Iordanis Koutsopoulos

NETWORK OPTIMIZATION LECTURE NOTES

Part 1 [1]

[1] Based on Notes from Fall 2006, and Fall 2010. Organized by Stella Gaki.

Contents

1 Introduction

1.1 State-of-the-art in networks and wireless networks

3G (WCDMA), 4G (MIMO-based)

- LTE, LTE-Advanced
- WLANs (IEEE 802.11/a/b/g/n)
- WiMAX (IEEE 802.16a/e/m/j): wireless mesh networks, relay based networks, femtocells
- WPANs (IEEE 802.15): Bluetooth
- Cognitive Radio Networks (CRNs): operate in an open spectrum market, by dynamic spectrum access and sharing
- Wireless sensor networks (various applications)
- Machine-to-machine communications: capture an event through a sensor network, relay it through a network to an application (software program) that translates captured event into meaningful information

Other classes of wireless networks

- Vehicular Networks

- Delay-tolerant Networks (DTNs)
- Peer-to-peer networks (P2P): for content distribution
- Other modern developments: cloud computing, resource virtualization,. . .

OPTIMIZATION: (i) control of adaptable parameters in communication networks, (ii) resource management.

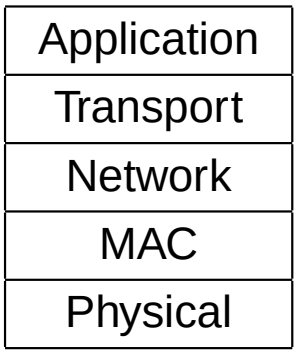
Resources to be managed:

- Spectrum
- Energy (especially in battery devices, but also for green communications)
- Storage capacity, cache memory
- Processing power (CPU)

Self-organizing networks: need for autonomous control, self-awareness, cognition, learning, self-coordination.

1.2 The mechanisms of OSI layers

The OSI layers are:



Mechanisms:

- Application layer: HTTP protocol, Source coding, Security (e.g. cryptography), compression (source coding).
- Transport layer: Flow control, Congestion control, TCP retransmission.
- Network layer: Routing, multi-casting, any-casting, admission control, end-to-end information transfer.
- Medium Access Control (MAC) layer: channel random access (probability of access), channel allocation (TDMA, FDMA, CDMA, OFDMA), MAC packet retransmission protocol, queue management, CRC check, scheduling.
- Physical (PHY) layer: Transmit power control, transmit rate control (modulation/demodulation, channel coding / decoding), transmit directionality control (e.g. beam-forming in smart antenna systems), MIMO, receiver design and functionality.

Digital Communication deals mostly with PHY.

1.3 Physical layer view (one link)

We examine a link between a source and a destination and not the entire network.

Procedures that take place before transmission:

- Source coding (e.g Quantization): describe source as succinctly as possible, i.e. with as few bits as possible.
- Channel coding: Some redundant bits are added to useful data bits. These bits can be a linear combination of useful data bits (if the code is linear). If some data bits are lost due to channel errors, we can retrieve them by using the redundant bits.
- Modulation: The signal is transformed into a continuous waveform.

1.4 Modes of information transport

- *Uni-cast*: if a separate message is conveyed to one destination.
- *Broadcast*: transmit the same data to *all* destinations in the network.

- *Multi-cast*: the delivery of a common message to a *group* of destination nodes (*multi-cast group*).

1.5 Performance Metrics to be optimized

- *Reception rate* in a hop (single-hop throughput).
- *End-to-end* amount of information per unit time that is received successfully (throughput).
- *Latency:* the amount of time it takes for data to travel from one location to another one across a network.
- *Energy Consumption:* Refers to the amount of energy is consumed for every bit is send. More general: *Energy efficiency*: energy consumption, network lifetime).
- *Network Reliability:* ability of network to provide communication in the event of failure of one or more components.
- *Deadline:*Accomplish to transmit data in a particular point in time.
- *Estimation / detection accuracy*: estimation error, probability of false alarm, . . .

1.6 Application areas of Optimization theory

- started from Operations Research
- Computer Science, Algorithm design
- Economics, Finance
- Transportation
- Data management, data mining
- Statistical Physics

2 Optimization preliminaries

Optimization problem: the problem of maximizing or minimizing an objective function with respect to a set of variables subject to given constraints to be fulfilled by these variables.

- Network consists of various elements that interact.
- Network operational objective viewed as an optimization problem.
- Network control variables $\rightarrow$ problem variables.

Remark: Variables may be *discrete* (combinatorial optimization problems) or *continuous*. We will deal with continuous variables.

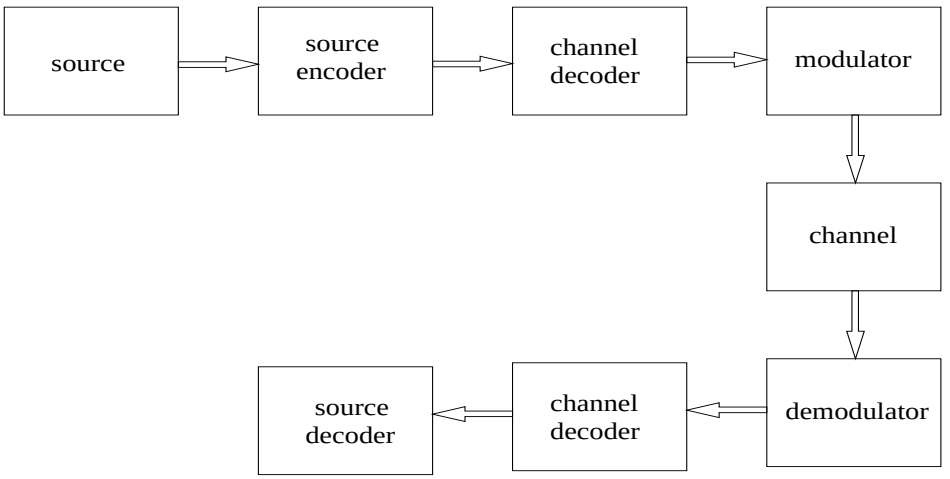


Figure 1: Transmission and reception.

An optimization problem (P) has the form:

$$\text{minimize } f_0(\mathbf{x}) \tag{1}$$

subject to:

$$g_i(\mathbf{x}) \leq 0, \text{ for } i = 1, 2, \ldots, m \tag{2}$$

- Vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is the vector of *optimization variables* of the problem.
- Function $f_0(\cdot) : \mathcal{R}^n \longrightarrow \mathcal{R}$ is the *objective function*.
- Functions $g_i(\cdot) : \mathcal{R}^n \longrightarrow \mathcal{R}, i = 1, 2, \ldots, m$, are called *constraint functions* or *constraints*.

A vector $\mathbf{x}_0$ is *feasible* or a *feasible solution* for problem (P) if its satisfies all constraints, i.e. if $g_i(\mathbf{x}_0) \leq 0$, for $i = 1, \ldots, m$.

A vector $\mathbf{x}^*$ is called *optimal*, or *an optimal solution* for problem (P), if it is feasible and it has the smallest objective function value among all feasible vectors. That is, for any feasible $\mathbf{x}$ with $g_i(\mathbf{x}) \leq 0, i = 1, \ldots, m$, it is $f_0(\mathbf{x}^*) \leq f_0(\mathbf{x})$.

2.1 Special cases of optimization problems

2.1.1 Least-squares problem (LS)

A least-squares problem is an optimization problem with no constraints and an objective function which is the sum of squares of K terms of the form $\mathbf{a}_i^T\mathbf{x} - b_i$,

$$\text{minimize } f_0(\mathbf{x}) = \|A\mathbf{x} - b\|^2 = \sum_{i=1}^{K} \left(\mathbf{a}_i^T\mathbf{x} - b_i\right)^2 \quad (3)$$

where $A \in \mathcal{R}^{k\times n}$ with ($k \geq n$) is a real matrix, $\mathbf{b} \in \mathcal{R}^K$ is a real vector, $\mathbf{a}_i^T$ for $i = 1, \ldots, k$ are the rows of A, and vector $\mathbf{x} \in \mathcal{R}^n$ is the vector of optimization variables.

- LS is one of few optimization problems that can be solved analytically. Its solution can be reduced to solving a set of linear equations,

  $$(A^TA)\mathbf{x} = A^T\mathbf{b}, \quad (4)$$

  with analytical solution $\mathbf{x}^* = (A^TA)^{-1}A^T\mathbf{b}$.

- Efficient algorithms exist.

2.1.2 Linear Programming (LP)

The objective and all constraint functions are linear:

$$\text{minimize } \mathbf{c}^T\mathbf{x} \tag{5}$$

subject to

$$\mathbf{a}_i^T\mathbf{x} \leq b_i, i = 1, \ldots, m. \tag{6}$$

Vectors $\mathbf{c}, \mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathcal{R}^n$ and scalars $b_1, \ldots, b_m \in \mathcal{R}$ are parameters that specify the objective function and constraints.

- LP problems do not have solution in analytical form.
- There exist algorithms to solve LP problems efficiently: the *Simplex* algorithm.

2.1.3 Convex optimization problems

$$\text{minimize } f_0(\mathbf{x}) \tag{7}$$

such that

$$g_i(\mathbf{x}) \leq 0, \text{for } i = 1, \ldots, m. \tag{8}$$

where $f_0(\cdot), g_1(\cdot), \ldots, g_m(\cdot)$ are convex functions i.e they satisfy:

$$f_i(a\mathbf{x} + (1-a)\mathbf{y}) \leq af_i(\mathbf{x}) + (1-a)f_i(\mathbf{y}) \tag{9}$$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{R}^n$ and all $a \in \mathbf{R}$ with $0 \leq a \leq 1$.

Note: The corresponding maximization problem where $f_0(\cdot)$ is concave is also called convex optimization problem (constraint functions must still be convex).

Non-linear programming problems (NLP): all problems that are not LP.

3 Convex Sets

Line segment between two points $\mathbf{x}_1$,and $\mathbf{x}_2$ is the set of points $\mathbf{x}$ which can be written as: $\mathbf{x} = \theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2$, with $0 \leq \theta \leq 1$. Due to this special constraint on θ, $\mathbf{x}$ is not a linear but a *convex* combination of $\mathbf{x}_1$, $\mathbf{x}_2$.

Convex Set: A set of points $\mathcal{C}$ is called *convex* if all points on the line segment between any two points of the set $\mathcal{C}$ also belong in $\mathcal{C}$. That is, $\mathcal{C}$ is convex set if

$$\forall \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{C} \text{ and } 0 \leq \theta \leq 1, \text{ it is } \theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2 \in \mathcal{C}$$

A set of discrete points is always non-convex.

For example, the set of points on the real line defined by $\{x : |x - a| \geq 1\}$ is not convex.

Note: A set of points that is not convex is called non-convex.

A *Convex combination* of K points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_K$ is ev-

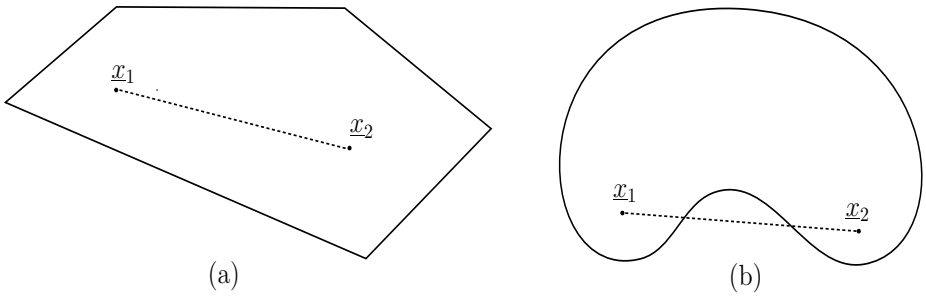


Figure 2: Examples: (a) a convex set, (b) a non-convex set

ery point $\mathbf{x}$ of the form:

$$\mathbf{x} = \sum_{i=1}^{K} \theta_i \mathbf{x}_i, \text{ with } \sum_{i=1}^{K} \theta_i = 1. \tag{10}$$

3.1 Convex Hull

For a set of points $\mathcal{C} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_K\}$, the *Convex hull* of $\mathcal{C}$, $Conv(\mathcal{C})$, is the set of all convex combination of points in $\mathcal{C}$:

$Conv(\mathcal{C}) = \left\{\mathbf{x} : \mathbf{x} = \sum_{i=1}^{K} \theta_i \mathbf{x}_i, \text{ for } \mathbf{x}_i \in \mathcal{C}\right\}$ with $0 \leq \theta \leq 1$ for $i = 1, \ldots, K$ and $\sum_{i=1}^{K} \theta_i = 1$.

Special Cases: If $\mathcal{C} = \{\mathbf{x}_1, \mathbf{x}_2\}$, then $Conv(\mathcal{C})$ is the line segment connecting $\mathbf{x}_1$ and $\mathbf{x}_2$. If $\mathcal{C} = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, then $Conv(\mathcal{C})$ is the triangle with these points as vertices.

Note: The Convex Hull of a set of points $\mathcal{C}$ is the smallest convex set that encloses all points of $\mathcal{C}$.

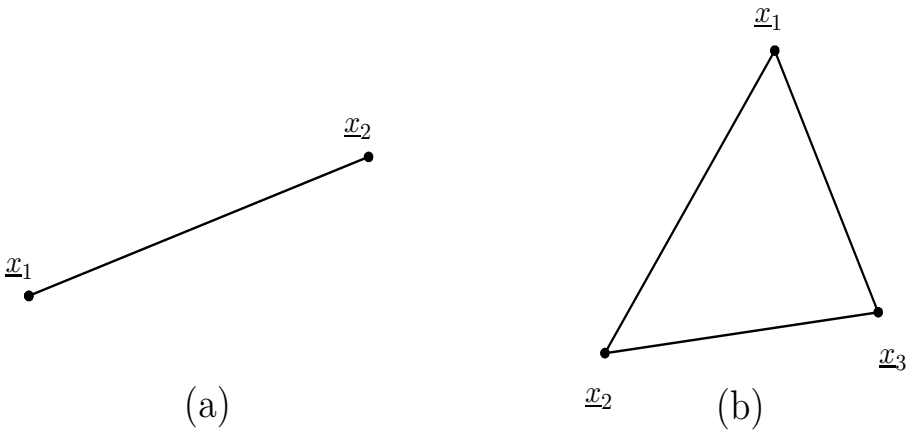


Figure 3: Convex hull of: (a) $\mathcal{C} = \{\mathbf{x}_1, \mathbf{x}_2\}$ (b) $\mathcal{C} = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$

Remarks:

- $Conv(\mathcal{C})$ is always a continuous and convex set irrespective of $\mathcal{C}$.
- If $\mathcal{C}$ is a convex continuous set, then $Conv(\mathcal{C}) = \mathcal{C}$.
- If $\mathcal{C}$ is non-convex and continuous, then $Conv(\mathcal{C}) \supset \mathcal{C}$.
- If $\mathcal{C}$ is a discrete set, then $Conv(\mathcal{C}) \supset \mathcal{C}$.

3.1.1 Example

Consider two wireless links: link 1 transmitting from A to B and link 2 from C to D. If link 1 only is active, it achieves reception

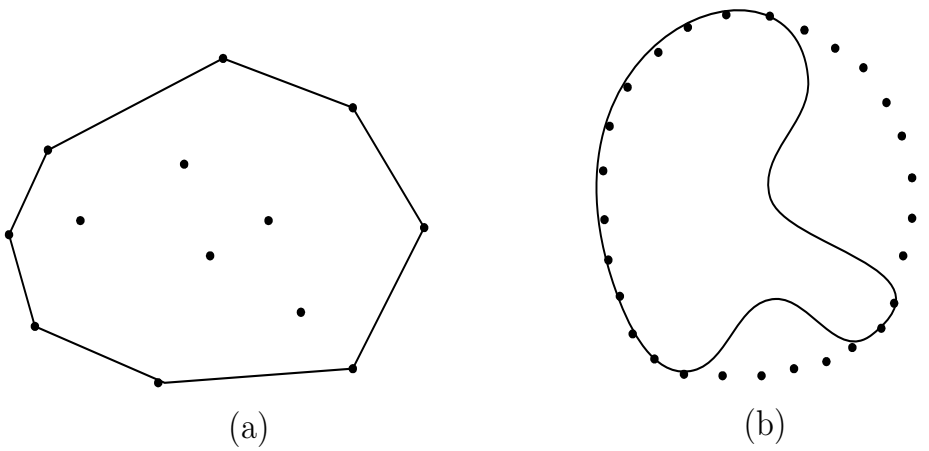


Figure 4: Convex hull of (a) a discrete set of points, (b) a continuous set of points.

of traffic at the receiver at rate R_1 bits/sec. If link 2 only is active, it achieves rate R_2 bits/sec. If both links are active, link i achieves $R_i' < R_i$ bits/sec, $i = 1, 2$. This is due to the interference caused by the other link.

Denote by $\mathcal{I}$ the set of activation policies, in that case there are 3 such policies (only link 1 active, only link 2 active, both links active).

An *average* link rate vector (r_1, r_2) is *achievable* if there exists an activation schedule for the links such that the long-term average rate vector is (r_1, r_2).

The set of achievable *average* rate vectors (r_1, r_2) is the convex hull of vectors $(R_1, 0)$, $(0, R_2)$ and (R_1', R_2'), i.e: $(r_1, r_2) = a_1(R_1, 0) + a_2(0, R_2) + a_3(R_1', R_2')$. for any

$a_1, a_2, a_3 \in [0, 1]$, with $a_1 + a_2 + a_3 = 1$. Note that a_1, a_2, a_3 can be viewed as the *percentages* of time that a certain activation policy is used.

A specific choice of (a_1, a_2, a_3) corresponds to an activation schedule for the 2 links.

More formally, let $\mathbf{s}$ be the (fixed) system state. Let $\mathcal{I}_{\mathbf{s}}$ be the set of possible activation policies when the system is in that state. Let $\mathbf{C}(I, \mathbf{s})$ be the rate vector for activation policy $I \in \mathcal{I}_{\mathbf{s}}$. The rate region, i.e the set of all possible rate vectors is given by $Conv(\mathbf{C}(I, \mathbf{s}) : I \in \mathcal{I}_{\mathbf{s}})$.

In order to check whether a certain link rate vector (ρ_1, ρ_2) is achievable, one has to check whether the corresponding set of linear equations with unknowns a_1, a_2, a_3 has solution.

Suppose now that $\mathcal{S}$ is the set of possible states of the system, and let $\pi_{\mathbf{s}}$ be the stationary distribution of $\mathbf{s} \in \mathcal{S}$. The rate region in that case is the set of vectors $\mathbf{R}$ such that:

$$\mathbf{R} = \sum_{\mathbf{s} \in \mathcal{S}} \pi_{\mathbf{s}} Conv(\mathbf{C}(I, \mathbf{s}) : I \in \mathcal{I}_{\mathbf{s}}).$$

E.g., suppose that the state of each link is $\{G, B\}$ where these stand for good and bad state.

Note that a vector in the rate region is the time-average of rate vectors corresponding to a given state, which in turn is the time average of rate vectors corresponding to a certain activation policy.

3.2 Properties of Convex Sets

1. If $\mathcal{C}$ is convex set and $b \in \mathcal{R}$ is real number, then the set
$$\mathcal{D} = b\mathcal{C} = \{\mathbf{x} : \mathbf{x} = b\mathbf{u} : \mathbf{u} \in \mathcal{C}\}. \quad (11)$$
is also convex (*scaling property*): when a convex set is multiplied by a real number, the resulting set remains convex.

2. If $\mathcal{C}_1, \mathcal{C}_2$ are convex sets then $\mathcal{C}_1 + \mathcal{C}_2$,
$\mathcal{C}_1 + \mathcal{C}_2 = \{\mathbf{x} : \mathbf{x} = \mathbf{u}_1 + \mathbf{u}_2\}$ where $\mathbf{u}_1 \in \mathcal{C}_1$ and $\mathbf{u}_2 \in \mathcal{C}_2$ is also convex (*addition* property). Example for set addition: If $\mathcal{C}_1 = \{1, 2\}$ and $\mathcal{C}_2 = \{10, 15, 18\}$, then $\mathcal{C}_1 + \mathcal{C}_2 = \{11, 16, 19, 12, 17, 20\}$.

3. If $\mathcal{C}_1, \mathcal{C}_2$ are convex sets, then $\mathcal{C}_1 \cap \mathcal{C}_2$ is convex (*intersection* property).

Remark: Generalization of convex combinations. The convex combination of a distinct set of K points are all points $\mathbf{x}$, such that:

$$\mathbf{x} = \sum_{i=1}^{K} \theta_i \mathbf{x}_i \text{ with } \sum_{i=1}^{K} \theta_i = 1 \quad (12)$$

How to generalize to continuous set of points $\mathcal{C}$ and continuous coefficients θ? Consider functions $P(\cdot) : \mathcal{R} \rightarrow \mathcal{R}^n$

such that $P(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathcal{C}$ and $\int_{\mathbf{x}\in\mathcal{C}} P(\mathbf{x}) = 1$. Then the convex combination is that case is given by the set of all points $\mathbf{y}$: $\mathbf{y} = \int_{\mathbf{x}\in\mathcal{C}} \mathbf{x}P(\mathbf{x})d\mathbf{x}$. If $\mathbf{x}$ is a *random* vector, then the above becomes $\mathbb{E}[X]$.

3.3 Hyperplanes and Polyhedra

Hyperplane: A *Hyperplane* $\mathcal{P}$ is a set of points $\mathbf{x}$ with a constant inner product to a given vector $\mathbf{a}$:
$\mathcal{P} = \{\mathbf{x} \in \mathcal{R}^n : \mathbf{a}^T\mathbf{x} = b\}$ where $b \in \mathcal{R}$ or $\mathcal{P} = \{(x_1, ..., x_n) : a_1x_1 + a_2x_2 + \ldots + a_nx_n = b\}$. A projection of a hyperplane in a two-dimensional plane is shown in figure 5.

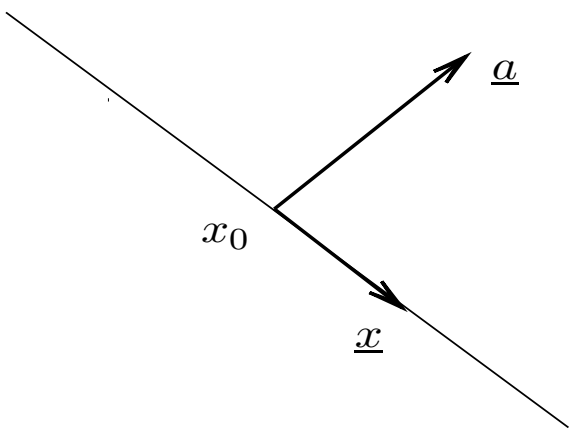


Figure 5: Projection of a hyperplane in two-dim.

The equation characterizing $\mathcal{P}$ is also written as $\mathbf{a}^T(\mathbf{x} - \mathbf{x}_0) = 0$ where $\mathbf{x}_0$ is any point on $\mathcal{P}$, (namely, it satisfies

$\mathbf{a}^T\mathbf{x}_0 = b$). Vector $\mathbf{a}$ *defines* a hyperplane $\mathcal{P}$ and is vertical to *all* points of the hyperplane.

The dimension of $\mathbf{x}$, n, specifies the type of hyperplane.

1. If $n = 2$, $\mathcal{P} = \{(x_1, x_2) : a_1x_1 + a_2x_2 = b\}$, which is a straight line with slope defined by vector (a_1, a_2).

2. If $n = 3$, (three-dimensional space), then $\mathcal{P} = \{(x_1, x_2, x_3)$ $a_1x_1 + a_2x_2 + a_3x_3 = b\}$. which is the usual plane.

Half-spaces: A hyperplane $\mathcal{P}$ satisfying $\mathbf{a}^T\mathbf{x} = b$ divides the space into two *half-spaces*. The one is all points $\mathbf{x}$ such that $\mathbf{a}^T\mathbf{x} \leq b$ and the other one consists of all points $\mathbf{x}$ such that $\mathbf{a}^T\mathbf{x} \geq b$.

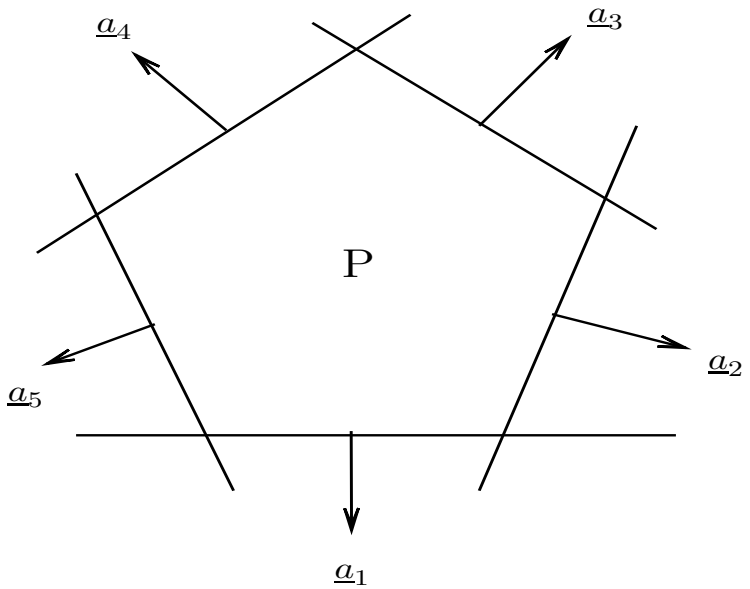
Polyhedron: A *Polyhedron* is the intersection of many half-spaces:

$\mathcal{P} = \{\mathbf{x} : \quad \mathbf{a}_j^T\mathbf{x} \leq b_j \quad , j = 1, \ldots, m \quad$ and $\mathbf{c}_j^T\mathbf{x} = d_j \quad , j = 1, \ldots, p \quad \}$

In this definition, there are m half-spaces and p hyperplanes that make up the polyhedron $\mathcal{P}$.

Example: A polyhedron defined as intersection of 5 half-spaces, $\mathbf{a}_j^T x \leq b_j \quad , j = 1, ..., 5$.

- If the polyhedron is closed (bounded), it is called a *polytope*.



4 Calculus Overview

4.1 Neighborhood

Given a point $\mathbf{x} \in \mathcal{R}^n$, the set of points $\{\mathbf{y} \in \mathcal{R}^n : \|\mathbf{y} - \mathbf{x}\|_2 < \epsilon\}$ is called *neighborhood* of $\mathbf{x}$, $N(\mathbf{x})$, where $\|\mathbf{a}\|_2 = \sqrt{\mathbf{a}^T\mathbf{a}}$ is the quadratic norm of vector $\mathbf{a}$ and ϵ is a small positive constant.

Depending on the dimension n of $\mathcal{R}^n$, the neighborhood can be:

- $n = 1$: the neighborhood of $x \in \mathcal{R}$ is the set of points $N(x) = \{y : y \in (x - \epsilon, x + \epsilon)\}$.
- $n = 2$: the neighborhood of $\mathbf{x}$ is a disk centered at $\mathbf{x}$

with radius ϵ.

- $n = 3$: the neighborhood of $\mathbf{x}$ is a sphere centered at $\mathbf{x}$ with radius ϵ.

- $n > 3$: the neighborhood of $\mathbf{x}$ is a "hyper-sphere".

4.2 Local and global solutions

A point $\mathbf{x}^*$ is called *local optimum (say minimum)* of function $f(\cdot)$ if

$$f(\mathbf{x}^*) \leq f(\mathbf{x}),\ \forall \mathbf{x} \in\ N(\mathbf{x}^*). \tag{13}$$

A point $\mathbf{x}^*$ is called *global optimum (minimum)* of function $f(\cdot)$ if

$$f(\mathbf{x}^*) \leq f(\mathbf{x}),\ \forall \mathbf{x} \in\ \Omega. \tag{14}$$

with Ω the definition set of $f(\cdot)$ (set of real vectors over which $f(\cdot)$ is defined.

Note: In the same spirit, *Local Maximum* and *Global Maximum* can be defined respectively by changing the inequality from $\leq$ to $\geq$.

4.3 Derivatives and Gradient

Reminder:

The *first derivative* $f'(x)$ of function f of one variable at point x is defined as:

$$\lim_{a \to 0} \frac{f(x+a) - f(x)}{a} = f'(x).$$

It denotes the value of the slope of the tangent line in the graph of $f(x)$ at point x or the *rate of change* of the value of f at point x.

Remark: The slope is the $tan\phi$, where ϕ is the angle that the tangent line above makes with the horizontal axis.

For a function $f : \mathcal{R}^n \to \mathcal{R}$ of many variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, the *partial derivative* of f with respect to x_i at point $\mathbf{x}$ is defined as:

$$\lim_{a \to 0} \frac{f(\mathbf{x} + a\mathbf{e}_i) - f(\mathbf{x})}{a}$$

and denoted as $\partial f(\mathbf{x})/\partial x_i$, where $\mathbf{e}_i$ is the vector with 1 in the i-th position and 0 otherwise.

For a function $f : \mathcal{R}^n \to \mathcal{R}$ of several variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, the *gradient* of $f(\cdot)$ at point $\mathbf{x}$, denoted as grad$f(\mathbf{x})$ or $\nabla f(\mathbf{x})$, is defined as the $n \times 1$ vector:

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}) \\ \frac{\partial f}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix}.$$

The gradient can be defined at point $\mathbf{x}$ only if partial derivatives of $f(\cdot)$ which respect to all variables exist at $\mathbf{x}$.

The i-th component of $\nabla f(\mathbf{x})$ denotes the rate of variation of the value of $f(\cdot)$ at point $\mathbf{x}$ when only variable x_i changes and the others remain fixed.

Example: For $f(\mathbf{x}) = x^2 + y^2$, calculate the gradient $\nabla f(\mathbf{x})$ at point $\mathbf{x_0} = (1, 2)$.

$$\nabla f(1,2) = \begin{pmatrix} 2x \\ 2y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

4.4 Hessian Matrix

The *Hessian Matrix* of a function of several variables $f(\cdot) : \mathcal{R}^n \rightarrow \mathcal{R}$ at point $\mathbf{x}$ is defined as:

$$\nabla^2 f(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{x}) & \dots & \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{x}) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_2^2}(\mathbf{x}) & \dots & \frac{\partial^2 f}{\partial x_n \partial x_2}(\mathbf{x}) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_2 \partial x_n}(\mathbf{x}) & \dots & \frac{\partial^2 f}{\partial x_n^2}(\mathbf{x}) \end{pmatrix}$$

also denoted by $F(\mathbf{x})$.

If $f(\cdot)$ has continuous second derivatives, then the Hessian Matrix is symmetric, i.e. $\partial^2 f / \partial x_i \partial x_j = \partial^2 f / \partial x_j \partial x_i$, for $i \neq j$.

Properties: Given a $N \times 1$ vector $\mathbf{x}$ a $N \times N$ matrix $\mathbf{A}$, the following properties hold:

1. $\nabla(\mathbf{x}^T\mathbf{A}\mathbf{x}) = (\mathbf{A} + \mathbf{A}^T)\mathbf{x}$, and if $\mathbf{A}$ is symmetric matrix $(\mathbf{A} = \mathbf{A}^T)$, then $\nabla(\mathbf{A} + \mathbf{A}^T)\mathbf{x} = 2\mathbf{A}\mathbf{x}$.
2. $\nabla(\mathbf{y}^T\mathbf{x}) = \mathbf{y}$.
3. $\nabla(\mathbf{x}^T\mathbf{x}) = 2\mathbf{x}$.
4. $\nabla(\mathbf{y}^T\mathbf{A}\mathbf{x}) = \mathbf{A}^T\mathbf{y}$

For example suppose $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2)$ and $\mathbf{A}$ is 2×2 matrix , then we have:

$$\nabla(x_1y_1 + x_2y_2) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathbf{y}.$$

$$\nabla(x_1^2 + x_2^2) = 2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2\mathbf{x}.$$

4.5 Taylor's expansion formula

Let $f : \mathcal{R}^n \to \mathcal{R}$ be twice continuously differentiable function, then for all $\mathbf{x}, \mathbf{x}_0$, Taylor's expansion formula of $f(\mathbf{x})$ around point $\mathbf{x}_0$ gives

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \nabla^T(f(\mathbf{x}_0))(\mathbf{x} - \mathbf{x}_0)$$

$$+\frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^T\nabla^2 f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + O(\|\mathbf{x} - \mathbf{x}_0\|^2)$$

or

$$f(\mathbf{x} + \Delta\mathbf{x}) = f(\mathbf{x}) + \nabla^T(f(\mathbf{x}))\Delta\mathbf{x}$$

$$+\frac{1}{2}\Delta\mathbf{x}^T\nabla^2 f(\mathbf{x})\Delta\mathbf{x} + O(\|\Delta\mathbf{x}\|^2)$$

where $\Delta\mathbf{x}$ is an infinitesimal change in $\mathbf{x}$. For functions of one variable, we get the known expressions:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)(x - x_0)^2$$

.

4.6 Square matrices and eigenvalues

1. The *characteristic polynomial* ϕ of an $n \times n$ matrix A is defined as $\phi(\lambda) = \det(\lambda I - A)$, where I is the identity matrix.

2. The n (possibly repeated, and, in general, complex) roots of $\phi(\lambda)$ are called the *eigenvalues* of A.

3. A vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$, where λ is an eigenvalue of A, is called an *eigenvector* of A corresponding to λ.

4. The eigenvalues of a triangular matrix are equal to its diagonal entries.

5. The eigenvalues of $cI + A$, $c \in \mathcal{C}$, are $c+\lambda_1, \ldots, c+\lambda_n$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A.

6. If A is non-singular, the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A.

7. The eigenvalues of A and A^T coincide.

8. The *spectral radius*, $\rho(A)$ of a square matrix A is the maximum (in absolute value) of the magnitudes of its eigenvalues.

4.7 Symmetric and positive definite matrices

1. A symmetric matrix A has real eigenvalues and n mutually orthogonal, real and nonzero eigenvectors.

2. Suppose that the eigenvectors of A are normalized, $\|\mathbf{x}_i\| = 1$ for $i = 1, \ldots, n$. Then $A = \sum_{i=1}^{n} \lambda_i \mathbf{x}_i \mathbf{x}_i^T$, where λ_i is the eigenvalue corresponding to $\mathbf{x}_i$. (Known as Singular Value Decomposition-SVD-)

3. A symmetric square matrix A is positive definite (symbolized as $A > 0$) in $\mathcal{R}^n$ if and only if the quadratic form
$\mathbf{x}^T A\mathbf{x} > 0 \; \forall \mathbf{x} \in \mathcal{R}^n$, with $\mathbf{x} \neq \mathbf{0}$.

4. A is called *positive semi-definite* if and only if
$\mathbf{x}^T A\mathbf{x} \geq 0 \; \forall \mathbf{x} \in \mathcal{R}^n$.

5. A square symmetric matrix $\mathbf{A} > 0$, if and only if all its eigenvalues are positive.

5 Convex functions

A function $f : \Omega \rightarrow \mathcal{R}, \Omega \subseteq \mathcal{R}^n$ is *convex* if and only if

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta)f(\mathbf{y}), \qquad (15)$$

$\forall \mathbf{x}, \mathbf{y} \in \mathcal{R}^n$ and $\theta \in [0, 1]$.

A convex function $f(\mathbf{x})$ of one variable x is shown in Figure

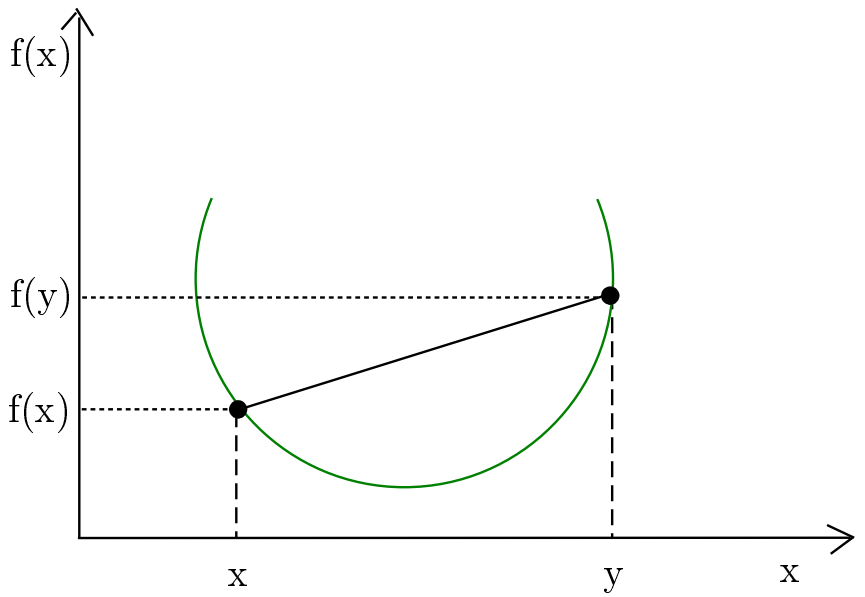


Figure 6: A convex function of one variable, $f(x)$.

That is, given any two points $\mathbf{x}, \mathbf{y} \in \Omega$ with values $f(\mathbf{x})$ and $f(\mathbf{y})$ respectively, the chord between points $(\mathbf{x}, f(\mathbf{x}))$ and $(\mathbf{y}, f(\mathbf{y}))$ lies above the graph of f.

5.1 Concave functions

A function $f : \Omega \to \mathcal{R}$ is *concave* if and only if

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \geq \theta f(\mathbf{x}) + (1-\theta)f(\mathbf{y}), \qquad (16)$$

$\forall \mathbf{x}, \mathbf{y} \in \mathcal{R}^n$ and $\theta \in [0, 1]$.

The schematic representation of a concave function $f(\mathbf{x})$ of one variable is shown below:

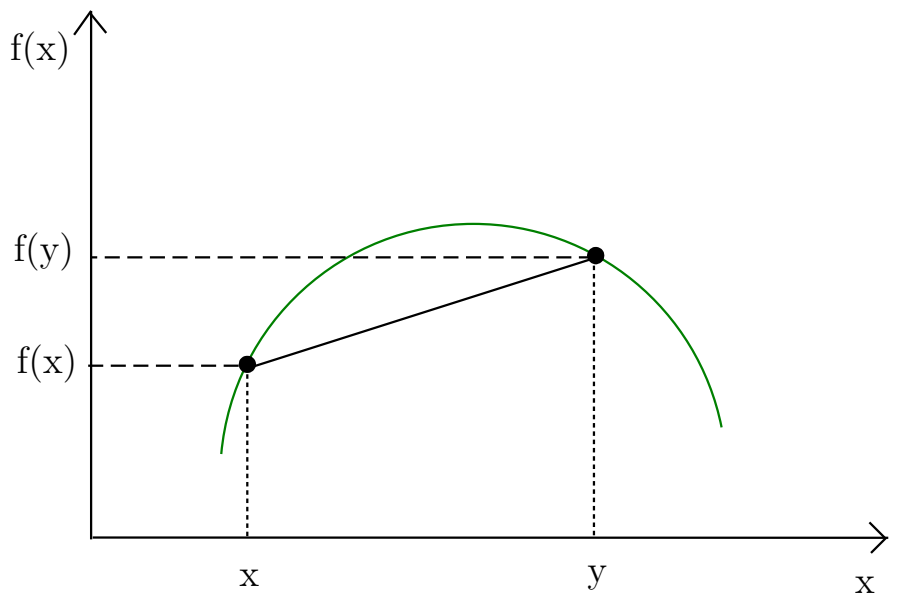


Figure 7: A concave function of one variable, $f(x)$.

That is, given any 2 points $\mathbf{x}, \mathbf{y} \in \Omega$ with values $f(\mathbf{x}), f(\mathbf{y})$ respectively, the chord between points $(\mathbf{x}, f(\mathbf{x}))$ and $(\mathbf{y}, f(\mathbf{y}))$ lies below the graph of f.

Remarks:

- A linear function is both convex and concave, since

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) = \theta f(\mathbf{x}) + (1-\theta)f(\mathbf{y}),$$

$\forall \mathbf{x}, \mathbf{y} \in \mathcal{R}^n$ and $\theta \in [0, 1]$.

- f is concave if and only if $(-f)$ is convex.

- For a convex function f,

$$f\left(\sum_{i=1}^{n} a_i \mathbf{x}_i\right) \leq \sum_{i=1}^{n} a_i f(\mathbf{x}_i)$$

for $a_1, \ldots, a_n \geq 0$ with $\sum_{i=1}^{n} a_i = 1$.

- The weighted sum of convex functions with positive weights is convex function.

5.2 Convexity Conditions

5.2.1 First-order convexity conditions

Suppose the first-order derivatives of $f(\mathbf{x})$ exist. Then f is *convex* at Ω if and only if

$$f(\mathbf{x}) \geq f(\mathbf{x}_0) + \nabla^T f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$$

$\forall \mathbf{x}, \mathbf{x}_0 \in \Omega \subseteq \mathcal{R}^n$.

For functions of one variable, x, the equation for the tangent line of f at point x_0 is $f(x_0) + f'(x_0)(x - x_0)$. For many dimensions, the equation for the tangent plane of f at point $\mathbf{x}_0$ is $f(\mathbf{x}_0) + \nabla^T f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$.

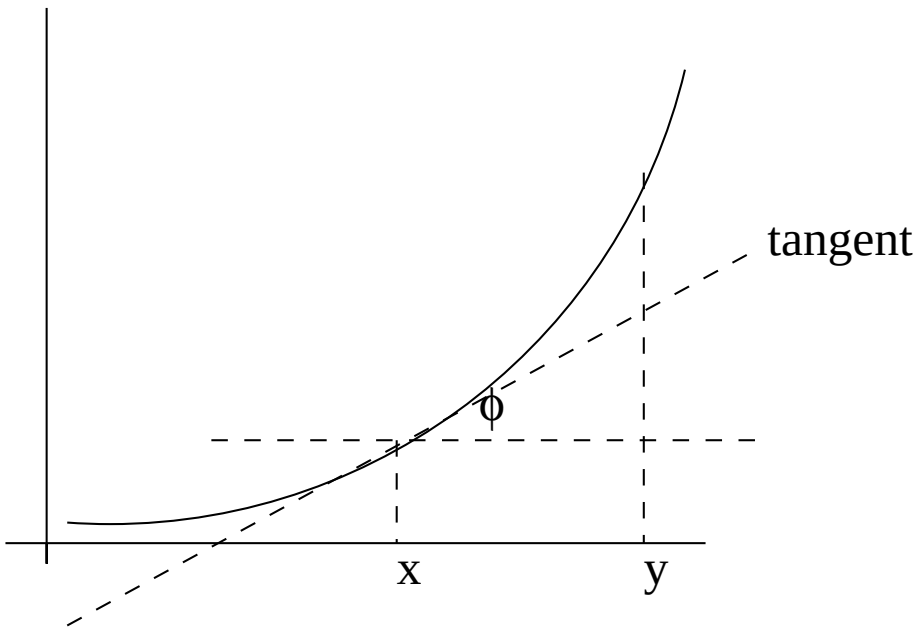


Figure 8: Demonstration of first-order convexity condition.

Note : We can say that, for convex functions, the first-order Taylor series linear approximation of the function value at every point *underestimates* (is a lower bound to) the value of the function at that point.

In the same spirit, f is *concave* at Ω if and only if

$$f(\mathbf{x}) \leq f(\mathbf{x}_0) + \nabla^T f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$$

$\forall \mathbf{x}, \mathbf{x}_0 \in \Omega \subseteq \mathcal{R}^n$.

5.2.2 Second-order Convexity Conditions

Suppose that f, defined on $\Omega \subseteq \mathcal{R}^n$, is twice differentiable (all second derivatives exist). If the Hessian matrix of f is positive semi-definite, i.e

$$\nabla^2 f(\mathbf{x}) \geq 0 \ \forall \mathbf{x} \in \Omega \subseteq \mathcal{R}^n$$

then f is convex in Ω.

If $\nabla^2 f(\mathbf{x})$ is positive-definite, then f is called *strictly convex*.

The inverse condition holds with the following difference: If Ω is an *open* set and f is convex function, then $\nabla^2 f(\mathbf{x})$ is positive semi-definite ($\nabla^2 f(\mathbf{x}) \geq 0$), for all $\mathbf{x} \in \Omega$.

Remark: If $f(\mathbf{x}) = \mathbf{x}^T Q \mathbf{x}$ where Q is symmetric matrix, then f is convex if and only if Q is positive semi-definite.

A function f is concave in Ω if $\nabla^2 f(\mathbf{x}) \leq 0 \ \forall \mathbf{x} \in \Omega \subseteq \mathcal{R}^n$.

If function f depends on *only one variable* the convexity condition becomes: if $f''(x) \geq 0$, then f is convex.

5.3 Global and local minima of convex functions

If f is a convex function, defined on a convex set Ω, then any local minimum of f is global minimum.

Proof: Suppose that x is a local minimum, but not a global minimum. Then there must exist a $y \neq x$ such that

$f(y) < f(x)$. Using the convexity of f:

$$f(ax + (1-a)y) \leq af(x) + (1-a)f(y) < f(x)$$

and this contradicts the fact that x is local minimum.

An equivalent proof: Since x is local minimum, $\nabla f(x) = 0$ (as we will see later). Then from the first-order convexity condition: $f(y) \geq f(x), \forall y \in \mathcal{R}^n$, so x is global optimum (minimum).

Also, if f is a concave function, defined on a convex set Ω, then any local maximum of f is also a global maximum.

5.4 Examples of convex and concave functions (one variable)

1. $f(x) = e^{\alpha x}$ is convex on $\mathcal{R}, \forall \alpha \in \mathcal{R}$.

2. $$f(x) = x^{\alpha} = \begin{cases} \text{convex on } R_+ \text{ if } a \geq 1 \text{ and } a \leq 0, \\ \text{concave if } a \in [0,1] \end{cases}$$

3. $$f(x) = x^{\alpha} = \begin{cases} \text{concave on } R_- \text{ if } a \geq 1 \text{ and } a \leq 0, \\ \text{convex if } a \in [0,1] \end{cases}$$

4.

$$f(x) = \log x \text{ is concave on } R_+.$$

The supremum and infimum of a set X are straightforward to define based on maximum and minimum.

5.5 Convex functions of several variables

- If J is an index set, and Ω a convex subset of $\mathcal{R}^n$, and functions $f_i : \Omega \to \mathcal{R}$, are convex, then the function:

  $$g(x) = \sup_{i \in J} f_i(x)$$

  is convex.

- $f(\mathbf{x}) = \sqrt{\mathbf{x}^T\mathbf{x}} = \left|\left|\mathbf{x}\right|\right|_2$ is a convex function of $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

- $f(\mathbf{x}) = \max\{x_1, x_2, \ldots, x_n\}$ is a convex function of $\mathbf{x} = (x_1, \ldots, x_n)$.

- $f(\mathbf{x}) = \log(e^{x_1} + \ldots + e^{x_n})$ is a convex function of $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ (although $f(\mathbf{x}) = \log(\mathbf{x})$ is a concave function of $\mathbf{x}$ in $\mathbb{R}_+$).

  Note: We will use this later to show that function

  $$f(\mathbf{P}) = \sum_{i=1}^{N} q_i \log \frac{G_{ii}P_i}{\sum_{j \neq i} G_{ji}P_j} = \sum_{i=1}^{N} q_i \log(\text{SIR}_i(\mathbf{P}) \tag{17}$$

is a concave function of $\mathbf{P}$, where SIR is the signal-to-interference ratio.

- $f(\mathbf{x}) = \left(\prod_{i=1}^{n} x_i \right)^{1/n}$ is a concave function of $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}_+$.

5.6 Jensen's Inequality

If f is a convex function, then

$$f\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) \leq \frac{f(\mathbf{x})+f(\mathbf{y})}{2}$$

If f is a convex function and X is a random variable, we can generalize the above to show that $f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$.

Sketch of Jensen's Inequality: First, we use the definition of convex function f for two points x, y :
$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$.

Then extend the inequality to more points: $x_1, x_2, \ldots, x_k$ as

$$f(\theta_1 x_1 + \ldots + \theta_k x_k) \leq \theta_1 f(x_1) + \ldots + \theta_k f(x_k), \quad (18)$$

where $\sum_{i=1}^{k} \theta_i = 1, \ \forall\, \theta_i \geq 0$ (convex combination of k discrete points).

Considering that $\int_S p(x)dx = 1$, where S is the set of points where f is defined, the inequality for continuous set of points becomes

$$f\left(\int_S p(x)xdx\right) \leq \int_S f(x)p(x)dx. \quad (19)$$

Thus the probability distribution $p(\cdot)$ is "similar" to a continuous distribution of θ's and declares the convex combinations. Eventually, we conclude that $f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$.

Notation:

$$\mathbf{x}^* = \arg\min_{\mathbf{x}\in\Omega} f(x) \Longleftrightarrow f(\mathbf{x}^*) \leq f(\mathbf{x}) \forall \mathbf{x} \in \Omega. \quad (20)$$

("arg" stands for argument of a function)

6 Example: Notion of the Utility function

The utility function $U(x)$ quantifies the amount of satisfaction from or the desirability of consumption of x amount of goods or resources. In networking, such resources may be the link or devoted service bandwidth or the allocated power or storage. The notion of utility function is central in resource allocation. The following example is from [1].

Suppose that a central authority has a divisible good of size C to be divided among N different users. For example, the

government has determined that a fixed quantity of ground water may be pumped in a certain region and would like to allocate quotas that may be pumped by different farms.One way of performing the allocation is to simply divide the resource into N parts and allocate C/N to each player. But such a scheme does not take into account the fact that each player might value the good differently. In our example, based on the type of crops being grown, the value of pumping a certain quantity water might be different for different farms. We refer to the value or utility obtained from an allocation x as $U(x)$. This utility is measured in any denomination common to all the players such as dollars.

What would be the properties of a utility function? It would be increasing in the amount of resource obtained. We might also expect that a law of diminishing returns applies. In our example of water resources, the return obtained by increasing the quota from 10 units to 20 units would make a large difference in the crop obtained, but an increase from 100 units to 110 units would not make such a significant difference. Such a law of diminishing returns is modeled by specifying that the utility function is a strictly concave function since the second derivative of a strictly concave function is negative. Thus, the first derivative (which is the rate at which the function increases) decreases.

The objective of the authority would be to maximize the system-wide utility. One commonly used measure of system-

wide utility is the sum of the utilities of all the players. Since the utility of each player can be thought of the happiness that he/she obtains, the objective of the central authority can be likened to maximizing the total happiness in the system, subject to the constraint that resources are fixed.

Remark: A similar example holds for the problem of dividing a pie of fixed amount C among two individuals, one poor and one rich. The poor will valuate a given portion x of the resource differently than the rich one. That means the different users to which the pie is to be allocated have different utility functions. That is, $U_1(x) \neq U_2(x), \forall x$.

6.1 Concave utility function

Given $U(x) = \log x$, its second derivative, $U''(x) = -\frac{1}{x^2} < 0, \forall x > 0$. Thus, U is concave function. The $\log$-function is useful in defining the utility function.

The first derivative in a point x of the curve gives the slope of the tangent line at x. Since $U''(x) < 0$, then $f'(x)$ is decreasing $(\downarrow)$. So, as x increases, the rate of increase of utility, dU/dx, decreases (this attribute holds only for concave functions) as figure 6. shows. For large enough values of power, the capacity is "saturated", in the sense that there are no marginal returns. Thus, the higher the power, the smaller the rate of increase of capacity.

In wireless networks, the first example of utility is capacity

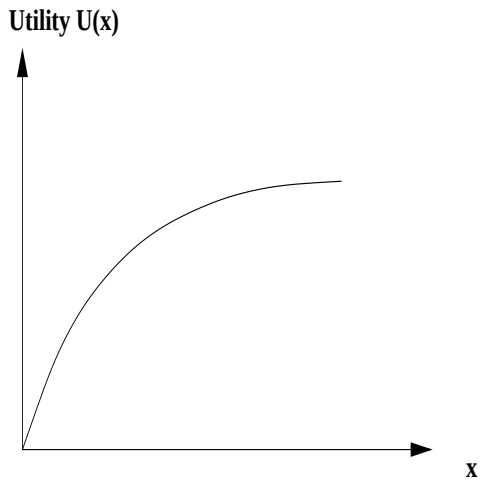


Figure 9: Concave utility function.

which depends on the amount of allocated power. The information-theoretic definition of *capacity* of a link is the largest number of bits per second that can be transmitted over a link with arbitrarily small probability of error,

$$C(P) = \log(1 + \mathrm{SNR}) = \log(1 + \frac{P}{N}), \qquad (21)$$

where P is the transmit power and N is the noise power. Function $C(P)$ is a concave function of P.

We have the following approximations:

- For large power P, $C(P) = \log(1 + \frac{P}{N}) \approx \log \frac{P}{N}$
- For small P, $C(P) \approx \frac{P}{N}$, since $\log(1 + x) \approx x$ for small x.

Another example is link bandwidth W dedicated to a user flow out of the ones traversing a link of total capacity C.

6.2 Convex utility function

Convex utility functions may quantify the amount of *cost* obtained as a function of the amount of allocated resource.

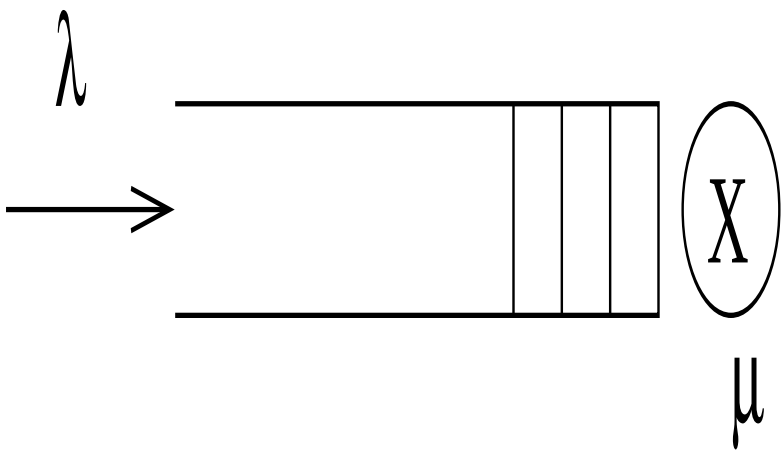


Figure 10: An M/M/1 queue.

A very important example of convex cost function arises in an M/M/1 queue. Let λ be the average customer arrival rate and μ be the customer service rate (both in customers/sec).

¿From Little's theorem, the average number of customers, $\mathbb{E}[N]$ in the queue is $\mathbb{E}[N] = \lambda \cdot \mathbb{E}[T]$, where $\mathbb{E}[T]$ is the average waiting time in the queue for a customer, $\mathbb{E}[T] = \frac{1}{\mu - \lambda}$ for an M/M/1 queue.

Is function $\mathbb{E}[N] = \frac{\lambda}{\mu - \lambda}$ *convex or concave ?* For fixed λ, $\mathbb{E}[N]$ is decreasing since $\mathbb{E}[N]' < 0$ and $\mathbb{N}''(\mu) = 2\lambda/(\mu - \lambda)^3 > 0$. Thus, for fixed λ, function $\mathbb{E}[N]$ is convex decreasing in μ.

For fixed μ, the function is increasing, since $\mathbb{N}'(\lambda) > 0$. Also, $N''(\lambda) = 2\mu/(\mu - \lambda)^3 > 0$, thus $\mathbb{N}(\cdot)$ is convex increasing in λ.

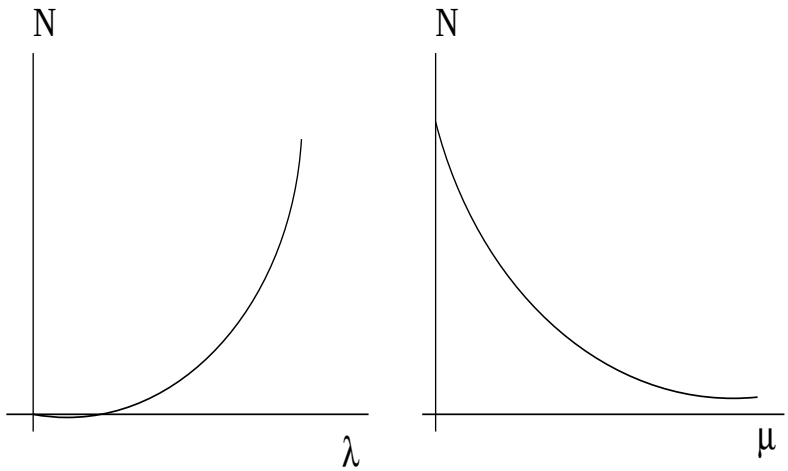


Figure 11: $N(\cdot)$ is convex increasing in λ and convex decreasing in μ.

7 Numerical methods for one-dimensional minimization

In this section, we provide numerical methods for finding the minimum of functions of one variable.

7.1 Bisection method

- Given a differentiable function $f : \Re \rightarrow \Re$, a point $x^\star$ that minimizes $f(x)$ has the property that $f'(x^\star) = 0$.

  Define $g(x) = f'(x)$. Then, we need to find a point $x^\star$ such that $g(x^\star) = 0$, i.e., a root of $g(\cdot)$. To find $x^\star$, we can use the *bisection* method.

  Pseudo-algorithm:

  STEP 1: Find two points a, b such that $g(a) \cdot g(b) < 0$ ($g(a)$, $g(b)$ should have opposite signs. Otherwise, function $g(\cdot)$ is increasing, or decreasing and the minimum within interval $[a, b]$ coincides with one of the two end points of the interval).

  STEP 2: Go to bisection point $y = \frac{b-a}{2} + a$.

  STEP 3:
  (a) If $g(a) \cdot g(y) < 0$, then set $b = y$,
  (b) If $g(y) \cdot g(b) < 0$, then set $a = y$,
  (c) If $g(y) \cdot g(a) = 0$ or $g(y) \cdot g(b) = 0$, then y is the minimum. **STOP**.
  STEP 4: If $|b - a| < \delta$ (where $\delta << 1$) **STOP**.

  Note: If we search for minimum in a closed interval $[c, d]$, the minimum is either c or d, depending on whether f is increasing or decreasing.

  In the algorithm above, the root is either found (hit) exactly, or the searching range constantly gets

narrower, until the minimizer is found with some pre-specified accuracy δ.

The first step of the algorithm is depicted in figure 8.

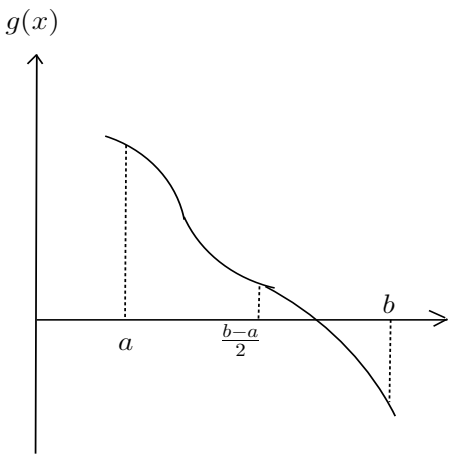


Figure 12: The first step of bisection method.

Number of iterations: At most $\log_2\left(\frac{|b-a|}{\delta}\right)$ iterations to find the minimum. Complexity depends on the initial search range $[a, b]$ and the error tolerance parameter δ.

7.2 Unimodal function method

Here we attempt to minimize function $f : \Re \rightarrow \Re$ directly, (namely without finding the root of f').

Uni-modal function: A function $f : \Re \rightarrow \Re$ defined in a closed interval $[a, b] \subset \Re$ is *unimodal* in $[a, b]$ if f has

only one local minimizer in $[a, b]$. Specifically, f is unimodal if, given $x^* \in [a, b]$, f is increasing for $x \geq x^*$ and decreasing for $x \leq x^*$, $x \in [a, b]$.

Conditions satisfied by uni-modal functions: Given points x_1, x_2 such that $a \leq x_1 < x_2 \leq b$, there exists a point x^* that:

(a) If $x_1 > x^*$, then $f(x_1) < f(x_2)$. Thus, f is decreasing, as x moves from x_1 towards x^* (Figure 9).

(b) If $x_2 < x^*$, then $f(x_1) > f(x_2)$. Thus, f is increasing, as x moves from x^* towards x_2 (Figure 10).

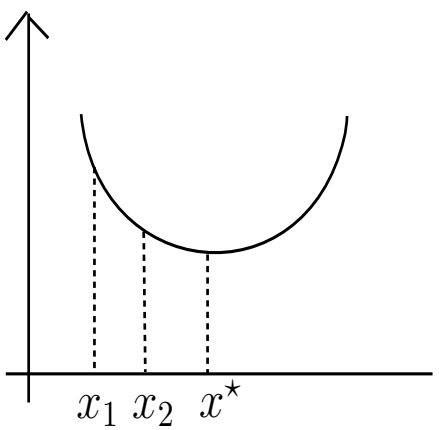


Figure 13: Conditions satisfied by a unimodal function.

The bisection method cannot be used in this case, because we cannot conclude about the sign of intermediate point. For example, f may have one of the two forms shown in figure 11 and the minimum can be either in the first interval or in the

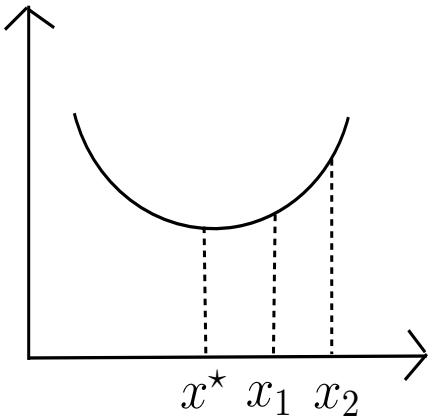


Figure 14: Conditions satisfied by a unimodal function.

second interval.

We proceed as follows: We find two points $\hat{x}_1, \hat{x}_2 \in [a, b]$ such that $a < \hat{x}_1 < \hat{x}_2 < b$ and we find $f(\hat{x}_1), f(\hat{x}_2)$. We can distinguish the following cases:

1. If $f(a) > f(\hat{x}_1) > f(\hat{x}_2)$, the interval $[a, \hat{x}_1]$ is excluded, as the minimum cannot be in that interval. Thus, we set the right point of search interval $\hat{x}_1 \leftarrow a$. The minimum should be somewhere in $[\hat{x_1}, b]$. This case is shown is figure 12.

2. If $f(b) > f(\hat{x}_2) > f(\hat{x}_1)$, the interval $[\hat{x}_2, b]$ is being excluded, as the minimum cannot be in that interval. The minimum should be somewhere in $[a, \hat{x}_2]$. Thus, we set the left point of search interval $\hat{x}_2 \leftarrow b$. This case is shown is figure 13.

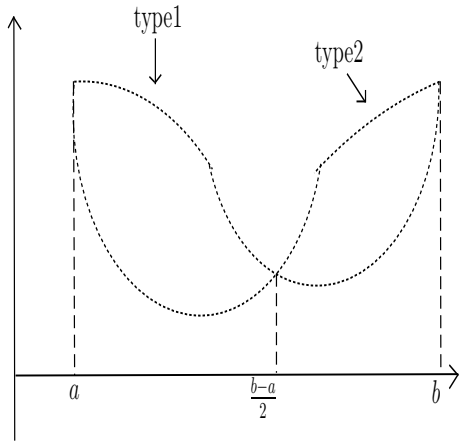


Figure 15: Bisection method cannot give any hint on where the minimum is.

3. Proceed in that fashion, until we find minimum x^* with some accuracy.

How do we choose the points $\hat{x}_1, \hat{x}_2$ at each iteration? We need to choose them so that the search interval (and thus the number of iterations) at each step are reduced fast. The following methods exist for choosing $\hat{x}_1, \hat{x}_2$: Golden ratio search and Fibbonacci search.

7.3 Newton's method

Newton's method is an iterative method that uses the second derivative of f. Thus, f needs to be twice differentiable.

Given a function $f(x)$ and a point $x^{(k)}$ the idea is to ap-

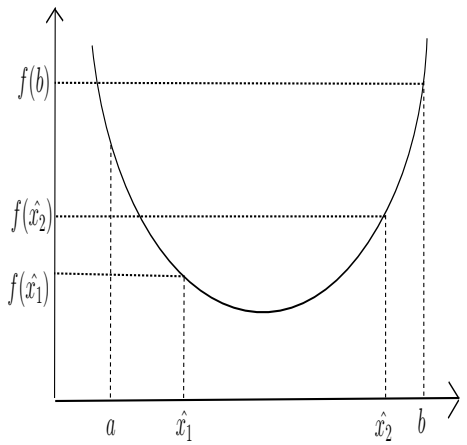


Figure 16: Case 1 in the method of directly finding the minimum.

proximate $f(x)$ with a quadratic function, namely a second degree polynomial, $q(x)$. Instead of minimizing f we then minimize its approximation $q(x)$,

$$q(x) = f(x^{(k)}) + f'(x^{(k)})(x - x^{(k)}) + \frac{1}{2} f''(x^{(k)})(x - x^{(k)})^2 \tag{22}$$

Note that polynomial $q(x)$ is chosen so as to satisfy:

$$q(x^{(k)}) = f(x^{(k)})$$

$$q'(x^{(k)}) = f'(x^{(k)})$$

$$q''(x^{(k)}) = f''(x^{(k)}).$$

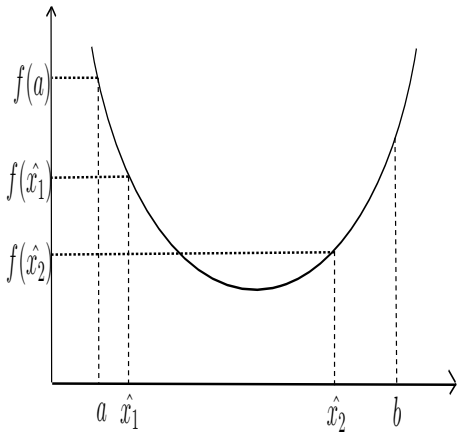


Figure 17: Case 2 in the method of directly finding the minimum.

By minimizing function $q(\cdot)$ we get: $q'(x) = 0 \Rightarrow f'(x^{(k)}) + f''(x^{(k)})(x - x^{(k)}) = 0$. Solving this equation to find the x that minimizes f,we get:

$$x = x^{(k)} - \frac{f'(x^{(k)})}{f''(x^{(k)})} \tag{23}$$

By setting the next point to be the minimizing x, i.e, $x \leftarrow x^{(k+1)}$ the equation above becomes:

$$x^{(k+1)} = x^{(k)} - \frac{f'(x^{(k)})}{f''(x^{(k)})} \tag{24}$$

and gives the form of iteration of Newton method.

As we will see later, Newton method belongs to the class of gradient methods.

- The iteration starts from an initial point x_0 and terminates either if $f'(x^{(k)}) = 0$ (in which case $x^{(k+n)} = x^{(k)}$, for $n > 0$), or if $|x^{(k+1)} - x^{(k)}| < \varepsilon$.

- In stopping condition $|x^{(k+1)} - x^{(k)}| < \varepsilon$, there exists a tradeoff: if ε is too small, then the minimum is found with more accuracy, but it takes more iterations to compute. If ε is larger, then the minimum is found faster but the error may be large.

- For functions of several variables, we will see that the Newton iteration becomes:

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \left[\nabla^2 f(x^{(k)})\right]^{-1} \nabla f(\mathbf{x}^{(k)}) \quad (25)$$

  where $\nabla^2 f(\mathbf{x}^{(k)})$ is the Hessian matrix of f at point $\mathbf{x}^{(k)}$.

8 Optimality conditions

Remark: Necessary and sufficient condition:

N is necessary condition for A, or $A \Rightarrow N$.

S is sufficient condition for A, or $S \Rightarrow A$.

8.1 Necessary Optimality Conditions

8.1.1 Condition on the Gradient

Let $\mathbf{x}^*$ be an unconstrained local minimum of $f : \mathcal{R}^n \to \mathcal{R}$. Then

$$\nabla f(\mathbf{x}^*) = \mathbf{0}.$$

Proof: From the Taylor expansion of first order around local minimum $\mathbf{x}^*$, we have that for small variations $\Delta\mathbf{x}$ around $\mathbf{x}^*$:

$$f(\mathbf{x}^* + \Delta\mathbf{x}) - f(\mathbf{x}^*) \approx \nabla^T f(\mathbf{x}^*)\Delta\mathbf{x}\,.$$

This means that:

$$\sum_{i=1}^{N} \frac{\partial f(\mathbf{x}^*)}{\partial x_i}\Delta x_i \geq 0 \tag{26}$$

for all $\Delta\mathbf{x} = (\Delta x_1, \ldots, \Delta x_N)$. Clearly, the value of f increases wherever we move to, since currently we are at $\mathbf{x}^*$, the local minimum.

Take $\Delta\mathbf{x}$ to be positive and negative multiples of the unit coordinate vectors $(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)$. The positive and negative ones are:

$$\Delta\mathbf{x} = \pm\varepsilon(1, 0, \ldots, 0)$$

$$\Delta\mathbf{x} = \pm\varepsilon(0, 1, \ldots, 0)$$

$$\vdots$$

$$\Delta\mathbf{x} = \pm\varepsilon(0, 0, \ldots, 1)$$

¿From equation above, we have for example for $\Delta\mathbf{x} = (1, 0, \ldots, 0)$:

$$\frac{\partial f(\mathbf{x}^*)}{\partial x_1}\varepsilon \geq 0 \text{ and } \frac{\partial f(\mathbf{x}^*)}{\partial x_1}\varepsilon \leq 0. \tag{27}$$

Thus, $\frac{\partial f(\mathbf{x}^*)}{\partial x_1}\varepsilon = 0$ and $\varepsilon > 0$. Similarly for the rest. So finally, $\frac{\partial f(\mathbf{x}^*)}{\partial x_i} = 0 \, \forall i$ or

$$\nabla f(\mathbf{x}^*) = \mathbf{0}$$

8.1.2 Condition on the Hessian matrix

If $\mathbf{x}^*$ is an unconstrained local minimum of f,
then $\nabla^2 f(\mathbf{x}^*) \geq 0$ (assuming that f has all partial second derivatives at $\mathbf{x}^*$). Equivalently, if $\mathbf{x}^*$ is a local minimum, then the Hessian matrix of f at $\mathbf{x}^*$, $\nabla^2 f(\mathbf{x}^*)$ is positive semi-definite.

Proof: From the Second Order Taylor series expansion of f around $\mathbf{x}^*$:

$$f(\mathbf{x}+\Delta\mathbf{x}) - f(\mathbf{x}) \approx \nabla^T f(\mathbf{x})\Delta\mathbf{x} + \frac{1}{2}(\Delta\mathbf{x})^T \nabla^2 f(\mathbf{x})(\Delta\mathbf{x})$$

.

If $\mathbf{x}^*$ is a local minimum, then the difference above is non-negative (≥ 0). Since $\nabla^T f(\mathbf{x}^*) = 0$, the above becomes:

$$f(\mathbf{x} + \Delta\mathbf{x}) - f(\mathbf{x}) \approx \frac{1}{2}(\Delta\mathbf{x})^T \nabla^2 f(\mathbf{x})(\Delta\mathbf{x}) \geq 0 \, \forall \Delta\mathbf{x}.$$

which implies that $\nabla^2 f(\mathbf{x}^*)$ is positive semi-definite.

Similar necessary conditions hold for local maximum.

8.2 Sufficient Optimality Conditions

Let $f : \mathcal{R}^n \rightarrow \mathcal{R}^n$ be twice continuously differentiable. If $\mathbf{x}^*$ satisfies $\nabla f(\mathbf{x}^*) = 0$ and $\nabla^2 f(\mathbf{x}^*) > 0$ (positive definite), then $\mathbf{x}^*$ is a local minimum of f.

Similarly, for a local maximum, we can prove that if $\mathbf{x}^* \in \Omega$ satisfies $\nabla f(\mathbf{x}^*) = 0$ and $\nabla^2 f(\mathbf{x}^*) < 0$ (negative definite), then $\mathbf{x}^*$ is a local maximum of f.

8.3 The case of convex / concave function

Recall that if f is convex function, every local minimum is also global minimum.

For f convex, the condition $\nabla f(\mathbf{x}^*) = 0$ is necessary and sufficient for optimality. Indeed, from a basic property of a convex function:

$$f(\mathbf{x}) \geq f(\mathbf{x}^*) + \nabla^T f(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*)$$

So, if $\nabla f(\mathbf{x}) = \mathbf{0}$, then $f(\mathbf{x}) \geq \mathbf{x}^*)$, for all $\mathbf{x}$ and $\mathbf{x}^*$ is global minimum.

If f is a concave function every local maximum is also global maximum and the condition $\nabla f(\mathbf{x}^*) = 0$ is necessary and sufficient for optimality.

Exercise: Given a set of vectors $\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(p)}\}$, with $\mathbf{x}^{(i)} \in \mathbb{R}^n$ for $i = 1, \ldots, p$, find vector $\mathbf{x} \in \mathcal{R}^n$ such that the average squared distance (norm) between $\mathbf{x}$ and all $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$,

$$\frac{1}{p} \sum_{i=1}^{p} \left|\left|\mathbf{x} - \mathbf{x}^{(i)}\right|\right|^2 \tag{28}$$

is minimized. Is the local minimum a global minimum?

Solution:

$$\mathbf{x}^* = \frac{1}{p} \sum_{i=1}^{p} \mathbf{x}^{(i)}. \tag{29}$$

References

[1] S. Shakkottai and R. Srikant, ”Network Optimization and Control”, Foundations and Trends in Networking, NoW Publishers, 2007.