Π.Θ.

THMMY

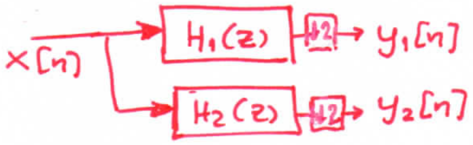
HY342 – D.S.P.

FINAL EXAM – SOLUTIONS

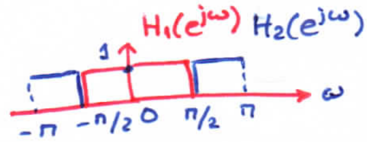
2015 – 16

27/01/16

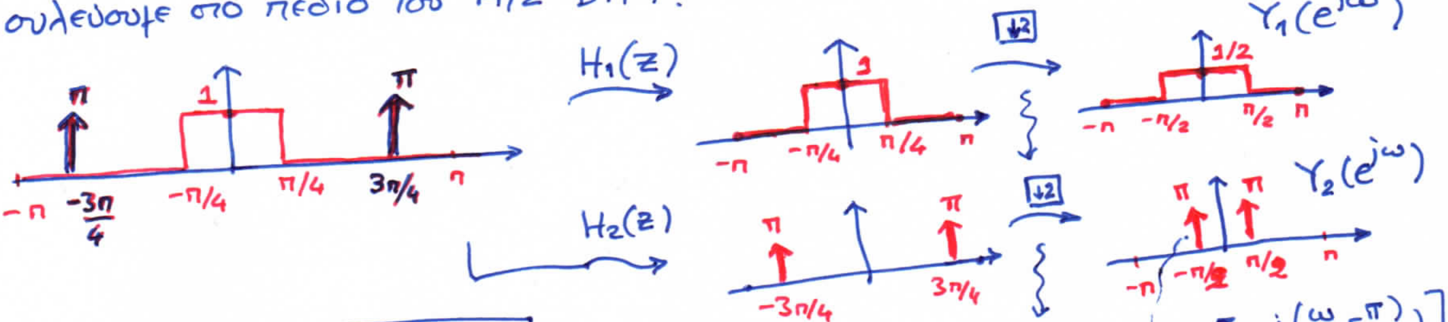
1



$$x[n] = \cos\frac{3\pi n}{4} + \frac{\sin\frac{\pi n}{4}}{\pi n} \Rightarrow y_{1,2}[n] = ?$$



- Δουλεύουμε στο πεδίο του Μ/Σ DTFT:



$$\frac{1}{2}\left[X'(e^{j\frac{\omega}{2}}) + X'(e^{j(\frac{\omega}{2}-\pi)})\right]$$

από τα $-\frac{3\pi}{4} + \pi = \frac{\pi}{4} \xrightarrow{*2} \pi/2$

$-\frac{5\pi}{4} + \pi = -\frac{\pi}{4} \xrightarrow{*2} -\pi/2$

(+ δ/σμός πλάτους δέλτα αναιρεί διαίρεση με 1/2)

- Άρα:

$$\boxed{y_1[n] = \frac{1}{2}\,\frac{\sin\left(\frac{\pi n}{2}\right)}{\pi n}}$$

$$\boxed{y_2[n] = \cos\left(\frac{\pi n}{2}\right)}$$

2

FIR – TYPE III

$h[n] \in \mathbb{R}$

$H(j\sqrt{2}) = 0$

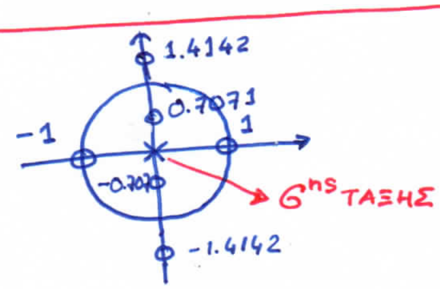
$H(e^{j\frac{\pi}{2}}) = -2$

$M = 6$

$\Rightarrow$

- POLES/ZEROS
- H(z)
- ΣΥΜΜΕΤΡΙΚΟ ΔΙΑΓΡΑΜΜΑ ΥΛΟΠΟΙΗΣΗΣ

(a)



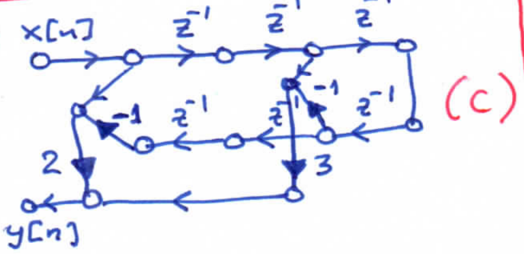
- Λόγω τύπου III έχουμε μηδενικά στα ±1.
- Λόγω γραμμ. φάσης + πραγματικής $h[n]$ έχουμε επίσης μηδενικά στα $\pm\frac{1}{\sqrt{2}}j$, $-\sqrt{2}j$

$$\Rightarrow H(z) = A(1+z^{-1})(1-z^{-1})\left(1+\frac{z^{-1}j}{\sqrt{2}}\right)\left(1-\frac{z^{-1}j}{\sqrt{2}}\right)\cdot(1+z^{-1}\sqrt{2}j)(1-z^{-1}\sqrt{2}j)$$

$$\Rightarrow H(z) = A(1-z^{-2})\left(1+\frac{z^{-2}}{2}\right)(1+2z^{-2}) = A\left(1+\frac{3}{2}z^{-2}-\frac{3}{2}z^{-4}-z^{-6}\right) \Rightarrow$$

Επίσης, $H(e^{j\frac{\pi}{2}}) = H(j) = A\left(1-\frac{3}{2}-\frac{3}{2}+1\right) = -A = -2 \Rightarrow A = 2$

$$\Rightarrow \boxed{H(z) = 2 + 3z^{-2} - 3z^{-4} - 2z^{-6}} \quad \text{(b)}$$



(c)

3 L.P. BUTTERWORTH, $N=3$, $\omega_c=\pi/2$, ΔΙΓΡΑΜΜΙΚΟΣ Μ/Σ $\Rightarrow$ $H(z)=?$ ΚΑΝΟΝΙΚΗ ΥΛΟΠΟΙΗΣΗ, POLES / ZEROS

- Βρίσκουμε πρώτα το $H_c(s)$.
- Διαλέγουμε από τις $2N=6$ ρίζες, αυτές του αριστερού ημιεπιπέδου:

$$s_k=\Omega_c\exp\left(j\pi\frac{2k+4}{6}\right)\Rightarrow \Omega_c e^{j2\pi/3},\ \Omega_c e^{j\pi}=-\Omega_c,\ \Omega_c e^{j4\pi/3}=\Omega_c e^{-j2\pi/3}$$

- Άρα,

$$H_c(s)=\frac{\Omega_c^3}{(s+\Omega_c)(s-\Omega_c e^{j2\pi/3})(s-\Omega_c e^{-j2\pi/3})}=\frac{\Omega_c^3}{(s+\Omega_c)(s^2+\Omega_c s+\Omega_c^2)}\quad [1]$$

$$-2\mathrm{Re}\{\Omega_c e^{-j2\pi/3}\}=-2\left(-\frac{\Omega_c}{2}\right)=\Omega_c$$

- Λόγω διγραμμικού μετασχηματισμού:

$$\Omega_c=\frac{2}{T}\tan\left(\frac{\omega_c}{2}\right)\underset{T=1,\ \omega_c=\pi/2}{=}2\tan\left(\frac{\pi}{4}\right)=2\quad [2]$$

- Από [1], [2] $\Rightarrow$

$$H_c(s)=\frac{8}{(s+2)(s^2+2s+4)}=\frac{8}{s^3+4s^2+8s+8}\Rightarrow$$

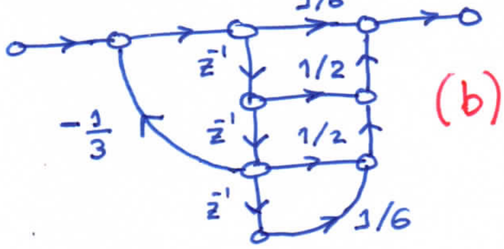
$$\underset{s=2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}{\Longrightarrow}\quad H(z)=\frac{1}{\left(\frac{1-z^{-1}}{1+z^{-1}}+1\right)\left(\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2+\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1\right)}\Rightarrow$$

$$\Rightarrow H(z)=\frac{(1+z^{-1})^3}{2\left(\underbrace{(1-z^{-1})^2+(1-z^{-1})(1+z^{-1})+(1+z^{-1})^2}\right)}=\frac{(1+z^{-1})^3}{2(3+z^{-2})}$$

$1-\cancel{2z^{-1}}+\cancel{z^{-2}}$
$1\ \cancel{-z^{-2}}$
$1+\cancel{2z^{-1}}+z^{-2}$

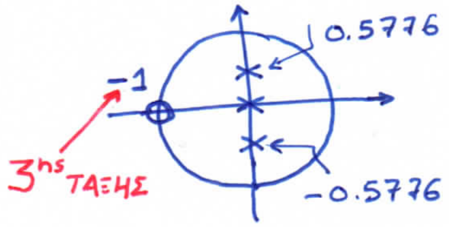
$$\Rightarrow \boxed{H(z)=\frac{1}{6}\frac{(1+z^{-1})^3}{(1+\frac{1}{3}z^{-2})}}=\frac{1}{6}\frac{1+3z^{-1}+3z^{-2}+z^{-3}}{1+\frac{1}{3}z^{-2}}\quad (a)$$

- DIRECT FORM II:



(b)

- ZEROS: $\{-1,-1,-1\}$
  POLES: $\{\pm\frac{1}{\sqrt{3}}j,\ 0\}$



(c)

4

$$x[n] \to \boxed{H_d(z)} \to y[n] \to \boxed{H_c(z)} \to x'[n]$$

$H_d(z)$: ZEROS ② $\{-\frac{7}{9}, \pm 2\}$

POLES ③ $\{0, \pm \frac{1}{3} j\}$

$(-1)^n \to (-1)^n$

$\Rightarrow$ (a) $H_d(z) = ?$

(b) $H_c(z) = ?$

s.t.: $|X'(e^{j\omega})| = |X(e^{j\omega})|$

(a) $$H_d(z) = A \cdot \frac{(1-4z^{-2})(1+\frac{7}{9}z^{-1})}{1+\frac{1}{9}z^{-2}} = -\frac{5}{3} \cdot \frac{(1-4z^{-2})}{1+\frac{1}{9}z^{-2}} \cdot (1+\frac{7}{9}z^{-1}) \quad [1]$$

$z=-1$

$$H_d(-1) = \frac{A(-3)}{\cancel{\frac{10}{9}}\ 5} \cdot \cancel{\frac{2}{9}} = -\frac{3A}{5} = 1 \Leftrightarrow A = -\frac{5}{3}$$

Μετά από πράξεις: $$H_d(z) = \frac{5}{3} \cdot \frac{-1-\frac{7}{9}z^{-1}+4z^{-2}+\frac{28}{9}z^{-3}}{1+\frac{1}{9}z^{-2}}$$

(b) Η πρώτη σκέψη είναι να θέσουμε $H_c(z) = 1/H_d(z)$, αλλά αυτό δεν είναι ευσταθές. Αντί αυτού, αναλύουμε το $H_d(z)$ σε min.phase + all.pass:

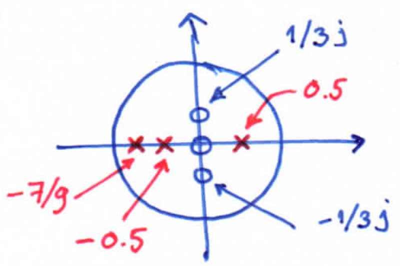
$$[1] \Rightarrow H_d(z) = \underbrace{-\frac{5}{3} \cdot \frac{1+\frac{7}{9}z^{-1}}{1+\frac{1}{9}z^{-2}} \cdot (-4)(1-\frac{1}{4}z^{-2})}_{H_{d,min}(z)} \cdot \underbrace{\frac{(1-4z^{-2})}{1-\frac{1}{4}z^{-2}} (-\frac{1}{4})}_{\text{ALL PASS } \|\cdot\|=1}$$

• Συνεπώς, διαλέγουμε $H_c(z) = 1/H_{d,min}(z)$, δηλ.

$$H_c(z) = \frac{20}{3} \cdot \frac{1+\frac{1}{9}z^{-2}}{(1+\frac{7}{9}z^{-1})(1-\frac{1}{4}z^{-2})} = \frac{20}{3} \cdot \frac{1+1/9\, z^{-2}}{1+\frac{7}{9}z^{-1}-\frac{1}{4}z^{-2}-\frac{7}{36}z^{-3}}$$

• POLES: $\{\pm\frac{1}{2}, -\frac{7}{9}\}$

ZEROS: $\{0, \pm\frac{1}{3}j\}$



5 A $\delta[n-6]$ ㉔ $(u[n]-u[n-24])\cos\frac{\pi n}{4}$

- Έστω $x[n]=\delta[n-6]$, $y[n]=\cos\frac{\pi n}{4}\cdot(u[n]-u[n-24])$, $z[n]=x[n]$ ㉔ $y[n]$.

- Τότε, $z[n]=IDFT_{24}\{X[k]\,Y[k]\}$, $0\le n\le 23$, [1]

όπου $X[k]=DFT_{24}\{x[n]\}=e^{-j\frac{2\pi}{24}6k}=e^{-j\frac{\pi}{2}k}=(-j)^k$ [2]

και $Y[k]=DFT_{24}\{y[n]\}=DFT_{24}\left\{\frac{1}{2}\left(e^{j\frac{2\pi}{24}3n}+e^{-j\frac{2\pi}{24}3n}\right)\right\}=$ $n=0,1,\dots,23$

$$=\frac{24}{2}\left(\delta[k-3]+\delta[\langle k-3\rangle_{24}]\right)=12\,\delta[k-3]+12\,\delta[k-21] \quad [3]$$

- Από [1], [2], [3] $\Rightarrow z[n]=IDFT_{24}\{\underbrace{12(-j)^3\delta[k-3]+12(-j)^{21}\delta[k-21]}_{Z[k]}\}$

$$\Rightarrow Z[k]=12j\,\delta[k-3]-12j\,\delta[k-21]=-\frac{24}{2j}\cdot\left(\delta[k-3]-\delta[k-21]\right),\quad 0\le k\le 23$$

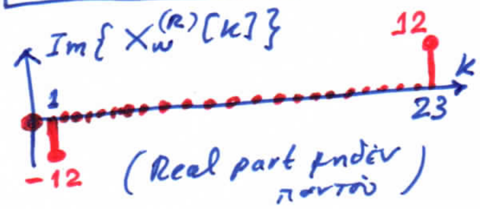
$\xRightarrow{IDFT_{24}} z[n]=-\sin\frac{\pi n}{4}$, $0\le n\le 23$ ή $\boxed{z[n]=\left(-\sin\frac{\pi n}{4}\right)(u[n]-u[n-24])}$

5 B $x[n]=\sin\left(\frac{\pi n}{12}\right)$, WINDOWING $(W_{23}^{(R)}, W_{24}^{(H')})$ $\Rightarrow X_W[k]=?$ $0\le k\le 23$

↳ MODIFIED HAMMING

- Ορθογώνιο παράθυρο: $\sin\left(\frac{\pi n}{12}\right)=\frac{1}{2j}\left(e^{j\frac{2\pi}{24}1n}-e^{-j\frac{2\pi}{24}1n}\right)$, $0\le n\le 23$ $\xleftrightarrow{DFT_{24}}$ $\frac{12}{j}\left(\delta[k-1]-\delta[\langle k+1\rangle_{24}]\right)$

$$\Rightarrow \boxed{X_W^{(R)}[k]=-12j\,\delta[k-1]+12j\,\delta[k-23]}$$



- Modified Hamming:

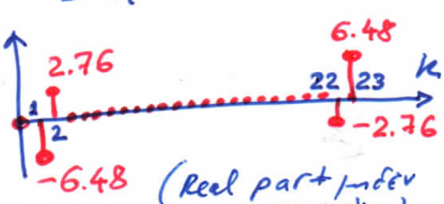
$x_W^{(H')}[n]=\left(0.54-0.46\cos\frac{2\pi n}{24}\right)\sin\frac{\pi n}{12}$, $0\le n\le 23$

$\Rightarrow x_W^{(H')}[n]=0.54\sin\frac{\pi n}{12}-0.46\cos\frac{\pi n}{12}\sin\frac{\pi n}{12}$

$\rightarrow x_W^{(H')}[n]=0.54\sin\frac{\pi n}{12}-0.23\sin\frac{\pi n}{6}\;\cancel{-0.23\sin 0}$

$\updownarrow$ DFT24

$$\Rightarrow \boxed{\begin{aligned}X_W^{(H')}[k]=&-0.54\cdot 12j\,\delta[k-1]+0.54\cdot 12j\,\delta[k-23]\\&+0.23\cdot 12j\,\delta[k-2]-0.23\cdot 12j\,\delta[k-22]\end{aligned}}$$



- Διαφοροποίηση λόγω μεγαλύτερου εύρους κύριου λοβού παραθύρου Hamming.