Lecture 14: Conjugates

Contents

Commutators provided us with a method for creating puzzle moves that affect only a small number of pieces. In this lecture we introduce "conjugation" which is a process for modifying existing moves to produce new moves that have similar structure.

Conjugation is discussed in Section 5.9 of Joyner's text.

14.1 Conjugates

When playing with permutation puzzles, a move sequence of the form "move 1, then move 2, then inverse of move 1" comes in handy. Moves of this form are called *conjugates*. You may have just realized that you frequently use "conjugate moves" when solving puzzles, if this is the case then you already have a working understanding of conjugation. As you read through this lecture, you will find it useful to have a puzzle on hand to try things out for yourself.

Definition 14.1 *If g, h are two elements of a group G, then we call the element*

$$g^h = h^{-1}gh$$

the ***conjugate*** *of g by h.*

Note that $g^h = g$ if and only if g and h commute. Therefore, much like the commutator, the conjugate g^h provides a measure of how much g and h fail to commute with each other. If g and h don't commute, then $g^h \neq g$, however g^h should be like g in some ways. In the case of permutations we can say exactly how they are similar. This is stated in Lemma 14.1 below.

The exponential notation is used because conjugation enjoys similar properties to that of exponentiation. See Exercise 5.

Example 14.1 *Consider the symmetric group S_3 and the elements $s_1 = (1,2)$, $s_2 = (1,3,2)$. Then the conjugate of s_1 by s_2 is*

$$s_1^{s_2} = s_2^{-1} s_1 s_2 = (1,2,3)(1,2)(1,3,2) = (1,3)$$

and the conjugate of s_2 by s_1 is

$$s_2^{s_1} = s_1^{-1} s_2 s_1 = (1,2)(1,3,2)(1,2) = (1,2,3).$$

Definition 14.2 *We say that two elements $g_1, g_2 \in G$ are **conjugate** (in G) if there is an element $h \in G$ such that $g_2 = g_1^h$.*

*The set of all elements in G that are conjugate to g is called the **conjugacy class of** g and denoted by $cl(g)$:*

$$cl(g) = \{xgx^{-1} \mid x \in G\}.$$

In the example above, we see that cycle structure seems to be preserved by conjugation. By this we mean, the conjugate of a 2-cycle was still a 2-cycle, the conjugate of a 3-cycle was still a 3-cycle. This is true in general, and we state it as the following remark. We prove this as a part of the subsequent lemma.

Remark 14.1 *For $\alpha, \beta \in S_n$, the two permutations α and $\beta^{-1}\alpha\beta$ have the same cycle structure.*

Lemma 14.1 (Conjugation preserves cycle structure) *Let α, β be any permutation in S_n, and suppose $\alpha(i) = j$. Then $\alpha^\beta = \beta^{-1}\alpha\beta$ sends $\beta(i)$ to $\beta(j)$:*

$$(\alpha^\beta)(\beta(i)) = \beta(j).$$

Moreover, if α has cycle structure

$$\alpha = (a_1, a_2, \ldots, a_{k_1})(b_1, b_2, \ldots, b_{k_2}) \cdots (c_1, c_2, \ldots, c_{k_m})$$

then α^β has the same cycle structure

$$\alpha^\beta = (\beta(a_1), \beta(a_2), \ldots, \beta(a_{k_1}))(\beta(b_1), \beta(b_2), \ldots, \beta(b_{k_2})) \cdots (\beta(c_1), \beta(c_2), \ldots, \beta(c_{k_m}))$$

Proof: To see $\alpha^\beta = \beta^{-1}\alpha\beta$ sends $\beta(i)$ to $\beta(j)$ we just compute it:

$$\alpha^\beta(\beta(i)) = (\beta^{-1}\alpha\beta)(\beta(i)) = \beta(\alpha(\beta^{-1}(\beta(i)))) = \beta(\alpha(i)) = \beta(j).$$

To show that the cycle structure is as described in the statement of the lemma first express α in disjoint cycle form: $\alpha = \sigma_1\sigma_2 \cdots \sigma_m$, where σ_i is a k_i-cycle. Observe that

$$\alpha^\beta = \beta^{-1}(\sigma_1\sigma_2 \cdots \sigma_m)\beta = (\beta^{-1}\sigma_1\beta)(\beta^{-1}\sigma_2\beta) \cdots (\beta^{-1}\sigma_m\beta),$$

so it suffices to prove the result for each of the cycles σ_i.

Consider the cycle $\sigma = (a_1, a_2, \ldots, a_k)$, and let $d_i = \beta(a_i)$. By the first part of the lemma, which we have already proved, σ^β contains the cycle $(d_1, d_2, \ldots, d_k)$. Moreover, if x is an element that is moved by σ^β then $(\beta^{-1}\sigma\beta)(x) \neq x$ and so $\sigma(\beta^{-1}(x)) \neq \beta^{-1}(x)$, which means $\beta^{-1}(x) = a_i$ for some i. Therefore, $x = d_i$ for some i. It follows that

$$\sigma^\beta = (d_1, d_2, \ldots, d_k).$$

This proves the lemma. □

As an example, this lemma and the preceding remark tells us that if we have a 3-cycle α, then no matter what the permutation β is, the conjugate α^β will be another 3-cycle. This is how we will use conjugation to modify existing puzzle moves.

As a consequence of Lemma 14.1 it is easy to see when two permutation $\alpha, \beta \in S_n$ are conjugate in S_n: they are conjugate if and only if the cycles in their respective disjoint cycle forms have the same length when arranged from shortest to longest (i.e. they have the same cycle structure). The "only if" part we have already proven. On the other hand, if two permutation α and β have the same cycle structure then arrange their disjoint cycle forms as follows (here we insert 1-cycles on the end):

$$\alpha = (a_{1,1}, a_{1,2}, \ldots, a_{1,k_1})(a_{2,1}, a_{2,2}, \ldots, a_{2,k_2}) \cdots (a_{m,1}, a_{m,2}, \ldots, a_{m,k_m})(a_{m+1,1}) \cdots (a_{m+s,1})$$
$$\beta = (b_{1,1}, b_{1,2}, \ldots, b_{1,k_1})(b_{2,1}, b_{2,2}, \ldots, b_{2,k_2}) \cdots (b_{m,1}, b_{m,2}, \ldots, b_{m,k_m})(b_{m+1,1}) \cdots (b_{m+s,1})$$

and construct a permutation γ such that $\gamma(a_{i,j}) = b_{i,j}$. It follows the $\alpha^\gamma = \beta$ and so α and β are conjugate. (Note, γ is not necessarily unique.)

For example, the permutations

$$\alpha = (1, 2, 3)(4, 5, 6, 7, 8)(9, 10), \quad \text{and} \quad \beta = (4, 5, 3)(1, 8, 2, 10, 11)(7, 12)$$

are conjugate in S_{12}. One possibility for γ is $(1, 4)(2, 5, 8, 11, 6)(3)(7, 10, 12, 9)$.

14.2 Modifying Puzzle moves with Conjugates

We've already made extensive use of conjugation while investigating the 15-puzzle. We showed in Lecture 9 that the solvable configurations of the 15-puzzle, where the empty space is in box 16, are precisely the even permutations. The way we argued this was we found one 3-cycle, namely $(11, 12, 15)$ and by conjugation we were able to modify this to produce any other 3-cycle.

In general, if we have a move α that does something useful, then we can modify using conjugates by first finding a set-up move β^{-1} that takes some pieces that we wish to affect and moves them to the positions affected by α. Applying α then affects these new pieces, and β then moves everything back. The result is that only the pieces moved by β^{-1} into M_α are affected, and they are permuted with the same structure that α has. This description may seem a little confusing, but once you've played around with conjugates you will see their actions are very intuitive. We'll look at many examples for the various puzzles over the next few sections.

14.2.1 Rubik's Cube

It is best if you have your Rubik's cube in hand while reading through this part.

Looking back at the commutators we constructed in Lecture 12 you will notice that many of the x moves were made up of conjugates.

We saw in Lecture 12 that the commutator $[LD^2L^{-1}, U]$ permuted three corner cubies as shown in Figure 1a.

We will modify this move so it permutes the three corner cubies as show in Figure 1b. To do this, first apply the set-up move B^{-1} which takes the urb corner piece to the rdb position. Then applying commutator $[LD^2L^{-1}, U]$ cycles the 3 corner cubies as shown in Figure 1a, though the piece in the rdb position is really the piece that started in the urb position. Undoing the set-up move results in the complete move sequence $B^{-1}[LD^2L^{-1}, U]B$ which moves the cubes as shown in Figure 1b.

As another example the commutator $[x, y]$ where

$$x = L^{-1}D^2LBD^2B^{-1}, \qquad \text{and} \qquad y = U$$

produced a twist of 2 corners as shown in Figure 2a. If we use the set-up move B, before apply the corner twist commutator, then undo the set-up move by taking B^{-1}, then we produce a new move which twists diagonally opposite corner cubies (see Figure 2b). This new move is the conjugate $B[L^{-1}D^2LBD^2B^{-1}, U]B^{-1}$.

(a) 3-cycle of corner cubies by commutator $LD^2L^{-1}ULD^2L^{-1}U^{-1}$

(b) conjugation of commutator by B

Figure 1: cycling 3 corner cubies

(a) 2 corner twist by commutator $xyx^{-1}y^{-1}$

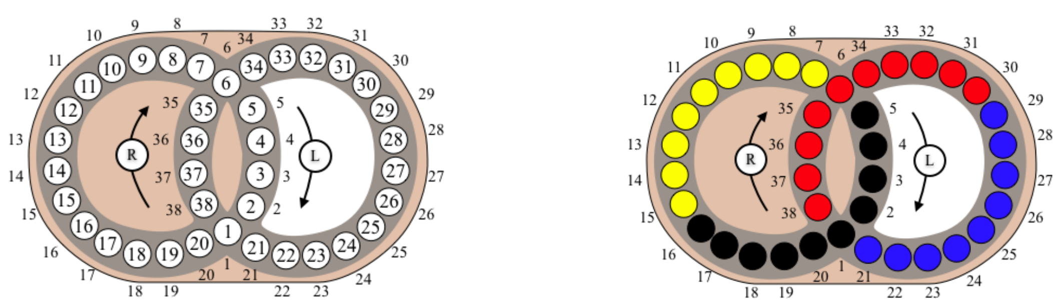
(b) conjugation of commutator by B^{-1}

Figure 2: twisting 2 corner cubies

14.2.2 Hungarian Rings

Using commutators we found some very useful moves on the Hungarian Rings puzzle:

$$[L^5, R^5] = (1, 25)(6, 11), \qquad [L^{-5}, R^{-5}] = (1, 16)(6, 30),$$
$$[L^5, R^{-5}] = (1, 6)(11, 30), \qquad [L^{-5}, R^5] = (1, 6)(16, 25).$$



(a) Hungarian Rings with numbers

(b) Hungarian Rings with colours

Figure 3: Hungarian Rings puzzle

Suppose we wanted to swap the contents of boxes 8 and 27, then we could move 8 to position 11, 27 to position 30. Call the move that does this γ. For example $\gamma = L^{-3}R^{-3}$ would achieve this. Then apply the commutator $[L^5, R^{-5}]$, which swaps disks 8 and 27, along with the disks in positions 1 and 6. Now undo the set-up move γ. The move sequence performed is $\gamma[L^5, R^{-5}]\gamma^{-1}$. Since the cycle structure is the same as $[L^5, R^{-5}]$ it swapped two pairs of disks, one pair being 8 and 27, and the other one was 32, 36.

In the coloured version of the Hungarian Rings puzzle, shown in Figure 3b, if two balls of the same colour are in the intersection positions (1 and 6) then applying one of the commutators, say $[L^5, R^{-5}] = (1,6)(11,30)$, would swap disks in positions 11 and 30, but the 1, 6-swap would go unnoticed since the disks were identical. This gives us a way to swap any two balls on the puzzle, and as we know from the theory of permutations, this is enough to construct any permutation of the disks.

In the numbered version of the puzzle, where every ball is distinct (Figure 3a) this is still not enough to solve every permutation. We will actually need to find a genuine 2-cycle. We'll pick up this topic in a later lecture. Though, armed with the tools of commutators and conjugates perhaps you can discover such a move for yourself! Next we'll use commutators to construct a 3-cycle.

Compound Commutator - Getting a 3-cycle.

We have seen that using commutators we can produce a product of two disjoint 2-cycles. For example $[L^5, R^5] = (1,25)(6,11)$. We now show that we are able to produce a move sequence which gives a 3-cycle by using *compound commutators*, that is, something of the form:

$$[[\alpha,\beta],\gamma] = (\alpha\beta\alpha^{-1}\beta^{-1})\gamma(\beta\alpha\beta^{-1}\alpha^{-1})\gamma^{-1}.$$

Since one of the transpositions in $[L^5, R^5]$ involves the lower point of intersection (position 1) and the right ring, while the other involves the upper point of intersection (position 6) and the left ring, we should be able to tweak one of the intersection points while leaving the other unchanged. We would like a move γ that has little overlap with $[L^5, R^5]$, where $M_{[L^5,R^5]} = \{1, 6, 11, 25\}$. Since each ring has 3 disks which are moved by $[L^5, R^5]$ we would like a move that temporarily moves the disks out of the intersection points, then moves the left ring (for example), and then moves disks back onto the intersection points. It would then follow that $M_{[L^5,R^5]} \cap M_\gamma = \{11\}$. Consider the move $\gamma = R^{-1}LR$. This leaves the disks in positions 1, 6 and 25 unchanged, but it moves the disk in position 11 to position 10. See Figures 4a and 4b. The circled positions in the figure are just to draw you attention to these positions. The pieces affected by the commutator $[[L^5, R^5], \gamma]$ are at most

$$\begin{aligned} M_{[[L^5,R^5],\gamma]} &\subset (M_{[L^5,R^5]} \cap M_\gamma) \cup [L^5,R^5]^{-1}(M_{[L^5,R^5]} \cap M_\gamma) \cup \gamma^{-1}(M_{[L^5,R^5]} \cap M_\gamma) \\ &= \{11\} \cup [L^5,R^5]^{-1}\{11\} \cup \gamma^{-1}\{11\} \\ &= \{11, 6, 12\} \end{aligned}$$

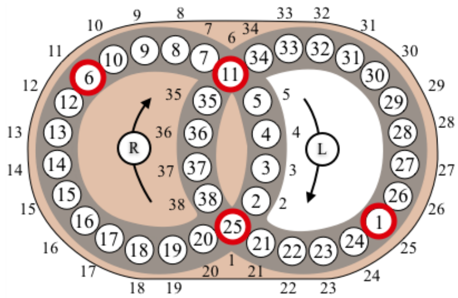
In fact, $[[L^5, R^5], \gamma]$ is the 3-cycle $(6, 11, 12)$.

14.2.3 Oval Track Puzzle

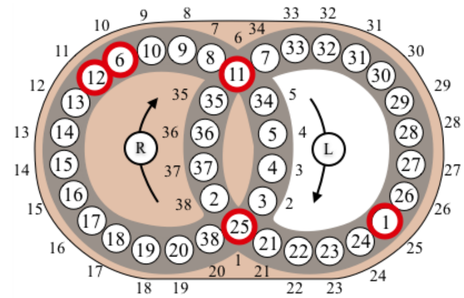
Conjugation is a very natural process on the Oval Track puzzle. If you have spent some time playing with the puzzle you undoubtedly use conjugation on almost every move. The reason for this is the turntable is located on one part of the puzzle, pieces will need to be moved into the turntable say by a move R^j, then they are rotated in the turntable T, and finally the pieces are moved back R^{-j}. The move sequence R^jTR^{-j} is conjugation.

Using commutators we found the 3-cycle $\gamma = [R^{-3}, T]^2 = (1, 7, 4)$. Any conjugate of this would also be a 3-cycle, so let's try to construct the 3-cycle $(1, 2, 3)$. To do this we would need to find a move sequence β that takes $\{1, 2, 3\}$ to $\{1, 7, 4\}$. The order doesn't much matter, for example we could find a move sequence that takes $1 \mapsto 1$, $2 \mapsto 4$, and $3 \mapsto 7$. What is important though is once we get $1, 2, 3$ into positions $1, 7, 4$ then we must cycle them appropriately: either γ of γ^{-1}. So before we do anything we make a mental note that to produce the 3-cycle $(1, 2, 3)$ we want "1 to chase 2". By this we mean, once we get disks $1, 2, 3$ into positions $1, 7, 4$ we then cycle them in the direction so the 1 goes to the current position of 2. The rest of the tiles will follow accordingly.

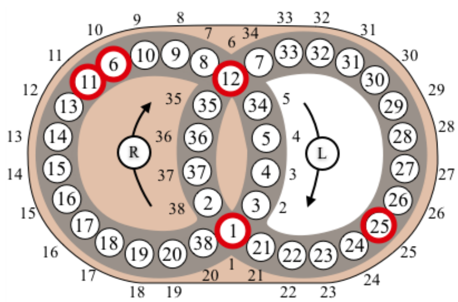
Since 1 is already in position 1 we leave it there. The move β just needs to take $2 \mapsto 4$ and $3 \mapsto 7$. We begin by pushing disk 3 away from the rest of the pack. To do this, move it to position 1 and apply T. It stills need to move one more unit to the right in order to be 6 units away from disk 1, so move it to position 2 and apply T.



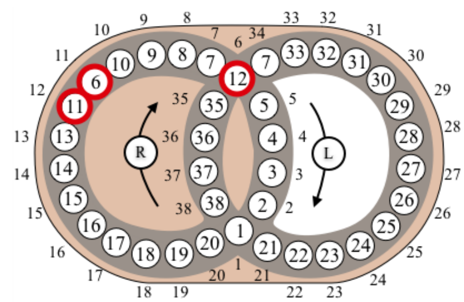
(a) Commutator $L^5R^5L^{-1}R^{-5}$



(b) $\gamma = R^{-1}LR$ tweaks disks around 11



(c) Undoing $L^5R^5L^{-1}R^{-5}$ restores 1 and 25



(d) Undoing γ undoes some changes made by γ

Figure 4: A compound commutator that uses the commutator $[L^5, R^5]$ to construct a 3-cycle $(6, 11, 12)$

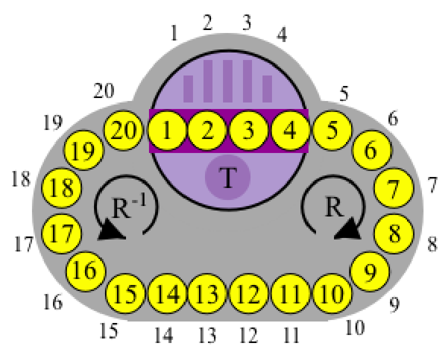
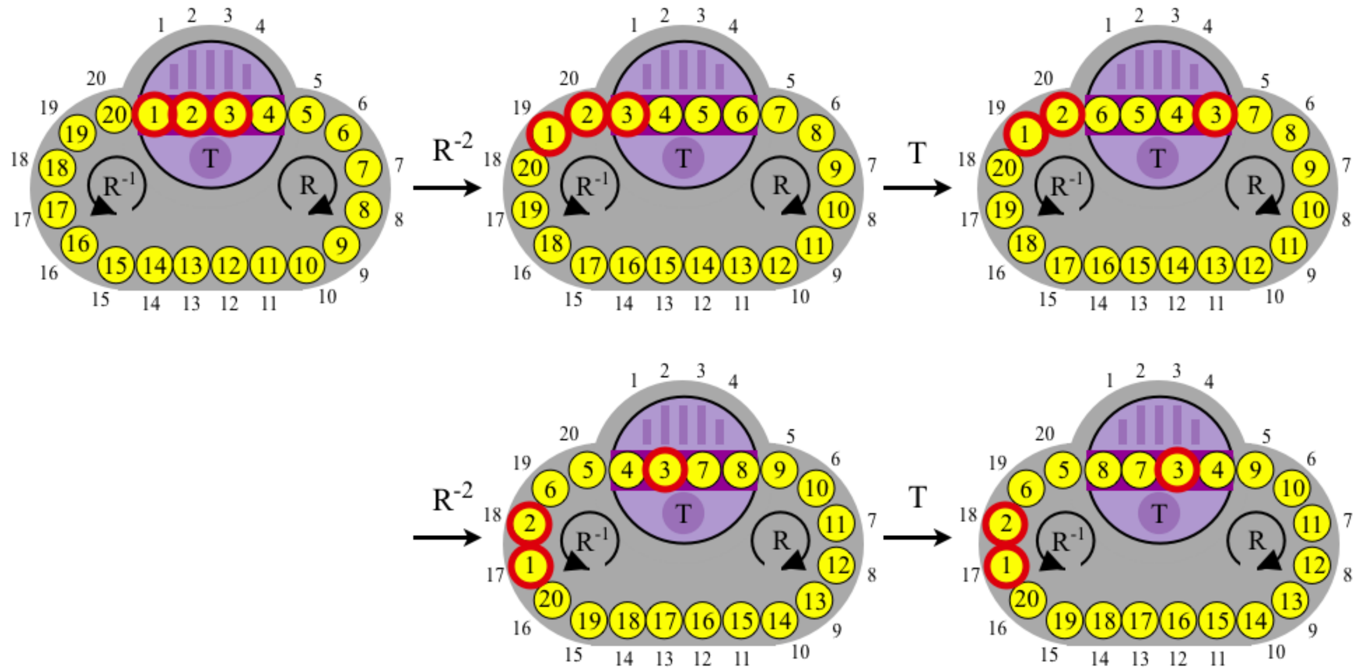
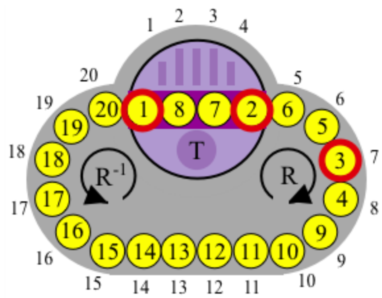


Figure 5: Oval Track puzzle.

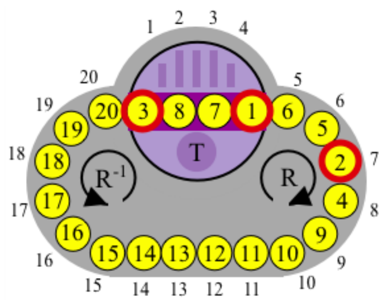
This move sequence $R^{-2}TR^{-2}T$ has now pushed disk 3 far enough away from 1 so that if 1 rotates to its home position disk 3 will be in position 7. See the following figure.



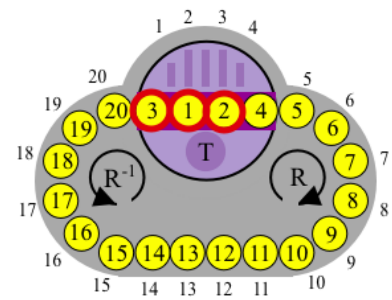
Now, using the space between disks 1 and 3 we push 2 two units to the right. This is done by putting it in position 2 applying T, then putting it again in position 2 and applying T. The move sequence to do this is $R^3TR^{-1}T$. The complete move sequence is $\beta = R^{-2}TR^{-2}TR^3TR^{-1}T$, and the puzzle now looks like this:



We now apply the 3-cycle $\gamma = [R^{-3}, T]^2 = (1, 7, 4)$, but we have to recall we wanted 1 to chase 2. Since 2 is in position 4 we want to apply the 3-cycle $(1, 4, 7)$ which is actually γ^{-1}. After applying γ^{-1} the puzzle now looks like this:



Finally, undoing β returns all pieces back to their original positions, except the pieces circled in red. These pieces have been moved since β was applied. β^{-1} will take the piece in position 1 back to 1, the piece in position 4 back to 2, and the piece in position 7 back to 3.



Therefore the move sequence $\beta\gamma\beta^{-1}$ produces the 3-cycle $(1, 2, 3)$.

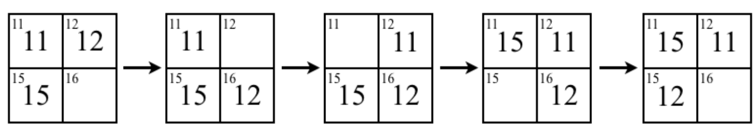
14.2.4 15-Puzzle

Here we revisit our results about the 15-puzzle using our new tool: conjugation. The proof of the solvability criteria, Theorem 9.1, which states that

> A configuration of the tiles, in which the empty space is in box 16, is solvable if and only if it is an even permutation.

relied on the ability to construct 3-cycles. The essence of the proof was based on conjugation.

Recall we can produce the 3-cycle $\sigma = (11, 12, 15)$ by focussing on the bottom right corner of the puzzle:



From this one 3-cycle σ, we can conjugate σ to construct any other 3-cycle we want. To do this we just need a way to move any 3 tiles down to the bottom right-hand corner, along with the empty space. Hiding any tiles you have already brought down in boxes 12 and 15, we can bring any other tile down using one of the two tours in Figure 6. Call the move sequence which brings all three tiles down β.

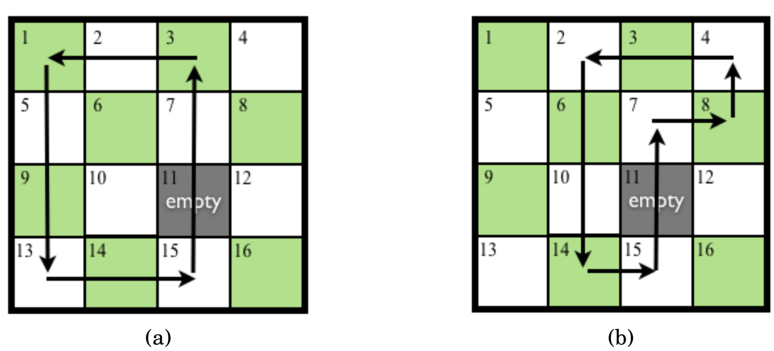


Figure 6: Tours for producing 3-cycles.

Applying σ cycles the tiles around (or if you want to cycle them in the other direction apply σ^{-1}). Then β^{-1} takes all the tiles back to where they started, but with the main three tiles now cycled. In other words $\beta\sigma\beta^{-1}$ is precisely the move that cycles the three tiles.

This was a purely theoretical argument, since in practice solving the puzzle in this way is completely inefficient. However, if one wants to produce a particular 3-cycle it is not necessary to push the 3 tiles down to the bottom right-hand corner, apply $(11, 12, 15)$, then reverse the moves. Instead, if we can apply a sequence of moves β which take the 3 tiles into any 2-by-2 array, along with the empty space, then we can perform a 3-cycle there, call it σ, then apply β^{-1}. The resulting move sequence $\beta\sigma\beta^{-1}$ will be a 3-cycle on the selected tiles.

14.3 Exercises

1. For each of the pairs of permutations $\alpha, \beta \in S_n$ calculate the conjugate $\alpha^{-1}\beta\alpha$. Note that it has the same cycle structure as β, and notice the each entry in the cycle is the image under α of the corresponding entry in the cycle of β.
   (a) $\alpha = (1,2,3,4,5,6,7)(8,9,10)$, $\quad\beta = (1,5,8)(2,6)(3,7,4)$
   (b) $\alpha = (1,5,8)(2,6)(3,7,4)$, $\quad\beta = (1,2,3,4,5,6,7)(8,9,10)$
   (c) $\alpha = (1,7,5,9,3,10,12)(4,6)(8,11)$, $\quad\beta = (1,6)(2,8)(4,7)$
2. For $\alpha = (1,2,3,4)(5,6)$ and $\beta = (1,6)(2,5,3)$ do the following.
   (a) Calculate $\beta^{-1}\alpha\beta$.
   (b) Calculate the values of $\beta(1)$, $\beta(2)$, $\beta(3)$, $\beta(4)$, $\beta(5)$, $\beta(6)$, then write down the product of cycles, $(\beta(1), \beta(2), \beta(3), \beta(4))(\beta(5), \beta(6))$.
   (c) Observe that the product of cycles in part (b) is the same as the answer to part (a). This is the essence of Lemma 14.1.
3. For each of the following pairs of permutations state whether they are conjugate in S_{10}. That is, determine whether there exists a $\gamma \in S_{10}$ so that $\alpha = \gamma^{-1}\beta\gamma$.
   (a) $\alpha = (1,2,3,4,5,6,7)(8,9,10)$, $\quad\beta = (1,5,8)(2,6,3,7,4,10,9)$
   (b) $\alpha = (1,5,8)(2,6)(3,7,4)$, $\quad\beta = (1,2)(7,3)(8,9,10)$
   (c) $\alpha = (1,7,5,9,3)$, $\quad\beta = (1,6,2,8,4)$
4. Let G be a group. Prove that every conjugate of a commutator is a commutator by showing that $g[a,b]g^{-1} = [gag^{-1}, gbg^{-1}]$ for all $a, b, g \in G$.
5. Show that for $g_1, g_2, h, h_1, h_2 \in G$ the following hold.
   (a) $(g_1g_2)^h == g_1^h g_2^h$
   (b) $g^{h_1h_2} = (g^{h_1})^{h_2}$
6. Show that g and g^h have the same order.
7. For permutations $\alpha, \beta \in S_n$, show that α and α^β have the same parity.
8. Show that the notion of conjugate defines an equivalence relation. That is, show that
   (a) any element of $g \in G$ is conjugate to itself (**relexive**)
   (b) if g is conjugate to h, then h is conjugate to g (**symmetry**)
   (c) if g is conjugate to h, and h is conjugate to k, then g is conjugate to k (**transitivity**)
9. Show that the conjugacy classes form a partition of G. That is, show that G can be expressed as a disjoint union of distinct conjugacy classes.

10. **Is the building of commutators associative?** (a) Explore the equation $[[\alpha,\beta],\gamma] = [\alpha,[\beta,\gamma]]$ by trying out these compound commutators on one of the puzzles. Show that this equation is not true for all permutations α, β, and γ. This show the operation of commutator building is not an associative operation. (b) Show that for any permutations α, β, and γ such that β commutes with both α and γ, this associativity equation is trivially true.

11. The expressions $[(\alpha\beta),\gamma]$ and $[\alpha,(\beta\gamma)]$ are commutators of products. Prove the following formulas which show a commutator of products is a product of commutators.

    (a) $[\alpha,(\beta\gamma)] = [\alpha,\beta](\beta[\alpha,\gamma]\beta^{-1}) = [\alpha,\beta][\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}]$

    (b) $[\alpha\beta,\gamma] = (\alpha[\beta,\gamma]\alpha^{-1})[\alpha,\gamma] = [\alpha\beta\alpha^{-1},\alpha\gamma\alpha^{-1}][\alpha,\gamma]$

15-Puzzle:

12. Starting with the 15-puzzle in the solved state write down a sequence of moves which will produce each of the following 3-cycles.

    (a) $(2,12,7)$ (b) $(3,8,12)$ (c) $(3,9,13)$.

    Either write the moves using transpositions, or use the words “up”, “down”, “left”, “right”, to indicate the direction the tile adjacent to the empty space is moved. It may help to use a physical or virtual version of the puzzle. See the “software” section of the course webpage for links to virtual versions of the puzzle. Rather than bringing the three tiles together in the lower right-hand corner, bring them together with the empty space into any 2-by2 array that is convenient.

Rubik’s Cube:

13. **Set-up Moves.** The move β^{-1} in the conjugate $\beta^{-1}\alpha\beta$ is called a *set-up* move. This is because it is the move that brings the desired pieces into the positions that are affected by α, once α is applied, the pieces are then restored by applying β. The important thing to keep in mind with these set-up moves is that it doesn’t matter how the other pieces are moved around, this will eventually be undone. All that matters is how a small subset of pieces are moved, this is where we are to focus our attention. To get some practice in creating set-up moves, find a sequence of moves which accomplishes each of the following. (See comment below for an explanation of the notation used.)

    (a) Moves the piece in the urf corner to the frd position.
    (b) Moves the piece in the rdf corner to the fur corner.
    (c) Moves the piece in the ur edge to the ul edge position, and the piece in the ul edge to the ur edge position.
    (d) Moves the piece in the ur edge to the ul edge position, and the piece in the ul edge to the ru edge position.
    (e) Moves the piece in the uf edge to the fu edge position (i.e. it flips the uf edge piece).
    (f) Moves the piece in the ufr corner to the fru corner position (i.e. it rotates the ufr corner piece counterclockwise).
    (g) Moves the piece in the ufr corner to the ulb corner, and the piece in the ulb corner to the ufr corner.

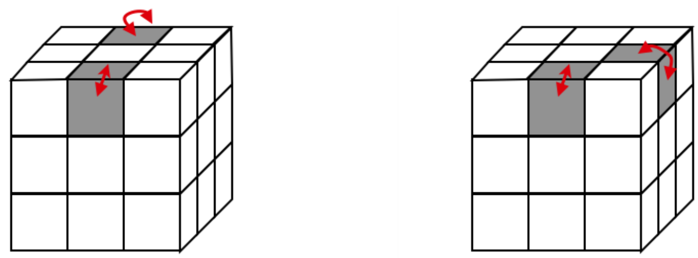
    Notation: Here order of how the positions are listed matters. For example $urf \mapsto rdl$ means the corner which is part of the *up*, *right*, and *front* faces is moved to the corner which is part of the *right*, *down*, and *left* faces, and moreover the facet in the *up* face moves to the *right* face, the facet in the *right* face moves to the *down* face, and the facet in the *front* face moves to the *left* face.

14. **More Set-up moves.** Find a sequence of moves which accomplishes the following:

$$ufr \mapsto bur, \quad bur \mapsto lfu, \quad lfu \mapsto ufr.$$

Do this so that all other cubies in the *up* layer remain in their home positions, but all other cubies in the *middle* and *down* layer may move around.

15. Suppose we know a move α which flips two opposite edges in the top layer, as shown in Figure 7a. Find a move β so the $\beta^{-1}\alpha\beta$ flips two adjacent edges in the top layer, as shown in Figure 7b.



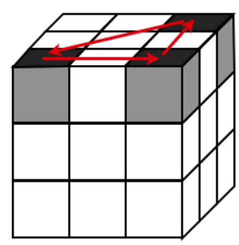
(a) Known move α flips opposite edges.

(b) Determine how to achieve this move from α.

Figure 7: Exercise 15

16. The commutator $[L^{-1}D^2L, U]$ permutes corner cubies as follows (ulb, ufl, frd). (Here we are using our cycle notation as a compact way to represent the movement of pieces.) What pieces does the conjugate $R^{-1}[L^{-1}D^2L, U]R$ permute?

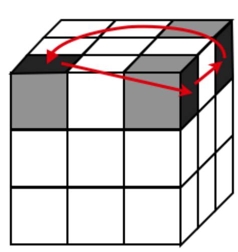
17. **Building a corner 3-cycle.** In this exercise we build a 3-cycle of corners in the *up* layer that preserves orientation (that is, the *up* facets remain in the *up* layer for each of the corners cubies being moved). The desired movement is shown in the figure, where black facets are moved to black facets.



(a) Verify the the conjugate ULU^{-1} brings one new corner cubie into the *right* face.

(b) Since ULU^{-1} brings one new corner cubie into the *right* face this makes a good candidate to form a commutator with R^{-1}. Verify the commutator $[ULU^{-1}, R^{-1}]$ moves the corner cubies as indicated in the diagram. The movement of pieces is also given notationally as follows

$$ulf \mapsto ruf, \quad ruf \mapsto rbu, \quad rbu \mapsto ulf.$$



(c) Unfortunately, the commutator $[ULU^{-1}, R^{-1}]$ twists the corners in addition to permuting them. We'd like to tweak this commutator a little bit so the it doesn't twist the corners. Find a set-up move γ which twists some of the corners in place, so that when the commutator $[ULU^{-1}, R^{-1}]$ is applied, followed by γ^{-1}, the corner cubies that were permuted still have their *up* facets in the *up* layer.

(Hint: a move which twists ufl counterclockwise, and urb clockwise should work.)

(d) Verify that $\gamma[ULU^{-1}, R^{-1}]\gamma^{-1}$ produces the desired 3-cycle of corners (as shown in the first figure above).

Oval Track:

18. By conjugating the 3-cycle $(1, 4, 7)$ produce three other 3-cycles, say $(1, 2, 4)$, $(2, 8, 14)$, and $(5, 10, 15)$.

19. (a) Verify that TR^{-1} is the product of a 17-cycle and a 2-cycle.
    (b) By raising TR^{-1} to the power of 17 the 17-cycle can be killed-off, leaving just a 2-cycle. Verify that $(TR^{-1})^{17} = (1, 3)$.
    (c) Find a move sequence β so that $\beta(TR^{-1})^{17}\beta^{-1} = (1, 2)$.
    (d) Using conjugation produce two other 2-cycles on this puzzle, say $(5, 15)$ and $(9, 12)$.
    (e) Convince yourself that you can produce any 2-cycle as a conjugate of $(TR^{-1})^{17}$. Since every permutation is a product of 2-cycles you have proven that every permutation is obtainable in this puzzle.