Coalitional Game Theory

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Outline

Coalitional Game Theory

Foundations laid in:
J. von Neumann, O. Morgenstern – *Theory of Games and Economic Behavior* (1944)

- mathematical models of cooperation in a given social environment
- coalition formation goes hand in hand with payoff negotiation
- analysis of coalition games is usually based only on payoff opportunities available to each coalition (no moves, no actions)

Coalition Game: Assumptions

Players are allowed to communicate before or during the game, they can even redistribute their final payoffs via side payments

Coalition acts in the common players' interest on specific issues

Worth of each coalition is the total amount that the players from the coalition can jointly guarantee themselves, it is measured in abstract units of utility

Two Fundamental Questions

1. Which coalitions are likely to form?
2. How will the members of a coalition allocate their joint payoff?

- We leave the behavioral aspects aside...
- ...and the attempt to answer the second question is in that follows!

Usual assumption: the society as a whole operates efficiently so that the coalition of all players arise in the end

Examples

Example (Selling a horse)

Player 1 *(a seller) has a horse which is worthless to him (unless he can sell it). Players* 2 *and* 3 *(buyers) value the horse at* 90 *and* 100, *respectively.*

Which contract will be accepted by all the players?

Example (The UNSC voting)

The United Nations Security Council consists of 5 permanent members and 10 other members. Every decision must be approved by 9 members including all the permanent members.

What is a "voting" power of the individual members?

Mathematical Model of Game

Definition

Let $N = \{1, \dots, n\}$ *be a finite set of players. A coalition is any subset of* N. *The set of all coalitions is denoted by* 2^N.
A (coalition) game is a mapping

$$\boxed{v : 2^N \to \mathbb{R}}$$

such that $v(\emptyset) = 0$. *For any coalition* $A \subseteq N$, *the number* $v(A)$ *is called the worth of* A.

$N \dots$ *the grand coalition*

Imputation

Which payoff will a rational player accept in a game v?

Assumptions

1. the players have the total amount $v(N)$ to divide
2. no player will accept less than the amount which he/she can attain alone

Definition

An imputation in a game v is a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that

$$\sum_{i \in N} x_i = v(N) \quad \textit{and} \quad x_i \geqslant v(\{i\}), \quad \textit{for every } i \in N.$$

$E(v) \ldots$ *the set of all imputations in a game v*

Existence of Imputations

Example

Three farmers produce crops. They want to merge their businesses. However, the grand coalition is not able to operate efficiently (technical difficulties, wrong communication, highly progressive taxes,...)

$N = \{1, 2, 3\}$
$v(\{i\}) = 1, \quad i = 1, 2, 3$
$v(\{i, j\}) = 2, \quad i \neq j$
$v(N) = -1$

Proposition

For every game v, we have $\boxed{E(v) \neq \emptyset}$ *iff* $v(N) \geqslant \sum_{i \in N} v(\{i\})$.

Superadditive Games

"Unity makes strength": a game v is superadditive when, for every pair of coalitions $A, B \subseteq N$ with $A \cap B = \emptyset$, we have

$$\boxed{v(A \cup B) \geqslant v(A) + v(B)}$$

Example (Selling a horse)

$N = \{1, 2, 3\}$

If 1 *sells the horse to* 2 *for the price* x*, he will effectively make a profit* x*, while* 2*'s profit is* $90 - x$*. The total profit of the coalition* $\{1, 2\}$ *is thus* 90*. Similarly for* $\{1, 3\}$*. The grand coalition* N *should assign the horse to* 3 *who can eventually give side payments to* 2*.*

$v(\{1, 2\}) = 90, \quad v(\{1, 3\}) = v(N) = 100$

$v(\{i\}) = v(\{2, 3\}) = 0, \quad i = 1, 2, 3$

Preference between Imputations

Given $x, y \in E(v)$, which imputation are players likely to choose?

1. A group of players will prefer x to y if they are better off with x
2. This group of players must be simultaneously strong enough to enforce the choice of x

Definition

Let $S \subseteq N$ and $x, y \in E(v)$. Coalition S prefers x to y ($x \succ_S y$) when

$$x_i > y_i, \text{ for every } i \in S, \quad \text{and} \sum_{i \in S} x_i \leqslant v(S).$$

Core of a Game

An imputation x is preferred to an imputation y (notation $x \succ y$) if there is a coalition S with $x \succ_S y$.

Definition

Let v be a game. The core of v is the set $Core(v) \subseteq \mathbb{R}^n$ such that

$$\boxed{Core(v) = \left\{ y \in E(v) \mid \forall x \in E(v),\ x \not\succ y \right\}}$$

Problems

- the relation $\succ$ is neither transitive nor antisymmetric
- how to describe the core?

Characterization of the Core

Theorem (Gillies, 1959)

Let

$$C(v) = \left\{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N),\ \sum_{i \in S} x_i \geqslant v(S),\ \text{for every } S \in 2^N\right\}.$$

Then:

- *for every game* v, *we have* $C(v) \subseteq Core(v)$
- *if* v *is superadditive, then* $\boxed{C(v) = Core(v)}$

The core of a superadditive game is a (possibly empty) convex polytope in $\mathbb{R}^n$ given by the intersection of an affine hyperplane with $2^n - 2$ halfspaces.

Example

Example (Selling a horse)

$N = \{1, 2, 3\}$
$v(\{1, 2\}) = 90, \quad v(\{1, 3\}) = v(N) = 100$
$v(\{i\}) = v(\{2, 3\}) = 0, \quad i = 1, 2, 3$

Core$(v) = C(v)$ *consists of vectors* $x \in \mathbb{R}^3$ *satisfying*

$$\begin{aligned} x_i &\geqslant 0, \quad i = 1, 2, 3 \\ x_1 + x_2 &\geqslant 90 \\ x_1 + x_3 &\geqslant 100 \\ x_1 + x_2 + x_3 &= 100 \end{aligned}$$

It follows that $C(v) = \{(t, 0, 100 - t) \in \mathbb{R}^3 \mid 90 \leqslant t \leqslant 100\}$
Player 3 *will purchase the horse at a price at least* 90, *player* 2 *is priced out of the market after bidding up the price to* 90.

A Superadditive Game with Empty Core

Example (Voting)

Three friends want to select one of the two restaurants for a dinner. The decision is made by a simple majority of votes.

$N = \{1, 2, 3\}$
$v(\{i\}) = 0, \quad i = 1, 2, 3$
$v(S) = 1,$ *whenever* $|S| \geqslant 2$

The game v is superadditive with Core$(v) = C(v) = \emptyset$.

Observe: some imputation of the type $(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2})$ is always preferred to any other imputation from $E(v)$

Stable Sets

Original solution concept of von Neumann and Morgenstern:

Definition

A stable set for a game v is a set $S(v) \subseteq E(v)$ such that

1. *if $x, y \in S(v)$, then $x \not\succ y$*
2. *if $x \in E(v) \setminus S(v)$, then there is $y \in S(v)$ with $y \succ x$*

Problem

- existence of a stable set is not guaranteed
- there can be more stable sets for a given game

Proposition

The core of every game is contained in any stable set of imputations for this game.

Simple Games

A game v is called simple when $v(S) \in \{0,1\}$ for each $S \in 2^N$.
A player $i \in N$ is said to be a veto player if

$$v(N \setminus \{i\}) = 0.$$

Proposition

Let v be a superadditive simple game such that $v(N) = 1$ and $v(\{i\}) = 0$, for each $i \in N$. Then

$$\boxed{Core(v) \neq \emptyset} \quad \text{iff} \quad \text{there is a veto player.}$$

Convex Games

A game v is said to be convex when

$$v(S \cup T) + v(S \cap T) \geqslant v(S) + V(T), \quad \text{for every } S, T \in 2^N.$$

Proposition

A game v is convex iff

$$\boxed{v(S \cup \{i\}) - v(S) \leqslant v(T \cup \{i\}) - v(T)}$$

for every $i \in N$ and every $S \subseteq T \subseteq N \setminus \{i\}$

Interpretation

- convexity of a game means nondecreasing marginal utility of coalition membership

Core Geometry of Convex Games

Theorem (Shapley, 1972)

Let v be a nontrivial convex game. Then:

- $\boxed{Core(v) \neq \emptyset}$
- *there are at most $n!$ extreme points (vertices) of $Core(v)$ and they coincide precisely with the set of imputations $x \in \mathbb{R}^n$ of the form*

$$x_i = v(S_i) - v(S_{i-1}), \quad \text{for each } i \in N,$$

where $S_i = \{j \in N \mid \pi(j) \leqslant i\}$ for some permutation π of N.

Core: Computational Issues

- the core is difficult to describe in general
- even when the game is superadditive, its core is the intersection of a large number of halfspaces
- for example, a game with 20 players amounts to solving the LP problem with more than 1 milion of affine constraints!
- a game is played as a one-shot affair: all fuzzy coalitions come up with their demands simultaneously

For some purposes it can be enough to

1. recover a single element from the core or
2. decide nonemptiness of the core

Bargaining for a Payoff

Motivation

- in real-world situations the coalitions bargain for a final payoff repetitively
- the bargaining process stops when no coalition raises objections against the payoff

Define

$$C_S = \begin{cases} \{x \in \mathbb{R}^n \mid \sum_{i \in S} x_i \geqslant v(S)\}, & S \neq N \\ \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N)\}, & S = N \end{cases}$$

If $x \notin C_S$, then the coalition S will contest the payoff x. The core of a superadditive game v is a set of imputations that is not contested by any coalition:

$$\boxed{\mathrm{Core}(v) = \bigcap_{S \in 2^N} C_S}$$

Bargaining Schemes

Definition

Let v be a game. A bargaining scheme for $Core(v)$ is an iterative procedure generating a sequence $x^0, x^1, x^2, \ldots$ of payoffs with $\lim_{k\to\infty} x^k \in Core(v)$, provided $Core(v) \neq \emptyset$.

Given any payoff $x \in \mathbb{R}^n$, each coalition S accepts the payoff

$$P_S(x) = \arg\min_{y \in C_S} \|y - x\|$$

But the payoff $P_S(x)$ can be contested by another coalition T!

Cimmino Type Bargaining Scheme

Let ω be a vector with $2^n - 1$ components $\omega_S > 0,\ S \in 2^N \setminus \emptyset$, such that

$$\sum_{S \in 2^N \setminus \emptyset} \omega_S = 1$$

The number ω_S is interpreted as a bargaining power of the coalition S in the game v.

Put $$\boxed{\mathbf{P}(x) = \sum_{S \in 2^N \setminus \emptyset} \omega_S P_S(x)}$$

Definition

The Cimmino type bargaining scheme for the game v is the following rule of generating sequences (x^k) in $\mathbb{R}^n$:

$$\boxed{x^0 \in \mathbb{R} \quad \textit{and} \quad x^{k+1} = \mathbf{P}(x^k),\ k \in \mathbb{N}_0}$$

Cimmino Type Bargaining Scheme (cont.)

Theorem

Let v be a superadditive game. For every $x^0 \in \mathbb{R}$ and every vector ω of the bargaining powers we have:

- *if $Core(v) \neq \emptyset$, then* $\boxed{\lim_{k\to\infty} x^k \in Core(v)}$
- *if $Core(v) = \emptyset$, then (x^k) converges to a fixed point of the function*

$$F(x) = \sum_{S \in 2^N \setminus \emptyset} \omega_S P_S(x)$$

Motivation

- is there a way how to distribute among the players the cooperative profit in a "fair" manner for every game?
- the need for a single-payoff solution concept
- Shapley value (1953)

Definition

A carrier for a game v is a coalition T such that $v(S) = v(S \cap T)$ for each coalition S.
Given a permutation π of the set of players N, put

$$\pi v(S) = v(\pi^{-1} S), \quad \text{for every } S \in 2^N.$$

Shapley Value

Definition

Shapley value is a mapping $\varphi : GAMES \to \mathbb{R}^n : v \mapsto \varphi[v] \in \mathbb{R}^n$ *satisfying the following axioms:*

Efficiency *If S is a carrier for a game v, then*

$$\sum_{i \in S} \varphi_i[v] = v(S)$$

Symmetry *For every permutation π of N and every $i \in N$,*

$$\varphi_{\pi i}[\pi v] = \varphi_i[v]$$

Additivity *If v_1, v_2 are games, then*

$$\varphi[v_1 + v_2] = \varphi[v_1] + \varphi[v_2]$$

Construction of the Shapley Value (1)

For any coalition S, put

$$w_S(T) = \begin{cases} 1, & S \subseteq T, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma (Value of unanimity games)

There exists a unique mapping $\varphi : \{w_S \mid S \in 2^N\} \to \mathbb{R}^n$ satisfying Efficiency and Symmetry axioms and we have

$$\varphi_i[w_S] = \begin{cases} \frac{1}{|S|}, & i \in S, \\ 0, & \textit{otherwise.} \end{cases}$$

Moreover, if $c > 0$, then

$$\varphi_i[c w_S] = \begin{cases} \frac{c}{|S|}, & i \in S, \\ 0, & \textit{otherwise.} \end{cases}$$

Construction of the Shapley Value (2)

Lemma (Hamel basis in the linear space of games)

For every game v there exist $2^n - 1$ uniquely determined real numbers c_S, $S \in 2^N \setminus \emptyset$, such that

$$v = \sum_{S \in 2^N \setminus \emptyset} c_S w_S.$$

Taking into account the two previous lemmata, the function φ is extended to the set of all games by linearity:

$$\boxed{\varphi[v] = \sum_{S \in 2^N \setminus \emptyset} c_S \varphi[w_s]}$$

Construction of the Shapley Value (3)

Theorem (Shapley, 1953)

There exists a unique mapping $\varphi : GAMES \to \mathbb{R}^n$ satisfying Efficiency, Symmetry, and Additivity axioms. For every game v and each player $i \in N$ we have

$$\varphi_i[v] = \sum_{T \in 2^N | i \in T} \frac{(|T|-1)!(n-|T|)!}{n!} \big(v(T) - v(T \setminus \{i\})\big)$$

The formula above becomes simple for a simple game v:

$$\varphi_i[v] = \sum_{T} \frac{(|T|-1)!(n-|T|)!}{n!}$$

where the sum runs over all "winning" coalitions T such that $T \setminus \{i\}$ is "loosing".

Examples

Example (Voting)

$N = \{1, 2, 3\}$
$v(\{i\}) = 0, \quad i = 1, 2, 3$
$v(S) = 1$, *whenever* $|S| \geqslant 2$
The game v *is simple with* $Core(v) = \emptyset$.

Shapley value is the same for each player:

$$\varphi_i[v] = \frac{1!1!}{3!} + \frac{1!1!}{3!} = \frac{1}{3}, \quad i = 1, 2, 3$$

Examples (cont.)

Example (Stockholders)

A company has 4 stockholders, each of them having 10, 25, 35, *and* 40 *shares of the company's stock. A decision is approved by a simple majority of all the shares.*

$N = \{1, 2, 3, 4\}$
v is a simple game in which the only winning coalitions are:
$\{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, N$

Shapley value:

$$\varphi_1[v] = 0, \quad \varphi_i[v] = \frac{1}{3}, \ i = 2, 3, 4$$

Properties

The Shapley value can be viewed as

- an index of power (in voting procedures,...)
- a prediction of a "fair" allocation of resources

Proposition

- *If a player* $i \in N$ *is dummy, that is,* $v(S) - v(S \setminus \{i\}) = 0$ *for every coalition* S *with* $i \in S$*, then* $\varphi_i[v] = 0$.
- *If* v *is a superadditive game, then* $\varphi[v]$ *is an imputation.*
- *If* v *is a convex game, then* $\varphi[v] \in Core(v)$.

Stochastic Interpretation of Shapley Value

- the players arrive randomly at a specified place, all orders of arrivals of the players have the same probability $\frac{1}{n!}$
- when a player $i \in N$ arrives and the players from some coalition T (with $i \in T$) are already there, he receives

$$X = v(T) - v(T \setminus \{i\})$$

- X is a random variable, its probability distribution P is

$$P\big[X = v(T) - v(T \setminus \{i\})\big] = \frac{(|T|-1)!(n-|T|)!}{n!}$$

and its expected value is $\boxed{E(X) = \varphi_i[v]}$

Computational Issues

- the computation of Shapley value for real-world problems requires a prohibitive number of calculations
- for example, in a game with 100 players the number of summands for one player can be

$$\sum_{i=1}^{99} \binom{99}{i} \approx 10^{30}$$

Enhancing the computations

- statistical estimation of Shapley value based on random sampling
- multilinear extension

Multilinear Extension

The set of all coalitions 2^N can be identified with the vertices of the cube $[0,1]^n$.
Can we extend a game $v : 2^N \to \mathbb{R}$ to an n-variable function

$$\boxed{\overline{v} : [0,1]^n \to \mathbb{R}}$$

with "nice" properties?

Proposition

There exists a unique multilinear function $\overline{v} : [0,1]^n \to \mathbb{R}$ that coincides with v on 2^N and we have

$$\overline{v}(x_1, \ldots, x_n) = \sum_{S \in 2^N} \Big(\prod_{i \in S} x_i \prod_{i \notin S} (1 - x_i) \Big) v(S)$$

for every $(x_1, \ldots x_n) \in [0,1]^n$.

Stochastic Interpretation of Multilinear Extension

- coalitions S are formed randomly, each x_i from $x = (x_1, \ldots x_n) \in [0,1]^n$ is a probability that the player i is available for participation in a coalitional activity
- for every (random) coalition S, the family of events

$$(i \in S)_{i \in N}$$

  is independent
- given $S \in 2^N$, the number

$$\prod_{i \in S} x_i \prod_{i \notin S} (1 - x_i)$$

  is then the probability that the coalition S will be formed
- the number $\overline{v}(x)$ is the expected value of the worth of the coalition that will actually arise

Example of Multilinear Extension

Example (Voting)

$N = \{1, 2, 3\}$
$v(\{i\}) = 0, \quad i = 1, 2, 3$
$v(S) = 1,$ *whenever* $|S| \geqslant 2$

The multilinear extension of v is

$$\overline{v}(x) = x_1 x_2 (1 - x_3) + x_1 x_3 (1 - x_2) + x_2 x_3 (1 - x_1) + x_1 x_2 x_3$$

Multilinear Extension and Shapley Value

Theorem (Diagonal formula)

Let v be a game with the multilinear extension $\overline{v}$. Then, for every player $i \in N$,

$$\varphi_i[v] = \int_0^1 \frac{\partial \overline{v}}{\partial x_i}(t, \dots, t)dt$$

- Shapley value is thus completely determined by behavior of the function $\overline{v}(x)$ in the neighborhood of the diagonal $\{(t, \dots, t) \in [0,1]^n \mid t \in [0,1]\}$
- $\varphi_i[v]$ can be interpreted as the expected value of the marginal contribution $\frac{\partial \overline{v}}{\partial x_i}$ when all n times of players' arrivals are iid random varibles with uniform distribution in $[0,1]$

Example

Example (Voting)

$N = \{1, 2, 3\}$
$v(\{i\}) = 0, \quad i = 1, 2, 3$
$v(S) = 1, \textit{ whenever } |S| \geqslant 2$

$\overline{v}(x) = x_1x_2(1 - x_3) + x_1x_3(1 - x_2) + x_2x_3(1 - x_1) + x_1x_2x_3$
We have

$$\frac{\partial \overline{v}}{\partial x_1}(x) = x_2 + x_3 - 2x_2x_3$$

so that $\frac{\partial \overline{v}}{\partial x_1}(t, t, t) = 2t - 2t^2$ *and thus*

$$\varphi_1[v] = \int_0^1 2t - 2t^2 dt = \left[t^2 - \frac{2t^3}{3}\right]_0^1 = \frac{1}{3}$$

The UNSC Voting (1)

Example

The United Nations Security Council consists of 5 permanent members and 10 other members. Every decision must be approved by 9 members including all the permanent members.

$N = \{1, \ldots, 15\}$
Representation as a weighted voting game:
$\omega_i = 7, \ i = 1, \ldots, 5$
$\omega_i = 1, \ i = 6, \ldots, 15$
quota=39

$$v(S) = \begin{cases} 1, & \sum_{i \in S} \omega_i \geqslant 39, \\ 0, & \text{otherwise}, \end{cases} \quad S \in 2^N.$$

Finding a weighted voting game representation for a simple game is equivalent to solving a system of linear inequalities!

The UNSC Voting (2)

Representation as a compound game: $N = N_1 \cup N_2$, where $N_1 = \{1, \ldots, 5\}$, $N_2 = \{6, \ldots, 15\}$
Define:

$$w_1(S) = \begin{cases} 1, & S = N_1, \\ 0, & S \neq N_1, \end{cases} \quad w_2(T) = \begin{cases} 1, & |T| \geqslant 4, \\ 0, & |T| \leqslant 3, \end{cases} \quad S \in 2^{N_1}, T \in 2^{N_2}$$

and

$$u(A) = \begin{cases} 1, & A = \{1, 2\} \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\boxed{v(S) = u\big(\{i \in \{1, 2\} \mid w_i(S \cap N_i) = 1\}\big), \quad S \in 2^N}$$

The UNSC Voting (3)

The multilinear extension $\overline{v} : [0,1]^{15} \to \mathbb{R}$ is composed in the same way as the game v. Defining $\overline{w} : \mathbb{R}^{15} \to \mathbb{R}^2$ as

$$\overline{w}(x_1, \dots, x_{15}) = (\overline{w_1}(x_1, \dots x_5), \overline{w}_2(x_6, \dots x_{15})),$$

we have $\boxed{\overline{v} = \overline{u} \circ \overline{w}}$

Due to Symmetry and Efficiency axioms, it suffices to calculate the Shapley value of one player (say $i \in N_2$). We get

$$\varphi_i[v] = \int_0^1 \frac{\partial u}{\partial y_2}(\overline{w}(\underbrace{t, \dots, t}_{15\times}))\frac{\partial \overline{w}_2}{\partial x_i}(\underbrace{t, \dots, t}_{10\times})dt$$

The UNSC Voting (4)

- $\overline{w}_1(x_1, \ldots, x_5) = x_1 x_2 x_3 x_4 x_5$
- $\frac{\partial \overline{w}_2}{\partial x_i}(t, \ldots, t) = \sum_S t^{|S|}(1-t)^{9-|S|} = \binom{9}{3} t^3 (1-t)^6$, where the first sum runs over all $S \subseteq N_2 \setminus \{i\}$ such that S loses but $S \cup \{i\}$ wins
- $u(y_1, y_2) = y_1 y_2$

Hence

$$\varphi_i[v] = \int_0^1 \overline{w}_1(t, \ldots, t) \frac{\partial \overline{w}_2}{\partial x_i}(t, \ldots, t) dt = \int_0^1 t^5 \cdot 84 t^3 (1-t)^6 dt = \frac{4}{2145}$$

The UNSC Voting (5)

Since there are 10 players in N_2, each player from the set N_1 has the Shapley value

$$\frac{1}{5} \cdot \left(1 - 10 \cdot \frac{4}{2145}\right) = \frac{1}{5} \cdot \frac{2105}{2145} = \frac{421}{2145}$$

The UNSC game has the Shapley value

$$\varphi_i[v] = \begin{cases} 0.1963, & \text{if } i \text{ is a permanent member,} \\ 0.0019, & \text{otherwise.} \end{cases}$$

Shapley value suggests that the permanent members of the UNSC have immense power in the voting!

Application: States' Power in the U.S. Presidential Election

Two-stage procedure for electing a president can be modeled as a compound game:

1. each state selects "Great Electors" for Electoral College
2. the Electoral College elects the president by simple majority rule

Assumptions

- the number of Great Electors of each state is in proportion to its census count
- each Great Elector votes for the candidate preferred by the majority of his/her state

What is a voting power of voters from different states?

Application: Airport Game

- a runway needs to be built for n planes of m different sizes
- costs $c_1 < c_2 < \cdots < c_m$
- $N = \bigcup_j N_j$, where N_J is the set of planes of size j
- define $j(S) = \max\{ j \mid S \cap N_j \neq \emptyset\}, \quad S \subseteq N$

Cost-sharing game c on 2^N:

$$c(S) = \begin{cases} c_{j(S)}, & S \in 2^N \setminus \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

How to allocate the cost of constructing/maintaining the runway among its users?

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