Lecture 2

Notes by Konstantinos Poularakis

Example

Battle of sexes

George \ Mary	Opera	Fight
Opera	(1, 2)	(0, 0)
Fight	(0, 0)	(2, 1)

- George and Mary want to go out this evening. George prefers to watch a fight film of the cinema and Mary prefers to visit opera. They both prefer to go anywhere together rather than stay alone.
- The above table represents the utilities George and Mary obtain by each of the four possible outcomes due to their strategies
- The Nash equilibrium points are the following:

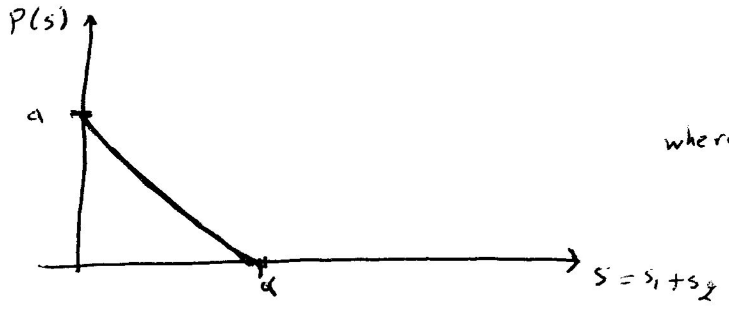
  (Opera, Opera) and (fight, fight)

Example

- 2 companies produce the same product. Each company wants to maximize it's personal benefit (utility).
- The strategy of each company is the amount of the product that it produces: $s_i \in [0, +\infty)$
- The utility function is of the form:

  utility = revenue − cost.
- The price per unit of product is a function of the total amount of the product that both companies produces:

  $p(s)$, where $s = s_1 + s_2$
- We assume that each company i will produce less than a parameter a units of the product.
- The function $p(s)$ is $p(s) = \max\{0, a - s\}$



where $0 \le s_1 < a$
$0 \le s_2 < a$

The utility function of each player (company) is

$$u_1(s_1, s_2) = s_1 \cdot p(s_1 + s_2) - c \cdot s_1$$
$$= s_1 \cdot (a - s_1 - s_2) - c\, s_1$$

where c is the cost of production of one unit of the product.

and $$u_2(s_1, s_2) = s_2\,(a - s_1 - s_2) - c\, s_2$$

a) Find the Nash Equilibrium point of the game: (s_1^*, s_2^*).

- To answer the previous question we have to define the best response function $Br_i(s_j)$, which represents the best way that player i have to response if the other player j does action s_j

$$s_1^* = Br_1(s_2) = \underset{0 \le s_1 \le a}{\arg\max}\; u_1(s_1, s_2)$$

and

$$s_2^* = Br_2(s_1) = \underset{0 \le s_2 \le a}{\arg\max}\; u_2(s_1, s_2)$$

- In order to define $Br_1(s_2)$ and $Br_2(s_1)$ we do the following:

$$\frac{\partial u_1(s_1, s_2)}{\partial s_1} = 0 \iff$$

$$\iff \frac{\partial}{\partial s_1}\left[s_1(a - s_1 - s_2) - c\, s_1 \right] = 0 \iff$$

$$\iff a - s_1 - s_2 - s_1 - c = 0 \iff$$

$$\iff \boxed{s_1^* = \frac{a - c - s_2}{2}} \quad (1)$$

Similarly,

$$\boxed{s_2^* = \frac{a - c - s_1}{2}} \quad (2)$$

- So, in order to find the Nash Equilibrium Point you can solve the system of (1)(2):

if you replace $s_2^* = \frac{a - c - s_1}{2}$ in (1)

you get: $$s_1^* = \frac{a - c - \frac{a - c - s_1^*}{2}}{2} = \frac{2a - 2c - a + c + s_1^*}{4} \iff$$

$$\iff 3 s_1^* = 2a - 2c - a + c \iff \boxed{s_1^* = \frac{a - c}{3}}$$

and by (2) $\Rightarrow \boxed{s_2^* = \frac{a - c}{3}}$

- The total utility at the Nash Equilibrium is:

$$u_1^*(s_1, s_2) + u_2^*(s_1, s_2) =$$

$$= s_1^*(a - s_1^* - s_2^* - c) + s_2(a - s_1^* - s_2^* - c) =$$

$$= (s_1^* + s_2^*)(a - c - s_1^* - s_2^*) =$$

$$= 2 \cdot \frac{a-c}{3} \left(a - c - 2\,\frac{a-c}{3}\right) =$$

$$= 2 \cdot \frac{(a-c)^2}{9}$$

b) Find the social optimum point of the system (s_1', s_2'):

- In order to find the social optimum point (s_1', s_2') we do:

$$\left.\begin{aligned} \frac{\partial}{\partial s_1}\left[u_1(s_1+s_2) + u_2(s_1+s_2)\right] &= 0 \\ \frac{\partial}{\partial s_2}\left[u_1(s_1+s_2) + u_2(s_1+s_2)\right] &= 0 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{aligned} a - c - s_1 - s_2 - s_1 - s_2 &= 0 \\ a - c - s_1 - s_2 - s_1 - s_2 &= 0 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \boxed{s_1' + s_2' = \frac{a-c}{2}}$$ (there are many social optimal points in the system).

- The total utility at the social optimal point system is:

$$u_1(s_1' + s_2') + u_2(s_1' + s_2') = \frac{(a-c)^2}{4}$$

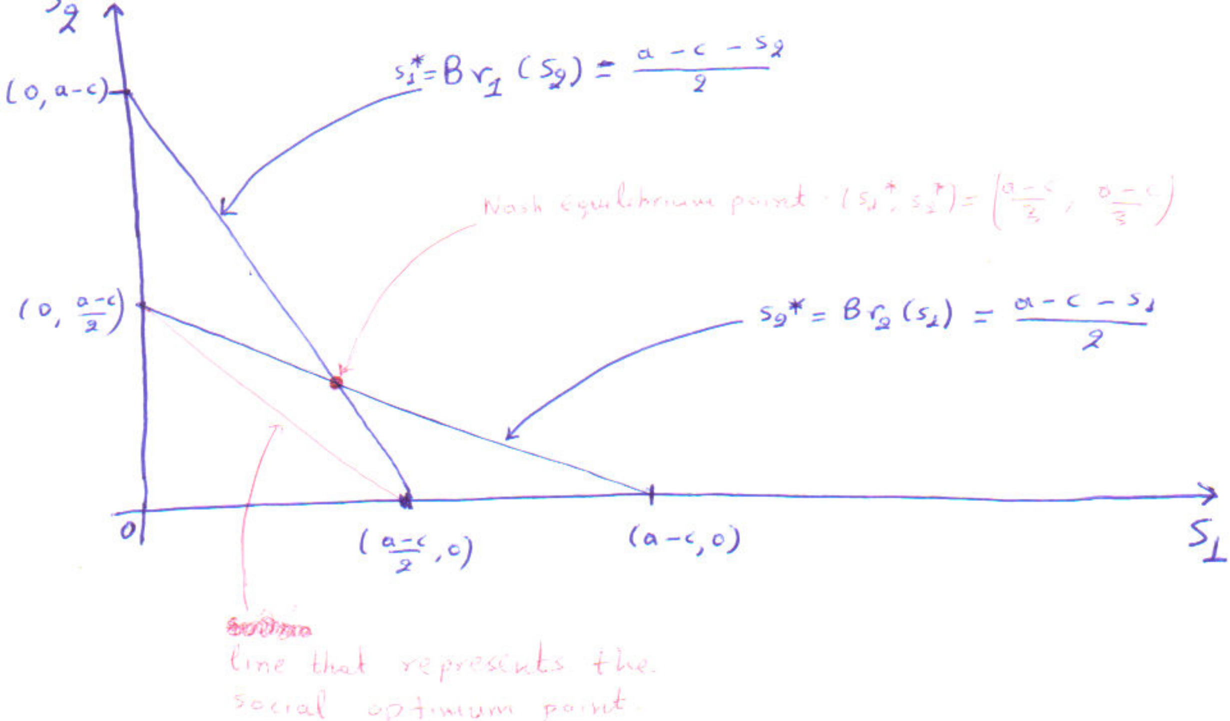
c) Define the Price of Anarchy (PoA):

$$PoA = \frac{u_{tot}(s_1', s_2')}{u_{tot}(s_1^*, s_2^*)} = \frac{\frac{(a-c)^2}{4}}{\frac{2(a-c)^2}{9}} = \frac{9}{8} > 1$$

d) If there is only one company, what is the best strategy to choose?

$$\frac{\partial}{\partial s_1} s_1(a - s_1 - c) = 0 \Leftrightarrow$$

$$\Leftrightarrow a - s_1 - c - s_1 = 0 \Rightarrow \boxed{s_1 = \frac{a-c}{2}}$$



example

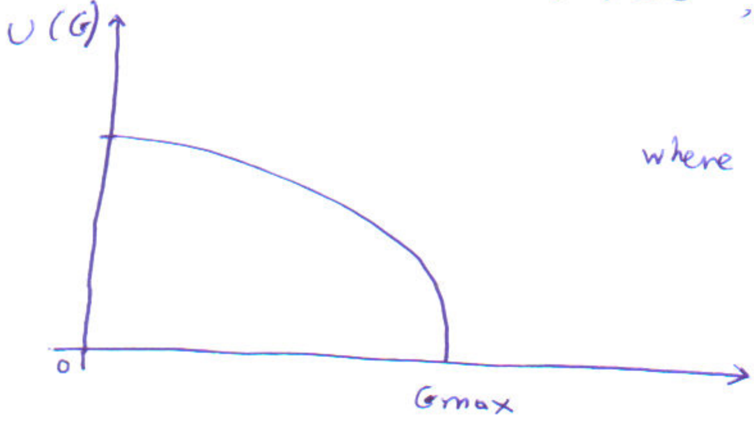
Tragedy of commons:

- Assume a village with N cattle-breeders. Each cattle-breeder chooses to buy $g_i \in [0, G_{max}]$ cattles.

We call G the total number of cattles that the N cattle-breeders chose to buy: $G = \sum_{i=1}^{N} g_i$, and we call c the cost per cattle for feeding.

We define as $U(G)$ the utility per cattle if there exist total G cattles.

We assume that $U(G) = 0$, if $G > G_{max}$.



where $U(G)$ = concave & diminishing

$U'(G) \downarrow$

$U''(G) < 0$

-- The utility that cattle-breeder i obtains by buying g_i cattles is:

$$U_i(g_1, \dots, g_N) = U_i(g_i, \underline{g}_{-i}) =$$
$$= g_i U(G) - c g_i =$$
$$= g_i U(g_i + \sum_{j \neq i} g_j) - c g_i \quad (1)$$

a) The point $(g_1^*, g_2^*, \dots, g_N^*)$ = Nash Equilibrium point

If: $\frac{\partial U_i}{\partial g_i}(g_i, \underline{g}^*_{-i}) = 0 \quad (2)$

$(1)(2) \Rightarrow U(g_i + \sum_{j \neq i} g_j^*) + g_i U'(g_i + \sum_{j \neq i} g_j^*) - c = 0 \Leftrightarrow$

I replace $g_i = g_i^*$, $G^* = \sum_{i=1}^{N} g_i^* \Rightarrow$

$(\Leftrightarrow) \left[U(G^*) + g_i^* U'(G^*) - c = 0 \right]$, for $i = 1, 2, \dots, N$

So, I found n equations for n unknown variables $g_1, g_2, \dots, g_n$.

If, I sum these n equations, I get:

$$N U(G^*) + G^* U'(G^*) - Nc = 0 \Leftrightarrow$$

$$\Leftrightarrow \boxed{U(G^*) + \frac{1}{N} G^* U'(G^*) - c = 0} \quad (3)$$

So, in Equilibrium point, relationship (3) holds.

b) The next step is to find the social optimum point; To do this we:

$$\underset{G}{\text{maximize}}\ G\,U(G) - cG \Leftrightarrow$$

$$\Leftrightarrow \boxed{U(G^o) + G^o U'(G^o) - c = 0} \quad (4)$$

where G^o is the total number of cattles that all cattle-breeders buy, in social optimum point.

c) So, the price of Anarchy is $\dfrac{G^o U(G^o) - cG^o}{G^* U(G^*) - cG^*}$

example

matched pennies

		P2	
		Head	Tails
P1	Head	(-1, 1)	(1, -1)
	Tails	(1, -1)	(-1, 1)

- 2 player flip a coin and if they both get the same outcome p2 wins, otherwise P1 wins.
- The above table represents the utilities the 2 players obtain by each of the four possible outcomes due to their strategies.
- We see that there is no pure Nash Equilibrium point, i.e. there are no specific strategies that satisfy the definition of Nash Equilibrium point.
- However, we can find mixed Nash Equilibrium points. A mixed N.E. point represents the uncertainness of each player to the strategy that other player(s) will choose.
- Assume $S_i = \{ s_{i1}, \dots, s_{ik} \}$ then:

  a mixed strategy is a pmf $= (P_{i1}, \dots, P_{ik})$

  with $\sum_{j=1}^{k} P_{ij} = 1 \quad \forall i$.

  If S_i = a continuous space $= (0, A_i)$ then a mixed strategy is a pdf in $(0, A_i)$
- In this example a mixed strategy is that the player1 owns a pmf: $(p, 1-p)$ and player2 $(q, 1-q)$.
- We will find a mixed Nash Equilibrium point. First we will find the best response function of the 2 players:

- Assume that P1 believes that P2 owns $(q, 1-q)$ pdf then:

  If $P1 \to H$: $(-1)q + 1(1-q) = 1-2q$ (utility of P1 if P1 does H)

  If $P1 \to T$: $1 \cdot q + (-1)(1-q) = 2q-1$ (utility of P1 if P1 does T)

So, if $1-2q > 2q-1 \Leftrightarrow q < \frac{1}{2} \Rightarrow H$

So, the best response of P1 is $\begin{cases} H, & \text{if } q < 1/2 \\ T, & \text{if } q > 1/2 \end{cases}$

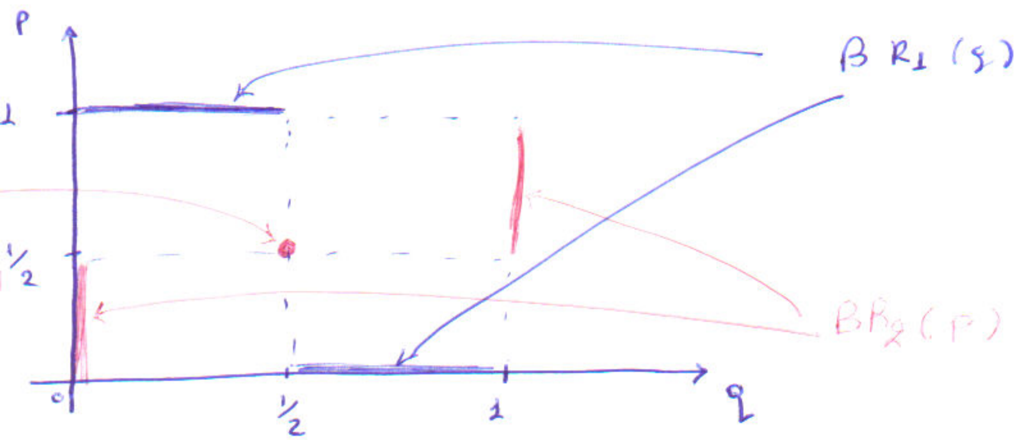
expected payoff of 1 is:

$$pq(-1) + p(1-q)1 + (1-p)q \cdot 1 + (1-p)(1-q)(-1) =$$
$$= p(\underbrace{2-4q}) + (2q-1)$$

So, we see that if $2-4q < 0$ in order to maximi[ze] the above value we need to choose $p=0$, else if $2-4q > 0$ we need to choose $p=1$.

So, the best response of P_1 is $\begin{cases} p=0, \text{ (tail)}, & \text{if } 2-4q<0 \Leftrightarrow q > \\ p=1 \text{ (head)}, & \text{if } 2-4q>0 \Leftrightarrow q < \end{cases}$

We can see the graphical represantation of the abo[ve] values:



where $BR_1(q) = p^*(q) = \begin{cases} 0, & q > 1/2 \\ 1, & q < 1/2 \end{cases}$

and $BR_2(p) = q^*(p) = \begin{cases} 1, & p > 1/2 \\ 0, & p < 1/2 \end{cases}$

[G]enerally if the 2 players posses pdf s:

$$\underline{P_1} = (P_{11}, P_{12}, \dots, P_{1J})$$

and $$\underline{P_2} = (P_{21}, P_{22}, \dots, P_{2K})$$

then: $$U_i(\underline{P_1}, \underline{P_2}) = \sum_{j=1}^{J} \sum_{k=1}^{K} P_{1j} P_{2k} U_i(S_{1j}, S_{2k})$$

and the Nash Equilibrium point is: $(\underline{P_1}^*, \underline{P_2}^*)$

if $$U_1(P_1^*, P_2^*) \geq U_1(P_1, P_2^*), \forall P_1$$

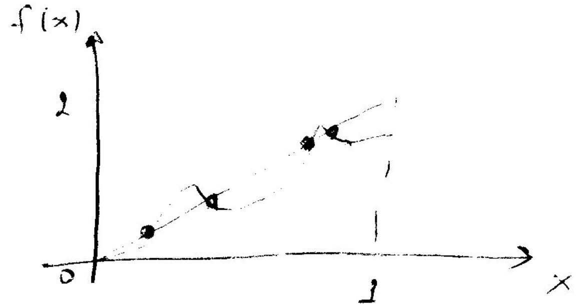
[a]nd if $$U_2(P_1^*, P_2^*) \geq U_2(P_1^*, P_2), \forall P_2$$

Theorem : Nash (1950)

Assume a game G with n players (n is finite) with finite S_i. Then there exists at least one Nash Equilibrium point with mixed strategies

Proof

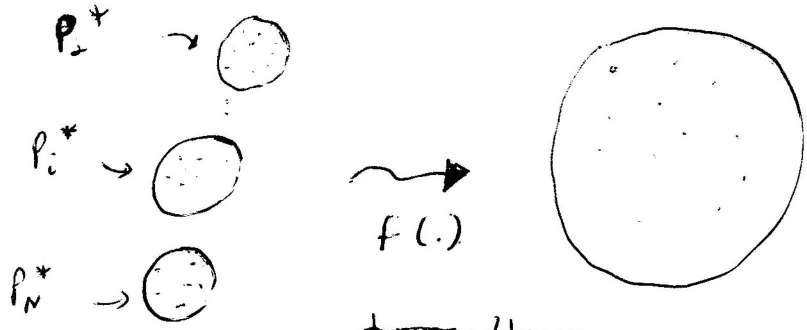
- the proof is based on the fixed point theorem:
  - assume a function $f(x)$ = continuous $[0,1] \to [0,1]$ then exists at least one x^* point such that $f(x^*) = x^*$



- the function f(x) cuts the line y = x at least one point

- We can intuitively prove the Theorem by saying:
  - for each player i there is the set of best response energies if the other players do $\underline{P}_{-i}$:

$P_i^* = BR(\underline{P}_{-i})$



~~during the game~~

- if we combine these sets of P_i^* due to a function f(.) we get a greater set
- It can be seen that f(.) has a fixed point, so there is equilibrium.

(The proof can be found in the book: Game theory for Applied Economists Gibbons, Robert)