left(\frac{\frac{\alpha}{2}-u}{\alpha}\right)\right)$$

$$= \mu mg\left(1 - \frac{\alpha-2u}{\alpha}\right) = \mu mg\left(\cancel{1}-\cancel{1}+\frac{2u}{\alpha}\right) = 2\mu mg\left(\frac{u}{\alpha}\right)$$

$$\cancel{m}\ddot{u} = 2\mu \cancel{m}g\left(\frac{u}{\alpha}\right) \Rightarrow \ddot{u} = 2\mu g\frac{u}{\alpha} \Rightarrow$$

$$\Delta E: \quad \boxed{\ddot{u} - \frac{2\mu g}{\alpha} u = 0}$$

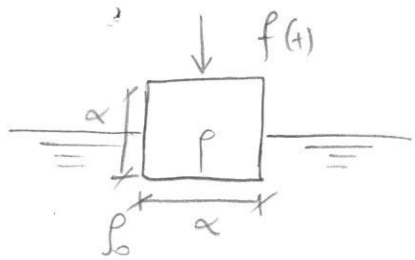
<u>X.E</u> $\quad x^2 + \frac{2\mu g}{\alpha} = 0$

$$\Delta = -4\,\frac{2\mu g}{\alpha} = -\frac{8\mu g}{\alpha} = i^2\,\frac{8\mu g}{\alpha}$$

$$\boxed{x_{1,2} = \frac{\pm \cancel{2}\sqrt{2\frac{\mu g}{\alpha}}}{\cancel{2}} = \pm\sqrt{\frac{2\mu g}{\alpha}}}$$

$$\boxed{u = A\cos\left(\sqrt{\frac{2\mu g}{\alpha}}\,t\right) + B\sin\left(\sqrt{\frac{2\mu g}{\alpha}}\,t\right)}$$

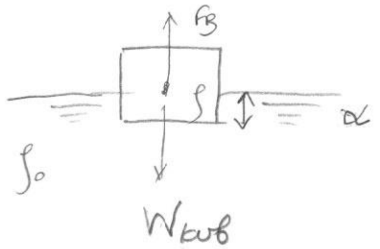
Εν 8 → ΑΣΚ 1

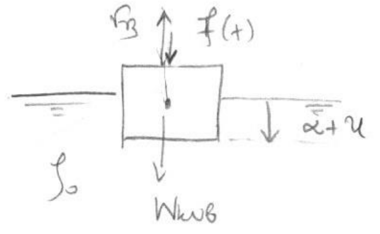


① εξ. κίνησης

② ιδιοσυχνότητα.

Πριν την διέγερση





1] $\left.\begin{array}{l} m = \rho V_{κύβου} \\ V_{κυβ} = \alpha^3 \end{array}\right\} \Rightarrow$

$$\boxed{W_{κυβ} = \rho \alpha^3 g}$$

$$\boxed{F_B = \rho_0 V_{βυθ}\, g}$$

$$\boxed{V_{βυθ} = \underbrace{A_{κυβ}}_{\alpha^2} (\alpha + x)}$$

$$f(t) + W_{κυβ} = F_B \Rightarrow$$

$$f(t) + \rho \alpha^3 g = \rho_0 \alpha^2 (\alpha + x)\, g$$

$$\Rightarrow f(t) + \rho \alpha^3 g = \rho_0 \alpha^3 g + \rho_0 \alpha^2 x\, g$$

$$\boxed{f(t) = \alpha^3 g (\rho_0 - \rho) + \rho_0 \alpha^2 x\, g}$$

→ Εξ. κίνησης.

* Όταν αφαιρεθεί η διέγερση έχω μόνο σύνδεση άνωσης + βάρους στην μάζα m.

2] από σύγκριση με $m\ddot{x}(t) + c\dot{x} + kx = f(t)$ → εξ. αυτού ταλ. με αποσβ.

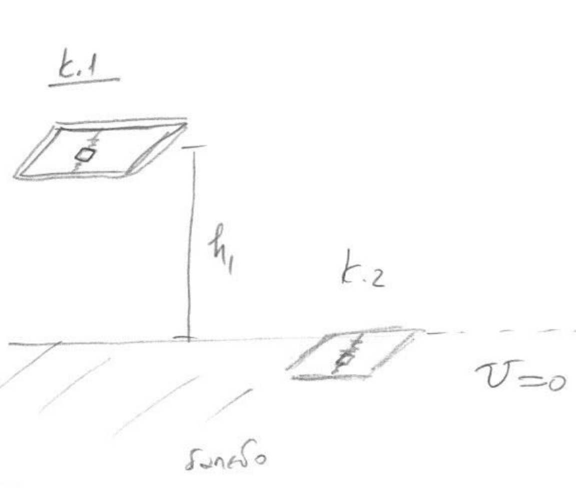
$$k = \rho_0 \alpha^2 g \rightsquigarrow \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{\rho_0 \alpha^2 g}{\rho \alpha^3}} \Rightarrow \boxed{\omega_0 = \sqrt{\frac{\rho_0 g}{\rho \alpha}}}$$

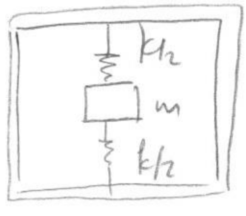
Ασκ 12 (Από ΣΕΤ. ασκ 11,12) * Το δάπεδο είναι επίπεδο μηδενικής δυναμικής ενέργειας

Α.Δ.Ε $E_{μηχ. αρχ} = E_{μηχ\, τελ} \Rightarrow \cancel{K_1}^{0} + U_1 = K_2 + \cancel{U_2}^{0}$

$$\cancel{m} g h_1 = \frac{1}{2} \cancel{m} \dot{x}^2 \Rightarrow$$

$$\dot{x}^2 = 2gh_1 \Rightarrow \boxed{\dot{x} = \sqrt{2gh_1}}$$





Ελατήρια σε σειρά

$$k_{eq} = \frac{k_1 k_2}{k_1 + k_2} =$$

$$\boxed{K_{eq} = \frac{\frac{k^2}{4}}{\cancel{2}\frac{k}{\cancel{2}}} = \frac{k}{4}}$$

α.) Περιγραφή της κίνησης.

Το σώμα φαίνεται να εκτελεί ελεύθερη πτώση με ταχύτητα $\dot{u} = \sqrt{2gh}$ στο δάπεδο.

Εάν τώρα θεωρήσω $t=0$ την θέση πρόσκρουσης.

Εξίσωση κίνησης ταλ. με απόσβεση

$$\boxed{u(t) = e^{-\zeta\omega_0 t}\left(\alpha \cos(\omega_\delta t) + \beta \sin(\omega_\delta t)\right)} \quad \text{I}$$

(Στό έδαφος $\rightarrow u(t_2) = h$, $\dot{u}(t_2) = \sqrt{2gh}$ | $h = \frac{1}{2}gt^2 \Rightarrow t_2 = \sqrt{\frac{2h}{g}}$: στιγμή που φθάνει στο έδαφος)

Αν θεωρήσω ως $\boxed{t=0}$ την θέση πρόσκρουσης:

$u(0) = 0$ (1)

και $\dot{u}(0) = \sqrt{2gh}$ (2)

$$(\mathrm{I}) \xrightarrow{(1)} 0 = \alpha \cos 0 + \cancel{\beta \sin 0} \Rightarrow \alpha = 0$$

$$\boxed{u(t) = e^{-\zeta\omega_0 t}\left(\beta \sin(\omega_\delta t)\right)}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = -\zeta\omega_0 e^{-\zeta\omega_0 t}\left(\cancel{\alpha\cos(\omega_\delta t)}^{\alpha=0} + \cancel{\beta\sin(\omega_\delta t)}^{0}\right) + e^{-\zeta\omega_0 t}\left(\cancel{-\alpha\omega_\delta \sin(\omega_\delta t)} + \beta\omega_\delta \cos(\omega_\delta t)\right)$$

$$\sqrt{2gh} = \beta\omega_\delta \cancel{\cos 0}^{1} \Rightarrow \boxed{\beta = \frac{\sqrt{2gh}}{\omega_\delta}}$$

$$\boxed{u(t) = e^{-\zeta\omega_0 t}\left(\frac{\sqrt{2gh}}{\omega_\delta}\sin(\omega_\delta t)\right)} \leftarrow \text{θέση}$$

$$\dot{u}(t) = -\zeta\omega_0 e^{-\zeta\omega_0 t}\left(\frac{\sqrt{2gh}}{\omega_\delta}\sin(\omega_\delta t)\right) + e^{-\zeta\omega_0 t}\left(\frac{\sqrt{2gh}}{\cancel{\omega_\delta}}\cancel{\omega_\delta}\cos(\omega_\delta t)\right)$$

$$\ddot{u}(t) = -\zeta\omega_0 \frac{\sqrt{2gh}}{\omega_\delta}(-\zeta\omega_0)e^{-\zeta\omega_0 t}\sin(\omega_\delta t) - \zeta\omega_0 \frac{\sqrt{2gh}}{\cancel{\omega_\delta}} e^{-\zeta\omega_0 t} \cancel{\omega_\delta}\cos(\omega_\delta t)$$

$$+ \sqrt{2gh}\left(-\zeta\omega_0 e^{-\zeta\omega_0 t}\cos(\omega_\delta t) - e^{-\zeta\omega_0 t}\sin(\omega_\delta t)\right)$$

$$\ddot{u}(t) = \sqrt{2gh}\, e^{-\zeta\omega_0 t}\left(\frac{(\zeta\omega_0)^2}{\omega_\delta}\sin(\omega_\delta t) - 2\zeta\omega_0\cos(\omega_\delta t) - \sin(\omega_\delta t)\right)$$

$$\boxed{\ddot{u}(t) = \sqrt{2gh}\, e^{-\zeta\omega_0 t}\left(\frac{(\zeta\omega_0)^2 - \omega_\delta}{\omega_\delta} - 2\zeta\omega_0\cos(\omega_\delta t)\right)}$$

Σχέση επιτάχυνσης.

<u>Max</u> επιτάχυνση έχω όταν η ταχύτητα μου είναι 0.

Αρα $\dot{u}(t_I) = 0 \Rightarrow$

$$\underbrace{\sqrt{2gh}\; e^{-\zeta\omega_0 t}}_{>0}\left(-\frac{\zeta\omega_0}{\omega_\delta}\sin(\omega_\delta t) + \cos(\omega_\delta t)\right) = 0$$

άρα $-\frac{\zeta\omega_0}{\omega_\delta}\sin(\omega_\delta t) + \cos(\omega_\delta t) = 0 \Rightarrow$

$$-\frac{\zeta\omega_0}{\omega_\delta}\tan(\omega_\delta) + 1 = 0 \Rightarrow \boxed{\tan(\omega_\delta t) = \frac{\omega_\delta}{\zeta\omega_0}}$$

$$\omega_\delta t_I = \arctan\left(\frac{\omega_\delta}{\zeta\omega_0}\right) = \arctan\left(\frac{\omega_0\sqrt{1-\zeta^2}}{\zeta\,\omega_0}\right)$$

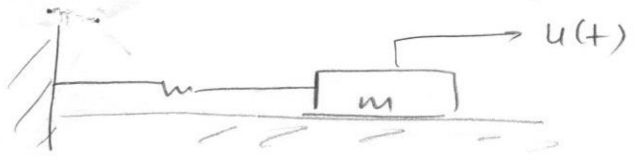
$$\boxed{t_I = \frac{1}{\omega_\delta}\arctan\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)}$$

Αρκεί τώρα να βρώ το <u>$\cos(\omega_\delta t)$</u> : $\cos(\omega_\delta t) = \zeta\omega_0$

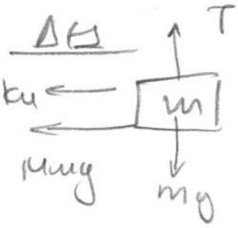
$$\rightarrow \boxed{\ddot{u}(t) = \sqrt{2gh}\; e^{-\zeta\omega_0 t_I}\left(\frac{(\zeta\omega_0)^2 - \omega_\delta}{\omega_\delta} - 2(\zeta\omega_0)^2\right)}$$

max επιταχ.

ΑΣΚ 1 Χιωτέλη Μαρία Δήμητρα



προωθεί



$$\boxed{m\ddot{u} + ku = \pm \mu m g}$$

$$u(0) = u_0$$
$$\dot{u}(0) = 0$$

Περ. 1 $\quad u > 0$ (κίνηση προς τα δεξιά) : $m\ddot{u} + ku = -\mu m g$

Για την ομογενή

$$\ddot{u} + \frac{k}{m} u = -\mu g$$

$$\lambda^2 + \frac{k}{m} = 0 \qquad \Delta = -4\left(\frac{k}{m}\right) < 0 \rightarrow 2 \text{ συζ. μιγαδικές}$$

$$\lambda_{1,2} = \frac{\pm\sqrt{4i^2\left(\frac{k}{m}\right)}}{2} = \frac{\pm \cancel{2} i \sqrt{\frac{k}{m}}}{\cancel{2}} = \pm i\omega, \qquad \sqrt{\frac{k}{m}} = \omega$$

$$u(t) = A e^{\lambda_1 t} + B e^{\lambda_2 t}$$

$$u(t) = A e^{i\omega t} + B e^{-i\omega t}$$

$$= A\left(\cos\omega t + i\sin\omega t\right) + B\left(\cos\omega t - i\sin\omega t\right)$$

$$= \cos(\omega t)\underbrace{(A+B)}_{\alpha} + i\sin(\omega t)\underbrace{(A-B)}_{\beta}$$

$$u(t) = \alpha\cos(\omega t) + \beta\sin(\omega t)$$
$$\dot{u}(t) = -\alpha\omega\sin(\omega t) + \beta\omega\cos(\omega t)$$
$$\ddot{u}(t) = -\alpha\omega^2\cos(\omega t) - \beta\omega^2\sin(\omega t)$$

$$\dot{u}(0) = 0 : \quad 0 = \beta\omega \rightsquigarrow \beta = 0$$
$$u(0) = u_0 \Rightarrow u_0 = \alpha$$

$$\boxed{u(t) = u_0\cos(\omega t)}$$

Μη ομογενής : $u_{εδ.} = -\frac{\mu m g}{k}$ ⇝ ικανοποιεί την Δ.Ε

$$\boxed{u(t) = u_0 \cos(\omega t) - \frac{\mu m g}{k}}$$ ⇝ Εξ. κίνησης.

2] $\boxed{u_{i+1} - u_i = ?}$

$\dot{u}(t) = -u_0 \omega \sin(\omega t)$

$\ddot{u}(t) = -u_0 \omega^2 \cos(\omega t)$

ακρότατα ⇒ $\dot{u}(t) = 0 \Rightarrow -u_0 \omega \sin(\omega t) = 0 \Rightarrow$

$\sin(\omega t) = 0 \Rightarrow$

$\omega t = n\pi \, , \, n \in \mathbb{Z}$

$$\boxed{t = \frac{n\pi}{\omega}}$$

$\boxed{n = 0}$ ⇝ $t_1 = 0$: $\ddot{u}(0) = -u_0 \omega^2 \cancelto{1}{\cos 0} < 0 \Rightarrow max$

άρα $\boxed{u_1(t) = u_0 - \frac{\mu m g}{k}}$

$\boxed{n = 2}$ ⇝ $t_2 = \frac{2\pi}{\omega}$: $\ddot{u}(t_2) = -u_0 \omega^2 \cancelto{1}{\cos\left(\omega \frac{2\pi}{\omega}\right)} < 0 \Rightarrow max$

$u_2(t) = u_0 \cancelto{1}{\cos\left(\omega \frac{2\pi}{\omega}\right)} - \frac{\mu m g}{k} = u_0 - \frac{\mu m g}{k}$

$$\boxed{u_{i+1} - u_i = u_2 - u_1 = 0}$$

3] Συνθήκη για να σταματήσει η ταλάντωση

$$\boxed{F_{ελ} = F_{τριβής}}$$

Το σώμα πάει προς τα αριστερά $\boxed{u(t) = u_0\cos(\omega t) + \frac{\mu m g}{k}}$

$$\dot{u}(t) = 0 \Rightarrow -u_0\omega\sin(\omega t) = 0 \Rightarrow \dots \Rightarrow \boxed{t = \frac{n\pi}{\omega}}$$

$$n = 0 \rightsquigarrow t_1 = 0 : \boxed{u_1(0) = u_0 + \frac{\mu m g}{k}}$$

$$n = 2 \rightsquigarrow t_2 = \frac{2\pi}{\omega} : \boxed{u_2(0) = u_0 + \frac{\mu m g}{k}}$$

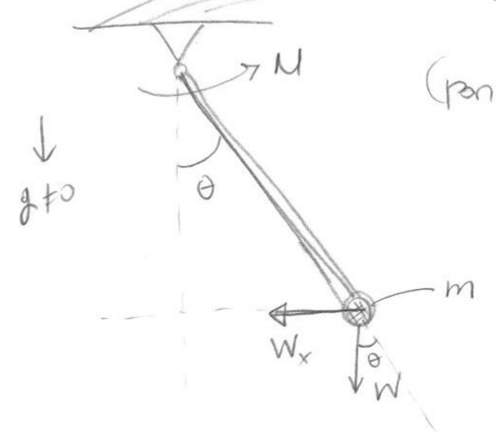
$$\boxed{u_{i+1} - u_i = u_2 - u_0 = 0}$$

δ] Το σύστημα σταματά όταν η δυν. του ελατηρίου γίνεται ίση με την τριβή δηλ.

$$ku = \mu m g \rightsquigarrow \boxed{u = \frac{\mu m g}{k}}$$

ΑΣΚ 2

$$\boxed{W_x = W\sin\theta = mg\sin\theta}$$



$I = mL^2$

(ροπής) $T = I\alpha = I\ddot{\theta} \Rightarrow$

$$-\cancel{m}g\sin\theta\, L = \cancel{m}\ddot{\theta}L^2 \Rightarrow$$

$$L^2\ddot{\theta} + g\sin\theta\, L = 0 \Rightarrow$$

$$\boxed{\ddot{\theta} + \frac{g}{L}\sin\theta = 0} \rightsquigarrow$$ από taylor. $\sin\theta \simeq \theta$ για $\theta \ll 1$

$$\boxed{\ddot{\theta} + \frac{g}{L}\theta = 0}$$ (ομογενής, 2ας τάξεως)

$\lambda = \kappa \pm \lambda i$

$$\lambda^2 + \frac{g}{L} = 0 \rightsquigarrow \Delta < 0 \rightsquigarrow \Delta = -4\frac{g}{L} \rightsquigarrow \sqrt{\Delta} = 2i\sqrt{\frac{g}{L}}^{=\omega}$$

$$\theta(t) = e^{\overset{kt=0}{kt}}(\alpha\cos(\omega t) + \beta\sin(\omega t)),$$

Α.Σ
$\theta(0) = \theta_0$
$\dot{\theta}(0) = 0$
$t = 0$

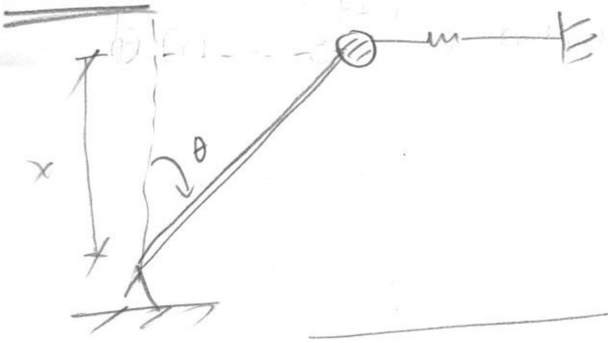
$$\dot{\theta}(0) = 0 \Rightarrow \beta\omega = 0 \Rightarrow \beta = 0 \qquad \dot{\theta}(t) = (-\alpha\omega\sin\omega t + \beta\omega\cos\omega t)$$

$$\cancel{\dot{\theta}(0) = ke^{k0}\alpha + e^{k0}\beta\omega \Rightarrow k\alpha + \beta\omega = 0 \quad (1)}$$

$\theta(0) = \alpha = \theta_0$
και $\dot{\theta} = 0$ $\Big\}$ $\Rightarrow$ $\boxed{\theta = \theta_0 \cos(\omega t)}$

με $\omega = \sqrt{\frac{g}{l}}$

Ασκ3



Δυν. Ενέργεια

$$V(\theta) = mgx + \frac{1}{2} k (l \sin\theta)^2$$

$$\rightarrow \boxed{V(\theta) = mgl\cos\theta + \frac{1}{2} k l^2 \sin^2\theta}$$

$$\boxed{\frac{dV}{d\theta} = -mgl\sin\theta + \cancel{\frac{1}{2}} k l^2 \cancel{2} \cos\theta}$$

$$s = l\theta \rightarrow \ddot{s} = l\ddot{\theta}$$

$$ml\ddot{\theta} = \frac{1}{l}\frac{dV}{d\theta} = \cancel{\frac{1}{l}}\left(-mg\cancel{l}\cos\theta + kl^{2}\cos\theta\right)$$

$$\Rightarrow ml\ddot{\theta} = -mg\sin\theta + kl\cos\theta$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta - \frac{k}{m}\cos\theta = 0$$

για $\theta \ll 1$: $\sin\theta \simeq \theta$ και $\cos\theta = 1$

$$\boxed{\ddot{\theta} + \frac{g}{l}\theta - \frac{k}{m} = 0}$$ $\leftarrow$ Εξίσωση κίνησης