Lecture 15:
Mastering the Oval Track Puzzle

Contents

We now have enough theory developed to give a thorough analysis of the Oval Track puzzle.

The interested reader may wish to see the book *Oval Track and other Permutation Puzzles* by J.O. Kiltinen for further reading. A link in the software section of our website will take you to his page where you can download a demo version of the Oval Track puzzle.

15.1 Oval Track with $T = (1,4)(2,3)$

In this section we focus on the standard Oval Track puzzle as shown in Figure 1. This version is also known as TopSpin and was once manufactured by Binary Arts (now ThinkFun).

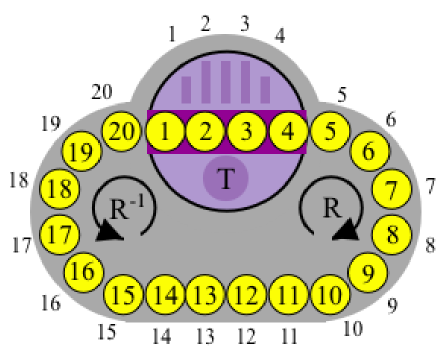


Figure 1: Oval Track puzzle.

The two basic moves of the Oval Track puzzle are R, and T, where R denotes a clockwise rotation of numbers around the track, where each number moves one space, and T denotes a rotation of the turntable.

The permutation corresponding to the legal moves R and T are as follows:

$$R = (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20)$$
$$T = (1,4)(2,3)$$

and the Oval Track puzzle group is $OT = \langle R, T \rangle$.

Note that $T^{-1} = T$ since T has order 2, and R^{-1} represents a counterclockwise rotation of the disks along the track.

Let's get right down to business and find out which permutations of the 20 disks are possible. We can set-up the corresponding puzzle group OT in SAGE and compute its order. Since the maximum possible number of permutation is $20!$ we'll ask if the order of OT is this value.

SAGE

```
sage: S20=SymmetricGroup(20)
sage: R=S20("(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20)")
sage: T=S20("(1,4)(2,3)")
sage: OT=S20.subgroup([R,T])
sage: OT.order()==factorial(20)
True
```

Therefore, *all* possible permutations of the puzzle pieces are possible. We could have instead asked SAGE if OT is the symmetric group S_{20} to achieve the same result.

SAGE

```
sage: OT==SymmetricGroup(20)
True
```

Theorem 15.1 (Solvability Criteria for Oval Track puzzle) *For the Oval Track puzzle with 20 disks and $T = (1,4)(2,3)$, every permutation $\alpha \in S_{20}$ is solvable. In other words, $OT = S_{20}$.*

Knowing that all permutation in S_{20} are obtainable is a start, but we actually would like to know how to solve the puzzle from any arrangement of the disks. Moreover, it would be nice to see exactly why SAGE is correct in stating $OT = S_{20}$; the algorithms implemented in SAGE to do these calculations are beyond the scope of this course.

The theory we have developed provides us with the answer as to why $OT = S_{20}$. In Lecture 13 we found a square commutator that produces a 3-cycle:

$$[R^{-3}, T]^2 = (1, 7, 4).$$

The puzzle provides us enough flexibility, or "wiggle room", to bring any 3 disks into positions $1, 7, 4$. See Exercise 3 for some practice in doing this. Therefore we may perform any 3-cycle by conjugation. See Section 15.1.2 for an example. This means we can produce any even permutation of the 20 disks, so $A_{20} < OT$. Also, OT contains an odd permutation: the 20-cycle R. This is enough to conclude that $OT = S_{20}$. See Exercise 6.

This gives a theoretical answer as to *why* every permutation of the disks is possible, but it doesn't provide us with a method, or strategy, to solve it. We still have some work to do to find out *how* to solve it.

We begin by looking for a 2-cycle, which we know must exist. Then we should be able to conjugate it to get all other 2-cycles, given that there seems to be enough "wiggle room".

15.1.1 2-cycles

The most basic combination of moves is TR^{-1}. (Here we use R^{-1} since this brings low numbered disks into the turntable, and this will be handy when we vary the number of disks used later.) This is the product of a 2-cycle and a 17-cycle. In other words, it is a move of order 34. The move $(TR^{-1})^{17}$ has order 2 and is in fact a 2-cycle. Let τ denote this 2-cycle:

$$\tau = (TR^{-1})^{17} = (1, 3).$$

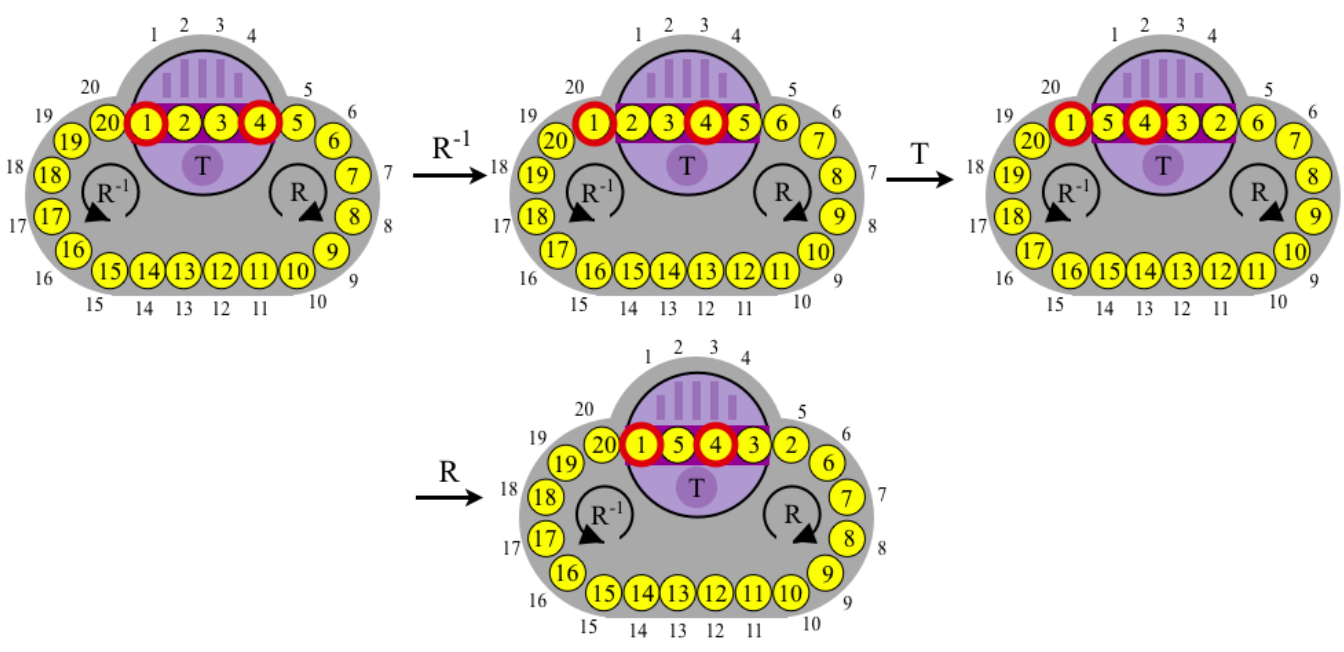
SAGE
```
sage: T*R^(-1)
(1,3)(4,20,19,18,17,16,15,14,13,12,11,10,9,8,7,6,5)
sage: (T*R^(-1))^(17)
(1,3)
```

Producing $\tau = (1,3)$ is a good first step. But it uses quite a few moves: 34 in total. Is it possible to perform a transposition using less moves? Notice that this move sequence sends *every* disk through the turntable, in some sense this sequence of moves is considered “global”. Maybe we could find a “local” move sequence, like the 3-cycle commutator: $[R^{-3}, T]^2 = (1,7,4)$, which only puts disks 1 through 7 in the turntable, all other disks are just rocked back-and-forth. Are we able to find a “local” move to produce a 2-cycle?

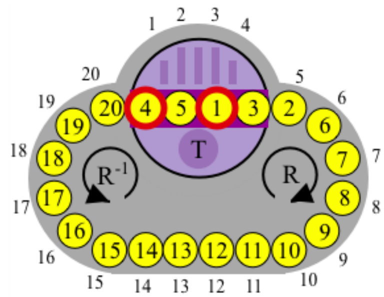
Well, our theory tells us: no! If we think about a local move sequence, it would only use disks 1 through m, all other disks ($m+1$ through 20) would just rock back-and-forth. This means, the same move sequence would produce a 2-cycle on the puzzle with 21 disks. Yes, we are changing the puzzle, but no this doesn't affect the 2-cycle, as long as it is “local”. But T is an even permutation, and R would be a 21-cycle, which is even too. Therefore $\langle T, R\rangle$ would only produce even permutations, hence no 2-cycle. Therefore, if we are able to get a 2-cycle in OT it must use a sequence of moves that puts every disk through the turntable at least once. Our move $\tau = (TR^{-1})^{17}$ does this: it sends each disk 5 through 20 through the turntable once. This seems to be the best we can do. This is an illustration of the power of the theory we have developed so far. We can answer questions about what we can, and cannot, do with the pieces of the puzzle.

Now that we have one 2-cycle we can conjugate it to get others.

For example, let's build $(1,4)$ as a conjugate of $\tau = (1,3)$. To do this, we will find a sequence of moves that takes 4 to position 3, while at the same time leaving 1 in position 1. The required movement is to push disk 4 one spot to the left (i.e. one spot closer to disk 1). If we rotate the track until 4 is in spot 3, then apply T, we have now moved 4 one spot closer to 1 on the right. Then rotate the track so 1 is back in position 1. You may have noticed we just applied a conjugate to do this: $R^{-1}TR$. See the following diagram.

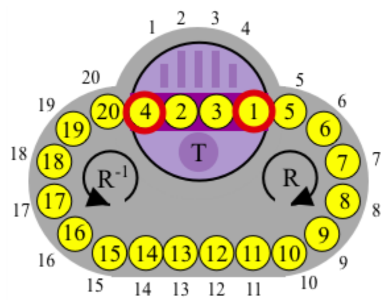


We now swap 1 and 4 using the transposition $\tau = (1,3)$, which puts the puzzle in following position.



Then undo the set-up moves above to produce $(1,4)$. To summarize, we performed the conjugate

$$(R^{-1}TR)\tau(R^{-1}TR)^{-1} = (1,4).$$



There was nothing special about 1 and 4 in this example. For any two disks a and b we can use turntable moves to bring them closer together, until there is only one disk between them, then we can rotate the track until they are in positions 1 and 2. This results in the set-up move β. Now apply τ, then undo the set-up move: β^{-1}. The result is $\beta\tau\beta^{-1} = (a,b)$. This proves the following.

Theorem 15.2 (2-cycles on Oval Track) *For the Oval Track puzzle with* 20 *disks and* $T = (1,4)(2,3)$*, every* 2*-cycle can be obtained as a conjugate of* $(TR^{-1})^{17} = (1,3)$.

Notice, Theorem 15.1 now follows from this theorem. We can be content with now knowing that SAGE was correct in its statement that $OT = S_{20}$.

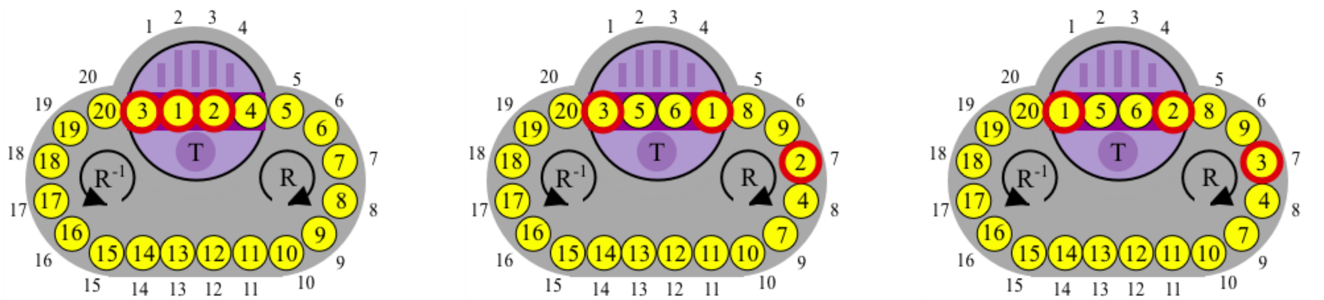
15.1.2 3-cycles

While investigating commutators in Lecture 13 we found a square commutator that produces a 3-cycle:

$$[R^{-3}, T]^2 = (1,7,4).$$

Having this one 3-cycle is valuable to us since we can conjugate it to get other 3-cycles. Note, we can't simply assume we can generate all 3-cycles as conjugates since we need to be able to perform a set-up move which takes any 3 disks to spots $1,7,4$. From the example below we'll see that the puzzle provides enough flexibility so that this is always possible.

For example suppose we are solving the puzzle and have brought it to an end-game position $(1,2,3)$. See Figure 2a. To solve the puzzle we need to apply the inverse 3-cycle $(1,3,2)$. To accomplish this we will use our fundamental 3-cycle $(1,7,4)$ by first performing a sequence of moves that puts disks 3, 1 and 2 into spots 1, 4 and 7. We will record the sequence of moves as β^{-1}.



(a) End-game position $(1, 2, 3)$. The cycle $(1, 3, 2)$ is needed to solve.

(b) Set-up by putting disks $3, 1, 2$ into spots $1, 4, 7$.

(c) Perform the 3-cycle $(1, 7, 4)$.

Figure 2: The steps for performing the 3-cycle $(1, 3, 2)$ as a conjugate of the 3-cycle $(1, 7, 4)$.

Before we start, we look at the current arrangement and make a mental note that "1 chases 3". By this we mean that disk 1 is to go to the spot where disk 3 is right now. This description will help us decide whether we should perform $(1, 7, 4)$ or $(1, 4, 7)$ at a later time.

It doesn't matter how you go about getting these three disks into spots 1, 4, and 7. Since the disks are already arranged in order $3, 1, 2$ we can keep them in this order and keep 3 in spot 1, move 1 to spot 4, and move 2 to spot 7. This means we need to space the disks out by adding two spaces between the pairs of disks.

To add spaces we proceed as follows. Move disk 3 to the left of the turntable and apply T to get some space between it and disk 1. Now there are three disks between 3 and 1 so cut this down by moving 1 into position 3 of the turntable and apply T. There are now two spaces between disks 3 and 1. Disk 2 is just to the right of disk 1. To add space between disk 1 and 2 we move 1 to the left of the turntable, apply T, which now puts three spaces between 1 and 2, so we close this gap by bringing 2 into spot 3 and applying T. Now the three disks are spaced out, and so we just move 3 to spot 1, and it follows that 1 is now in spot 4, and 2 is in spot 7. See Figure 2b. The move sequence we used to do this was $\beta^{-1} = R^{-1}TR^{-1}TR^{-2}TR^{-1}TR^{5}$.

Now we are ready to apply our fundamental 3-cycle: $(1, 7, 4)$, but we need to know whether we are to apply it or its inverse. This is where our mental note comes in: "1 chases 3". We need to send disk 1 to where disk 3 is now, this means we should apply $(1, 7, 4)$. The puzzle is now in the position shown in Figure 2c.

Finally we undo the set-up move by applying β, and the puzzle is solved.

This example provides the general technique for producing 3-cycles.

Guide for producing a 3-cycle:

Step 1. Pick the three disks you wish to cycle: (a, b, c). Make a mental note that "a chases b".

Step 2. Move the disks to positions $1, 4, 7$, in any way whatsoever. Call this move β^{-1}.

Step 3. Apply the fundamental 3-cycle $(1, 4, 7)$ or its inverse $(1, 7, 4)$, depending on locations of a and b.

Step 4. Undo the set-up move by applying β. The result is the 3-cycle (a, b, c).

15.1.3 Strategy for Solution

We are now ready to describe a strategy for solving the Oval Track puzzle. Since we can perform any 2-cycle we already have a method at hand. However, the fundamental 2-cycle is $(TR^{-1})^{17} = (1, 3)$ is 34 moves long, and any other 2-cycle obtained by conjugation will use more moves. So solving by swapping two pieces at a time is an inefficient way to solve the puzzle.

Similarly, we can create any 3-cycle by conjugating the fundamental 3-cycle $[R^{-3}, T]^2 = (1, 7, 4)$. But again this will result in pretty long move sequences.

Instead, we will just approach the puzzle by first setting pieces 20 through 5 in order, which is fairly straightforward since there is enough "wiggle room" to move things around. This brings the puzzle to its end-game position, that is, a position where only disks $1, 2, 3, 4$ are permuted. It is at this point where 2-cycles and 3-cycles will be useful. Moreover, we will try to use 3-cycles since the move sequence is significantly shorter, but if forced we may need to use a 2-cycle, which we have ready and waiting.

Will we ever be forced to use a 2-cycle? In the end game all permutations in S_4 are possible. For example, we may be faced with an end-game configuration which is an odd permutation. Since 3-cycles are even we won't be able to solve it using 3-cycles alone, we will be forced to use at least one 2-cycle.

Guide to solve the puzzle:

(a) Starting with disk 20 put disks 20 through 5 in numerical order.

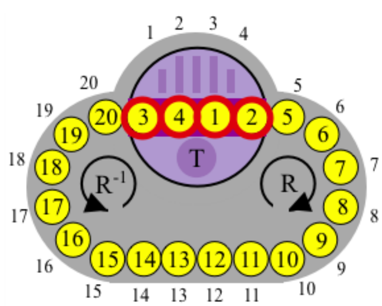
(b) The permutation of the final 4 disks is either even or odd. This is the *end-game* phase.

  (a) If the permutation is even, express it is either a 3-cycle or a product of two 2-cycles.
  3-**cycle**: Use a conjugate of the fundamental 3-cycle $[R^{-3}, T]^2 = (1, 7, 4)$ to solve it.
  two 2-**cycles**: Check whether it is $(1, 4)(2, 3)$, if it is apply T and you're done. Otherwise, express it as the product of 3-cycles, and use conjugates of the fundamental 3-cycle $[R^{-3}, T]^2 = (1, 7, 4)$ to solve it.

  (b) If the permutation is odd, it is either a 2-cycle or a 4-cycle.
  2-**cycle**: Use a conjugate of the fundamental 2-cycle $(TR^{-1})^{17} = (1, 3)$ to solve it.
  4-**cycle**: First check whether apply T reduces the 4-cycle to a 2-cycle. Otherwise, there is a 3-cycle that does. Once you have reduced it to a 2-cycle use a conjugate of the fundamental 2-cycle $(TR^{-1})^{17} = (1, 3)$ to solve it.

Let's now practice a few end-game configuations.

Example 15.1 *Solve the end-game configuration* $(1, 3)(2, 4)$.



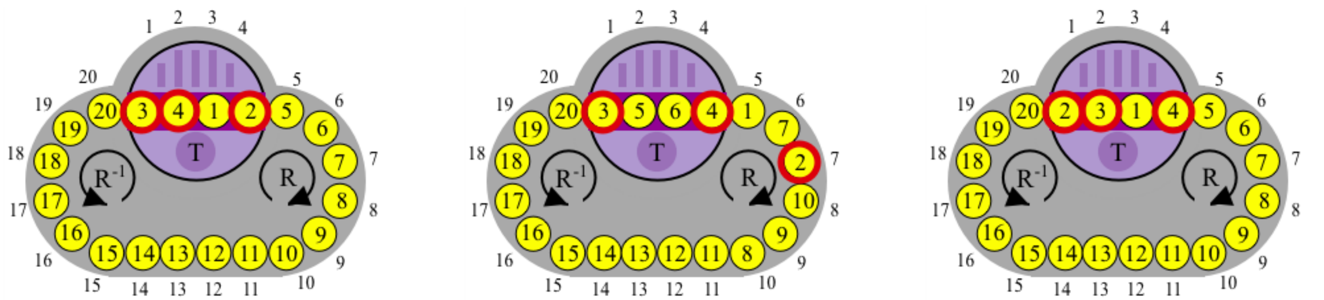
To solve the puzzle we need to produce the inverse permutation, which is just itself, $(1, 3)(2, 4)$*. Since* $(1, 3)(2, 4)$ *is an even permutation we can write it as a product of* 3*-cycles:* $(1, 2, 4)(1, 2, 3)$*. We focus on constructing each* 3*-cycle as a conjugate of the fundamental* 3*-cycle.*

$\mathbf{(1, 2, 4)}$*: This* 3*-cycle involves disks* $3, 4, 2$ *and results in putting* 4 *in its home position. See Figure 3a.*

The direction we want to cycle these disks is summarized by "3 *chases* 4*". Apply the strategy of spacing out the disks by making sure there are two disks between the middle and each outer disk. A move sequence that does this is*

$$\beta^{-1} = R^{-1}TR^{-1}TR^{-3}TR^{-1}TRTR^5.$$

The puzzle will be as shown in Figure 3b.



(a) Initial configuration $(1,3)(2,4)$. The cycle $(1,2,4)$ is to be produced.

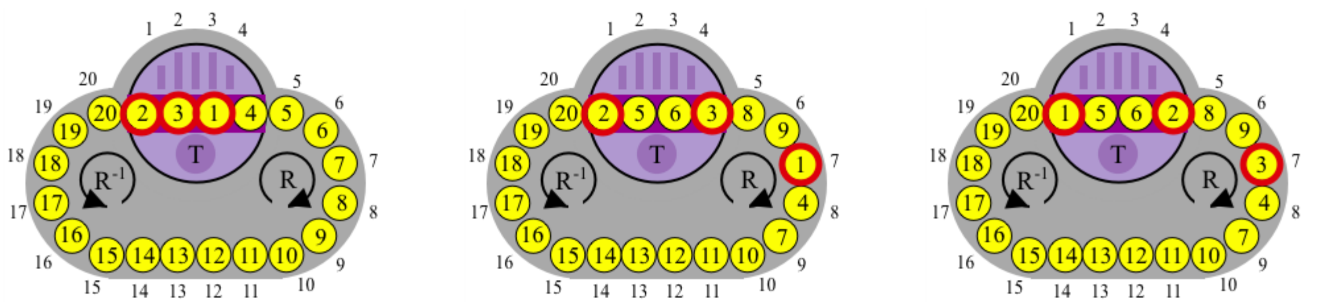
(b) Set-up by putting disks $3,4,2$ into spots $1,4,7$.

(c) Perform the 3-cycle $(1,4,7)$, then undo the set-up move.

Figure 3: The steps for performing the 3-cycle $(1,2,4)$ as a conjugate of the 3-cycle $(1,4,7)$.

Recalling that 3 *chases* 4*, the fundamental* 3*-cycle we should apply is* $[R^{-3},T]^{-2}=(1,4,7)$*. Then applying* β *to undo the set-up, we end up with the puzzle in the configuration shown in Figure 3c.*

$\mathbf{(1,2,3)}$*: This* 3*-cycle involves disks* $2,3,1$ *and the direction we want to cycle these disks is summarized by "*2 *chases* 3*". See Figure 4a.*



(a) Initial configuration $(1,3,2)$. The cycle $(1,2,3)$ is to be produced.

(b) Set-up by putting disks $2,3,1$ into spots $1,4,7$.

(c) Perform the 3-cycle $(1,4,7)$.

Figure 4: The steps for performing the 3-cycle $(1,2,3)$ as a conjugate of the 3-cycle $(1,4,7)$.

We can space out the disks, making sure there are two disks between the middle and each outer disk, by using the move sequence

$$\delta^{-1}=R^{-1}TR^{-1}TR^{-2}TR^{-1}TR^{5}.$$

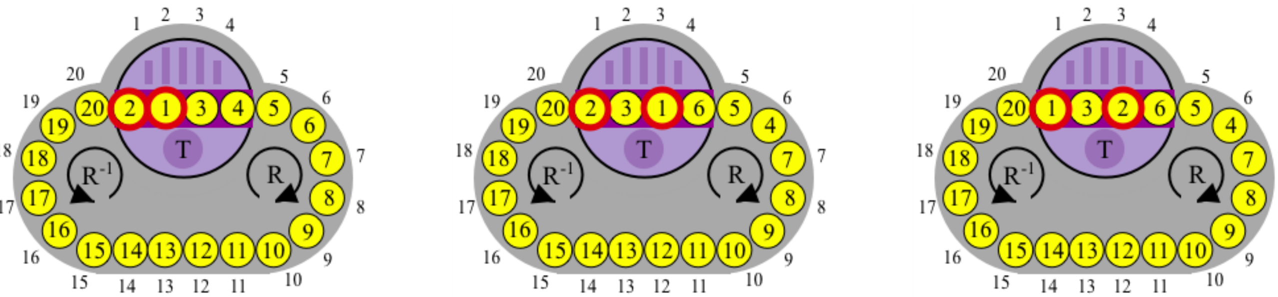
The resulting position is shown in Figure 4b.

Since 2 *is to chase* 3*, the fundamental* 3*-cycle we should apply is* $[R^{-3},T]^{-2}=(1,4,7)$*. See Figure 4c. Applying* δ *to undo the set-up move solves the puzzle.*

In the next example we consider the case when the end-game permutation is a 2-cycle.

Example 15.2 *Solve the end-game configuration* $(1,2)$*. See Figure 5a.*

To solve the puzzle we need to produce the inverse permutation, which is just itself, $(1,2)$*. Since* $(1,2)$ *is a* 2*-cycle we construct it as a conjugate of the fundamental* 2*-cycle* $(TR^{-1})^{17}=(1,3)$*. Apply a set-up move which leaves* 2 *in spot* 1*, and moves* 1 *to spot* 3*. One such move sequence is* $\beta^{-1}=R^{-1}TR^{-1}TRTR$*. Recall that to do this you just want to insert two disks between disks* 2 *and* 1*. The puzzle should now look like Figure 5b. Apply the*



(a) Initial configuration $(1, 2)$. The cycle $(1, 2)$ is to be produced.

(b) Set-up by putting disks $3, 4, 2$ into spots $1, 4, 7$.

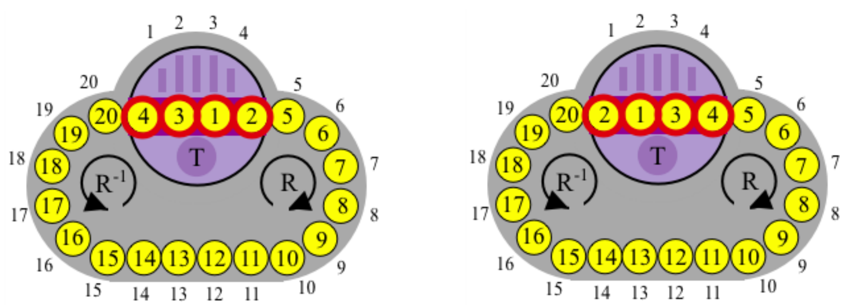
(c) Perform the 3-cycle $(1, 4, 7)$, then undo the set-up move.

Figure 5: The steps for performing the 2-cycle $(1, 2)$ as a conjugate of the 2-cycle $(1, 3)$.

fundamental 2*-cycle* $(TR^{-1})^{17} = (1, 3)$*, which results in swapping disks* 2 *and* 1*. This is shown in Figure 5c. Undoing the set-up move by applying* β *solves the puzzle.*

The end-game permutation could be a 4-cycle, which is an odd permutation. If we are lucky a move T will take it to a transposition as the next example illustrates.

Example 15.3 *Solve the end-game configuration* $(1, 3, 2, 4)$*. See Figure 6a.*



(a) Initial configuration $(1, 3, 2, 4)$. The cycle $(1, 4, 2, 3)$ is to be produced.

(b) Start by performing T to put as many disks in their home positions as possible.

Figure 6: The 4-cycle $(1, 3, 2, 4)$ is only one move T away from the 2-cycle $(1, 2)$.

Every disk is out of place, but disk 4 *can be moved to spot* 4 *by move* T*. This also brings disk* 3 *home as well. The permutation of the puzzle pieces is now* $(1, 2)$ *(see Figure 6b) which we already solved in the last example.*

Contrary to the last example, it may happen the an end-game 4-cycle cannot be immediately converted into a 2-cycle by performing move T. In this case there will always be a 3-cycle that does. Just use a 3-cycle to send any piece home, it follows that some other piece must also be sent home as well. The reason for this is the product of an odd permutation and an even permutation is odd, and the only odd permutations on 3 objects are transpositions. See Exercise 5 for one such end-game.

15.1.4 Changing the number of disks

What happens if we change the number of disks in the puzzle. For example, suppose we used only 19 disks instead of 20. Would we expect our results to be the same. For example, does Theorem 15.1 remain true for 19 disks? Let's ask SAGE .

SAGE
```
sage: S20=SymmetricGroup(20)
sage: R=S20("(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,19)")
sage: T=S20("(1,4)(2,3)")
sage: OT19=S20.subgroup([R,T])
sage: OT19.order == factorial(19)
False
```

Let OT_{19} be the Oval Track group on 19 disks. Then we determined that $|OT_{19}| \neq 19!$, so OT_{19} does not contain every permutation of the 19 disks. This shouldn't come as a surprise though, since the rotation move R is a 19-cycle, which is even, and the turntable move T is also even. Therefore we can only generate even permutations, and at best we could get the group of all even permutations A_{19}. Let's see if we get all of A_{19}.

SAGE
```
sage: OT19 == AlternatingGroup(19)
True
```

We do! This means that for the Oval Track puzzle on 19 disks the solvable permutations are precisely the even permutations.

What happened to our fundamental 2-cycle $(TR^{-1})^{17}$? It was built from TR^{-1} so let's see what TR^{-1} is now.

SAGE
```
sage: T*R^(-1)
(1,3)(4,19,18,17,16,15,14,13,12,11,10,9,8,7,6,5)
```

It is now a product of a 2-cycle and a 16-cycle. Unlike the 20 disk case, there is no way to take a power of this to kill-off the 16-cycle and leave the 2-cycle alone. However, we still have our fundamental 3-cycle: $[R^{-3}, T]^2 = (1, 7, 4)$ so we can use conjugates of this to solve the end-game of this puzzle.

What about changing the number of disks even more? Let n be the number of disks, and let OT_n be the Oval Track group on n disks. Notice the move R, which is an n-cycle, will be even if and only if n is odd. Therefore OT_n will contain only even permutations: $OT_n \leq A_n$. On the other hand, if n is even then R is an odd permutation, so OT_n will contain some odd permutations. The questions are then: (i) for n odd is $OT_n = A_n$, and (ii) for n-even is $OT_n = S_n$?

We can use SAGE to help us answer these questions. Here we consider the number of disks $4 \leq n \leq 20$.

SAGE
```
sage: for n in (4..20):
sage:     Rn=S20([tuple(range(1,n+1))])      # creates n-cycle (1,2,3,...,n)
sage:     OTn=S20.subgroup([Rn,T])          # creates OTn: the Oval Track group on n disks
sage:     if is_even(n):
sage:         print n, OTn==SymmetricGroup(n)     #check if OTn is the full symmetric group
sage:     else:
sage:         print n, OTn==AlternatingGroup(n)   #check if OTn is the full alternating group
4 False
5 False
6 True
7 True
8 True
9 True
10 True
11 True
12 True
13 True
14 True
```

```
15 True
16 True
17 True
18 True
19 True
20 True
```

Therefore, for $n \geq 6$ the answers to our questions are: yes. However, for small values of n the answer is: no. It seems like there just isn't enough "wiggle room" to get all the permutations when there is a small number of disks.

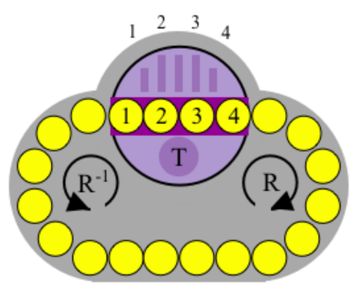
Let's investigate this further.

If $n \geq 6$, the product TR^{-1} consists of a 2-cycle and an $(n-3)$-cycle: disk 2 remains fixed, disks 1 and 3 are swapped, and the remaining $n-3$ disks are cycled to the left around the track. If n is even, then $n-3$ is odd so $(TR^{-1})^{n-3}$ is a 2-cycle $(1,3)$. Having this 2-cycle, and using conjugation, indicates why $OT_n = S_n$ when n is even.

If $n \geq 7$ we still have the fundamental 3-cycle $[R^{-3}, T]^2 = (1,7,4)$, so we can conjugate it to get other 3-cycles. This indicates why $OT_n = A_n$ when n is odd.

The remaining cases are $n = 4, 5$.

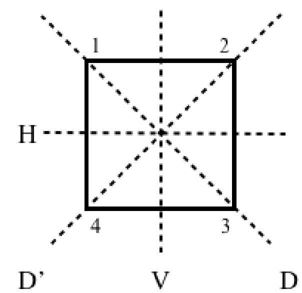
$\boldsymbol{n = 4}$: We can view this puzzle as in the diagram, where only the labeled disks are in play and they are free to move around the track by the rotation $R = (1,2,3,4)$.



We use SAGE to work out the order of OT_4.

SAGE
```
sage: S4=SymmetricGroup(4)
sage: R=S4("(1,2,3,4)")
sage: T=S4("(1,4)(2,3)")
sage: OT4=S4.subgroup([R,T])
sage: OT4.order()
8
sage: OT4.list()
[(), (2,4), (1,2)(3,4), (1,2,3,4), (1,3), (1,3)(2,4), (1,4,3,2), (1,4)(2,3)]
```

It is a group of order 8, and these elements look very familiar. They remind us of another group of order 8 we know, the dihedral group D_4, which is the group of symmetries of a square. If we label the vertices of the square by $1,2,3,4$, then the group of symmetries D_4 can be viewed as a subgroup of S_4.



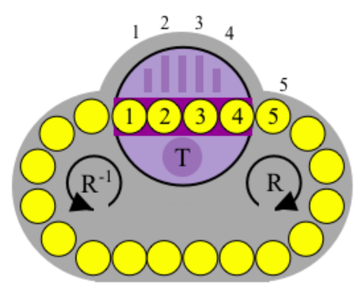
SAGE
```
sage: D4=DihedralGroup(4)
sage: D4.list()
[(), (2,4), (1,2)(3,4), (1,2,3,4), (1,3), (1,3)(2,4), (1,4,3,2),(1,4)(2,3)]
```

A rotation R of the pieces along the track, corresponds to a rotation of the square. A turntable move correspond to a reflection about the horizontal axis.

Since OT_4 and D_4 are essentially the same group, it is just the context that is different, we say these groups are **isomorphic** and write

$$OT_4 \approx D4.$$

$\boldsymbol{n = 5}$: We can view this puzzle as in the diagram, where only the labeled disks are in play and they are free to move around the track by rotation the $R = (1,2,3,4,5)$.

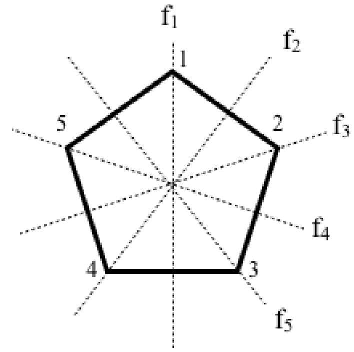


SAGE
```
sage: S5=SymmetricGroup(5)
sage: R=S5("(1,2,3,4,5)")
sage: T=S5("(1,4)(2,3)")
sage: OT5=S5.subgroup([R,T])
sage: OT5.order()
10
```

Based on our experience with $n = 4$, and the fact that the dihedral group of a regular pentagon has order 10, we may suspect that $OT_5 \approx D_5$. Checking with SAGE we see this is indeed the case.

SAGE
```
sage: OT5==DihedralGroup(5)
True
```

Since spot 5 is not on the turntable, move T is analogous to reflection f_1 of the pentagon in the digram below. This analogy indicates that the symmetries of the pentagon are generated by a clockwise rotation and the reflection f_1.



We summarize our results in the following theorem.

Theorem 15.3 *On the Oval Track puzzle with* $T = (1,4)(2,3)$ *, any scrambling can be solved if the number* n *of disks is even and* $n \geq 6$*. If* $n \geq 7$ *and odd then every even scrambling can be solved. Under these latter conditions, odd permutations can be brought down to a single transposition, but cannot be completely solved. In particular, if* OT_n *denotes the group of permutations achievable by the Oval Track puzzle with* n *disks then:*

$OT_4 \approx D_4$,

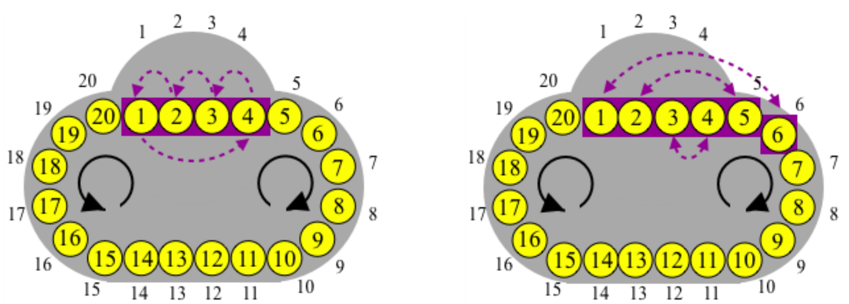
$OT_5 \approx D_5$,

$OT_n \approx S_n$ *if* $n \geq 6$ *and even,*

$OT_n \approx A_n$ *if* $n \geq 6$ *and odd,*

15.2 Variations of the Oval Track T move

Variations of the Oval Track puzzle can be created by changing the turntable move T. Figure 7 shows two different variations.



(a) $T = (1,4,3,2)$ (b) $T = (1,6)(2,5)(3,4)$

Figure 7: Some variation of the turntable move T for the Oval Track puzzle.

We can use SAGE to investigate the groups associated with these variations.

SAGE
```
sage: S20=SymmetricGroup(20)
sage: R=S20("(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20)")
sage: T=S20("(1,4,2,3)")
sage: OTv1=S20.subgroup([R,T])
sage: OTv1==SymmetricGroup(20)
True
```

SAGE
```
sage: S20=SymmetricGroup(20)
sage: R=S20("(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20)")
sage: T=S20("(1,6)(2,5)(3,4)")
sage: OTv2=S20.subgroup([R,T])
sage: OTv2==SymmetricGroup(20)
True
```

Therefore, on both these puzzles, all permutations of the pieces are possible.

Coming up with a strategy to solve these puzzles is similar to how we approached the original Oval Track puzzle. Use commutators to create moves, and conjugates to modify them. Try finding a fundamental 3-cycle or 2-cycle.

For the first variation, where $T = (1,4,3,2)$, we have commutators $[R^{-1},T] = (1,2,5)$, and $[T^{-1},R^{-1}] = (1,5,4)$. The product of these two is

$$[R^{-1},T][T^{-1},R^{-1}] = (1,2,4).$$

What is interesting about this is that combining this with T gives a 2-cycle:

$$[R^{-1},T][T^{-1},R^{-1}]T = (2,3).$$

We know how important having a 2-cycle is for solvability.

15.3 Exercises

1. **Getting to the end-game position.** Go to the course website and under the "software" link go to Jaaps Puzzle page and play with the javascript "Top Spin" puzzle. Mix up the disks and try to restore disks 20 through 5. That is, reduce the puzzle down to the end-game position. Do this a number of times until you are confident that getting to the end-game position is fairly straightforward. Don't worry about solving the end-game just yet.

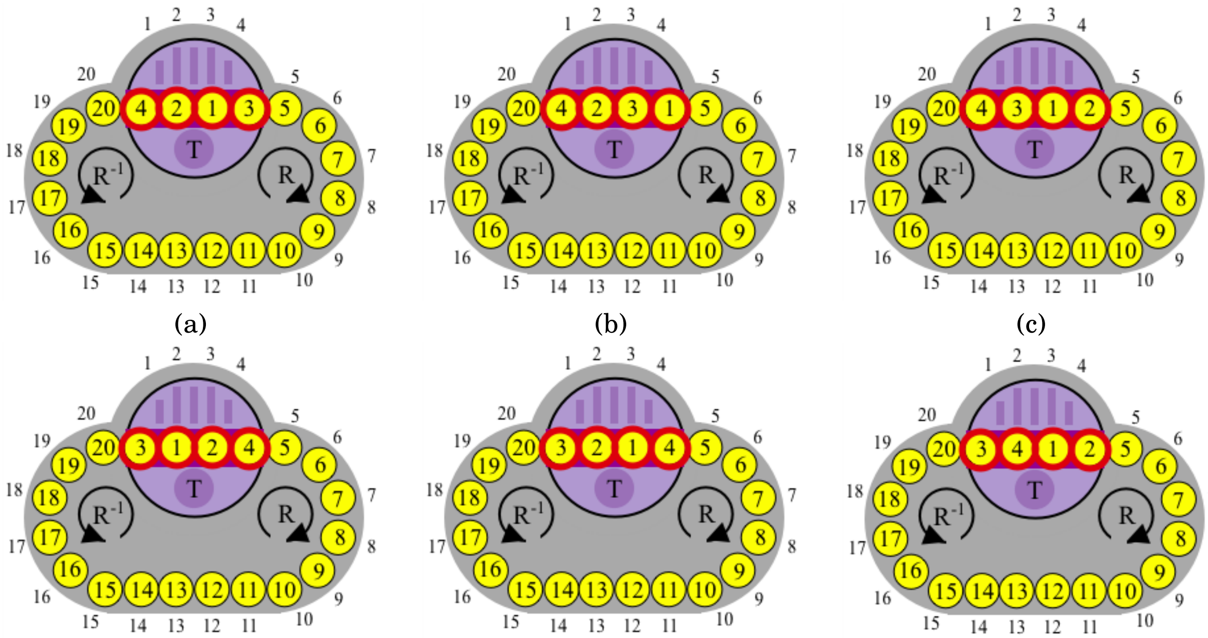
2. **2-cycles on OT with $T = (1,4)(2,3)$.** For each of the following 2 cycles, find a conjugate of $\tau = (1,3)$ which produces the 2-cycle. That is, find a sequence of moves β^{-1} so the $\beta^{-1}\tau\beta$ produces the desired 2-cycle.

   (a) $(1,9)$ (b) $(1,2)$ (c) $(3,14)$ (d) $(2,11)$

3. **3-cycles on OT with $T = (1,4)(2,3)$.** For each of the following 3 cycles, find a conjugate of the fundamental 3 cycle $\sigma = (1,4,7)$, or its inverse σ^{-1} which produces the 3-cycle. That is, find a sequence of moves β^{-1} so the $\beta^{-1}\sigma\beta$ produces the desired 3-cycle.

   (a) $(1,4,3)$ (b) $(1,3,4)$ (c) $(2,3,4)$

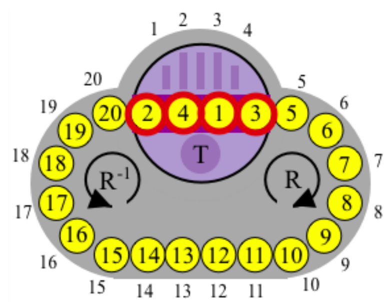
4. There are six end-game configurations shown below. (a) Write out each in cycle notation. (b) Plan a strategy for solving the end-game. (c) Implement your strategy and solve each of the puzzles. You may find it useful to use the virtual puzzles on the course website to try out your move sequences.



(a) (b) (c)

(e) (f) (g)

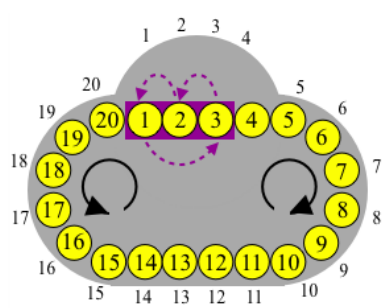
5. Solve the end-game configuration $(1, 3, 4, 2)$, which is shown in the diagram.



6. **Getting all permutations from one odd, and A_n.** Let $G < S_n$ be a group of permutation which contains all even permutations (i.e. $A_n < G$). and has at least one odd permutation $\beta \in G$. Show that $G = S_n$.

   (Hint: We already know the set of odd permutation O_n is the same size as the set of even permutations A_n. It suffices to show we can get all the elements of O_n from A_n and β. Show $O_n = \beta A_n := \{\beta\alpha \mid \alpha \in A_n\}$.)

7. Consider the variation $T = (1, 3, 2)$ of the turntable move on the Oval Track puzzle with 20 disks. Are all permutations of the puzzle pieces possible?



8. Consider the variation $T = (1,3)(2,4)$ of the turntable move on the Oval Track puzzle with 20 disks. Are all permutations of the puzzle pieces possible?