Linear optimization: Theory, methods, and extensions

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Chapter 1

Introduction

This book is mainly about linear programming which is to minimize a linear function over a set of linear equalities and inequalities. Hence, the book is about the problem

$$\begin{array}{lll} \text{minimize} & c^T x & \\ \text{subject to} & Ax = b, & \\ & \hat{A}x \leq \hat{b}. & \end{array} \tag{1.1}$$

Although this problem does not seem too general, because no nonlinearities in the functions are allowed, then this type problem occurs frequently in practice.

Chapter 2

Theory

2.1 The standard LP

An LP is defined as minimizing or maximizing a linear function subject to linear constraints. This is a general definition and for convenience we will mainly work with LPs in standard form that is

$$\begin{array}{lll} (P) & \text{minimize} & c^T x \\ & \text{subject to} & Ax = b, \\ & & x \geq 0, \end{array}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

are vectors and

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

is a matrix. c, b, and A are given parameters and x is the decision variables. Hence, there are n variables, n inequalities, and m equality constraints. In the next section we shall demonstrate there is no loss in generality by only considering LPs on standard form, because any LP can be stated on this form using some simple transformations.

2.1.1 Definitions

The set of feasible solutions to (P) is denoted

$$\mathcal{X} := \{x \in \mathbf{R}^n : \ Ax = b, \ x \geq 0\}.$$

In the case that $\mathcal{X}$ nonempty we say that the problem (P) is **feasible**. Moreover, any solution $x \in \mathcal{X}$ is said to be a feasible solution. On the other hand if $\mathcal{X}$ is empty, then (P) is said to be **infeasible**, because then the problem does not have any solution.

The definition of an optimal solution $x^* \in \mathcal{X}$ to (P) is

$$c^T x^* \leq c^T x, \ \forall x \in \mathcal{X}. \tag{2.1}$$

Hence, the optimal solution is a feasible solution such that no other feasible has a lower objective value $c^T x$.

In the case the optimal objective value to (P) is unbounded, then we say (P) is undounded. An example of an unbounded problem is

$$\text{minimize} \quad x_1 - x_2 \quad \text{subject} \quad x_1 = 1, \quad x_1, x_2 \geq 0.$$

2.1.2 Inequalities

In practice an LP does not consists of equalities only, but may contain several inequalities such as

$$\bar{a}^T x \leq \bar{b}, \tag{2.2}$$

where $\bar{a} \in \mathbf{R}^n$ and $\bar{b} \in \mathbf{R}$. A less than equal inequality can be converted into an equality by introducing an additional positive slack variable denoted $\bar{x} \in \mathbf{R}$ as follows

$$\begin{aligned} \bar{a}^T x + \bar{x} &= \bar{b}, \\ \bar{x} &\geq 0. \end{aligned} \tag{2.3}$$

Let $(x, \bar{x})$ be a feasible solution to (2.3), then

$$\bar{a}^T x = \bar{b} - \bar{x} \leq \bar{b},$$

because $\bar{x}$ is nonnegative. This implies a feasible solution to (2.3) is a feasible solution to the original inequality (2.2). Now if x is a feasible solution to (2.2), then

$$\bar{x} = \bar{b} - \bar{a}^T x \geq 0.$$

Hence, for a suitable chosen $\bar{x}$, then x is a feasible solution to (2.3).

In summary in terms of the x variables, then the constraints (2.2) and (2.3) has the same set of feasible solutions, which implies they are equivalent. Therefore, in general all less than equal inequalities can be converted to equalities by introducing one slack variable per inequality.

$\bar{x}$ is called a slack variable, because $\bar{a}^T x$ may be interpreted as the requirement of a given resource for a given x and $\bar{b}$ is the available quantity of the resource. Using this interpretation $\bar{x}$ denotes the unused part of the available resource or in other words the slack. If the slack is zero, then this is the same as the inequality is binding or the whole resource is used.

A greater than equal inequality

$$\bar{a}^T x \geq \bar{b} \tag{2.4}$$

can by converted into an equality by introducing an additional positive surplus variable denoted $\bar{x}$ as follows

$$\begin{aligned} \bar{a}^T x - \bar{x} &= \bar{b}, \\ \bar{x} &\geq 0. \end{aligned} \tag{2.5}$$

Using the same analysis as for the less than equal inequality it is easy to verify that (2.4) and (2.5) are equivalent.

2.1.3 Maximization

The standard problem is stated as a minimization problem, but it could equally well have been a maximization problem that is

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0. \end{array} \tag{2.6}$$

However, maximizing $c^T x$ subject to some constraints is equivalent to minimizing $-c^T x$ subject to the same constraints. These two problem must have the same set optimal solutions. Therefore, (2.6) is equivalent to

$$\begin{array}{ll} -\text{minimize} & -c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0. \end{array} \tag{2.7}$$

2.1.4 Free variables

A more general form than (P) is

$$\begin{array}{ll} \text{minimize} & c^T x + \hat{c}^T \hat{x} \\ \text{subject to} & Ax + \hat{A}\hat{x} = b, \\ & x \geq 0, \end{array} \tag{2.8}$$

where $\hat{x}$ is a vector of free variables. The variables $\hat{x}$ are denoted free variables, because they can take any value from minus infinity to plus infinity. This problem can be converted to standard form by using the transformation

$$\begin{array}{rcl} \hat{x} & = & \hat{x}^+ - \hat{x}^-, \\ \hat{x}^+, \hat{x}^- & \geq & 0, \end{array} \tag{2.9}$$

where $\hat{x}^+$ and x^- are two additional vector of variables. Clearly for any value of $\hat{x}$ it is possible to choose a value for $\hat{x}^+$ and $\hat{x}^-$ such that (2.9) holds. Therefore, the problem (2.8) is equivalent to

$$\begin{array}{ll} \text{minimize} & c^T x + \hat{c}^T(\hat{x}^+ - \hat{x}^-) \\ \text{subject to} & Ax + \hat{A}(\hat{x}^+ - \hat{x}^-) = b, \\ & x, \hat{x}^+, \hat{x}^- \geq 0. \end{array} \tag{2.10}$$

This technique for handling free variables is called splitting, because the free variables are split into their positive and negative part. In general the free variables should never be split in practice, because this increases the problem size.

2.2 Duality

In practice an LP problem is solved using a computer program. However, such a program only reports the optimal solution, but not the actual computations to find the optimal solution. **Therefore, how is optimality of the computed solution verified?** This is main question we are going to study in this section.

One way of checking optimality of a computed solution x^* is to check the condition

$$c^T x^* \leq c^T x, \ \forall x \in \mathcal{X} := \{x \in \mathbf{R}^n : \ Ax = b, \ x \geq 0\}.$$

However, this condition is very difficult to check, because $\mathcal{X}$ may contain an infinite number of elements.

An alternative way of proving optimality is to generate a lower bound denoted z on the objective value. By definition a valid lower bound z must satisfies the condition

$$c^T x \geq z, \quad \forall x \in \mathcal{X}.$$

Now clearly if a solution $x^* \in \mathcal{X}$ has been generated such that

$$c^T x^* = z,$$

then x^* is an optimal solution due to the fact z is lower bound on the optimal objective value. Note it is very easy to check whether a given feasible solution has the same objective value as the lower bound.

Now the main question is how to generate lower bounds. Fortuantely, in the subsequent section we will develop the so-called LP duality theory which give an method to generate valid lower bounds.

2.2.1 The dual problem

The problem

$$\begin{array}{lll} (P) & \text{minimize} & c^T x \\ & \text{subject to} & Ax = b, \\ & & x \geq 0. \end{array}$$

is denoted the primal problem and corresponding to each primal there is a so-called dual problem corresponding to (P) is

$$\begin{array}{lll} (D) & \text{maximize} & b^T y \\ & \text{subject to} & A^T y + s = c, \\ & & s \geq 0, \end{array}$$

where $y \in \mathbf{R}^m$ and $s \in \mathbf{R}^n$. y and s are called dual multipliers and dual slacks respectively. It can be seen that the dual problem is constructed from the same data as the primal problem, but uses different variables. Note there is one dual multiplier y_i associated with each constraint and one dual slack s_j for each variable x_j in the primal problem.

The s variables in (D) are essentially slack variables and can be removed. Hence, an alternative form of (D) is

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c. \end{array} \tag{2.11}$$

(y, s) is a dual feasible solution if it satisfies the constraints in (D). Similarly y is dual feasible if it satisfies the constraints of (2.11). A pair (x, y, s) is said to be primal and dual feasible if x and (y, s) is primal and dual feasible respectively. Such a pair is called a primal-dual feasible pair.

Any LP problem has a dual problem, because as demonstrated in the previous section, then an arbitrary LP can be converted to the standard form LP. Using this principle we study the dual corresponding to different LPs.

The first problem is

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b, \\ & x \geq 0. \end{array} \tag{2.12}$$

Introducing slack variables $\bar{x} \in \mathbf{R}^n$ gives the problem

$$\begin{array}{ll} \text{minimize} & (c;0)^T (x;\bar{x}) \\ \text{subject to} & [A\ I](x;\bar{x}) = b, \\ & x, \bar{x} \geq 0 \end{array} \tag{2.13}$$

stated on matrix from. The dual of this problem is

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c, \\ & y \leq 0. \end{array} \tag{2.14}$$

Similarly, it can be verified that the dual of

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b, \\ & x \geq 0 \end{array} \tag{2.15}$$

is

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c, \\ & y \geq 0. \end{array} \tag{2.16}$$

Hence, if the primal problem contains inequalities, then it implies that the dual variables y are not free. An $\leq$ ($\geq$) inequality implies that the corresponding dual multiplier y_i is $\leq$ ($\geq$)0.

Next study the problem

$$\begin{array}{ll} \text{minimize} & c^T x + \hat{c}^T \hat{x} \\ \text{subject to} & Ax + \hat{A}\hat{x} = b, \\ & x \geq 0, \end{array} \tag{2.17}$$

where $\hat{x}$ is a vector of free variables. Splitting the free variables leads to the problem

$$\begin{array}{ll} \text{minimize} & c^T x + \hat{c}^T (\hat{x}^+ - \hat{x}^-) \\ \text{subject to} & Ax + \hat{A}(\hat{x}^+ - \hat{x}^-) = b, \\ & x, \hat{x}^+, \hat{x}^- \geq 0, \end{array} \tag{2.18}$$

on standard form. The dual of this problem is

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c, \\ & \hat{A}^T y \leq \hat{c}, \\ & -\hat{A}^T y \leq -\hat{c}, \end{array} \tag{2.19}$$

Note that

$$\hat{A}^T y \leq \hat{c} \quad \text{and} \quad -\hat{A}^T y \leq -\hat{c}$$

implies $\hat{A}^T y = \hat{c}$. Hence, the dual is equivalent to

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c, \\ & \hat{A}^T y = \hat{c}. \end{array} \tag{2.20}$$

or alternatively

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & \hat{A}^T y + \hat{s} = \hat{c}, \\ & s \geq 0,\ \hat{s} = 0. \end{array} \tag{2.21}$$

In other words the dual slacks corresponding to the free variables should be zero.

Finally, let us investigate the dual problem corresponding to (D). (D) is equivalent to

$$\begin{array}{ll} -\text{minimize} & -b^T y \\ \text{subject to} & A^T y \leq c, \end{array} \tag{2.22}$$

and the dual corresponding to this problem is

$$\begin{array}{ll} -\text{maximize} & c^T \bar{x} \\ \text{subject to} & A\bar{x} = -b, \\ & \bar{x} \leq 0. \end{array} \tag{2.23}$$

Using the transformation $x = -\bar{x}$ we obtain

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0. \end{array} \tag{2.24}$$

The problem (2.24) is exactly the primal problem (P). Hence, the dual of the dual is equivalent to the primal problem. This symmetry between the primal and dual problem is occasionally very useful.

2.2.2 Properties of the dual problem

The first fact we will prove is that there is a close relationship between the optimal objective value of (P) and (D). Indeed given a dual feasible solution y, then $b^T y$ provides a lower bound on the optimal objective value to (P) as stated in the weak duality theorem 2.2.1.

Theorem 2.2.1 *(Weak duality) Let (x, y, s) be a primal-dual feasible pair, then*

$$c^T x \geq b^T y.$$

Proof: From feasibility we have

$$\begin{array}{rcll} Ax & = & b, & x \geq 0, \\ A^T y + s & = & c, & s \geq 0. \end{array}$$

Hence,

$$\begin{aligned} c^T x - b^T y &= c^T x - (Ax)^T y \\ &= (c - A^T y)^T x \\ &= s^T x \\ &\geq 0. \end{aligned}$$

The last equality follows from $x^T s = \sum_{j=1}^{n} x_j s_j$ and $x, s \geq 0$.

□

The difference

$$c^T x - b^T y$$

is denoted the **duality gap** and by Theorem 2.2.1 it is positive for any primal-dual feasible pair. Moreover, it follows from the proof of Theorem 2.2.1 that the dual gap is identical to the **complementary gap** $x^T s$ for any primal-dual feasible pair.

Now if a primal-dual feasible pair has zero duality gap, then x must be an optimal solution, because $c^T x$ is identical to the lower bound $b^T y$. Hence, it is impossible to find another primal feasible solution x which has better objective value than $b^T y$.

Corollary 2.2.1 *If (x^*, y^*, s^*) is a primal-dual feasible pair and $c^T x^* = b^T y^*$, then x^* is an optimal solution to (P).*

An obvious question to ask is if there always exists a a feasible primal-dual pair having zero duality gap, because then optimality is automatically by computing a primal-dual feasible pair having zero duality gap. The answer to question yes as we going to prove.

Now first study the problem

$$\begin{array}{ll} \text{minimize} & 0^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0. \end{array} \tag{2.25}$$

This problem is special in the sense its objective function is zero. The dual corresponding to (2.25) is

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq 0. \end{array} \tag{2.26}$$

It can be observed that any feasible solution to (2.25) is optimal and the optimal objective value is always zero. Moreover, (2.26) always has a feasible solution, because $y = 0$ is a feasible solution.

Now suppose that

$$\exists y: \ A^T y \leq 0, \ b^T y > 0,$$

then (2.25) cannot be feasible, because this is a contradiction to the weak duality theorem (Why?). Hence, we have proved one part of the following lemma.

Lemma 2.2.1 *(Farkas lemma). Either*

$$\exists x: \ Ax = b, \ x \geq 0$$

or

$$\exists y: A^T y \leq 0, \ b^T y > 0$$

is true.

Note that the Lemma says that exactly one of the statements is true. However, we have only proved that at most one of the them can be true. Unfortunately proving that exactly one of the statements is true is somewhat difficult and we will skip this part.

Farkas lemma is a very useful lemma, which we are going to use several times subsequently.

A natural question to investigate is under which conditions (D) is infeasible.

Lemma 2.2.2 *If (D) is infeasible, then (P) is either infeasible or unbounded.*

Proof: If (D) is infeasible, then

$$\nexists y: \quad A^T y \leq c$$

does not have solution. Equivalently we have

$$\nexists y^+, y^- : \quad A^T(y^+ - y^-) + s = c, \quad y^+, y^-, s \geq 0.$$

Using Farkas lemma this implies

$$\exists \bar{x} : \quad \begin{bmatrix} A \\ -A \\ I \end{bmatrix} \bar{x} \leq 0, \; c^T \bar{x} > 0,$$

which is identical to

$$\exists \bar{x} : \; A\bar{x} = 0, \; \bar{x} \leq 0, \; c^T \bar{x} > 0.$$

Hence, we obtain

$$Ad_x = 0 \quad \text{and} \quad c^T d_x < 0,$$

where $d_x := -\bar{x} \geq 0$. If (P) is infeasible, then the lemma is trivially true. On the other hand assume x^0 is a feasible solution to (P) then for all α sufficiently large we have that

$$\begin{aligned} A(x^0 + \alpha d_x) &= b, \\ x^0 + \alpha d_x &\geq 0, \end{aligned}$$

showing $x^0 + \alpha d_x$ is a primal feasible solution to (P). Moreover,

$$\lim_{\alpha \to \infty} c^T(x^0 + \alpha d_x) = -\infty$$

implying (P) is unbounded.

□

In summary if (P) has an optimal solution, then (P) is feasible and bounded, which implies (D) is feasible. Correspondingly, if (P) is unbounded, then (D) must be infeasible. The reason is the primal objective value can be decreased to minus infinity and at the same time the primal objective value is bounded below by $b^T y$ for any dual feasible y leading to a contradiction. Similarly, we can prove if (D) is unbounded, then (P) is infeasible. Hence, we have the lemma.

Lemma 2.2.3 *i. If (P) is unbounded, then (D) is infeasible.*

ii. If (D) is unbounded, then (P) is infeasible.

Suppose (P) has an optimal solution, which is equivalent to assuming (D) is feasible. Then the hypothesis is that a primal-dual feasible pair having a zero duality gap does not exists. Subsequently we prove this leads to contradiction. This hypothesis is equivalent to assuming that the system

$$\begin{array}{rcll} Ax & = & b, & x \geq 0, \\ A^T y + s & = & c, & s \geq 0, \\ c^T x - b^T y & \leq & 0, & \end{array} \tag{2.27}$$

is infeasible. Adding the slack variable κ to the inequality and splitting the free variables $y(= y^+ - y^-)$ we obtain the equivalent system

$$\begin{array}{rcll} Ax & = & b, & x \geq 0, \\ A^T(y^+ - y^-) + s & = & c, & s, y^+, y^- \geq 0, \\ c^T x - b^T(y^+ - y^-) + \kappa & = & 0, & \kappa \geq 0. \end{array} \tag{2.28}$$

Clearly, this system is also infeasible. Therefore, using Farkas lemma we obtain

$$\exists \mu, \lambda, \tau : \begin{bmatrix} A^T & & c \\ & A & -b \\ & -A & b \\ & I & \\ & & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \lambda \\ \tau \end{bmatrix} \leq 0, \quad b^T \mu + c^T \lambda > 0. \tag{2.29}$$

From which it follows

$$A\mu - b\tau \leq 0 \quad \text{and} \quad -A\mu + b\tau \leq 0,$$

which implies $A\mu - b\tau = 0$. This leads to the more compact system

$$\begin{array}{rcll} A^T \mu + c\tau & \leq & 0, & \\ A\lambda - b\tau & = & 0, & \lambda, \tau \leq 0, \\ b^T \mu + c^T \lambda & > & 0. & \end{array} \tag{2.30}$$

In the following we will investigate this system, where two different cases appear depending on the value of τ.

Case 1: Assume $\tau = 0$ then we have

$$\begin{array}{rcll} A^T \mu & \leq & 0, & \\ A\lambda & = & 0, & \lambda \leq 0, \\ b^T \mu + c^T \lambda & > & 0. & \end{array} \tag{2.31}$$

At least one of $b^T \mu$ or $c^T \lambda$ must be strictly positive. Assume $b^T \mu > 0$ and then from the fact $A^T \mu \leq 0$ and Farkas lemma, it follows (P) is infeasible. This is a contradiction to the assumption that (P) has an optimal solution. Similarly, it can be verified if $c^T \lambda > 0$, then this implies (D) is infeasible, which is also a contradiction to the assumptions.

Case 2: Suppose $\tau < 0$, then we have

$$\begin{array}{rcll} A^T \mu/\tau + c & \geq & 0, & \\ A\lambda/\tau - b & = & 0, & \lambda/\tau \geq 0, \\ b^T \mu/\tau + c^T \lambda/\tau & < & 0. & \end{array} \tag{2.32}$$

Note this system is equivalent to (2.30), where each side of the equations have been divided by τ. It is easy to verify that

$$(x, y, s) = (\lambda/\tau, -\mu/\tau, c + A^T\mu/\tau) \tag{2.33}$$

is a primal-dual feasible pair. From feasibility of the solution (2.33) and the weak duality theorem it follows

$$c^T x - b^T y = c^T\lambda/\tau + b^T\mu/\tau \geq 0,$$

which is a contradiction to (2.32). Hence, we obtain a contradiction again.

In summary if (P) and (D) are feasible, then the system (2.27) has a solution that is we have proved the following theorem.

Theorem 2.2.2 *(Strong duality) If (P) has an optimal solution, then a primal-dual feasible pair (x^*, y^*, s^*) exists such that $c^T x^* = b^T y^*$.*

The strong duality theorem is a very important theorem in LP, because it gives a method for checking optimality. Indeed if somebody claims (x^*, y^*, s^*) is an optimal primal-dual pair, then it is easy to check this claim by verifying the following three conditions are satisfied.

- Primal feasibility:
$$Ax^* = b,\ x^* \geq 0.$$
- Dual feasibility:
$$A^T y^* + s^* = c,\ s^* \geq 0.$$
- Optimality:
$$c^T x^* - b^T y^* = 0.$$

These three conditions can easily be checked, because they essentially reduces to performing some matrix with vector multiplications.

Virtually all algorithms for LP generates a dual solution either explicitly or implicitly. Hence, in practice the optimality of a computed solution can always be verified by checking primal and dual feasibility and that the dual gap is zero.

2.3 The complementarity conditions

An alternative formulation of the optimality conditions exists, because a primal-dual pair (x, y, s) satisfying

$$\begin{aligned} Ax &= b, \quad x \geq 0, \\ A^T y + s &= c, \quad s \geq 0, \\ x_j s_j &= 0, \quad j = 1, \ldots, n, \end{aligned} \tag{2.34}$$

is an optimal solution. These conditions contain three components, which is primal and dual feasibility and the complementarity conditions

$$x_j s_j = 0,\ j = 1, \ldots, n.$$

Assume (x, y, s) satisfies (2.34), then the duality gap is given by

$$\begin{aligned} c^T x - b^T y &= x^T s \\ &= \textstyle\sum_{j=1}^{n} x_j s_j \\ &= 0. \end{aligned}$$

Demonstrating any primal-dual pair satisfying (2.34) is an optimal primal-dual pair.

The complementarity conditions states that **not** both x_j and s_j can be nonzero at the same time in an optimal solution. Hence, if for example $x_j > 0$, then this implies the corresponding dual slack is zero that is $s_j = 0$. However, there may exists an optimal primal-dual pair such both $x_j = s_j = 0$.

A feasible primal-dual pair (x, y, s) satisfying the conditions

$$x_j s_j = 0 \quad \text{and} \quad x_j + s_j > 0$$

is said to be a strictly complementary solution, because either x_j or s_j is zero, but not both. Perhaps slightly unexpected it is possible to prove that any LP has a strictly complementary solution as stated in following theorem.

Theorem 2.3.1 *Assume both (P) and (D) are feasible, then a strictly complementary solution to (P) exists.*

We will not prove this theorem, but refer the reader to [17].

2.3.1 Interpretation of the dual variables

It should be clear by now that the dual problem is very important, because the optimal dual solution makes it possible to check optimality of a proposed primal solution easily.

However, the dual solution is not only useful with respect to verifying optimality, but it can be shown that the optimal dual variables conveys important information as demonstrated below.

Assume that the ith component of b_i denotes the available amount of a certain raw material in a production planning model and the objective function denotes the total production costs to be minimized. Moreover, assume that it is possible to buy more of the raw material for given price and more of this raw material may lead to a more efficient production. However, it is only worthwhile to buy the raw material if the reduction in the production cost is larger than the price that has to be paid for the raw material. Therefore, information about how the optimal objective value changes with changes in b_i is highly valuable in this case.

Any optimal solution much satisfy the optimality conditions implying an optimal primal-dual pair (x, y, s) must satisfy the feasibility conditions and

$$c^T x - b^T y = x^T s. \tag{2.35}$$

This implies (using sloppy mathematics) that

$$\frac{\partial c^T x}{\partial b_i} = \frac{\partial (b^T y + x^T s)}{\partial b_i} = y_i.$$

Hence, if the ith component of b_i is changed by Δb_i, then the optimal objective value is changed by

$$\Delta b_i y_i.$$

In other words it is not worthwhile to buy more of the raw material unless it can bought for a price less than $-y_i$. It should be emphasized that this is only true for sufficiently small values of Δb_i.

Once again using sloppy mathematics we have that

$$\frac{\partial c^T x}{\partial x_j} = \frac{\partial (b^T y + x^T s)}{\partial x_j} = s_j.$$

Hence, if the jth variable is changed by Δx_j, then the optimal objective value is changed by

$$\Delta x_j s_j.$$

In other words s_j is the change in the optimal objective value per unit change in x_j.

These observations helps motivating the complementarity conditions, because recall that the complementarity conditions are

$$x_j s_j = 0, \quad \text{for} \quad j = 1, \ldots, n,$$

In addition to (x, y, s) should be primal and dual feasible. These conditions implies if $x_j > 0$, then $s_j = 0$ must be true implying that the optimal objective value is not changed by a marginal change in x_j. Indeed if this was not the case, then either a small increase ($s_j < 0$) or decrease ($s_j > 0$) in x_j would lead to a decrease in the optimal objective value and hence contradicting the fact x is an optimal solution. (Note that the jth variable can be both increased or decreased without violating the positivity requirement, because $x_j > 0$).

Now if $x_j = 0$, then $\Delta x_j \geq 0$ is required to maintain feasibility. However, $s_j \geq 0$ implies nothing is gained by increasing x_j and hence verifying the optimality of x.

To conlude if the complementarity conditions are not satisfied, then it may be possible to obtain a better primal solution in terms of the objective value. Therefore, an optimal solution must satisfy the complementarity conditions.

Moreover, then y_i (s_j) can be interpretated as the change in the optimal objective value per unit increase in b_i (x_j).

This last observation is quite useful, when the dual problem to an arbitrary primal problem should be stated. Assume we are asked to state the dual problem to

$$\begin{array}{lllll} \text{maximize} & c^T x, & & & \\ \text{subject to} & A^1 x & \leq & b^1, & \\ & A^2 x & = & b^2, & \\ & A^3 x & \geq & b^3, & \\ & x & \geq & 0. & \end{array} \tag{2.36}$$

and let y^j for $j = 1, \ldots 3$ be dual multipliers for the three blocks of constraints respectively. Now it may be difficult to recall the sign restriction on y^j i.e. should y^j be positive or negative.

However, if for example b_i^1 is increased by $\Delta b_i^1 \geq 0$, then the set of feasible solutions to (2.36) is expanded. This implies that the optimal objective value must decrease or possible be unchanged (see excercise 1). Hence, if y_i^1 denote the change in the optimal objective value for an increase in b_i^1, then $y_i^1 \leq 0$. Using an identical argument for all the less than equal constraints we obtain $y^1 \leq 0$[1]. Moreover, using a similar argument we

[1]How would this statement change if the problem (2.36) was a maximization problem.

obtain $y^3 \geq 0$. Whereas if b^2 is changed nothing can be said about the direction of change in the optimal objective value i.e. y^2 can be both negative and positive. Now let us state the dual problem to (2.36) which is

$$\begin{array}{lll} \text{minimize} & \sum_{j=1}^{3} (b^j)^T y^j, & \\ \text{subjext to} & \sum_{j=1}^{3} A^j y^j + s & = \quad c, \\ & y^1 \leq 0, \quad y^3 \geq 0, & \end{array} \tag{2.37}$$

verifying that our intuitive argument for the sign restriction on y^j is correct.

2.4 Exercises

1. Define the problem

$$\begin{array}{ll} \text{minimize} & -4x_1 - 5x_2 \\ \text{subject to} & 2x_1 + 2x_2 \leq 20, \\ & 3x_1 + 7x_2 \leq 42, \\ & x_1, x_2 \geq 0. \end{array} \tag{2.38}$$

   (a) Find the optimal solution to (2.38).

   (b) State the dual problem to (2.38).

   (c) Find the optimal solution to the dual problem of (2.38).

   (d) Verify that the primal solution is optimal using the dual solution.

2. Let $\mathcal{X}^1, \mathcal{X}^2 \subset \mathbf{R}^n$ and define

$$\begin{array}{lll} z^j & = & \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in \mathcal{X}^j, \quad j = 1, 2. \end{array} \end{array} \tag{2.39}$$

   (a) Given $\mathcal{X}^1 \subseteq \mathcal{X}^2$ then prove that $z^1 \geq z^2$.

   (b) Define $\mathcal{X}^1 := \{x \in \mathbf{R}^n : Ax \leq b\}$ and $\mathcal{X}^2 := \{x \in \mathbf{R}^n : Ax \leq b + v\}$, where $v \geq 0$. Prove that $\mathcal{X}^1 \subseteq \mathcal{X}^2$.

   (c) Let $I^1 = \{1, \ldots, m\}$ and $I^2 \subseteq I^1$ and define $\mathcal{X}^j := \{x : A_{Ij:} x \leq b_{Ij}\}$ for $j = 1, 2$. ($A_{Ij:}$ is submatrix of A corresponding to rows indicies of I^j.) Prove that $\mathcal{X}^1 \subseteq \mathcal{X}^2$.

   (d) Given $X^1 \subseteq X^2$, then let $x^j \in X^j$ and $z^j = c^T x^j$ for $j = 1, 2$. Moreover, it is given $z^1 = z^2$ and x^2 is an optimal solution to (2.39) for $j = 2$, then prove x^1 is an optimal solution to (2.39) for $j = 1$.

Chapter 3

The simplex method

In the previous chapters the basic properties of LP have been studied, but of course LP is not interesting if we cannot solve LPs. Therefore, in this section we are going to present the primal simplex method for solution of an LP. This is the classical solution method for LP and it has been around since G. B. Dantzig invented it in the late forties [6, 7]. Indeed until approximately 1984 all LP problems were solved by the simplex method.

3.1 Derivation of the primal simplex method

The presentation of the primal simplex method deviates from the conventional practice, because the method is first presented from a general algebraic point view. This is followed by a geometric interpretation of the algebra. Usually the presentation is the other way around.

The problem of study is the primal LP problem

$$\begin{array}{lll} (P) & \text{maximize} & z = c^T x \\ & \text{subject to} & Ax = b, \\ & & x \geq 0. \end{array}$$

As in the previous chapter we assume there are m equalities and n variables. For simplicity we maintain the assumptions that $n \geq m$ and $\text{rank}(A) = m$.

The derivation of the simplex method is best started by introducing a partitioning of the variables into m basic and $n - m$ nonbasic variables which leads to the following definition.

Definition 3.1.1 *$(\mathcal{B}, \mathcal{N})$ is a basic and nonbasic partition of the variables if*

i) $\mathcal{B} \subseteq \{1, \ldots, n\}, \quad |\mathcal{B}| = m.$

ii) $\mathcal{N} = \{1, \ldots, n\} \setminus \mathcal{B}.$

iii) *rank*$(B) = m.$

Using the definition $B := A_{:\mathcal{B}} = [a_{:\mathcal{B}_1}, \ldots, a_{:\mathcal{B}_m}]$ *and similarly* $N := A_{:\mathcal{N}}$.

The variables in $\mathcal{B}$ and $\mathcal{N}$ are called basic and nonbasic variables respectively. B is denoted the basis and is by definition a nonsingular matrix. This is going to be important subsequently, but does there always exists a basic and nonbasic partition of the variables? As shown in the subsequent lemma a partition always exists if A is of full row rank.

Lemma 3.1.1 *Given rank$(A) = m$ then a basic and nonbasic partition of the variables exists.*

Proof: It is well-known that using Gaussian elimination, then m linearly independent columns from A can be identified. The variables corresponding to those columns are basis variables.

□

Assume $(\mathcal{B}, \mathcal{N})$ is a basic and nonbasic partition of the variables, then (P) is equivalent to

$$\begin{array}{ll} \text{maximize} & z = c_{\mathcal{B}}^T x_{\mathcal{B}} + c_{\mathcal{N}}^T x_{\mathcal{N}} \\ \text{subject to} & Bx_{\mathcal{B}} + Nx_{\mathcal{N}} = b, \\ & x_{\mathcal{B}}, x_{\mathcal{N}} \geq 0. \end{array} \tag{3.1}$$

The only difference between (P) and (3.1) is that the variables have been partitioned explicitly.

Using the equalities in (3.1) and that B is nonsingular the basic variables may be expressed as a function of the nonbasic variables that is

$$x_{\mathcal{B}} = B^{-1}(b - Nx_{\mathcal{N}}). \tag{3.2}$$

This implies that the nonbasic variables may be seen as *independent* variables, whereas the basic variables are *dependent* variables.

A key fact proved in the following theorem is, if (P) is feasible then the problem also has a feasible basic solution such that $x_{\mathcal{N}} = 0$. Hence, there exists a basis such that $B^{-1}b \geq 0$.

Theorem 3.1.1 *If (P) is feasible, then a basic and nonbasic partition of the variables exists such that*

$$(x_{\mathcal{B}}, x_{\mathcal{N}}) = (B^{-1}b, 0) \geq 0.$$

Proof: The theorem is proved by construction. Given the assumptions and Lemma 3.1.1 then a basic and nonbasic partition of the variables $(\mathcal{B}, \mathcal{N})$ exists. Furthermore, a feasible solution x^0 exists. Therefore, by assumption we have that

$$Ax^0 = Bx^0 + Nx^0 = b, \; x^0 \geq 0$$

which implies

$$x_{\mathcal{B}} = B^{-1}(b - Nx_{\mathcal{N}}^0) = x_{\mathcal{B}}^0.$$

Now define the set

$$\bar{\mathcal{N}} := \{j \in \mathcal{N} : x_j^0 > 0\},$$

which is called the set of superbasic variables. Clearly, if $\bar{\mathcal{N}} = \emptyset$, then the theorem is proved. Therefore, assume there is one or more superbasic variables and let $j \in \bar{\mathcal{N}}$. Now if the jth variable is decreased to zero, then the number of superbasic variables is decreased by one, but relation (3.2) shows this implies the basic variables should be changed too. Let us investigate this formally by introducing

$$x_{\mathcal{N}}^+ = x_{\mathcal{N}}^0 + \alpha e_j. \tag{3.3}$$

x^+ denotes the new solution, α is a step size scalar, and e_j is a special vector defined as follows

$$A_{:j} = Ne_j = N \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3.4}$$

Hence, e_j is identical to a zero vector except one component is identical to 1. The position of the one is chosen such that the relation (3.4) is satisfied. Therefore, when α in (3.3) is decreased, then x_j is decreased. It follows from relation (3.2) and (3.3) that

$$\begin{aligned} x_{\mathcal{B}}^+ &= B^{-1}(b - Nx_{\mathcal{N}}^+) \\ &= B^{-1}(b - N(x_{\mathcal{N}}^0 + \alpha e_j)) \\ &= B^{-1}(b - Nx_{\mathcal{N}}^0) + \alpha B^{-1}Ne_j \\ &= x_{\mathcal{B}}^0 - \alpha B^{-1}Ne_j. \end{aligned} \tag{3.5}$$

This relation shows the change in the basic (dependent) variables as a function α such that $Ax^+ = b$. In summary we have

$$\begin{bmatrix} x_{\mathcal{B}}^+ \\ x_{\mathcal{N}}^+ \end{bmatrix} = \begin{bmatrix} x_{\mathcal{B}}^0 \\ x_{\mathcal{N}}^0 \end{bmatrix} + \alpha \begin{bmatrix} -B^{-1}Ne_j \\ e_j \end{bmatrix}.$$

Now if α is decreased, then two cases may happen.

> **Case 1:** A basic variable hits its lower bound zero before the nonbasic variable j becomes zero. In which case the binding basic variable are made nonbasic at level zero and the nonbasic variable j is made basic.
>
> **Case 2:** x_j becomes zero before one of the basic variables hits its lower bound zero.

In both cases the number of superbasic variables is reduced by one. Therefore, after repeating this operation at most $|\bar{\mathcal{N}}|$ times we obtain a basic solution such that $x_{\mathcal{N}} = 0$.

Note we have not proved in Case 1 that the new basis matrix B is nonsingular as is required. However, this is proved later in connection with the derivation of the simplex method.

□

In conclusion Theorem 3.1.1 states that if an LP problem is feasible, then it has a feasible basic solution such that all the nonbasic variables are zero. It is also possible to prove that if (P) has an optimal solution, then an optimal solution exists such that it is a basic solution. Therefore, the main idea of the simplex method is to start from a feasible basic solution and then move to another feasible basic solution having a better (larger) objective value. In the following we will develop the specifics of this idea.

First using (3.2) the objective function in (3.1) can be rewritten as follows

$$\begin{aligned} z &= c^T x \\ &= c_{\mathcal{B}}^T x_{\mathcal{B}} + c_{\mathcal{N}}^T x_{\mathcal{N}} \\ &= c_{\mathcal{B}}^T B^{-1}(b - N x_{\mathcal{N}}) + c_{\mathcal{N}}^T x_{\mathcal{N}} \\ &= c_{\mathcal{B}}^T B^{-1} b + (c_n^T - c_{\mathcal{B}}^T B^{-1} N) x_{\mathcal{N}} \\ &= y^T b^T + (c_{\mathcal{N}}^T - y^T N)^T x_{\mathcal{N}} \\ &= b^T y + s_{\mathcal{B}}^T x_{\mathcal{B}} + s_{\mathcal{N}}^T x_{\mathcal{N}} \\ &= b^T y + s^T x, \end{aligned} \tag{3.6}$$

where we use the definitions[1]

$$y := B^{-T} c_{\mathcal{B}} \tag{3.7}$$

and

$$s := c - A^T y. \tag{3.8}$$

Note this definition implies

$$s_{\mathcal{B}} = c_{\mathcal{B}} - B^T y = 0.$$

s_j is referred to as *reduced cost* of the jth variable and it shows the change in the objective function per unit change in the x_j variable. Indeed differentiation gives

$$\frac{\partial z}{\partial x_j} = \frac{b^T y + x^T s}{\partial x_j} = s_j.$$

Hence, if for example $s_3 = 7$ and x_3 is increased by one unit, then the objective value is increased by 7 units. This is a very important observation because it shows that by increasing the value of a nonbasic variable having a positive reduced cost, then the objective value is increased. Due to the fact that the reduced cost of the basic variables are zero, then a change in those variables does not change the objective value. This implies by increasing a nonbasic variable having a positive reduced cost the objective value is increased. Even though the basic variables are changed to maintain feasibility.

Using these observations the principle of the simplex method can be stated as follows. First select a nonbasic variable having a positive reduced cost that is $s_j > 0$. Second increase this nonbasic variable as much as possible. In general the variable cannot be increased indefinitely, because the basic variables must remain nonnegative. Indeed due to (3.2), then an increase in a nonbasic variable may imply that one or more basic variables move towards their lower bounds. Third the value of the nonbasic variable is increased until a basic variable becomes zero. Fourth this basic variable and the profitable nonbasic variable are exchanged implying a change in the basic and nonbasic partition of the variables. Finally, these operations are repeated until all the reduced costs are nonpositive in which case the current basic solution is optimal, because no further improvement is possible.

Subsequently this idea is presented formally. First assume a feasible basic and nonbasic partition $(\mathcal{B}, \mathcal{N})$ of the variables is known such that

$$x_{\mathcal{B}}^0 = B^{-1} b \geq 0 \quad \text{and} \quad x_{\mathcal{N}}^0 = 0.$$

[1] $B^{-T} := (B^T)^{-1} = (B^{-1})^T$.

In general a feasible basic solution is of course not known, but in Section 3.2 it is shown how this assumption can be handled. Moreover, assume for some $j \in \mathcal{N}$ that $s_j > 0$. Hence, by increasing this variable the objective value z can be improved. Therefore, let

$$x_{\mathcal{N}}^{+} = x_{\mathcal{N}}^{0} + \alpha e_j,$$

then similar to the construction used in Theorem 3.1.1 we obtain

$$\begin{bmatrix} x_{\mathcal{B}}^{+} \\ x_{\mathcal{N}}^{+} \end{bmatrix} = \begin{bmatrix} x_{\mathcal{B}}^{0} \\ x_{\mathcal{N}}^{0} \end{bmatrix} + \alpha \begin{bmatrix} -B^{-1}Ne_j \\ e_j \end{bmatrix}.$$

Define

$$\begin{bmatrix} d_{x_{\mathcal{B}}} \\ d_{x_{\mathcal{N}}} \end{bmatrix} := \begin{bmatrix} -B^{-1}Ne_j \\ e_j \end{bmatrix},$$

then we may write compactly

$$x^{+} = x^{0} + \alpha d_x.$$

The interpretation is x^0 is the current solution and d_x is a search direction in which we look for a better solution. Clearly by moving in the direction d_x then feasibility should be maintained of the solution and the objective value should be improved. Let us investigate whether this is the case. First, we have

$$\begin{aligned} Ad_x &= Bd_{x_{\mathcal{B}}} + Nd_{x_{\mathcal{N}}} \\ &= -BB^{-1}Ne_j + Ne_j \\ &= 0, \end{aligned}$$

which shows d_x belongs to the null space of A. Therefore, we obtain

$$\begin{aligned} Ax^{+} &= A(x^0 + \alpha d_x) \\ &= Ax^0 + \alpha Ad_x \\ &= b. \end{aligned}$$

This shows the new point satisfies the equality constraints in (P) for all α and by choosing α sufficiently small the new point satisfies the inequalities in (P).

Furthermore, the new objective value is given by

$$\begin{aligned} z^{+} &= c^T x^{+} \\ &= c^T(x^0 + \alpha d_x) \\ &= c^T x^0 + \alpha c^T d_x \\ &= c_{\mathcal{B}}^T x_{\mathcal{B}}^0 + c_{\mathcal{N}}^T x_{\mathcal{N}}^0 - \alpha(c_{\mathcal{B}}^T B^{-1}Ne_j + c_{\mathcal{N}}^T e_j) \\ &= c_{\mathcal{B}}^T B^{-1}b + \alpha(c_{\mathcal{N}}^T - c_{\mathcal{B}}^T B^{-1}N)e_j \\ &= y^T b^T + \alpha(c_{\mathcal{N}}^T - y^T N)^T e_j \\ &= b^T y + \alpha s_{\mathcal{N}}^T e_j \\ &= b^T y + \alpha s_j, \end{aligned} \tag{3.9}$$

Therefore, if $s_j > 0$, then the increase in the objective value is proportional to s_j and α.

In summary we have verified that the new point satisfies the equality constraints in (P) and for a suitable α it also satisfies the inequalities. Moreover, it follows from (3.9) that a large α implies a large increase in the objective value. This leads to choose α maximal such that the new point satisfies the inequalities, which is the solution to the following problem

$$\begin{aligned} &\text{maximize} && \alpha \\ &\text{subject to} && x^{+} = x^0 + \alpha d_x \geq 0. \end{aligned} \tag{3.10}$$

The problem (3.10) may not have a finite solution. Indeed the problem (3.10) is unbounded if and only if $d_x \geq 0$. If this is the case we can conclude that the problem (P) is unbounded and d_x is an extreme ray along which the objective value tends to plus infinity. Assume this is not the case, then it can be observed that

$$d_{x_{\mathcal{N}}} = e_j \geq 0.$$

Hence, (3.10) is equivalent to

$$\begin{array}{ll} \text{maximize} & \alpha \\ \text{subject to} & x^0_{\mathcal{B}} \geq -\alpha d_{x_{\mathcal{B}}}. \end{array} \tag{3.11}$$

Let α^* denotes the optimal solution to this problem, then it can be computed as follows

$$\alpha^* = \min_i \{-x_{\mathcal{B}_i}/d_{x_{\mathcal{B}_i}} : \ d_{x_{\mathcal{B}_i}} < 0\}. \tag{3.12}$$

The final step of the primal simplex method is to determine which one of the basic variables that first hits its lower bound zero. Hence, for which $\mathcal{B}_i$ is

$$\alpha^* = -x_{\mathcal{B}_i}/d_{x_{\mathcal{B}_i}} \quad \text{and} \quad d_{x_{\mathcal{B}_i}} < 0.$$

This basic variable is denoted the **leaving** variable, because it is leaving the basis. The nonbasic variable that is introduced into basis is denoted the **entering** variable. Note several basic variables may be binding in which case any of those variables can be removed from the basis.

All the elements to state the simplex method has now been developed and therefore in Algorithm 3.1.1 the simplex method can be stated formally.

Algorithm 3.1.1

1. *Choose a basic and nonbasic partition* $(\mathcal{B}, \mathcal{N})$ *such that* $(x^0_{\mathcal{B}}, x^0_{\mathcal{N}}) = (B^{-1}b, 0) \geq 0$. $k := 0$.

2. $y^k := B^{-T} c_{\mathcal{B}}$.

3. *If*
$$\exists j^k \in \mathcal{N} : s^k_{j^k} = c_{j^k} - A^T_{:j^k} y^k > 0,$$
then continue else exit because x^k *is an optimal solution.*

4. *Let*
$$\begin{bmatrix} d_{x_{\mathcal{B}}} \\ d_{x_{\mathcal{N}}} \end{bmatrix} := \begin{bmatrix} -B^{-1}Ne_{j^k} \\ e_{j^k} \end{bmatrix}.$$

5. *If* $d_{x_{\mathcal{B}}} \geq 0$, *then terminate because* (P) *is unbounded.*

6. *Let*
$$\alpha^k := \min_i \{-x^k_{\mathcal{B}_i}/d_{x_{\mathcal{B}_i}} : \ d_{x_{\mathcal{B}_i}} < 0\}$$
and choose an $i^k \in \{i : \ d_{x_{\mathcal{B}_i}} < 0, \ \alpha^k = -x^k_{\mathcal{B}_i}/d_{x_{\mathcal{B}_i}}\}$.

7. $x^{k+1} = x^k + \alpha^k d_x$

8. $\mathcal{B} := (\mathcal{B} \setminus \{\mathcal{B}_{i^k}\}) \cup \{j^k\}, \quad \mathcal{N} := (\mathcal{N} \setminus \{j^k\}) \cup \mathcal{B}_{i^k}$.

9. *k := k+1.*

10. *Goto 2.*

Let us discuss each step of the primal simplex method in some detail. In step 1 an initial feasible basic solution is chosen. This is discussed in detail in Section 3.2. In step 2 the vector y is computed. In step 3 the entering variable is chosen. Clearly, there is a lot of freedom with respect to the choice of the entering variable. The inventor of the simplex method G. B. Dantzig suggests the choice

$$j^k = \text{argmax}_{j \in N}\{s_j = c_j - A_{:j}^T y\}.$$

Hence, he proposes to choose the nonbasic variable, which has the largest reduced cost. The motivation for this choice comes from (3.9), which indicates a larger reduced cost implies a larger increase in the objective value. Although it should be noted that the actual increase in the objective value also depends on α. In step 4 the search direction is computed and in step 5 it is checked, whether the problem is unbounded. If the problem is not unbounded, then a finite α^k is computed in step 6 and the leaving basic variable is determined. In step 7 and 8 the solution and the basic and nonbasic partition of the variables is updated. Finally, the iteration counter k is incremented and the whole process is repeated. Execution of the steps 2 to 10 of the primal simplex method is denoted one iteration.

There are several unresolved issues related to the primal simplex algorithm. They are

- Is the final solution optimal as claimed?
- Does the basis remains nonsingular?
- How is the method initialized?
- How many iterations are required to converge?

In the following sections we deal with these issues as well as a few others.

3.1.1 Is the optimal solution really optimal

Clearly, the simplex method is not of much use if the optimal solution is not really optimal. However, using LP duality it is easy to verify the optimality of the final solution. In the case the simplex method terminates in step 2, then a primal feasible basic solution is known such that

$$(x_{\mathcal{B}}^*, x_{\mathcal{N}}^*) = (B^{-1}b, 0) \geq 0 \text{ and } Ax^* = b.$$

Moreover,

$$y^* = B^{-T}c_{\mathcal{B}} \text{ and } (s_{\mathcal{B}}^*, s_{\mathcal{N}}^*) = (c_{\mathcal{B}} - B^T y^*, c_{\mathcal{N}} - N^T y^*) \leq 0$$

and it follows trivially $s_{\mathcal{B}}^* = 0$. This implies $(y, s) = (y^*, s^*)$ is a feasible solution to the dual problem coresponding to (P). Moreover, the duality gap is given by

$$b^T y^* - c^T x^* = (x^*)^T s^* = (x_{\mathcal{B}}^*)^T s_{\mathcal{B}}^* + (x_{\mathcal{N}}^*)^T (s_{\mathcal{N}}^*) = 0$$

proving x^* is an optimal solution.

3.1.2 The update of the basis matrix

Next we will investigate whether the basis matrix remains nonsingular in all the iterations of the primal simplex method.

We know that the initial basis matrix denoted B is nonsingular. Moreover, if B^+ denotes the new basis matrix, then by construction

$$\begin{aligned} B^+ &= B + [0, \ldots, 0, A_{:j} - B_{:i}, 0, \ldots, 0] \\ &= B + (A_{:j} - B_{:i})e_i^T. \end{aligned} \tag{3.13}$$

It is assumed that the ith basic variable leaves the basis and the jth variable enters the basis. e_i is the ith column of the identity matrix. Note (3.13) states the new basis is equivalent to the old basis, but the ith column is replaced by the column of the jth nonbasic variable.

(3.13) is equivalent to

$$\begin{aligned} B^+ &= B(I + [0, \ldots, 0, B^{-1}A_{:j} - B^{-1}B_{:i}, 0, \ldots, 0]) \\ &= B(I + [0, \ldots, 0, B^{-1}A_{:j} - e_i, 0, \ldots, 0]) \end{aligned}$$

because $B^{-1}B = I$ implies $B^{-1}B_{:i} = e_i$. Now define

$$\begin{aligned} E &:= (I + [0, \ldots, 0, B^{-1}A_{:j} - B^{-1}B_{:i}, 0, \ldots, 0] \\ &= [e_1, \ldots, e_{i-1}, B^{-1}A_{:j}, e_{i+1}, \ldots, e_m] \\ &= [e_1, \ldots, e_{i-1}, \tilde{A}_{:j}, e_{i+1}, \ldots, e_m] \\ &= \begin{bmatrix} 1 & & \tilde{A}_{1j} & & \\ & \ddots & \vdots & & \\ & & \tilde{A}_{ij} & & \\ & & \vdots & \ddots & \\ & & \tilde{A}_{mj} & & 1 \end{bmatrix} \end{aligned}$$

where

$$\tilde{A}_{:j} := B^{-1}A_{:j}.$$

The matrix E clearly has a special structure because it is almost identical to the identity matrix except the ith column has been replaced by another column $\tilde{A}_{:j}$. Assuming $\tilde{A}_{ij} \neq 0$, then

$$E^{-1} = \left[e_1, \ldots, e_{i-1}, e_i - \frac{1}{\tilde{A}_{ij}}(\tilde{A}_{:j} - e_i), e_{i+1}, \ldots, e_m \right],$$

which is verfied by the computation

$$EE^{-1} = \begin{bmatrix} 1 & & \tilde{A}_{1j} & & \\ & \ddots & \vdots & & \\ & & \tilde{A}_{ij} & & \\ & & \vdots & \ddots & \\ & & \tilde{A}_{mj} & & 1 \end{bmatrix} \begin{bmatrix} 1 & & -\tilde{A}_{1j}/\tilde{A}_{ij} & & \\ & \ddots & \vdots & & \\ & & 1/\tilde{A}_{ij} & & \\ & & \vdots & \ddots & \\ & & -\tilde{A}_{mj}/\tilde{A}_{ij} & & 1 \end{bmatrix} = I.$$

We are now ready to prove the following lemma.

Lemma 3.1.2 *B^+ is nonsingular.*

Proof: It follows from the above arguments that

$$B^+ = BE, \tag{3.14}$$

which implies

$$\det(B^+) = \det(B)\det(E).$$

B is nonsingular[2]. Now if E is nonsingular, then $\det(B^+) \neq 0$ implying B^+ is nonsingular. E^{-1} exists if and only if $\tilde{A}_{ij} \neq 0$, but

$$\tilde{A}_{ij} = e_i^T \tilde{A}_{:j} = e_i^T B^{-1} N e_j = e_i^T(-d_{x_{\mathcal{B}}}) = -d_{x_{\mathcal{B}_i}} > 0.$$

The strict inequality follows from the choice of the leaving basis variable in the primal simplex method. In conclusion both B and E are nonsingular and therefore B^+ is nonsingular.

□

As the simplex method is stated in Algorithm 3.1.1, then the inverse of B must be recomputed each iteration. However, from (3.14) it follows that

$$(B^+)^{-1} = (BE)^{-1} = E^{-1}B^{-1}.$$

The inverse of the new basis is given by

$$(B^+)^{-1} = E^{-1}B^{-1} = \begin{bmatrix} B_{1:}^{-1} - B_{i:}^{-1}\tilde{A}_{1j}/\tilde{A}_{ij} \\ \vdots \\ B_{(i-1):}^{-1} - B_{i:}^{-1}\tilde{A}_{(i-1)j}/\tilde{A}_{ij} \\ B_{i:}^{-1}/\tilde{A}_{ij} \\ B_{(i+1):}^{-1} - B_{i:}^{-1}\tilde{A}_{(i+1)j}/\tilde{A}_{ij} \\ \vdots \\ B_{m:}^{-1} - B_{i:}^{-1}\tilde{A}_{mj}/\tilde{A}_{ij} \end{bmatrix}. \tag{3.15}$$

This result has a nice interpretation. First form the augmented matrix

$$[B^{-1}\ \tilde{A}_{:j}] = \left[\begin{array}{c|c} B_{1:}^{-1} & \tilde{A}_{1j} \\ \vdots & \\ B_{i:}^{-1} & \tilde{A}_{ij} \\ \vdots & \\ B_{m:}^{-1} & \tilde{A}_{mj} \end{array}\right]. \tag{3.16}$$

The matrix (3.16) is identical to the inverse of the old basis augmented with the column $\tilde{A}_{:j}$. Now using the row operations known from Gaussian elimination on the augmented matrix to transform the augmenting column to e_i and we obtain

$$\left[\begin{array}{c|c} B_{1:}^{-1} - B_{i:}^{-1}\tilde{A}_{1j}/\tilde{A}_{ij} & 0 \\ \vdots & \vdots \\ B_{(i-1):}^{-1} - B_{i:}^{-1}\tilde{A}_{(i-1)j}/\tilde{A}_{ij} & 0 \\ B_{i:}^{-1}/\tilde{A}_{ij} & 1 \\ B_{(i+1):}^{-1} - B_{i:}^{-1}\tilde{A}_{(i+1)j}/\tilde{A}_{ij} & 0 \\ \vdots & \vdots \\ B_{m:}^{-1} - B_{i:}^{-1}\tilde{A}_{mj}/\tilde{A}_{ij} & 0 \end{array}\right] = [(B^+)^{-1}\ e_i].$$

[2] G is nonsingular, if and only if $\det(G) \neq 0$.

It easy to see that after the row operations have been performed, then the inverse of the new basis appears on the lefthand side in the augmented matrix. This demonstrates it is not necessary to compute the inverse of the basis from scratch in every iteration, but it can be updated cheaply from one iteration and to the next.

3.2 Initialization

3.2.1 The two-phase approach

The primal simplex method should be initiated with a feasible basic solution. In general such a solution is not known and it may not be easy to "guess" such a solution. Therefore, we need a systematic method for constructing an initial feasible basic solution.

The main idea of most methods for constructing an initial feasible basic solution is to formulate an artificial problem, which has a known feasible basic solution. Moreover, the optimal solution to the artificial problem is a feasible basic solution to (P). In the following one such method is suggested.

First define the matrix $\bar{A} \in R^{m \times m}$ given by

$$\bar{A}_{ij} := \begin{cases} 1, & i = j,\ b_i \geq 0, \\ -1, & i = j,\ b_i < 0, \\ 0, & \text{otherwise.} \end{cases}$$

The matrix $\bar{A}$ is clearly nonsingular because it is a square matrix having either 1 or -1 on its main diagonal. Moreover, let

$$\bar{x}_i^0 := \begin{cases} b_i, & b_i \geq 0, \\ -b_i, & b_i < 0. \end{cases}$$

Then by construction

$$\bar{A}\bar{x}^0 = b \quad \text{and} \quad \bar{x}^0 \geq 0.$$

Therefore, an artificial LP problem

$$\begin{array}{lrll} \text{maximize} & \bar{z} & = & 0^T x - e^T \bar{x} \\ \text{subject} & & & Ax + \bar{A}\bar{x} = b, \\ & & & x, \bar{x} \geq 0, \end{array} \tag{3.17}$$

can be formulated, where $e = [1, \ldots, 1]^T$. The problem (3.17) is almost identical to (P) except a new objective function and the **artificial** variables $\bar{x}$ have been introduced. The problem (3.17) is an LP having a known feasible solution, because $(x, \bar{x}) = (0, \bar{x}^0)$ is a feasible solution to (3.17). There are m artificial variables and $\bar{A}$ is nonsingular. Therefore, $\mathcal{B} = \{\bar{x}\}$ and $\mathcal{N} = \{x\}$ is a feasible basic solution to (3.17), because

$$B^{-1}b = \bar{A}^{-1}b = \bar{A}b = \bar{x}^0 \geq 0.$$

The objective function in (3.17) is identical to

$$e^T \bar{x} = \sum_{j=1}^{m} \bar{x}_j.$$

Hence, the purpose of (3.17) is to maximize the negative sum of the artificial variables. This is the same as minimizing the sum of the artificial variables. This sum is bounded below by zero because all the artificial variables are nonnegative variables. This leads to the conclusion that the purpose of the objective function in (3.17) is to force the artificial variables to zero. Let us make some further observations about (3.17):

i) It must be true that

$$\bar{z} \leq 0$$

for all feasible solutions to (3.17) that is the objective value in (3.17) is bounded from above by zero.

ii) Assume (3.17) has been solved to optimality using the simplex method and $(x^*, \bar{x}^*)$ is the optimal solution. Moreover, assume the optimal objective value is zero that is $\bar{z}^* = 0$. Then we have

$$\bar{z}^* = -\sum_{j=1}^{m} \bar{x}_j^* = 0 \quad \text{and} \quad x_j^* \geq 0,$$

which implies

$$\bar{x}_1^* = \ldots = \bar{x}_m^* = 0.$$

Therefore, we have

$$b = Ax^* + \bar{A}\bar{x}^* = Ax^* + \bar{A}0 = Ax^* \quad \text{and} \quad x^* \geq 0$$

showing x^* is a feasible solution to (P).

iii) Assume (P) has a feasible solution that is

$$\exists x^0 : \ Ax^0 = b, \ x^0 \geq 0.$$

Now let $(x, \bar{x}) = (x^0, 0)$ then we obtain

$$Ax + \bar{A}\bar{x} = Ax^0 + \bar{A}0 = Ax^0 = b \quad \text{and} \quad (x, \bar{x}) \geq 0.$$

Moreover, we have

$$\bar{z} = e^T\bar{x} = e^T 0 = 0.$$

Therefore if (P) has a feasible solution, then there exists an optimal solution to (3.17) with a zero objective value ($\bar{z} = 0$). It is known from i) that the objective value to (3.17) must be less than zero. Therefore, any feasible solution to (3.17) having objective value zero is an optimal solution, because it is not possible to find a solution with a better objective value. This leads to the conclusion that any feasible solution (P) is an optimal solution to (3.17).

We can now conclude given (P) has a feasible solution, then it follows from iii) that an optimal solution to (3.17) having zero objective value exists. Moreover, from ii) it follows that any feasible solution to (3.17) with zero objective value gives a feasible solution to (P). This implies if the problem (3.17) is solved to optimality, then the optimal objective value is zero if and only if (P) is feasible. Clearly, if (3.17) is solved using the simplex method, then the optimal solution is also a basic solution. Furthermore, if the optimal objective value is zero, then the optimal basic solution is a feasible basic solution to the

original problem (P). Hence, the optimal basic solution to the problem (3.17) may be used to initialize the simplex method.

This approach is called a two-phase method because in phase 1 we are searching for a feasible solution by solving the problem (3.17). If the problem (3.17) has an optimal objective value strictly less than zero, then we can conclude (P) is infeasible. If this is not the case, then the optimal solution to (3.17) may be used to initialize the simplex method leading to a second optimization phase. The purpose of the second phase is to search for an optimal solution to (P) starting from a feasible solution.

There are some loose ends of this two-phase approach, which we are now going to discuss.

The purpose of phase 1 is to drive the value of the artificial variables to zero because whenever all the artificial variables are zero, then a feasible solution to (P) is known. Hence, whenever an artificial variables becomes zero, then it is beneficial it remains zero. Therefore, if an artificial variable becomes nonbasic, then it is by construction zero. Moreover, if a nonbasic artificial variable is dropped from the problem the basis remains nonsingular and feasible. Hence, it is beneficial to drop artificial variables from the problem whenever they become nonbasic.

Second, it is important to note that the optimal basis after phase 1 may contain several artificial variables. These artificial variables is of course zero and should remain zero, because otherwise the solution becomes infeasible. However, the problem with the artificial variables in the initial basis does not cause any problems in phase 2 if the choice of the leaving variable is slightly modified. Recall the step size α is chosen maximal such that

$$x^0 + \alpha d_x \geq 0.$$

However, in this case α should be chosen maximal satisfying the constraints

$$\begin{aligned} x^0 + \alpha d_x &\geq 0, \\ \bar{x}^0 + \alpha d_{\bar{x}} &= 0 \end{aligned}$$

because the artificial variables should remain zero. This implies that whenever $\left|d_{\bar{x}_j}\right| > 0$ for an artificial variable $\bar{x}_j$, then this variable is removed from the basis and $\alpha = 0$. This small change of the algorithm secures that the artificial variables remain zero.

Finally, it should be noted that artificial variables are appended to the x variables. Hence, $x_{n+j} = \bar{x}_j$ for $j = 1, \ldots, m$.

3.2.2 A Big-M method

The approach in the previous section is called a two-phase method, because it consists of two separate phases. In the first phase a feasible basic solution is determined. Next starting from this feasible basic solution an optimal basic solution is computed in the second phase.

A disadvantage of the two phase approach is that the optimal solution to the phase 1 problem may be far from the optimal solution to (P) because any feasible solution to (P) is an optimal solution to (3.17). Therefore, it seems to be more efficient to combine phase 1 and phase 2 such that a feasible and optimal solution is obtained simultaneously.

This goal can be achieved by solving

$$\begin{array}{lll} \text{maximize} & z = & c^T x - M e^T \bar{x} \\ \text{subject} & & Ax + \bar{A}\bar{x} = b, \\ & & x, \bar{x} \geq 0, \end{array} \tag{3.18}$$

where M is a big positive constant. If M is chosen sufficiently large, then $\bar{x}$ will be zero in the optimal solution to (3.18) as desired.

The main disadvantage of this approach is it is difficult to choose the big M, because a very big M may introduce numerical problems such as rounding errors. On the other hand if M is chosen too small, then the artificial variables may not be forced to zero. Hence, we do not recommend the Big-M approach, but suggest to use the two phase approach presented in the previous section.

3.2.3 A numerical example

Before we head on to the remaining issues regarding the simplex method it is useful to study a small numerical example.

We use the example

$$\begin{array}{lrrcrcrcl} \text{maximize} & z = & 40x_1 & + & 30x_2 & & & & \\ \text{subject to} & & 2/5x_1 & + & 1/2x_2 & \leq & 20, \\ & & & - & 1/5x_2 & \geq & -5, \\ & & 3/5x_1 & + & 3/10x_2 & \leq & 21, \\ & & & & x & \geq & 0. \end{array}$$

First slack and surplus variables are introduced to convert the problem to standard form

$$\begin{array}{lrrcrcrcrcl} \text{maximize} & z = & 40x_1 & + & 30x_2 & & & & & & \\ \text{subject to} & & 2/5x_1 & + & 1/2x_2 & + & x_3 & = & 20, \\ & & & & -1/5x_2 & - & x_4 & = & -5, \\ & & 3/5x_1 & + & 3/10x_2 & + & x_5 & = & 21, \\ & & & & & & x & \geq & 0. \end{array} \tag{3.19}$$

The first step of the simplex algorithm is to setup the phase 1 problem by introducing artificial variables as follows

$$\begin{array}{lrcl} \text{maximize} & -x_6 - x_7 - x_8 & & \\ \text{subject to} & 2/5x_1 + 1/2x_2 + x_3 + x_6 & = & 20, \\ & -1/5x_2 - x_4 - x_7 & = & -5, \\ & 3/5x_1 + 3/10x_2 + x_5 + x_8 & = & 21, \\ & x & \geq & 0. \end{array} \tag{3.20}$$

x_6, x_7, x_8 are the artificial variables.

Iteration 0: The standard approach is to let the initial basis consist of all artificial variables. However, recall we can start from any feasible basis. One such basis is $\mathcal{B} = \{3, 7, 5\}$ and $\mathcal{N} = \{1, 2, 4\}$. Starting from this basis reduces the number of iterations spend in phase 1. However, we leave one artificial variable (x_7) in basis to illustrate how phase 1 works. Both x_6 and x_8 are nonbasic artificial variables and therefore they are dropped from the problem.

Please note in phase 1 we use the objective function $c = [0\ 0\ 0\ 0\ 0\ -1\ -1\ -1]^T$ because we are solving the phase 1 problem (3.20).

In summary

$$\mathcal{B} = \{3, 7, 5\}, \quad \mathcal{N} = \{1, 2, 4\}, \quad x^0_{\mathcal{B}} = \begin{bmatrix} x^0_3 \\ x^0_7 \\ x^0_5 \end{bmatrix} = \begin{bmatrix} 20 \\ 5 \\ 21 \end{bmatrix}, \quad z^0 = -5,$$

and

$$B = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix}, \quad B^{-1} = B.$$

First step is to compute

$$y^0 = B^{-T} c_{\mathcal{B}} = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Next the reduced cost of the nonbasic variables are computed

$$s^0_{\mathcal{N}} = \begin{bmatrix} s^0_1 \\ s^0_2 \\ s^0_4 \end{bmatrix} = c_{\mathcal{N}} - N^T y^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 2/5 & & 3/5 \\ 1/2 & -1/5 & 3/10 \\ & -1 & \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/5 \\ 1 \end{bmatrix}.$$

The entering variable is chosen among the nonbasic variable having a positive reduced cost. Both the second and the third nonbasic variable have positive reduced cost and they are therefore possible candidates to enter the basis. We choose the variable x_4 to enter the basis because it has the largest positive reduced cost. Hence, the entering variable is x_4. This gives

$$d_{x_{\mathcal{B}}} = -B^{-1} A_{:4} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad \alpha^0 = \min\{-, -5/(-1), -\} = 5.$$

The second basic variable is binding. Hence, the leaving variable is x_7. Finally, the solution

$$x^0_{\mathcal{B}} + \alpha^0 d_{x_{\mathcal{B}}} = \begin{bmatrix} 20 \\ 0 \\ 21 \end{bmatrix}, \quad x^0_4 = 5, \quad B = \{3, 4, 5\}, \quad \mathcal{N} = \{1, 2\}.$$

is updated. Note the artificial variable x_7 now becomes nonbasic and can therefore be discarded from the problem by not including the variable in $\mathcal{N}$.

Iteration 1: It can be seen that the phase 1 objective value is zero implying that the value of the artificial variables have been driven to zero. Indeed all the artificial variables are nonbasic. Therefore, we now have a feasible basic solution to the original problem. Hence, we may terminate phase 1 and start phase 2. Phase 2 is identical to phase 1 except we now use the real c. Hence, we now use $c = [40\ 30\ 0\ 0\ 0]^T$. Moreover, we do not want the artificial variables to become nonzero again, which in turn implies that they are not allowed to enter the basis.

Therefore, the artificial variables are dropped from the problem from now on by not including them in $\mathcal{N}$.

In summary

$$B = \{3,4,5\}, \quad \mathcal{N} = \{1,2\}, \quad x^1_{\mathcal{B}} = \begin{bmatrix} x^1_3 \\ x^1_4 \\ x^1_5 \end{bmatrix} = \begin{bmatrix} 20 \\ 5 \\ 21 \end{bmatrix}, \quad z^1 = 0,$$

and

$$B = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix}, \quad B^{-1} = B,$$

so

$$y^1 = B^{-T} c_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad s^1_{\mathcal{N}} = \begin{bmatrix} 40 \\ 30 \end{bmatrix}.$$

The first nonbasic variable is chosen as the entering variable because it has the largest positive reduced cost. Entering variable: x_1. Hence,

$$d_{x_{\mathcal{B}}} = \begin{bmatrix} -2/5 \\ 0 \\ -3/5 \end{bmatrix}, \quad \alpha^1 = \min\{20/(2/5), -, 21/(3/5)\} = 35,$$

and the third basic variable are binding. Leaving variable: x_5. Update

$$x^1_{\mathcal{B}} + \alpha^1 d_{x_{\mathcal{B}}} = \begin{bmatrix} 6 \\ 5 \\ 0 \end{bmatrix}, \quad x^1_1 = 35, \quad \mathcal{B} = \{3,4,1\}, \quad \mathcal{N} = \{2,5\}.$$

Iteration 2:

$$\mathcal{B} = \{3,4,1\}, \quad \mathcal{N} = \{2,5\}, \quad x^2_{\mathcal{B}} = \begin{bmatrix} x^2_3 \\ x^2_4 \\ x^2_1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 35 \end{bmatrix}, \quad z^2 = 1400,$$

and

$$B = \begin{bmatrix} 1 & & 2/5 \\ & -1 & \\ & & 3/5 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & & -2/3 \\ & -1 & \\ & & 5/3 \end{bmatrix},$$

so

$$y^2 = B^{-T} c_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 66\frac{2}{3} \end{bmatrix}, \quad s^2_{\mathcal{N}} = \begin{bmatrix} 10 \\ 66\frac{2}{3} \end{bmatrix}.$$

Entering variable: x_2 ($s_2 > 0$). Hence,

$$d_{x_{\mathcal{B}}} = \begin{bmatrix} -3/10 \\ -1/5 \\ -1/2 \end{bmatrix}, \quad \alpha^2 = \min\{6/(3/10), 5/(1/5), 35/(1/2)\} = 20.$$

Leaving variable: x_3. So,

$$x^2_{\mathcal{B}} + \alpha^2 d_{x_{\mathcal{B}}} = \begin{bmatrix} 0 \\ 1 \\ 25 \end{bmatrix}, \quad x^2_1 = 20.$$

Before we proceed to the next iteration we will update the inverse of the basis using the method presented in Section 3.2.3:

$$\left[\begin{array}{ccc|c} 1 & & -2/3 & 3/10 \\ & -1 & & 1/5 \\ & & 5/3 & 1/2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 10/3 & & -2\frac{2}{9} & 1 \\ -2/3 & -1 & 4/9 & \\ -5/3 & & 2\frac{7}{9} & \end{array}\right].$$

Iteration 3:

$$\mathcal{B} = \{3, 4, 1\}, \quad \mathcal{N} = \{2, 5\}, \quad x^3_{\mathcal{B}} = \begin{bmatrix} x^3_2 \\ x^3_4 \\ x^3_1 \end{bmatrix} = \begin{bmatrix} 20 \\ 1 \\ 25 \end{bmatrix}, \quad z^3 = 1600,$$

and

$$B = \begin{bmatrix} 1/2 & & 2/5 \\ -1/5 & -1 & \\ 3/10 & & 3/5 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 10/3 & & -2\frac{2}{9} \\ -2/3 & -1 & 4/9 \\ -5/3 & & 2\frac{7}{9} \end{bmatrix}.$$

So,

$$y^3 = B^{-T} c_{\mathcal{B}} = \begin{bmatrix} 33\frac{1}{3} \\ 0 \\ 44\frac{4}{9} \end{bmatrix}, \quad s^3_{\mathcal{N}} = \begin{bmatrix} -33\frac{1}{3} \\ -44\frac{4}{9} \end{bmatrix}.$$

Now all the reduced costs are negative, so the current solution is optimal.

The optimal solution is presented in the following summary:

$$x = \begin{bmatrix} 25 \\ 20 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad y = \begin{bmatrix} 33\frac{1}{3} \\ 0 \\ 44\frac{4}{9} \end{bmatrix}, \quad s = \begin{bmatrix} 0 \\ 0 \\ -33\frac{1}{3} \\ 0 \\ -44\frac{4}{9} \end{bmatrix}, \quad z = 1600.$$

Let us end this section by noting that solving LP problems by hand is one of the less entertaining exercises. Indeed the process is prune to error and is time consuming. Fortunately, in practice it is much more efficient to solve LP problems using a computer. However, when solving LP problems by hand it beneficial to check the results carefully at the end of each iteration. For example by checking if $Bx_{\mathcal{B}} = b$, $BB^{-1} = I$ and so forth. This helps spotting an error early in the solution process.

3.2.4 Geometric interpretation

The numerical example presented in the previous section is essentially in two variables. Hence, it is possible to illustrate the path taking by the simplex method from the initial solution and to the optimal solution in a two dimensional diagram.

Table 3.1 presents the relevant information with respect to the two variables x_1 and x_2 for each iteration. First the value of each variable at the beginning of each iteration is shown. Second the value of the search direction for those two variables is shown.

In Figure 3.1 the same information is presented graphically. It can be observed:

Iteration(k)	x_1^k	x_2^k	d_{x_1}	d_{x_2}
1	0	0	0	0
2	0	0	1	0
3	35	0	-1/2	1
4	25	20	0	0

Table 3.1: Iteration information.

Figure 3.1: The solution path.

- The shaded region is the feasible region of the LP example.
- All the solutions x^k generated by the simplex method are corner point solutions. This fact is in general true.
- The search direction d_x is an edge of the feasible region.

3.3 Convergence and complexity

An important question is how many iterations the simplex method requires to solve an LP. This the main issue for this section.

If x^0 and x^+ is the current and new solution respectively, then if the step size α is positive it follows from the previous sections that

$$z = c^T x^0 < c^T x^+ = z^+.$$

Hence, the objective value is strictly increased. Clearly, this implies that x^0 and x^+ are two different basic solutions. Now assume that α^k is positive in all iterations then we obtain

$$c^T x^1 < c^T x^2 < \ldots < c^T x^k$$

and that none of iterates x^k are identical to another iterate. Therefore, in the case the step size is strictly positive in all iterations, then the simplex method generates a sequence different basic solutions. Hence, no basic solution are repeated. Furthermore, the number of basic solutions is bounded by

$$C_{n,m} = \left(\begin{array}{c} n \\ m \end{array} \right) = \frac{n!}{m!(n-m)!} \tag{3.21}$$

because corresponding to each basic solution is a basis. Note each basis consists of m variables taken from the n variables. Hence, in the worst case $C_{n,m}$ basic solutions exists.

Unfortunately the quantity $C_{n,m}$ grows very fast as a function of n and for example $C_{30,10} \approx 10^7$, which implies that the simplex method may require 10^7 iterations to solve an LP having 10 equality constraints and 30 variable. Fortunately, in practice the simplex method requires significantly fewer iterations. One reason is that whenever a basic solution having a certain objective value is reached, then the simplex method never evaluates a basic solution having a lower objective value because the objective value is nondecreasing in each iteration. This implies after each iteration of the simplex method many basic solution are effectively excluded from any further consideration.

In summary if the step size is positive in all the iterations, then the simplex method is finitely convergent. Although the number of iterations in the worst case may be very large. Unfortunately the assumption about a strictly positive step size α may not be satisfied, because

$$\alpha^k = \min_i \{-x^k_{\mathcal{B}_i}/d_{x_{\mathcal{B}_i}} : \ d_{x_{\mathcal{B}_i}} < 0\}.$$

Therefore, if one or more of the basic variables are zero, then α^k may be zero. A basic solution, where one or more of the variables in $x_{\mathcal{B}}$ is zero, is said to be degenerate. Hence, if the current solution is degenerate, then it cannot be guaranteed that α^k is positive and the simplex method converge. Indeed it is possible to create an LP such that the simplex method repeats a finite sequence of basic solutions. In which case the simplex method is said to be cycling. Cycling is a rare phenomen in practice, but most large scale problems are degenerate to some degree.

It is possible to suggest rules for the choice of the entering and leaving variable such that cycling is prevented.

3.4 Extensions

3.4.1 General bounds

In Section 3.1 the simplex method is developed for problems on standard form. This development is without loss of generality, because any problem can be put on standard form using some simple transformations. However, in this section we modify the simplex method to solve a slightly more general problem. The advantage of the modification is a potential large computational saving in the computational cost for some LP problems.

One particular special problem structure occuring frequently in practice is simple upper bounds such as

$$x_j \leq 100.$$

Such a constraint can be handled by introducing an additional constraint in the LP problem and hence leads to an increase in the row dimension of A by one.

Another type of variable is a free variable that is a variable without any bounds. For example if a variable denotes the change in a certain quantity, then it can assume both negative and positive values. Hence, it is free. Free variables can be handle using the splitting free variable technique leading to the introduction of more variables.

In summary both upper bounded and free variables leads to an increase in the problem dimension and hence to an increase in the computational cost of solving the problem. In

the following it is shown that this increase in the dimension can be avoided if the simplex method is modified to deal with a more general problem.

Therefore, we will study solution of the more general problem

$$\begin{array}{ll} \text{maximize} & z = c^T x \\ \text{subject to} & Ax = b, \\ & l \leq x \leq u, \end{array} \tag{3.22}$$

where $l, u \in \mathbf{R}^m$ are the vectors of lower and upper bounds respectively. The values in l and u may assume any value including plus and minus infinity i.e. $l_j = -\infty$ and $u_j = \infty$ are valid assignments. However, for simplicity we will assume that $l \leq u$ because otherwise the problem is trivially infeasible.

First step of deriving the simplex method for LPs with general bounds is to partition the variables into basic and nonbasic variables $(\mathcal{B}, \mathcal{N})$ such that the basis matrix is nonsingular. However, it is **not** assumed that the nonbasic variables have the value zero. Indeed the only requirement is that they satisfies their bounds that is

$$l_{\mathcal{N}} \leq x_{\mathcal{N}} \leq u_{\mathcal{N}}.$$

In general it is advantageous to let the nonbasic variables be identical to one of their bounds provided the bound is finite. Otherwise for free variables zero is suitable value.

By construction we have

$$x_{\mathcal{B}} = B^{-1}(b - Nx_{\mathcal{N}}),$$

and such a solution is feasible if

$$l_{\mathcal{B}} \leq x_{\mathcal{B}} \leq u_{\mathcal{B}}.$$

In the ordinary simplex method the reduced costs are used to deduce which of the nonbasic variables should enter the basis. Therefore, let us compute the reduced cost

$$\begin{array}{rcl} z & = & c^T x \\ & = & c_{\mathcal{B}}^T x_{\mathcal{B}} + c_{\mathcal{N}}^T x_{\mathcal{N}} \\ & = & c_{\mathcal{B}}^T B^{-1}(b - Nx_{\mathcal{N}}) + c_{\mathcal{N}}^T x_{\mathcal{N}} \\ & = & b^T y + (c_{\mathcal{N}} - y^T N)^T x_{\mathcal{N}} \\ & = & b^T y + s_{\mathcal{N}}^T x_{\mathcal{N}} \\ & = & b^T y + s^T x, \end{array} \tag{3.23}$$

where y and s are defined by (3.7) and (3.8) respectively.

The interpretation of (3.23) is that a reduced cost s_j shows the change in the objective value per unit change in a variable x_j. This implies if a reduced cost is positive, then the variable should be increased. On the other hand if a reduced cost of a variable is negative, then the variable should be decreased. Therefore, if the jth nonbasic variable has a positive reduced cost $s_j > 0$ and $x_j < u_j$, then it is possible to increase this variable leading to an increase in the objective value. Note if $x_j = u_j$, then it is not possible to increase the variable and therefore even though the reduced cost is positive the variable is not a candidate for entering the basis. Similarly, if the reduced cost is negative ($s_j < 0$) and $x_j > l_j$, then by decreasing this variable the objective value can be improved. This gives the conditions for choosing the entering nonbasic variable.

Whenever an entering variable j has been chosen, then the search direction must be computed by

$$\begin{bmatrix} d_{x_{\mathcal{B}}} \\ d_{x_{\mathcal{N}}} \end{bmatrix} := \begin{bmatrix} -B^{-1}Ne_j \\ e_j \end{bmatrix}.$$

Hence, the search direction is unchanged from previously. Next the new point is given by

$$x^+ = x^0 + \alpha d_x,$$

for a suitable chosen step size α. Clearly, the step size should be chosen such that the new solution x^+ remains feasible i.e.

$$l \leq x^+ \leq u.$$

Therefore, if s_j is positive (negative) α should be maximized (minimized) subject to maintaining feasibility because this leads to the largest improvement in the objective value. This implies the optimal step size α is given as the optimal solution to the problem

$$\begin{array}{ll} \text{maximize} & s_j\alpha \\ \text{subject to} & u \geq x^0 + \alpha d_x \geq l. \end{array} \tag{3.24}$$

The optimal solution to (3.24) may be unbounded in which case the problem (3.22) is unbounded. If this is not the case, then at least one variable must hit one of its bounds, when α is changed. If the binding variable is a basic variable, then the entering nonbasic variable and the binding basic variable are exchanged as usual. However, it can occur that the binding variable is the nonbasic variable itself. Indeed if a variable both has a finite lower and upper bound and its current value is identical to the lower bound, then it may happen that the constraint

$$x_j^0 + \alpha d_{x_j} \leq u_j$$

is the binding constraint in (3.24). This implies that the value of the nonbasic variable is moved from its lower bound and to the upper bound. In this case the basic and nonbasic partition of the variables is unchanged. Such an operation is called a *move*. The reverse may of course also happen that is a variable moves from its upper bound and to its lower bound.

An important observation is that if a variable is free that is $-l_j = u_j = \infty$, then the constraint corresponding to this variable in (3.24) can never be binding. The consequence is if a free variable has entered the basis it will never leave the basis again.

Based on the previous discussion the primal simplex method is restated for the case with general bounds in Algorithm 3.4.1.

Algorithm 3.4.1

1. *Choose a basic and nonbasic partition* $(\mathcal{B},\mathcal{N})$ *such that* x^0 *satisfies* $u \geq x^0 \geq l$ *and* $Ax = b$. $k := 0$.

2. $y^k := B^{-T}c_{\mathcal{B}}, \quad s^k := c - A^T y^k$.

3. *Choose* j^k *such that*

$$j^k \in \{j \in \mathcal{N} : s_j^k > 0,\ x_j^k < u_j\}$$

or

$$j^k \in \{j \in \mathcal{N} : s_j^k < 0,\ x_j^k > l_j\}$$

if possible. Otherwise terminate (optimal).

4. *Let*

$$\begin{bmatrix} d_{x_{\mathcal{B}}} \\ d_{x_{\mathcal{N}}} \end{bmatrix} := \begin{bmatrix} -B^{-1}Ne_{j^k} \\ e_{j^k} \end{bmatrix}.$$

5. *Let* α^k *be the optimal solution to*

$$\begin{array}{ll} \textit{maximize} & s_{j^k}\alpha \\ \textit{subject to} & u \geq x^k + \alpha d_x \geq l. \end{array}$$

6. *If* $\left|\alpha^k\right| = \infty$, *then terminate (unbounded).*

7. *If* $s^k_{j^k} > 0$ *then*

$$\mathcal{L} := \{j : l_j = x^k_j + \alpha^k d_{x_j},\ d_{x_j} < 0\} \cup \{j : u_j = x^k_j + \alpha^k d_{x_j},\ d_{x_j} > 0\}$$

else

$$\mathcal{L} := \{j : l_j = x^k_j + \alpha^k d_{x_j},\ d_{x_j} > 0\} \cup \{j : u_j = x^k_j + \alpha^k d_{x_j},\ d_{x_j} < 0\}.$$

8. $x^{k+1} := x^k + \alpha^k d_x$

9. *If* $j^k \notin \mathcal{L}$ *then*

   Choose $\mathcal{B}_{i^k} \in \mathcal{L}$.

   $\mathcal{B} := (\mathcal{B} \setminus \{\mathcal{B}_{i^k}\}) \cup \{j^k\}, \quad \mathcal{N} := (\mathcal{N} \setminus \{j^k\}) \cup \{\mathcal{B}_{i^k}\}.$

10. *k := k+1.*

11. *Goto 2.*

Next Algorithm 3.4.1 is discussed in some detail.

In step 1 an initial feasible basic solution is chosen. In step 3 the entering nonbasic variable is chosen or it is concluded that the current solution is optimal. In step 4 the search direction is computed and this is followed by computing the optimal step size in step 5. In step 6 it is checked, whether the optimal step size is finite and if not it is concluded that (3.22) is unbounded. In step 7 the set $\mathcal{L}$ is computed, which is the set of binding variables. In step 8 the solution is updated and in step 9 the partition of the variables is updated if necessary. Finally, the iteration counter is incremented and the whole process is repeated.

Note that both upper bounded variables and free variables is handled by Algorithm 3.4.1 without introducing additional constraints or variables. Hence, the problem dimension is not increased. Moreover, in the case of free variables, then whenever a free variable has entered the basis it will never leave it again. This is likely to lead to fewer iterations. Note in particular that a free variable cannot be degenerate and hence many free variables in the problem leads to less degeneracy. Therefore, it advantageous to formulate LP problems with many free variables if the problem is solved using the simplex method.

3.5 Reoptimization and warm-start

In practice an LP problem is solved several times slightly modified for example to evaluate the optimal solutions sensitivity to to certain parameters. Clearly, one method for doing the reoptimization is to solve the modified problem from scratch each time. However, in many cases it is possible to do better.

For example assume that the objective function c is changed. This implies that the optimal basic solution to the previous problem is still feasible but not necessarily optimal. Hence, starting from the previous optimal basis, then the phase 1 can be completely avoided. Similarly, if a new variable is introduced into the problem, then the old optimal basis is still a feasible basis. Hence, reoptimization from the old optimal basis is likely to faster than starting from scratch.

However, in the case b is modified, then the old basis is not necesarily feasible. However, if b is also slightly modified, then it can be expected that the old optimal basis is nearly feasible to the new problem. Therefore, it should be advantageous to start phase 1 from the old optimal basis.

3.6 References

The presentation of the simplex method in this chapter is based on the ideas presented in Nazareth [15]. Nazareth also presents in great details the issues related to implementing the simplex method on a computer.

Furthermore, for readers interested in further details about the simplex method, then the book of Chvátal [5] is recommended.

Chapter 4

Sensitivity analysis

A basic assumption in LP is that the parameters of the problem is known and fixed. This is clearly not always the case in practice. For example a c_j may reflect price, which is likely to fluctuate over time. Therefore, it is important to investigate the sensitivity of the optimal solution to changes in the parameters.

The kind of information which is useful is the optimal solutions sensitivity to changes in the parameters. Indeed it might be that even a large change in a parameter only leads to a small change in the optimal solution. The reverse might also be the case that is a small change in a parameter leads to a large change in the optimal solution. Such a parameter is said to be critical and more attention should be given to the critical parameters.

One way of studying changes in the optimal solution given a change in the data is of course to modify the problem and then reoptimize. The difference in the optimal solution to the modified problem and the original problem is measure for the sensitivity to changes in the data. Unfortunately such a sensitivity analyzes is computationally expensive, because the problem has to be reoptimized for each set possible parameters. Actually this is not neccesarily the case any more because large LPs can solved fast and cheaply on todays fast PCs.

4.1 Sensitivity analyses based on an optimal basic and nonbasic partition

The main question dealt with in this section is how much a component in c or b can change before the optimal basic and nonbasic partition of the variables changes.

If (P) has been solved to optimality, then an optimal basic and nonbasic partition $(\mathcal{B},\mathcal{N})$ of the variables is known such that

$$\begin{array}{rcll} x^*_{\mathcal{B}} & = & B^{-1}b \geq 0, & x^*_{\mathcal{N}} = 0, \\ y^* & = & B^{-T}c_{\mathcal{B}}, & \\ s^*_{\mathcal{N}} & = & c_{\mathcal{N}} - N^T y^* \leq 0, & s^*_{\mathcal{B}} = 0. \end{array} \tag{4.1}$$

This follows from the termination criteria of the primal simplex method. Even though the data is changed, then the optimal partition $(\mathcal{B},\mathcal{N})$ does not necessarily change. Indeed if a c_j for one of the nonbasic variables is decreased, then this implies a decrease in s_j. Hence, the current basic and nonbasic partition remains optimal.

It should be observed that the solution (x, y, s) change, when the parameters are modified.

In the following we study how to compute ranges for each b_i and c_j such that as long as they lie within these ranges, then the current optimal basic and nonbasic partition of the variables remains optimal.

4.2 Analyzing modifications to b

The first question to be answered is how much b_i can be decreased and increased such that the current basic and nonbasic partition remains optimal. Therefore, define the modified b^+ by

$$b^+ := b + \theta e_i,$$

where e_i is the ith unit vector. It follows from (4.1) that only the basic variables are affected by a change b. Indeed we have

$$\begin{aligned} x_{\mathcal{B}}^+ &= B^{-1}b^+ \\ &= B^{-1}(b + \theta e_i) \\ &= x_{\mathcal{B}}^* + \theta B^{-1}e_i, \end{aligned} \tag{4.2}$$

whereas y, s, and $x_{\mathcal{N}}$ remain unchanged. Note the vector

$$B^{-1}e_i$$

is nothing but the ith column of the optimal basis inverse. Therefore, the interpretation of (4.2) is that the new value of the basis variables is a function of the current optimal value $x_{\mathcal{B}}^*$, the ith column of the optimal basis inverse, and θ.

It can be observed that if θ is chosen such $x_{\mathcal{B}}^+ \geq 0$, then the current basic and nonbasic partition remains optimal. Hence, we would like to compute the minimal and maximal θ for which $x_{\mathcal{B}}^+ \geq 0$ is satisfied. This gives exactly the range we are looking for.

Therefore, define the problems

$$\begin{aligned} \theta_l^i := \quad & \text{minimize} \quad \theta \\ & \text{subject to} \quad x_{\mathcal{B}}^* + \theta B^{-1}e_i \geq 0 \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} \theta_u^i := \quad & \text{maximize} \quad \theta \\ & \text{subject to} \quad x_{\mathcal{B}}^* + \theta B^{-1}e_i \geq 0. \end{aligned} \tag{4.4}$$

In summary for all

$$b_i^+ \in [b_i + \theta_l^i e_i, b_i + \theta_u^i e_i]$$

then the current basic and nonbasic partition of the variables remains optimal.

The two problems (4.3) and (4.4) are easy to solve, because computing the optimal solution to the problems is equivalent to the problem of computing the maximal step size α in the primal simplex method. It should observed that the optimal solution to one or both of these problems can be unbounded.

Now assume a b_i represents the available quantity of some resource and it might be possible to buy more of the resource by paying a price p. Now if the objevtive value increase by more than p for each unit bought of the resource, then it is worthwhile to

buy some units of the resource. Therefore, let us investigate the change in the optimal objective value as a function of θ. We have that

$$\begin{aligned} z^+ &= c_{\mathcal{B}}^T x_{\mathcal{B}}^+ \\ &= c_{\mathcal{B}}^T B^{-1}(b + \theta e_i) \\ &= (y^*)^T(b + \theta e_i) \\ &= b^T y^* + \theta y_i^* \end{aligned} \tag{4.5}$$

which implies for each additional unit of b_i then the objective value increases by y_i^* units. (Actually if y_i^* is negative then it decreases.) Hence, if the price p is less than y_i^*, then it is worthwhile to but more of the ith resource. In this case the value θ_u^i is the maximum amount we are willing to buy for a price less than y_i^*. After this threshold the basic and nonbasic partition of the variables change and the price y_i^* is likely to change too. Although this is not necessarily the case.

It should now clear that y_i^* shows the change in the objective value if b_i is increased slightly. Due this interpretation of y_i^*, then the ys are called **shadow** prices.

This form of sensitivity analysis is potentially very powerful, but the reader should be aware of some pitfalls:

- In general it is only possible to analyze a change in one b_i at the time. Hence, the intervals $[\theta_l^i, \theta_u^i]$ and $[\theta_l^j, \theta_u^j]$ are not the correct intervals if both b_i and b_j are changed simultaneously

- The optimal basis is not neccesarily unique. Therefore, the computed intervals are depended on the optimal basis reported by the simplex method. Moreover, the dual prices y are not necessarily unique. Hence, the conclusions obtained from from the sensitivity analysis may be somewhat arbitrary depended on the optimal basis.

The above assumes that only one component in b in changed. However, it might be that to increase one component of b, then another component should be decreased. Therefore, let us analyze this case briefly. Assume the vector $\delta b \in \mathbf{R}^m$ denotes the direction of change in b. Hence,

$$b^+ := b + \theta \delta b.$$

The minimal and maximal possible values of θ such that

$$x_{\mathcal{B}}^* + \theta B^{-1} \delta b \geq 0$$

gives the for interval for which the basic and nonbasic partition of the variables remains optimal.

4.2.1 Analyzing modifications to c

Using the techniques used in previous section, then it is possible to analyze the effect of changes in c easily.

Now let the modified objective function be given by

$$c^+ := c + \theta e_j,$$

where e_j is the jth unit vector and $\theta \in R$. Hence, by varying θ we modifying the jth component of c.

First, assume that $j \in \mathcal{N}$ that is the jth variable is a nonbasic variable then the current basic and nonbasic partition remains optimal for all θ such that

$$\begin{aligned} s^+_{\mathcal{N}} &= c^+_{\mathcal{N}} - N^T y^* \\ &= c_{\mathcal{N}} + \theta e_j - N^T y^* \\ &= s^*_{\mathcal{N}} + \theta e_j \leq 0, \end{aligned}$$

Therefore, a c_j corresponding to a nonbasic variable can be decreased indefinitely. Whereas θ can only be increased until s^+_j becomes positive because then jth variable should enter the basis. Hence, for all

$$c^+_j \in [-\infty, c_j - s^*_j]$$

then the current optimal basic and nonbasic partition of the variables remains unchanged.

Next assume a c_j for a $j \in \mathcal{B}$ is changed. Let $\mathcal{B}_i = j$ and if e_i is the ith unit vector, then by definition

$$c^+_{\mathcal{B}} := c_{\mathcal{B}} + \theta e_i$$

implying

$$\begin{aligned} y^+ &= B^{-T} c^+_{\mathcal{B}} \\ &= B^{-T}(c_{\mathcal{B}} + \theta e_i) \\ &= y^* + \theta B^{-T} e_i. \end{aligned}$$

The relation

$$B^{-T} e_i = (e_i^T B^{-1})^T$$

shows that $B^{-T} e_i$ is the ith row of the optimal basis inverse.

In this case a change in θ implies a change in y, which in turn implies a change in the dual slacks of the nonbasic variables i.e. in $s_{\mathcal{N}}$. However, the basic and nonbasic partition remains optimal if $s^+_{\mathcal{N}} \leq 0$ is satisfied. Therefore, the partition remains optimal for all θ such that

$$\begin{aligned} s^+_{\mathcal{N}} &= c_{\mathcal{N}} - N^T y^+ \\ &= c_{\mathcal{N}} - N^T(y^* + \theta B^{-T} e_i) \\ &= s^*_{\mathcal{N}} - \theta N^T B^{-T} e_i \\ &\leq 0 \end{aligned} \tag{4.6}$$

is satisfied. Similar to case with a change in b_i it is now easy to compute the minimal and maximal θ such that (4.6) is satisfied. Let $[\theta^j_l, \theta^j_u]$ denote this interval and then for all

$$c^+_j \in [\theta^j_l + c_j, \theta^j_u + c_j]$$

then the current optimal basic and nonbasic partition of the variables remains unchanged. This gives the maximal changes in c_j.

The new objective value is given by

$$\begin{aligned} z^+ &= (c^+_{\mathcal{B}})^T x^*_{\mathcal{B}} \\ &= (c_{\mathcal{B}} + \theta e_i)^T x^*_{\mathcal{B}} \\ &= z^* + \theta x^*_{\mathcal{B}_i}. \end{aligned}$$

Hence, for each unit change in c_j the objective value is changed by $x^*_{\mathcal{B}_i}$ units.

Chapter 5

Interior-point methods

During the last 10 years there has been a rapid development in the interior-point methods for linear programming (LP). Indeed for large LP problems interior-point methods have a superior performance compared to the simplex algorithm. Therefore, interior-point methods are an important new tool in operations research.

In this chapter one of the most efficient variants of the interior-point methods are presented. This method is called the primal-dual infeasible-interior-point algorithm or the primal-dual algorithm for short.

5.1 Introduction

The modern era of interior-point methods started with Karmarkar's 1984 paper [11]. This paper gained a lot of interest, because the interior-point method Karmarkar presented had excellent (theoretical) convergence properties compared to the simplex method.

The name interior-point methods arises from the methods move trough the interior of the feasible region towards the optimal solution. This is in contrast to the simplex algorithm which follows a sequence of adjacent extreme points to the optimal solution. Unfortunately this path of extreme points may contain an exponentially large number of points. Therefore, by moving trough the interior, the problem with a large number of extreme points is avoided.

Among the different interior-point algorithms the primal-dual algorithm has gained a reputation for being the must efficient method for practical LP problems. Therefore, this algorithm is presented in this chapter. Moreover, by exploiting the primal and dual properties in linear programming, then the primal-dual algorithm can be stated in a simple and elegant way.

5.2 Review

In this section we will review some relevant theory from nonlinear optimization, which will be useful later. The section may be skipped by readers who is already familiar with the optimization.

5.2.1 The first order optimality conditions

An equality constraint nonlinear optimization problem has the form

$$\begin{array}{ll} \text{minimize} & c(x) \\ \text{subject to} & g(x) = 0, \end{array} \tag{5.1}$$

where $c : \mathbf{R}^n \to \mathbf{R}$ and $g : \mathbf{R}^n \to \mathbf{R}^m$ are general smooth functions. The optimality conditions to (5.1) can be formulated using the so-called Lagrange function

$$L(x, y) = c(x) - y^T g(x). \tag{5.2}$$

y are known as Lagrange multipliers. Now the first order optimality conditions to (5.1) are

$$\begin{array}{rcrcl} \nabla_x L(x, y) & = & \nabla c(x) - \nabla g(x)^T y & = & 0, \\ \nabla_y L(x, y) & = & -g(x) & = & 0. \end{array} \tag{5.3}$$

The optimality conditions (5.3) are in general only necessary for optimality. Hence, an optimal solution to (5.1) must satisfy (5.3), but a solution to (5.3) is not necessarily an optimal solution to (5.1). However, we have the following important theorem.

Theorem 5.2.1 *Assume f is a convex function and g is an affine function, then (5.3) is sufficient.*

The theorem says if the objective function is convex and the constraints are linear, then the first order optimality conditions (5.3) are sufficient.

In summary it has been shown how a nonlinear programming problem can be reduced to a set of nonlinear equations. Therefore, a solution method for the nonlinear programming problem (5.1) is to compute a solution to the first order optimality conditions. In general this solution is not necessarily an optimal solution. Although in some cases it is. Hence, we needed a method for solution of nonlinear equations and this is the subject of the next section.

5.2.2 Newton's method

There exists several methods for solution of nonlinear equations, but in this section we will restrict the attention to Newton's method.

Assume $f : \mathbf{R}^n \to \mathbf{R}^n$ is a general smooth nonlinear function and a solution to the system

$$f(x) = 0 \tag{5.4}$$

is required. It follows from Taylor's theorem that

$$f(x^0 + d_x) \approx f(x^0) + \nabla f(x^0) d_x, \tag{5.5}$$

where $x^0, d_x \in \mathbf{R}^n$. If x^0 is an initial guess for the solution to (5.4), then we want to compute d_x such that $f(x^0 + d_x) = 0$. In general this is impossible, but in view of (5.5) we may obtain an approximation from

$$f(x^0) + \nabla f(x^0) d_x = 0. \tag{5.6}$$

The system (5.6) defines a set of linear equations in the variables d_x, which can easily be solved. Indeed if it is assumed that $\nabla f(x^0)$ is nonsingular, then d_x is given by

$$d_x = -\nabla f(x^0)^{-1} f(x^0).$$

d_x defines a search direction and a new point x^+ is obtained from

$$x^+ = x^0 + \alpha d_x,$$

where α is a step size scalar. The plain Newton method chooses $\alpha = 1$. Unfortunately this does not secure convergence and therefore α has to be chosen in the interval $(0, 1]$. One possible choice of α is given by

$$\alpha^* = \text{argmin}_{\alpha \in [0,1]} \left\| f(x^0 + \alpha d_x) \right\| \tag{5.7}$$

and $\alpha = \alpha^*$. The advantage of this choice is that $\|f(x^+)\| \leq \|f(x^0)\|$ and therefore the new point (x^+) in some sense is closer to the solution.

In summary the idea of Newton's method is to linearize the function. Next the root of the linearized function is computed and is used as a new guess for the root to the real nonlinear problem. Clearly this leads to an iterative method for solution of a set of nonlinear equations, which is terminated when $\|f(x)\| \approx 0$. In general Newton's method does not necessarily converge unless the initial guess is close to the solution. However, for specific classes of functions convergence can be proved. On the other hand if the initial guess is close to the solution, then Newton's method is known to converge very rapidly. Indeed Newton's method is known to have excellent local convergence properties.

Obviously Newton's method can be applied to the first order optimality conditions (5.3). Let (x^0, y^0) be an initial guess for the solution. Then the Newton search direction to the first order optimality conditions is defined by

$$\begin{bmatrix} \nabla^2 c(x^0) - \sum_{i=1}^m y_i \nabla^2 g_i(x^0) & -\nabla g(x^0)^T \\ -\nabla g(x^0) & 0 \end{bmatrix} \begin{bmatrix} d_x \\ d_y \end{bmatrix} = - \begin{bmatrix} \nabla c(x^0) - \nabla g(x^0)^T y^0 \\ -g(x^0) \end{bmatrix}. \tag{5.8}$$

Finally, the new point is given by

$$\begin{bmatrix} x^+ \\ y^+ \end{bmatrix} = \begin{bmatrix} x^0 \\ y^0 \end{bmatrix} + \alpha \begin{bmatrix} d_x \\ d_y \end{bmatrix} \tag{5.9}$$

for a suitable chosen step size α.

This Newton based method for nonlinear programming inherits the good local convergence properties of Newton's method. Indeed if the initial guess is sufficient close to the optimal solution, then this method converge rapidly. Therefore, Newton's method for nonlinear programming is considered very powerful. The main disadvantages of the method are:

- Solution of the linear equation system (5.8) might be computationally expensive.
- Second order information is required, that is the Hessian of the involved functions are required.
- Newton's method is not globally convergent. Hence, for some initial guesses (x^0, y^0) Newton's method may not converge or at least converge very slowly.

5.3 Interior-point methods

If Newton's method for nonlinear programming is powerful, then why not apply it to the primal or dual LP problem? Unfortunately this is not possible, because both (P) and (D) contain some **inequalities**, which cannot be handled by Newton's method. Therefore, the main topic in the following is a method for handling inequalities within Newton's method.

5.3.1 The primal approach

Assume we want to solve the problem (P) then we can get rid of the inequalities as follows

$$\begin{array}{lll} (PB) & \text{minimize} & c^T x - \mu \sum_{j=1}^n \ln(x_j) \\ & \text{subject to} & Ax = b, \\ & & x > 0, \end{array}$$

where μ is a positive parameter. Re-call

$$\lim_{x_j \to 0} \ln(x_j) = -\infty. \tag{5.10}$$

Therefore, the logarithmic term in the objective function acts as a barrier which penalize nonpositive solutions. This implies any optimal solution to (PB) will satisfy the inequalities $x \geq 0$ strictly, because we are minimizing. Using this observation, the $x > 0$ inequalities in (PB) may be dropped and an equality constrained nonlinear optimization problem is obtained.

An optimal solution to (PB) is not necessarily an optimal solution to (P), but it will be proved later that an optimal solution to (PB) for a sufficiently small μ is a good approximate solution to (P). This can be proved from the optimality conditions to (PB).

First define the Lagrange function

$$L(x, y) = c^T x - \mu \sum_{j=1}^{n} \ln(x_j) - y^T (Ax - b), \tag{5.11}$$

where $y \in \mathbf{R}^m$ are Lagrange multipliers corresponding to the equality constraints in (PB). Now differentiation gives

$$\frac{\partial L}{\partial x_j} = c_j - \mu x_j^{-1} - A_{:j}^T y \quad \text{and} \quad \frac{\partial L}{\partial y_i} = b_i - A_{i:} x.$$

In vector notation

$$\begin{array}{rclcl} \nabla_x L(x, y) & = & c - \mu X^{-1} e - A^T y & = & 0, \\ \nabla_y L(x, y) & = & b - Ax & = & 0, \quad x > 0. \end{array} \tag{5.12}$$

Let us refresh the notation that $A_{:j}$ and $A_{i:}$ is the jth column and the ith row of A respectively. Moreover, $e := (1, \ldots, 1)^T$. A very important notation used subsequently is if x is a vector, then capital X is a diagonal matrix with x on the diagonal that is

$$X := \operatorname{diag}(x) := \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix}. \tag{5.13}$$

Note if $x > 0$, then the inverse of X exists and

$$X^{-1}e = \begin{bmatrix} x_1^{-1} & 0 & \cdots & 0 \\ 0 & x_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} x_1^{-1} \\ x_2^{-1} \\ \vdots \\ x_n^{-1} \end{bmatrix}. \tag{5.14}$$

The first order optimality conditions (5.12) can be rewritten by introducing the vector $s = \mu X^{-1}e$ leading to

$$\begin{aligned} c - s - A^T y &= 0, \\ b - Ax &= 0, \qquad x > 0, \\ s &= \mu X^{-1} e. \end{aligned} \tag{5.15}$$

If both sides of the last equality are multiplied by X and using a minor reordering we obtain

$$\begin{aligned} A^T y + s &= c, \\ Ax &= b, \quad x > 0, \\ Xs &= \mu e. \end{aligned} \tag{5.16}$$

The conditions (5.16) have a very nice interpretation. Indeed the first set of equalities enforces dual feasibility (see (D)) and the second set of equalities enforces primal feasibility. Finally, the last set of equalities are identical to

$$Xs = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} x_1 s_1 \\ x_2 s_2 \\ \vdots \\ x_n s_n \end{bmatrix} = \begin{bmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{bmatrix} = \mu e,$$

which is the complementarity conditions perturbed by μ. In fact if $\mu = 0$ they are identical to the complementarity conditions.

Let $(x(\mu), y(\mu), s(\mu))$ be a solution to (5.16) for some $\mu > 0$. Then $x(\mu)$ is primal feasible. Furthermore,

$$A^T y(\mu) + s(\mu) = c \quad \text{and} \quad s(\mu) = \mu X(\mu)^{-1} e > 0,$$

which shows $(y(\mu), s(\mu))$ is dual feasible. In other words $(x(\mu), y(\mu), s(\mu))$ is a primal-dual feasible pair. Therefore, the duality gap can be computed as follows

$$\begin{aligned} c^T x(\mu) - b^T y(\mu) &= x(\mu)^T s(\mu) \\ &= e^T X(\mu) s(\mu) \\ &= e^T (\mu e) \\ &= \mu e^T e \\ &= \mu n. \end{aligned}$$

In conclusion any solution $(x(\mu), y(\mu), s(\mu))$ which satisfies (5.16) and hence is an optimal solution to (PB) defines a primal-dual feasible pair. Moreover, the duality gap is given by $n\mu$. This verifies that an optimal solution to (PB) for a sufficiently small μ is an approximate optimal solution to (P), because using the duality theorem then

$$c^T x(\mu) - c^T x^* \leq c^T x(\mu) - b^T y(\mu) \leq n\mu,$$

where x^* is an optimal solution to (P).

5.3.2 Convexity of the barrier function

An important question is whether the objective function in the problem (PB) is convex, because in this case the optimality conditions are sufficient. Hence, any solution to the first order optimality conditions is also a solution to (PB).

This question leads to study the barrier function

$$\begin{aligned} B_\mu(x) &:= c^T x - \mu \textstyle\sum_{j=1}^n \ln(x_j) \\ &= \textstyle\sum_{j=1}^n (c_j x_j - \mu \ln(x_j)). \end{aligned}$$

The function $\ln(x)$ is concave, which implies $-\mu \ln(x)$ is convex. Therefore, the barrier function is a positive sum of convex functions, which implies the barrier function is convex.

This fact can also be verified from the facts

$$\nabla B_\mu(x) = c - \mu X^{-1} e \quad \text{and} \quad \nabla^2 B(x) = \mu X^{-2}.$$

Now let $v \in \mathbf{R}^n \setminus \{0\}$. Then

$$\begin{aligned} v^T \nabla^2 B_\mu(x) v &= \mu v^T X^{-2} v \\ &= \mu \textstyle\sum_{j=1}^n (v x_j^{-1})^2 \\ &> 0 \end{aligned}$$

for $x > 0$ and $\mu > 0$. This shows the Hessian of $B_\mu(x)$ is positive definite which implies $B_\mu(x)$ is strictly convex.

5.3.3 A dual approach

In the previous section we applied the logarithmic barrier transformation to the primal problem, but it could equally well has been applied to the dual problem. In this section we investigate this possibility.

Introducing the barrier term into the dual problem gives

$$\begin{array}{lll} (DB) & \text{maximize} & b^T y + \mu \sum_{j=1}^n \ln(s_j) \\ & \text{subject to} & A^T y + s = c, \\ & & s > 0. \end{array}$$

Note the barrier term is added to the objective function, because the dual problem is a maximization problem.

If we let x denote the Lagrange multipliers corresponding to the equalities in (DB), then the Lagrange function is given by

$$L(x, y, s) = b^T y + \mu \sum_{j=1}^n \ln(s_j) - x^T (A^T y + s - c). \tag{5.17}$$

The optimality conditions are

$$\begin{array}{rcccl} \nabla_x L(x,y,s) &=& c - s - A^T y &=& 0, \quad s > 0, \\ \nabla_y L(x,y,s) &=& b - Ax &=& 0, \\ \nabla_s L(x,y,s) &=& \mu S^{-1} e - x &=& 0. \end{array} \tag{5.18}$$

After some rearrangements it is seen that (5.18) is equivalent to

$$\begin{array}{rcll} A^T y + s &=& c, & s > 0, \\ Ax &=& b, & \\ Xs &=& \mu e. & \end{array} \tag{5.19}$$

Perhaps not surprisingly these conditions are essentially the same conditions as in the primal case. Hence, they are the perturbed KKT conditions to (P).

5.3.4 The primal-dual approach

We have now shown how to transform the primal and dual problem to remove the inequalities. In both cases we obtain a set of first order optimality conditions to the barrier problem. Combining these two set of optimality conditions gives

$$\begin{array}{lcll} Ax & = & b, & x > 0, \\ A^T y + s & = & c, & s > 0, \\ Xs & = & \mu e. & \end{array} \tag{5.20}$$

These conditions are called the perturbed KKT conditions, because they are identical to the KKT conditions to (P) except the complementarity conditions have been perturbed by μ. Therefore, a solution to (5.20) for a sufficiently small μ is a good approximation to the optimal solution to (P).

Now the system (5.20) defines a set of nonlinear equations which can be solved using Newton's method. This is exactly the main idea of the so-called primal-dual algorithm, which we are going to discuss.

First define the nonlinear function

$$F_\gamma(x, y, s) := \begin{bmatrix} Ax - b \\ A^T y + s - c \\ Xs - \gamma\mu e \end{bmatrix},$$

where $\mu := x^T s/n$ and $\gamma \geq 0$. Note instead of letting μ be a parameter to be chosen we have introduced γ and defined μ to be the average complementary product. This redefinition is convenient in the subsequent development.

Assume $(\bar{x}, \bar{y}, \bar{s})$ is given such that $\bar{x} > 0$ and $\bar{s} > 0$, then one iteration of Newton's method applied to the system

$$F_\gamma(x, y, s) = 0$$

is identical to

$$\nabla F_\gamma(\bar{x}, \bar{y}, \bar{s}) \begin{bmatrix} d_x \\ d_y \\ d_s \end{bmatrix} = -F_\gamma(\bar{x}, \bar{y}, \bar{s}).$$

Using the fact

$$\nabla F_\gamma(\bar{x}, \bar{y}, \bar{s}) = \begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ \bar{S} & 0 & \bar{X} \end{bmatrix}$$

we obtain

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ \bar{S} & 0 & \bar{X} \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_s \end{bmatrix} = \begin{bmatrix} \bar{r}_P \\ \bar{r}_D \\ -\bar{X}\bar{s} + \gamma\bar{\mu}e \end{bmatrix}, \tag{5.21}$$

where

$$\bar{r}_P := b - A\bar{x}, \tag{5.22}$$

and

$$\bar{r}_D := c - A^T\bar{y} - \bar{s}. \tag{5.23}$$

(5.22) and (5.23) are denoted the primal and dual residuals respectively. If the residual vectors are zero, then the current point $(\bar{x}, \bar{y}, \bar{s})$ is primal and dual feasible respectively, because by construction $(\bar{x}, \bar{s}) \geq 0$.

The first step of the primal-dual algorithm consist of solving (5.21), and then a new point

$$\begin{bmatrix} x^+ \\ y^+ \\ s^+ \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{s} \end{bmatrix} + \alpha \begin{bmatrix} d_x \\ d_y \\ d_s \end{bmatrix},$$

is obtained for a suitable choice of α. Before discussing the choice of α, then let us study some basic convergence properties.

The goal is to compute a solution such that the primal and dual residuals and the complementary gap ($x^T s$) all are zero. Therefore, let us investigate if the new point is closer to this goal. The new primal residuals are given by

$$\begin{aligned} r_P^+ &= b - Ax^+ \\ &= b - A(\bar{x} + \alpha d_x) \\ &= b - A\bar{x} - \alpha A d_x \\ &= \bar{r}_P - \alpha \bar{r}_P \\ &= (1-\alpha)\bar{r}_P, \end{aligned} \tag{5.24}$$

where the fact $Ad_x = \bar{r}_P$ given by (5.21) is used. This shows that the new residuals are identical to the old residuals multiplied by the factor $(1-\alpha)$. Indeed if $\alpha \in (0,1]$, then the residuals are reduced. In particular if $\alpha = 1$, we have that $r_P^+ = 0$ showing the new point x^+ satisfies the primal equality constraints exactly.

In summary the primal residuals are reduced by a factor $(1-\alpha)$ and therefore a large step size is beneficial. Finally, it can be seen that $\bar{r}_P = 0$ implies $r_P^+ = 0$. Hence, if the current solution is primal feasible, then the new solution is primal feasible too.

We will leave it as an exercise to the reader to verify the relation

$$r_D^+ = (1-\alpha)\bar{r}_D \tag{5.25}$$

showing the dual residuals are also reduced.

Next the new duality gap is identical to

$$\begin{aligned} (x^+)^T s^+ &= (\bar{x} + \alpha d_x)^T (\bar{s} + \alpha d_s) \\ &= \bar{x}^T \bar{s} + \alpha(\bar{x}^T d_s + \bar{s}^T d_x) + \alpha^2 d_x^T d_s \\ &= \bar{x}^T \bar{s} + \alpha e^T (\bar{X} d_s + \bar{S} d_x) + \alpha^2 d_x^T d_s \\ &= \bar{x}^T \bar{s} + \alpha e^T (-\bar{X}\bar{s} + \gamma \bar{\mu} e) + \alpha^2 d_x^T d_s \\ &= \bar{x}^T \bar{s} - \alpha(\bar{x}^T \bar{s} - \gamma \bar{x}^T \bar{s}) + \alpha^2 d_x^T d_s \\ &= (1 - \alpha(1-\gamma))\bar{x}^T \bar{s} + \alpha^2 d_x^T d_s. \end{aligned} \tag{5.26}$$

The fourth equality is obtained using (5.21). This implies

$$\begin{aligned} \lim_{\alpha \to 0} \frac{(x^+)^T s^+ - \bar{x}^T \bar{s}}{\alpha} &= \lim_{\alpha \to 0} \frac{-\alpha(1-\gamma)\bar{x}^T \bar{s} + \alpha^2 d_x^T d_s}{\alpha} \\ &= -(1-\gamma)\bar{x}^T \bar{s}. \end{aligned} \tag{5.27}$$

The interpretation of (5.27) is for $\gamma \in [0,1)$ and for a sufficiently small α, then the complementary gap is reduced.

We have now shown that if the step size α is positive, then the primal and dual residuals are reduced. Moreover, for a sufficiently small step size the complementary gap is reduced too. In conclusion by an appropriate choice of the step size, then the algorithm should converge.

Let us for a moment assume that the primal and dual solution are feasible that is $\bar{r}_P = 0$ and $\bar{r}_D = 0$. Now using (5.21) this implies

$$Ad_x = 0 \quad \text{and} \quad A^T d_y + d_s = 0$$

leading to the conclusion

$$d_x^T d_s = -d_x^T A^T d_y = 0.$$

In this case (5.26) simplifies to

$$(x^+)^T s^+ = (1 - \alpha(1-\gamma))\bar{x}^T \bar{s}$$

and it follows the new complementary gap decreases with an increase in α. Furthermore, the smaller γ the larger is the decrease in the gap. In particular if $\alpha = 1$ then

$$(x^+)^T s^+ = \gamma \bar{x}^T \bar{s}. \tag{5.28}$$

The important observations from the convergence analysis is, if the step size is strictly positive, then the primal and dual residuals are reduced by a factor $(1-\alpha)$. Moreover, from (5.27) it can be seen that a small γ leads to a large reduction rate in the complementary gap. However, it should be noted that the search direction (d_x, d_y, d_s) is a function of γ. Hence, implicitly the step size α is also a function of γ.

5.3.5 Update of the variables

The choice of the step size α has not yet been specified. However, from the previous section it is known that a large step size implies a large reduction in the primal and dual residuals. Also the reduction in the complementary gap tend to be proportional to the step size. Unfortunately, the step size cannot be chosen arbitrarily large because the new point must satisfy the conditions $x^+ > 0$ and $s^+ > 0$. Therefore, the step size has to be strictly less than $\alpha^{\max}$, where $\alpha^{\max}$ is defined by

$$\alpha^{\max} := \text{argmax}_{0 \leq \alpha} \left\{ \begin{bmatrix} \bar{x} \\ \bar{s} \end{bmatrix} + \alpha \begin{bmatrix} d_x \\ d_s \end{bmatrix} \geq 0 \right\}.$$

$\alpha^{\max}$ is the maximum possible step size until one of the primal or dual variables hits its lower bound exactly. Therefore, a possible choice of α is

$$\alpha = \min(1, \theta \alpha^{\max}),$$

where $\theta \in (0, 1)$. For example $\theta = 0.9$ implies 90% of the maximum possible step size to the boundary is taken and this guarantees that $x^+ > 0$ and $s^+ > 0$. The step size is not allowed to be larger than one, because ideally Newton's method never take steps larger than one.

This choice of the step size does **not** guarantee convergence, but it usually works well in practice. In practice $\alpha^{\max}$ is computed as follows. First let

$$\alpha_P^{\max} = \min_j \{-x_j/(d_x)_j : (d_x)_j < 0\}$$

and

$$\alpha_D^{\max} = \min_j \{-s_j/(d_s)_j : (d_s)_j < 0\},$$

then

$$\alpha^{\max} = \min(\alpha_P^{\max}, \alpha_D^{\max}).$$

5.4 Termination criteria

An important issue is of course when to terminate the primal-dual algorithm. Clearly the algorithm should be terminated if the primal and dual residuals and the complementary gap all are zero. In practice a relaxed version of this termination criteria is used. Hence, the algorithm is terminated if

$$\begin{aligned} ||Ax - b|| &\leq \varepsilon_P, \\ ||A^T y + s - c|| &\leq \varepsilon_D, \\ x^T s &\leq \varepsilon_G, \end{aligned}$$

where ε_P, ε_D, and ε_G are small positive constants.

In practice an exact optimal solution cannot be obtained, because computations are performed in finite precision on a computer. Moreover, an exact optimal solution might not be necessary, because the data (c, A, b) often only have limited accuracy. Therefore, a highly accurate solution is not necessarily much better than a reasonable approximate solution.

5.4.1 The algorithm

Finally, all the components of the primal-dual algorithm have been presented in Algorithm 5.4.1.

Algorithm 5.4.1

1. *Choose* (x^0, y^0, s^0) *such that* $(x^0, s^0) > 0$ *and* $\varepsilon_P, \varepsilon_D, \varepsilon_C > 0$.

2. $k := 0$

3. *LOOP: Let*
$$\begin{aligned} r_P^k &:= b - Ax^k, \\ r_D^k &:= c - A^T y^k - s^k, \\ \mu^k &:= (x^k)^T s^k / n. \end{aligned}$$

4. *If*
$$||r_P^k|| \leq \varepsilon_P, \quad ||r_D^k|| \leq \varepsilon_D, \quad (x^k)^T s^k \leq \varepsilon_G,$$
then terminate.

5. *Pick* $\gamma \in [0, 1)$ *and solve*
$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S^k & 0 & X^k \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_s \end{bmatrix} = \begin{bmatrix} r_P^k \\ r_D^k \\ -X^k s^k + \gamma \mu^k e \end{bmatrix}.$$

6. *Compute*
$$\alpha^{\max} = argmax_{0 \leq \alpha} \left\{ \begin{bmatrix} x^k \\ s^k \end{bmatrix} + \alpha \begin{bmatrix} d_x \\ d_s \end{bmatrix} \geq 0 \right\}.$$

7. *Let* $\alpha := \min(\theta\alpha^{\max}, 1)$ *for some* $\theta \in (0,1)$ *and update*

$$\begin{aligned} x^{k+1} &:= x^k + \alpha d_x, \\ y^{k+1} &:= y^k + \alpha d_y, \\ s^{k+1} &:= s^k + \alpha d_s. \end{aligned}$$

8. $k = k + 1$

9. *GOTO LOOP*

In step 1 an initial point is chosen. The restrictions on the initial point are very moderate and a possible choice is $(x^0, y^0, s^0) = (e, 0, e)$. Also the termination tolerances are selected.

Next the iteration counter is initialized, the residuals are computed, and the termination criteria is checked. This is followed by picking a value for γ. Afterwards the Newton equations system is solved. This system of linear equations can of course be solved directly, but it is also possible to reduce it as follows. The system is equivalent to

$$\begin{aligned} Ad_x &= r_P^k, \\ A^T d_y + d_s &= r_D^k, \\ S^k d_x + X^k d_s &= -X^k s^k + \gamma\mu^k e. \end{aligned} \tag{5.29}$$

The second equation in (5.29) gives

$$d_s = r_D^k - A^T d_y.$$

Plugging this result into the third equation of (5.29) gives

$$S^k d_x + X^k(r_D^k - A^T d_y) = -X^k s^k + \gamma\mu^k e. \tag{5.30}$$

Now multiplying each side of (5.30) by $A(S^k)^{-1}$ gives

$$Ad_x + A(S^k)^{-1}X^k(r_d^k - A^T d_y) = A(S^k)^{-1}(-X^k s^k + \gamma\mu^k e).$$

Finally, using $Ad_x = r_P^k$ we obtain

$$\begin{aligned} (AX^k(S^k)^{-1}A^T)d_y &= r_P^k + A((S^k)^{-1}(X^k r_D^k - \gamma\mu^k e) + x^k) \\ &= b - Ax^k + A((S^k)^{-1}(X^k r_D^k - \gamma\mu^k e) + x^k) \\ &= b + A(S^k)^{-1}(X^k r_D^k - \gamma\mu^k e). \end{aligned} \tag{5.31}$$

Let

$$M := (AX^k(S^k)^{-1}A^T) \tag{5.32}$$

and

$$r := b + A(S^k)^{-1}(X^k r_D^k - \gamma\mu^k e). \tag{5.33}$$

Then

$$Md_y = r,$$

which shows d_y is a solution to a linear equation system. When the system (5.31) has been solved for d_y, then d_s and d_x can be recovered from the relations

$$\begin{aligned} d_s &= r_D^k - A^T d_y, \\ d_x &= -x^k + (S^k)^{-1}(\gamma\mu^k e - X^k d_s). \end{aligned} \tag{5.34}$$

In step 6 and 7 the maximum step size to the boundary is computed. Afterwards the new point is obtained by taking a fixed percentage (θ) of the maximum possible step size. For example let $\theta = 0.9$. Finally, the whole process is repeated.

5.5 Convergence analysis

An important issue is the convergence of the primal-dual algorithm that is will the primal-dual algorithm ever reach the termination criteria. In this section we will discuss some changes to the primal-dual algorithm, which guarantees it's convergence.

Let (x^k, y^k, s^k) be the kth iterate of the primal-dual algorithm, then we have

$$r_P^{k+1} = (1-\alpha) r_P^k \quad \text{and} \quad r_D^{k+1} = (1-\alpha) r_D^k.$$

It follows if the step size is strictly positive then the residuals are reduced. Moreover, we have

$$(x^{k+1})^T s^{k+1} = (1 - \alpha(1-\gamma))(x^k)^T s^k + \alpha^2 d_x^T d_s.$$

Clearly if the step size is zero, then the complementary gap is not reduced. On the other hand if $d_x^T d_s$ is large and positive, then only for a very small step size the complementary gap is reduced. Therefore, the main idea in a convergence proof is to show that there exists a strictly positive step size in every iteration such that the complementary gap is reduced. Unfortunately this is not easy to prove, but nevertheless some of the ingredients in the proof are discussed.

Let (x^k, y^k, s^k) be the kth iterate in the primal-dual algorithm, then some key quantities are

$$\frac{\left\|Ax^k - b\right\|}{\|Ax^0 - b\|}, \quad \frac{\left\|A^T y^k + s^k - c\right\|}{\|A^T y^0 + s^0 - c\|}, \quad \text{and} \quad \frac{(x^k)^T s^k}{(x^0)^T s^0}$$

which measure the rate of reduction in the primal residuals, the dual residuals, and the complementary gap respectively. Next define the neighborhood

$$\begin{aligned} \mathcal{N}(\beta) = \{(x,y,s): \ & (x,s) > 0, \\ & \|Ax - b\| \, (x^0)^T s^0 \le \|Ax^0 - b\| x^T s, \\ & \left\|A^T y + s - c\right\| (x^0)^T s^0 \le \left\|A^T y^0 + s^0 - c\right\| x^T s, \\ & x_j s_j \ge \beta x^T s / n\}, \end{aligned}$$

where $\beta \in (0,1)$. If it is assumed $(x^k, y^k, s^k) \in \mathcal{N}(\beta)$, then this implies the following.

- The primal infeasibility has been reduced faster than the complementary gap, because

$$\frac{\left\|Ax^k - b\right\|}{\|Ax^0 - b\|} \le \frac{(x^k)^T s^k}{(x^0)^T s^0}.$$

- The dual infeasibility has been reduced faster than the complementary gap.

- The condition

$$x_j^k s_j^k \ge \beta (x^k)^T s^k / n$$

  implies each complementary product $(x_j^k s_j^k)$ is greater than a fraction of the average complementary product. Hence, a complementary product cannot be arbitrarily small which also prevents prevents x_j and s_j from being arbitrarily small.

A key ingredient in a convergence proof is to keep all the iterates in this neighborhood that is

$$(x^k, y^k, s^k) \in \mathcal{N}(\beta), \quad \forall k.$$

Therefore, primal and dual feasibility is obtained before complementarity. This is advantageous, because assume (x^k, y^k, s^k) is a nearly complementary solution ($(x^k)^T s^k \approx= 0$), but is infeasible. Now from the definition of the search direction we obtain

$$x_j^k (d_s)_j + s_j^k (d_x)_j = -x_j^k s_j^k + \gamma \mu^k.$$

Furthermore assume $x_j^k \approx \mu^k \approx 0$ and $x_j^k s_j^k \approx 0$, then this implies

$$(d_x)_j = \frac{-x_j^k s_j^k + \gamma \mu^k - x_j^k (d_s)_j}{s_j^k} \approx 0.$$

If the primal solution is infeasible, then to achieve feasibility it may be that x_j has to be large. Therefore, even for a large step size

$$x_j^k + \alpha (d_x)_j$$

is far from the optimal value. Hence, the algorithm converges very slowly. Therefore, feasibility should be achieved simultaneously or before complementarity. Otherwise, the algorithm might end up with a complementary solution from which it is difficult to move to a feasible solution.

Furthermore, it can seen from the computation of the step size that if x_j^k or s_j^k is small, then they might limit the step size significantly, because only for small α the quantity

$$x_j^k + \alpha (d_x)_j \quad \text{and} \quad s_j^k + \alpha (d_s)_j$$

is positive. Therefore, until the current iterate is close to the optimal solution it is beneficial to keep the two variables x_j and s_j away from zero. This is the main purpose of the last condition

$$x_j^k s_j^k \geq \beta (x^k)^T s^k / n,$$

which prevents x_j^k and s_j^k from being small before the complementary gap is small.

Assuming $(x^k, y^k, s^k) \in \mathcal{N}(\beta)$, then it is possible to show there exists an $\alpha \geq \bar{\alpha}$ such that

$$(x^{k+1}, y^{k+1}, s^{k+1}) \in \mathcal{N}(\beta) \quad \text{and} \quad (x^{k+1})^T s^{k+1} \leq (1 - \bar{\alpha}(1 - \bar{\gamma}))(x^k)^T s^k,$$

where $\bar{\alpha}, \bar{\gamma} \in (0, 1)$ are constants. Hence, the new point remains in the neighborhood and the complementary gap is strictly reduced.

The proof requires the assumption $\left\| (x^k, s^k) \right\| \leq M$ for all k and M is a large positive constant. We refer the reader to [12] for details.

The important conclusion from this section is that feasibility must be achieved before complementarity. Fortunately for the majority LP problems it not necessary to enforce this condition explicitly. The reason is the feasibility conditions are linear equations and the complementarity conditions are quadratic. Therefore, it is automatically easier for Newton's method to generate a feasible solution than a complementary solution. Moreover, it is important not to approach the boundary before the complementary gap is small.

5.6 A numerical example

In this section we will perform one iteration of the primal-dual algorithm on a small example.

Let

$$c = [-1 \ -2 \ 0]^T, \quad A = [1 \ 1 \ 1], \quad \text{and} \quad b = 1.$$

This problem is equivalent to

$$\begin{array}{ll} \text{minimize} & -1x_1 - 2x_2 \\ \text{subject to} & x_1 + x_2 \leq 1, \\ & x_1, x_2 \geq 0, \end{array} \tag{5.35}$$

because x_3 is a slack variable.

First let $(x^0, y^0, s^0) = (e, 0, e)$ then

$$\begin{array}{rcl} r_P^0 & := & b - Ax^0 = 1 - (1 + 1 + 1) = -2, \\ r_D^0 & := & c - A^T y^0 - s^0 = [-1 \ -2 \ 0]^T - [1 \ 1 \ 1]^T = [-2 \ -3 \ -1]^T \\ \mu^0 & = & e^T e/3 = 1. \end{array}$$

Now letting $\gamma = 0.1$, then by definition

$$A(X^0)(S^0)^{-1}A^T d_y = b + A(S^0)^{-1}(X^0 r_D^0 - \gamma\mu^0 e).$$

Therefore,

$$[\ 1\ 1\ 1\ ] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} d_y = 1 + [1 \ 1 \ 1]([-2 \ -3 \ -1]^T - 0.1[1 \ 1 \ 1]^T).$$

Therefore, $d_y = 0.7$ and this implies

$$\begin{array}{rcl} d_y & = & -1.77, \\ d_s & = & r_D^0 - A^T d_y \\ & = & [1 \ 1 \ -1]^T - [1 \ 1 \ 1]^T(-1.77) \\ & = & [-0.23 \ -1.23 \ 0.77]^T \\ d_x & = & -x^0 + (S^0)^{-1}(\gamma\mu^0 e - X^0 d_s) \\ & = & -[1 \ 1 \ 1]^T + (0.1[1 \ 1 \ 1]^T - [-0.23 \ -1.23 \ 0.77]^T \\ & = & [-0.667 \ 0.333 \ -1.67]^T \end{array}$$

Next we compute $\alpha^{\max}$ by

$$\alpha_P^{max} := \min(1/0.667, -, 1/1.67) \quad \text{and} \quad \alpha_D^{max} := \min(1/0.23, 1/1.23, -).$$

Therefore, $\alpha^{\max} := 1/1.67$ and $\alpha := 0.9\alpha^{\max}$. Finally, we obtain

$$\begin{array}{rcl} x^+ & = & [1 \ 1 \ 1]^T + 0.9/1.67[-0.667 \ -0.333 \ -1.67]^T = [0.64 \ 1.18 \ 0.1]^T \\ y^+ & = & 0 + (0.9/1.67)(-1.77) = -0.954 \\ s^+ & = & [1 \ 1 \ 1]^T + 0.9/1.67[-0.23 \ -1.23 \ 0.77]^T = [0.874 \ 0.334 \ 1.41]^T. \end{array}$$

A good exercise is to continue the computations for a couple of more iterations. Nevertheless, in Table 5.1 we have shown the sequence of iterates generated by the primal-dual algorithm until a fairly good approximation to the optimal solution is reached. From Table 5.1 it can been seen that the primal-dual algorithm converges towards the optimal solution. It requires 8 iterations to reach a highly accurate solution, but already in 4 iterations a good approximation is reached. In this case the primal-dual algorithm is vastly

k	x_1^k	x_2^k	$(x^k)^T s^k$
0	1.00e0	1.00e0	3.0e0
1	6.40e-1	1.18e0	1.1e0
2	0.905e-2	1.13e0	2.5e-1
3	0.905e-3	9.93e-1	4.1e-2
4	1.91e-3	9.98e-1	6.1e-3
5	2.42e-4	1.00e0	7.2e-4
6	2.45e-5	1.00e0	7.3e-5
7	2.45e-6	1.00e0	7.4e-6
8	2.56e-7	1.00e0	7.3e-7

Table 5.1: Iterations sequence to example (5.35) .

k	x_1^k	x_2^k	$(x^k)^T s^k$
0	1.00e0	1.00e0	3.0e0
1	1.00e0	1.00e0	1.3e0
2	6.53e-1	6.53e-1	5.6e-1
3	4.96e-1	4.96e-1	1.4e-1
4	4.99e-1	4.99e-1	1.9e-2
5	0.50e0	0.50e0	1.9e-3
6	0.50e0	0.50e0	2.0e-4
7	0.50e0	0.50e0	2.0e-5
8	0.50e0	0.50e0	2.0e-6
9	0.50e0	0.50e0	2.0e-7

Table 5.2: Iterations sequence to example (5.36) .

inefficient compared to the simplex algorithm, because the simplex algorithm solves the example in 1 iteration.

Note that in the final iterations, then the duality gap is reduced by a factor 0.1. The reason is we choose $\gamma = 0.1$ and the algorithm takes unit steps ($\alpha = 1.0$).

The example (5.35) has a unique optimal solution. Therefore, it is interesting to investigate which solution the primal-dual algorithm generates in the case the problem has multiple optimal solutions. Therefore, study the example

$$\begin{array}{ll} \text{minimize} & -2x_1 - 2x_2 \\ \text{subject to} & x_1 + x_2 \leq 1, \\ & x_1, x_2 \geq 0. \end{array} \tag{5.36}$$

It can be observed that the problem has multiple optimal solutions. Indeed for all $\lambda \in [0, 1]$ then

$$x_1 = \lambda \quad \text{and} \quad x_2 = 1 - \lambda$$

is an optimal solution. In Table 5.2 is the iterates generated by the primal-dual algorithm shown. It can be observed that in this case, then the primal-dual algorithm does not converge towards one of the optimal vertex solutions $(0, 1)$ or $(1, 0)$. Rather it converge towards the "center" $(1/2, 1/2)$ of these two optimal solutions. In other words whenever an LP has multiple optimal solutions, then the interior-point algorithm will generate a

sequence of x^k converging towards an optimal solution x^*. This particular optimal solution x^* will have as many components strictly positive as possible.

5.7 Comparison with the simplex algorithm

It is interesting to compare the primal-dual algorithm to the well-known simplex algorithm.

One disadvantage of the primal-dual algorithm is it cannot detect a possible infeasible or unbounded status of the problem (P). Therefore, in one sense the primal-dual algorithm is not complete. Fortunately this problem can handled using a homogeneous model, see [21, 18, 3].

Another disadvantage of the primal-dual algorithm is each iteration is computationally much more expensive than one iteration of the simplex algorithm. However, the total work performed to solve an LP problem is a product of the number of iterations and the work performed in each iteration.

It can be observed that the primal-dual algorithm does not has any problems with degeneracies and the number iterations is not related to the number of vertices in the feasible region. Therefore, in practice for large LP problems the interior-point methods uses significantly fewer iterations than the simplex algorithm. Most implementations of interior-point methods usually solve an LP problem in less 100 iterations even though the problem may contain millions of variables.

Finally, the optimal solution reported by the simplex algorithm is a basic solution. In many practical applications this advantageous. For example the traditional sensitivity analysis requires a basic solution. Unless the LP problem has a unique optimal solution then the primal-dual algorithm does not produce an optimal solution which is also a basic solution as demonstrated in the previous section. However, there exists a method for purifying the interior-point solution into a basic solution, see [14, 4].

5.8 The homogeneous and self-dual method

The primal-dual algorithm dicussed in the previous section has the major drawback that it cannot detect whether the problem is primal and dual infeasible. To overcome this problem it has been suggested to solve the homogeneous and self-dual model

$$\begin{array}{lllllll} \text{minimize} & 0 & & & & & \\ \text{subject to} & & Ax & -b\tau & = & 0, & \\ & -A^T y & & +c\tau & \geq & 0, & \\ & b^T y & -c^T x & & \geq & 0, & \\ & & x \geq 0, & \tau \geq 0 & & & \end{array} \tag{5.37}$$

instead of (P) using for example the primal-dual algorithm. Clearly, (5.37) is a homogeneous LP and is self-dual which essentially follows from constraints form a skew-symmetric system. The interpretation of (5.37) is τ is a homogenizing variable and the constraints represent primal feasibility, dual feasibility, and reversed weak duality.

The homogeneous model (5.37) was first studied by Goldman and Tucker [10] in 1956 and they proved (5.37) always has a nontrivial solution (x^*, τ^*) statisfying

$$\begin{array}{rclcrcl} x_j^* s_j^* & = & 0, & x_j^* + s_j^* & > & 0, & \forall j, \\ \tau^* \kappa^* & = & 0, & \tau^* + \kappa^* & > & 0, & \end{array} \tag{5.38}$$

where $s^* := \tau^* c - A^T y^* \geq 0$ and $\kappa^* := b^T y^* - c^T x^* \geq 0$. A solution to (5.37) satisfying the condition (5.38) is said to be *strictly complementary solution*. Moreover, Goldman and Tucker showed that if $(x^*, \tau^*, y^*, s^*, \kappa^*)$ is any strictly complementary solution then exactly one of the two following situations occur:

- $\tau^* > 0$ if and only if (P) has an optimal solution. In this case $(x^*, y^*, s^*)/\tau^*$ is an optimal primal-dual solution to (P).

- $\kappa^* > 0$ if and only if (P) is primal or dual infeasible. In the case $b^T y^* > 0$ ($c^T x^* < 0$) then (P) is primal (dual) infeasible.

The conclusion is that a strictly complementary solution to (5.37) provides all the information required, because in the case $\tau^* > 0$ then an optimal primal-dual solution to (P) is trivially given by $(x, y, s) = (x^*, y^*, s^*)/\tau^*$. Otherwise, the problem is primal or dual infeasible. Therefore, the main algorithmic idea is to compute a strictly complementary solution to (5.37) instead of solving (P) directly.

Ye, Todd, and Mizuno [21] suggested to solve (5.37) by solving the problem

$$\begin{array}{lrrrrcl} \text{minimize} & & & & n^0 z & & \\ \text{subject to} & & Ax & -b\tau & -\bar{b}z & = & 0, \\ & -A^T y & & +c\tau & +\bar{c}z & \geq & 0, \\ & b^T y & -c^T x & & +\bar{d}z & \geq & 0, \\ & \bar{b}^T y & -\bar{c}^T x & -\bar{d}\tau & & = & -n^0, \\ & & x \geq 0, & \tau \geq 0, & & & \end{array} \tag{5.39}$$

where

$$\begin{array}{rcl} \bar{b} & := & Ax^0 - b\tau^0, \\ \bar{c} & := & -c\tau^0 + A^T y^0 + s^0, \\ \bar{d} & := & c^T x^0 - b^T y^0 + \kappa^0, \\ n^0 & := & (x^0)^T s^0 + \tau^0 \kappa^0. \end{array}$$

It can proved that the problem (5.39) always has an optimal solution. Moreover, the optimal value is identical to zero and and it is easy to verify that if (x, τ, y, z) is an optimal strictly complementary solution to (5.39), then (x, τ, y) is a strictly complementary solution to (5.37). Hence, the problem (5.39) can solved using any method that generates an optimal strictly complementary solution, because the problem always has a solution. Note by construction then $(x, \tau, y, z) = (e, 1, 0, 1)$ is an interior feasible solution to (5.39). (e is a n dimensional vector of all ones). This implies that the problem (P) can be solved by most feasible-interior-point algorithms.

Xu, Hung, and Ye [19] suggest an alternative solution method which is also an interior-point algorithm, but specially adapted to the problem (5.37). The algorithm can be stated as follows:

Algorithm 5.8.1

1. *Choose* $(x^0, \tau^0, y^0, s^0, \kappa^0)$ *such that* $(x^0, \tau^0, s^0, \kappa^0) > 0$. *Choose* $\varepsilon_f, \varepsilon_g > 0$ *and* $\gamma \in (0, 1)$ *and let* $\eta := 1 - \gamma$.

2. $k := 0$.

3. Compute:

$$\begin{aligned} r_p^k &:= b\tau^k - Ax^k, \\ r_d^k &:= c\tau^k - A^T y^k - s^k, \\ r_g^k &:= \kappa^k + c^T x^k - b^T y^k, \\ \mu^k &:= \tfrac{(x^k)^T s^k + \tau^k \kappa^k}{n+1}. \end{aligned}$$

4. If

$$||(r_p^k; r_d^k; r_g^k)|| \leq \varepsilon_f \quad \textit{and} \quad ((x^k)^T s^k + \tau^k \kappa^k) \leq \varepsilon_g,$$

then terminate.

5. Solve the linear equations

$$\begin{aligned} Ad_x - bd_\tau &= \eta r_p^k, \\ A^T d_y + d_s - cd_\tau &= \eta r_d^k, \\ -c^T d_x + b^T d_y - d_\kappa &= \eta r_g^k, \\ S^k d_x + X^k d_s &= -X^k s^k + \gamma \mu^k e, \\ \kappa^k d_\tau + \tau^k d_\kappa &= -\tau^k \kappa^k + \gamma \mu^k \end{aligned}$$

for $(d_x, d_\tau, d_y, d_s, d_\kappa)$.

6. For some $\theta \in (0,1)$ *let*

$$\begin{aligned} \alpha^k := \quad & \textit{maximize} \quad \theta\alpha \\ & \textit{subject to} \quad \begin{bmatrix} x^k \\ \tau^k \\ s^k \\ \kappa^k \end{bmatrix} + \alpha \begin{bmatrix} d_x \\ d_\tau \\ d_s \\ d_\kappa \end{bmatrix} \geq 0, \\ & \qquad\qquad\qquad\quad \alpha \qquad\quad \leq \theta^{-1}. \end{aligned}$$

7.

$$\begin{bmatrix} x^{k+1} \\ \tau^{k+1} \\ y^{k+1} \\ s^{k+1} \\ \kappa^{k+1} \end{bmatrix} := \begin{bmatrix} x^k \\ \tau^k \\ y^{k+1} \\ s^{k+1} \\ \kappa^{k+1} \end{bmatrix} + \alpha^k \begin{bmatrix} d_x \\ d_\tau \\ d_y \\ d_s \\ d_\kappa \end{bmatrix}.$$

8. $k = k + 1$.

9. goto 3

The following facts can be proved about the algorithm

$$\begin{aligned} r_p^{k+1} &= (1 - (1-\gamma)\alpha^k) r_P^k, \\ r_d^{k+1} &= (1 - (1-\gamma)\alpha^k) r_D^k, \\ r_g^{k+1} &= (1 - (1-\gamma)\alpha^k) r_G^k, \end{aligned} \tag{5.40}$$

and

$$((x^{k+1})^T s^{k+1} + \tau^{k+1}\kappa^{k+1}) = (1 - (1-\gamma)\alpha^k)((x^k)^T s^k + \tau^k \kappa^k) \tag{5.41}$$

which shows that the primal residuals (r_p), the dual residuals (r_d), the gap residual (r_g), and the complemantary gap ($x^T s + \tau\kappa$) all are reduced strictly if $\alpha^k > 0$ and at the same rate. This shows that $(x^k, \tau^k, y^k, s^k, \kappa^k)$ generated by the algorithm converges towards an optimal solution to (5.37) (and the termination criteria in step 4 is ultimately reached). In principle the initial point and the stepsize α^k should be should be chosen such that

$$\min_j(x_j^k s_j^k, \tau^k \kappa^k) \geq \beta\mu^k, \quad \forall k = 0, 1, \ldots$$

is satisfied because this guarantues $(x^k, \tau^k, y^k, s^k, \kappa^k)$ converges towards a strictly complementary solution.

Further details about the homogeneous algorithm can be seen in [20]. Issues related to implementing the homogeneous algorithm are discussed in [1, 19].

5.9 Notes

There exists a large literature on interior-point methods which it has not been possible to cover in this Chapter. In this section we will give a few reference to additional literature.

First it should be noted that interior-point methods have been known for long time. Indeed they were studied extensively in the sixties by Fiacco and McCormick [8].

Some recent surveys papers about interior-point methods in general are [9, 16]. The book of Wright [17] presents the primal-dual algorithms and the related theory in great detail. The primal-dual algorithm presented in this paper is the basis for almost all commercially available interior-point based software. However, in practice a lot refinements are added to the algorithm. For further details on implementing the primal-dual algorithm we refer to [13, 2]

5.10 Exercises

1. Solve the nonlinear equation

   $$x^2 = 18$$

   using Newton's method. (Note you are essentially finding $\sqrt{18}$). Show graphically how Newton's method works in this case.

2. Define the problem

   $$\begin{array}{lll} (QP) & \text{minimize} & 0.5x^T Qx + c^T x, \\ & \text{subject} & Ax = b. \end{array} \tag{5.42}$$

   (a) State the Lagrange function to (QP).

   (b) What is the first order optimality conditions to (QP). (Hint let $f(x) = 0.5x^T Qx$. Then $\nabla f(x) = Qx$).

   (c) What is the Newton equation system to the first order optimality conditions in this case.

3. Prove (5.25).

4. Plot

$$f(x) = 2x - 3ln(x)$$

in a diagram. Is $f(x)$ convex?

5. Assume $x = [1\ 2\ 4\ 5]^T$ and $d_x = [-3.0\ 0.0\ 2.0\ -20.9]^T$ then find the maximum α such that $x^+ = x + \alpha d_x \geq 0$. Next verify $x + \theta\alpha d_x > 0$ for $\theta = 0.99$.

6. Let

$$\begin{array}{lrcrcr} \text{maximize} & 10x_1 & & 9x_2, & & \\ \text{subject to} & 7/10x_1 & + & 1x_2 & \leq & 630, \\ & 1/2x_1 & + & 5/6x_2 & \leq & 600, \\ & 1x_1 & + & 2/3x_2 & \leq & 708, \\ & 1/10x_1 & + & 1/4x_2 & \leq & 135, \\ & x_1, & & x_2 & \geq & 0. \end{array} \tag{5.43}$$

   (a) Solve the problem (5.43) graphically.

   (b) Put the problem (5.43) on standard form and convert it to a minimization problem.

   (c) Perform two iterations of the primal-dual method starting from $(x^0, y^0, s^0) = (e, 0, e)$. Let $\theta = 0.9$ and $\gamma = 0.1$.

7. In Section 5.3.4 we developed the primal-dual algorithm. However, we could equally well have developed a primal algorithm. The idea of this algorithm is to solve (5.12) using Newton's method. In this exercise we will investigate this idea.

   (a) Compute one step of Newton's method to (5.12).

   (b) Obtain a closed form expression for d_y and d_x similar to (5.31) and (5.34).

   (c) How is the new point x^+ obtained.

   (d) What assumptions must the initial guess x^0 satisfies.

8. Verify that the matrix M defined by (5.32) is a symmetric and positive-definite matrix.

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