Non Linear Equations (4) - Optimization

Πέμπτη, 26 Φεβρουαρίου 2015 10:06 πμ

Θέλω να υπολογίσω

$$\min_x F(x) \text{ ή } \max_x F(x)$$

??

$$F'(x) = 0$$

Αν $x \in$ (ανοικτό διάστημα) αρκεί

αρκεί $F'(x) = 0$

Αν $x \in$ (κλειστό διάστημα) τότε

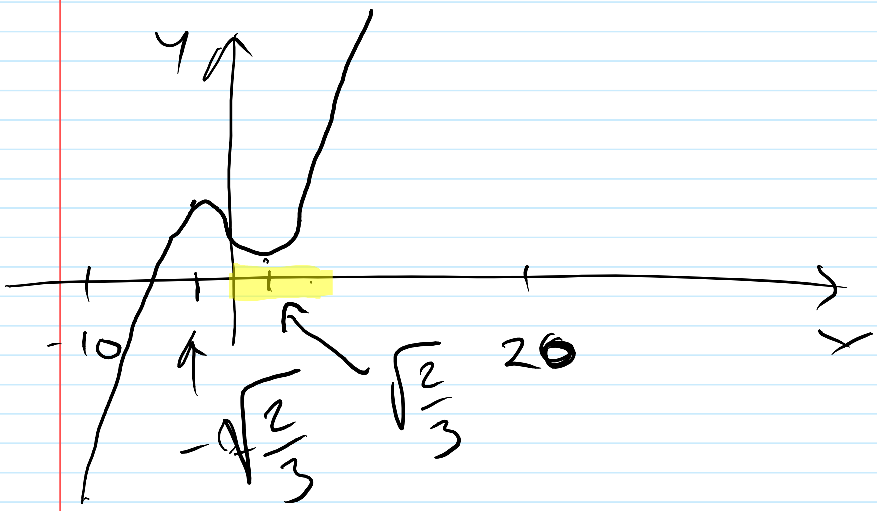
$F'(x) = 0$ + έλεγχος στα άκρα.

$F'(x) = 0$ και στις 2 περιπτώσεις.

Π.χ.:

$\min_{x} F(x)$, $F(x) = x^3 - 2x + 5$

$x \in (-10, 20)$



$F'(x) = 0$

$F'(x) = 3x^2 - 2$

$F'(x) = 0 \iff$

$3x^2 - 2 = 0 \iff$

$x^2 = \frac{2}{3}$

$x = \pm\sqrt{\frac{2}{3}}$

$F''(x) = 6x$

$F''\left(-\sqrt{\frac{2}{3}}\right) = -6\sqrt{\frac{2}{3}} \Rightarrow$ τοπ. μεγ.

$F''\left(\sqrt{\frac{2}{3}}\right) = 6\sqrt{\frac{2}{3}} \Rightarrow$ τοπ. ελάχ.

Έστω ότι θέλω να υπολογίσω το ελάχιστο στο (0, 1)

$\min F(x)$ ή $\min (x^3 - 2x + 5)$

$$\min_{x\in(0,1)} F(x) \quad ή \quad \min_{x\in(0,1)} (x^3-2x+5)$$

$$\Longleftrightarrow F'(x)=0 \quad x\,(0,1)$$

$$F'(x) = 3x^2-2$$

$\Downarrow$ χρήση Newton

$$x_0 = 1$$

$$x_{n+1} = x_n - \frac{h(x_n)}{h'(x_n)} \quad \left(\begin{matrix}\text{μέθοδος}\\ \text{Newton}\end{matrix}\right)$$

$$x_{n+1} = x_n - \frac{F'(x_n)}{F''(x_n)} =$$

$$= x_n - \frac{3x_n^2-2}{6x_n} =$$

$$= \frac{6x_n^2 - 3x_n^2+2}{6x_n} = \frac{3x_n^2+2}{6x_n}$$

$x_0 = 1$

$$x_1 = \frac{3 \cdot 1^2 + 2}{6 \cdot 1} = \frac{5}{6}$$

$$x_2 = \frac{3\left(\frac{5}{6}\right)^2 + 2}{6 \cdot \frac{5}{6}} = \dots$$

$\vdots$

x_n

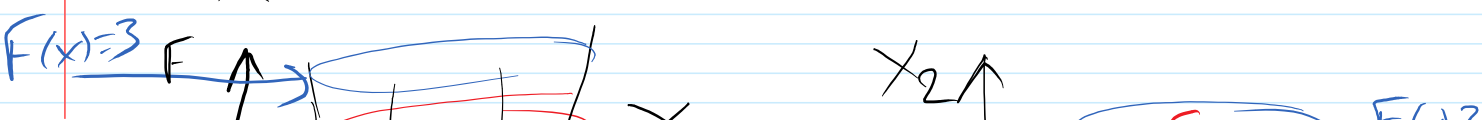
Συμπεράσματα: (1-D)

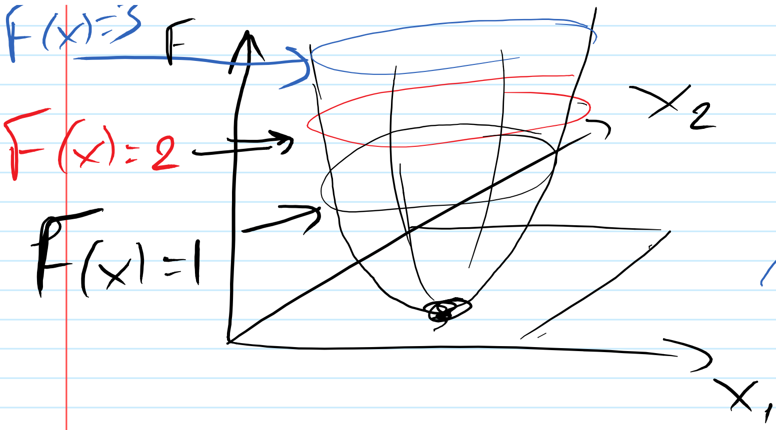
$$\min_x F(x) \xrightarrow{\text{Newton}} x_{n+1} = x_n - \frac{F'(x_n)}{F''(x_n)} \quad x_0$$

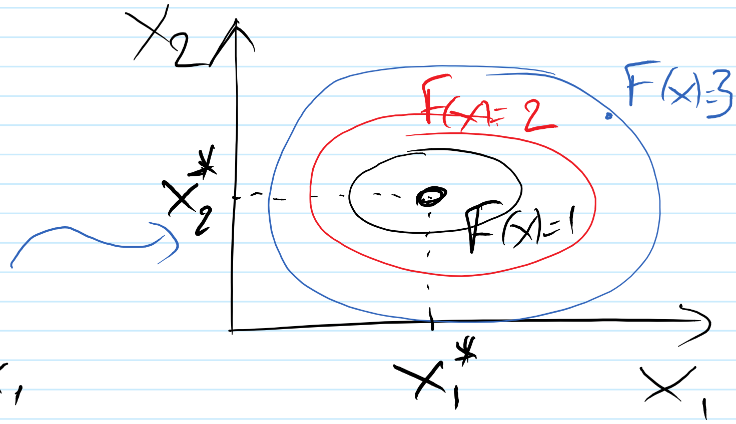
(2-D)

$$F(x_1, x_2) = x_1^2 + (\log x_2)^2$$

$$\min_x F(x) = ? \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$$







Ελάχιστο στο
$x^* = (x_1^*, x_2^*)$

Όπως πριν θα προσπαθήσω να υπολογίσω την "Παράγωγο" της F

$$F(x) = F(x_1, x_2) = x_1^2 + (\log x_2)^2$$

$$\nabla F(x) = \begin{bmatrix} \frac{\partial F}{\partial x_1}(x) \\ \\ \frac{\partial F}{\partial x_2}(x) \end{bmatrix} = \begin{bmatrix} 2x_1 \\ \\ \frac{2\log x_2}{x_2} \end{bmatrix}$$

Άρα θα λύσω το $\nabla F(x) = 0$

$2x_1 = 0$ (Newton σε

$$2x_1 = 0$$

$$\frac{2\log x_2}{x_2} = 0$$

(Newton σε 2 διαστάσεις)

⇓

$$x^{(0)}$$

$$x^{(n+1)} = x^{(n)} - J^{-1}(x^{(n)}) \cdot h(x^{(n)})$$

$$x^{(n+1)} = x^{(n)} + S$$

όπου S λύση του

$$J_h(x^{(n)}) \cdot S = h(x^{(n)})$$

(η συνάρτηση που εφαρμόζω τη Newton)

Στο παράδειγμα

$$J_{\nabla F}(x) = \begin{bmatrix} \frac{\partial}{\partial x_1}\left(\frac{\partial F}{\partial x_1}\right)(x) & \frac{\partial}{\partial x_2}\left(\frac{\partial F}{\partial x_1}\right)(x) \\ \frac{\partial}{\partial x_1}\left(\frac{\partial F}{\partial x_2}\right)(x) & \frac{\partial}{\partial x_2}\left(\frac{\partial F}{\partial x_2}\right)(x) \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\partial}{\partial x_1}(2x_1) & \frac{\partial}{\partial x_2}(2x_1) \\ \end{bmatrix}.$$

$$= \begin{bmatrix} \frac{\partial}{\partial x_1}(\ ) & \frac{\partial}{\partial x_2}(\ ) \\ \frac{\partial}{\partial x_1}\left(\frac{2\log(x_2)}{x_2}\right) & \frac{\partial}{\partial x_2}\left(\frac{2\log(x_2)}{x_2}\right) \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & \frac{2\frac{1}{x_2}x_2 - 2\log x_2 \cdot 1}{x_2^2} \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & \frac{2 - 2\log x_2}{x_2^2} \end{bmatrix} = M(x)$$

$$x^{(0)}$$

$$x^{(n+1)} = x^{(n)} - M^{-1}(x^{(n)}) \cdot \nabla F(x^{(n)})$$

αφού $M(x) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{2 - 2\log(x_2)}{x_2^2} \end{bmatrix}$

$$M^{-1}(x) = \begin{bmatrix} 1/2 & 0 \\ \ldots \end{bmatrix}$$

$$M^{-1}(x) = \begin{bmatrix} 1/2 & 0 \\ 0 & \dfrac{x_2^2}{2 - 2\log(x_2)} \end{bmatrix}$$

Άρα η επαναληπτική μέθοδος Newton παίρνει την εξής μορφή:

$X^{(0)}$

$$X^{(n+1)} = X^{(n)} - \begin{bmatrix} 1/2 & 0 \\ 0 & \dfrac{(x_2^{(n)})^2}{2 - 2\log x_2^{(n)}} \end{bmatrix} \begin{bmatrix} 2x_1^{(n)} \\ 2\dfrac{\log x_2^{(n)}}{x_2^{(n)}} \end{bmatrix}$$

Λάθος: ~~$X = X - \dfrac{F(x)}{F'(x)}$~~

Προσοχή στους δείκτες

Έστω $X^{(0)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$X^{(1)} = X^{(0)} - \begin{bmatrix} 1/2 & 0 \\ 0 & \dfrac{(x_2^{(0)})^2}{\quad} \end{bmatrix} \begin{bmatrix} 2x_1^{(0)} \\ 2\dfrac{\log(x_2^{(0)})}{x_2^{(0)}} \end{bmatrix}$$

$$\begin{bmatrix} \cdot & \cdot \\ 0 & \frac{(x_2^{(0)})}{2-2\log(x_2^{(0)})} \end{bmatrix} \begin{bmatrix} \cdot \\ 2\frac{\log(x_2^{(0)})}{x_2^{(0)}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1/2 & 0 \\ 0 & \frac{4}{2-2\cdot\log 2} \end{bmatrix} \begin{bmatrix} 2 \\ 2\frac{\log(2)}{2} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ \frac{4\log 2}{2-2\log 2} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 - \frac{2\log 2}{1-\log 2} \end{bmatrix}$$

Παράδειγμα:

$$F(x) = x_1^2 \cdot \sin(x_2) + x_2^2 x_1$$

$$\min_x F(x)$$

$$\nabla F(x) = \begin{bmatrix} 2x_1 \cdot \sin(x_2) + x_2^2 \\ x_1^2 \cos x_2 + 2x_1 x_2 \end{bmatrix}$$

$$\nabla\nabla F \quad \left[\partial f(\quad) \quad \partial(\quad) \right]$$

$$J_{\nabla F(x)} = \nabla\nabla F(x) = \begin{bmatrix} \frac{\partial(\ )}{\partial x_1} & \frac{\partial(\ )}{\partial x_2} \\ \frac{\partial(\ )}{\partial x_1} & \frac{\partial(\ )}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} 2\sin(x_2) & 2x_1\cos(x_2)+2x_2 \\ 2x_1\cos(x_2)+2x_2 & -x_1^2\sin(x_2)+2x_1 \end{bmatrix}$$

Έστω $x^{(0)} = \begin{bmatrix} \pi \\ \pi/2 \end{bmatrix}$

$$x^{(1)} = x^{(0)} + S$$

$$J_{\nabla F}(x^{(0)}) \cdot S = \nabla F(x^{(0)})$$

$$\begin{bmatrix} 2 & +\pi \\ & \end{bmatrix} \begin{bmatrix} S_1 \\ \end{bmatrix} = \begin{bmatrix} 2\pi+\pi^2 \\ \end{bmatrix}$$

$$\begin{bmatrix} 2 & +\pi \\ +\pi & -\pi^2+2\pi \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \begin{bmatrix} \\ -\pi^2+2\pi \end{bmatrix}$$

Παράδειγμα από διαφάνειες:

$f(x) = x^2 - x - 2$, $x = -1, 2$

$g(x) = x^2 - 2$

$g(x) = \sqrt{x+2}$

$g(x) = 1 + 2/x$

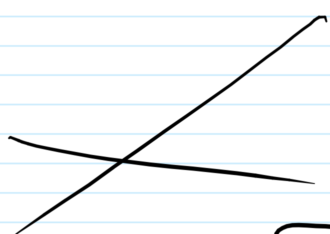
$g(x) = \dfrac{x^2+2}{2x-1}$

(a) $g(x) = x^2 - 2$, $g'(x) = 2x$

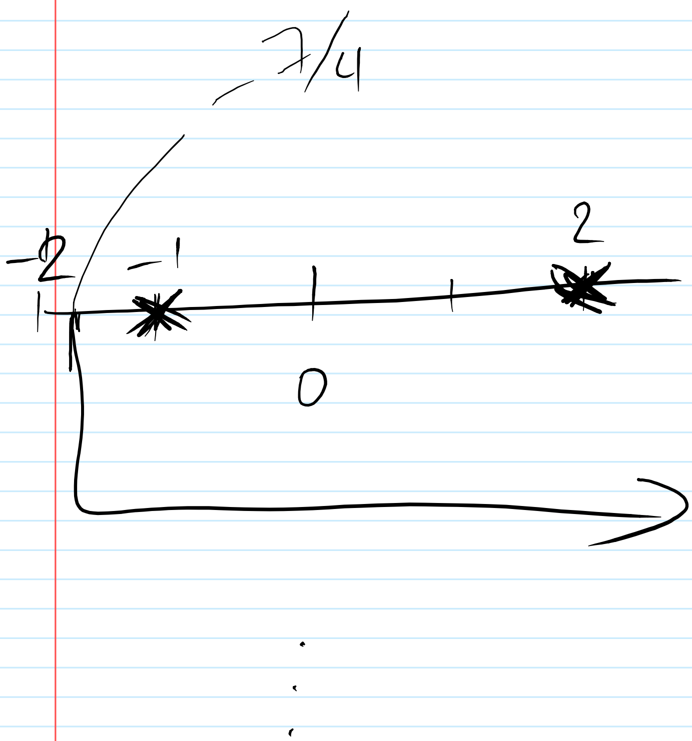
$|g'(x)| < 1 \Leftrightarrow |2x| < 1$

$|x| < 1/2$

$-1/2 < x < 1/2$



(b) $g(x) = \sqrt{x+2}$, $g'(x) = \dfrac{1}{2\sqrt{x+2}}$



$$|g'(x)| < 1 \Leftrightarrow$$

$$\frac{1}{2\sqrt{x+2}} < 1 \Leftrightarrow$$

$$2\sqrt{x+2} > 1$$

$$\sqrt{x+2} > 1/2$$

$$x+2 > 1/4$$

$$x > -2 + 1/4 = -7/4$$