Lecture 1:
Permutation Puzzles

Contents

1.1 Introduction

Imagine a mixed up Rubik's Cube™ (for example see Figure 1a), or better yet, mix up your own cube. As you begin to try to solve the cube you should notice a few things. Solving a single face (i.e. getting all 9 pieces of the same colour onto the same face) isn't that difficult. There seems to be enough room to move things around. Continuing in this manner, you then begin to solve the next layer. You'll soon notice that certain moves will undo your previous work. If you twist a face that contains some pieces that you had previously put in their correct place, then these pieces now move out of place. And this is where the puzzle becomes challenging. It seems the more pieces that are in the correct place, the harder it is to move the remaining ones into place. This is known as the *end-game* of the Rubik's Cube, and solving the puzzle requires a thorough understanding this part of the puzzle.

Rubik's Cube is probably one of the most well known puzzles to date. It is estimated that over 350 million have been sold since it's creation by Ernö Rubik around 1980. What has made it so popular is not certain. Perhaps it looks seemingly innocent, then once a few sides have been twisted, and the colours begin to mix, the path back home is not so easy to see. The more you twist it the further you seem to be taken away from the solution. Perhaps it is that the number of ways to mix up the cube seems endless. Or perhaps to others, it doesn't seem endless at all. Despite the reasons for its appeal, it has become one of the most popular puzzles in history.

It is rare to find a puzzle, or toy, that has captured the imagination of millions, is accessible to all age levels, is challenging, yet satisfying, and is so *mathematically rich*. The Rubik's Cube is one such puzzle. Others examples include the 15-puzzle, TopSpin, Hungarian Rings, and Lights Out.

What do I mean by mathematically rich? Well it turns out that one area of mathematics that has had an impact on all areas of science, and has even popped up in art, is the area called *group theory*. Often referred to as the *language of symmetry*, group theory has led to many new discoveries in theoretically physics, chemistry,

(a) A mixed up Rubik's Cube

(b) An end-game for Rubik's Cube. Only the bottom layer remains to be solved.

Figure 1: A mixed up Rubik's cube and end-game.

and mathematics itself. It underlies the techniques in cryptography (sending private information over public channels), and coding theory (digital communications, digital storage and retrieval of information). It is no surprise that a mathematical theory developed to understand symmetry so adequately describes the Rubik's cube, however, for us, we are more interested in the opposite. We will use Rubik's cube, and these various other puzzles, to provide us a window into group theory. But most of all, we plan to have a lot of fun doing it.

Our goal in this course is to uncover some pretty fascinating mathematics, while playing with puzzles. We will not be too concerned with solving the puzzles, though strategies for solution will fall out of our investigations, but instead we want to see how we can model these puzzles mathematically and see what the mathematics has to tell us about the puzzles. In this sense we want to *understand* these puzzles.

This is the theme for all these puzzles. There is a certain stage in solving the puzzle where a simple strategy, and trial and error, can't get you any further. This is typically referred to as the *end-game* for the puzzle. For Rubik's Cube the end-game is usually when two layers are solved, and the last layer remains (see Figure 1b).

It is understanding the *end-game* of these puzzles that mathematics becomes such a useful tool.

1.2 A Collection of Puzzles

We begin by briefly describing the puzzles we will be investigating in this course. One thing to observe is that all puzzles have a common theme: the pieces of the puzzle are rearranged, and the goal is to return the pieces to their proper (original) arrangement.

1.2.1 A basic game, let's call it *Swap*

Imagine a set of objects laid out in front of you and ordered in some way. This puzzle can be played with any number of objects, but the more objects that are used the more challenging it becomes.

It doesn't matter what the objects are, they could all be different, or some could be the same. For starters we will just use 5 distinct objects, and for simplicity we will just take the objects to be the numbers 1, 2, 3, 4, and 5. Figure 3 shows the objects laid out in front of us:

1	2	3	4	5

Figure 2: *Solved state* of *Swap* with 5 objects.

This arrangement, where the numbers appear in order from left to right, is called the *home position* or *solved state*. Since, as we'll see shortly, we will be moving the numbers around the boxes so it will be nice to have a little reminder of whose home is whose. We do this by putting a little number in the top left corner of each box.

1	2	3	4	5
1	2	3	4	5

Figure 3: *Solved state* of *Swap* with 5 objects. Boxes labeled.

The way this puzzle is played is first the numbers are randomly arranged in the boxes. Then using only *legal moves*, one tries to move the numbers back to their home positions (i.e. return the puzzle to its solved state).

What are the *legal moves*? This is where different version of the puzzle can be created. For now, let us simply say there is one type of legal move called a *swap*, and it consists of picking any two boxes and swapping the contents (the numbers in large font).

Example 1.1 *Consider the starting position the Figure 4. Our goal is to return the numbers to the solved state using only legal moves. Notice that objects* 2 *and* 4 *are in the correct positions (correct boxes). This is where the*

1	2	3	4	5
3	2	5	4	1

Figure 4: Staring position for Example 1.1.

little numbers in the top left corners come in handy. As for the numbers 1*,* 3 *and* 5*, we need to move these to their correct positions. Since the legal moves consist of swapping the contents of two boxes at a time, we'll focus first on getting* 5 *into it's correct position. To do this we swap the contents of boxes* 3 *and* 5*, since object* 5 *is in box* 3*.*

1	2	3	4	5
3	2	1	4	5

Now 2*,* 4 *and* 5 *are in the correct positions. Lastly we swap the contents of boxes* 1 *and* 3 *and solve the puzzle.*

1	2	3	4	5
1	2	3	4	5

■ [1]

This is a pretty basic puzzle in some sense, but it is good to have this as our starter puzzle. In some sense, every puzzle we will investigate will just be some variation of Swap. Either we will increase the number of objects, or we will change the *legal moves*. We now consider some possible variations of the puzzle.

Variations of Swap: There are many ways to vary this puzzle. One way is to increase the number of objects that are used. Here we used 5, but we could use 10, 20, 48, or any number we wish.

Another way to vary the puzzle is to choose a different collection of legal moves. Our legal moves consisted of swapping the contents of two boxes. Instead we could have stated that legal moves only consisted of swapping the contents of box 1 and any other box. If this was the case then the solution in Example 1.1 would be illegal, since started by swapping the contents of boxes 3 and 5, which is a move that doesn't involve box 1.

Exercise 1.1 *Beginning with the starting position in Figure 4, solve the puzzle using only legal moves of the form: the contents of any box can only be swapped with box* 1*. In other words, any swap must involve box* 1*.*

[1]The black square symbol ■ is used to indicate the example is finished. Later, when we prove theorems, lemmas, etc. we will use a hollow square □ to denote the end of a proof.

We could also extend the notion of a legal move beyond "swaps". For instance we could restrict ourselves to use only moves of the form: *pick three boxes and cycle the contents either to the right (clockwise) or to the left (counterclockwise)*.

For example, consider 6 objects in Figure 5, we could cycle the contents of boxes 2, 3 and 5 to the left (other boxes are shaded to allow us to focus on what is changing).

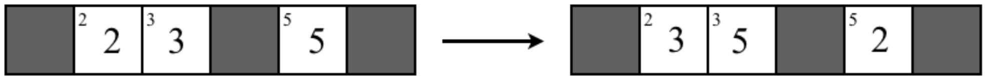


Figure 5: Legal move variation: 3-cycle to the left.

Exercise 1.2 *Beginning with the starting position in Figure 4, solve the puzzle using only legal moves consisting of* 3*-cycles: pick three boxes and cycle the contents either to the right (clockwise) or to the left (counterclockwise).*

We can now describe the puzzle **Swap**, including all possible variations.

Rules of Swap (n objects):

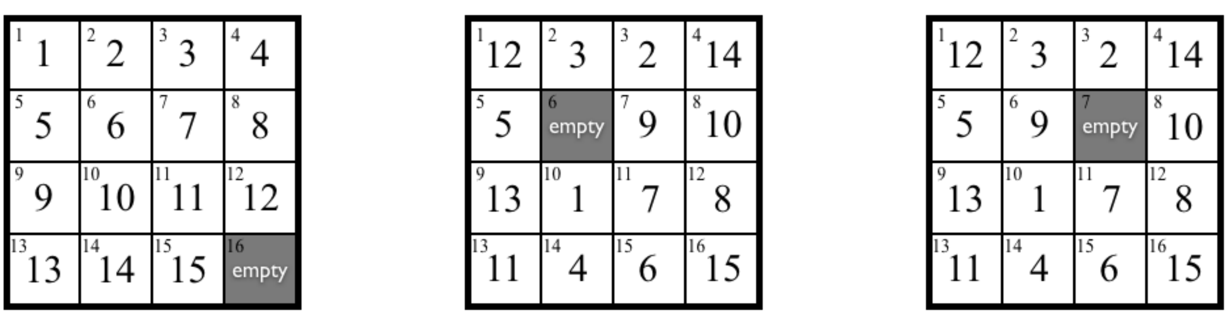
Let T be a set of n objects, and suppose the objects have been ordered in some way. For example, the objects can be the numbers 1 to n and the ordering could be their natural ordering from smallest to largest written from left to right. When the objects are in their proper order, we say they are in their *home positions* and the puzzle is in the *solved state*. Let $\mathcal{M}$ be a collection of *legal moves*.

1. **Puzzle Start**: Randomly arrange the numbers 1 through n from left to right.
2. **Puzzle Play**: Using only legal moves (i.e. moves in $\mathcal{M}$), return the puzzle to the solved state.

Stated here in its most general form we'll see that most Rubik's Cube-like puzzles are just variations of Swap. Of course, this connection with Swap doesn't make the Rubik's Cube any easier to solve, at least not yet, but it will provide us a way to investigate and understand the cube and other puzzles.

1.2.2 The 15-Puzzle

The 15-puzzle consists of a 4×4 grid with numbered tiles from 1 to 15 placed in the grid. The space where the 16 tile would go is left empty. See Figure 6a.



(a) The 15-Puzzle in the solved state

(b) A random arrangement of the 15-Puzzle.

(c) Obtained from 6b by moving the tile in box 7 (tile number 9) to box 5.

Figure 6: The 15 Puzzle

The little numbers in the top left corner of each box are not present on any of the manufactured puzzles, nor are they present on the software versions of the puzzle. But these little numbers proved to be so handy in reminding us where each tiles home position is in Swap that we'll use them here to.

The object of this puzzle is to randomly arrange the tiles on the grid, and then through a sequence of legal moves which consist of sliding a tile into the empty space (which results in the empty space moving around the board), one tries to return all tiles to their home positions.

Current manufactured versions of the puzzle are assembled in such a way that the pieces are not removable. There is a tongue-and-grove design which allows the pieces to slide around but doesn't allow them to be removed. However, the original versions of the puzzle (manufactured in the 1880's) consisted of removable wooden pieces. This little difference in puzzle constructions has significant impact on the ability to solve the puzzle. This puzzle started a craze that swept across the nation, and across the world, from January to April of 1880. All the fuss was centred around the fact that after randomly putting the wooden blocks back into the box, solving the puzzle seemed to take you to one of two places: either you solved it completely, or you got every number in its correct position except the 14 and 15 were switched (see Figure 7). In the case when the last two tiles were switched it seemed the puzzle wasn't solvable. Cash rewards of $1000 were offered for a solution, and one dentist even offered a set of teeth to the person who could produce a sequence of moves swapping the 14 and 15 tiles.

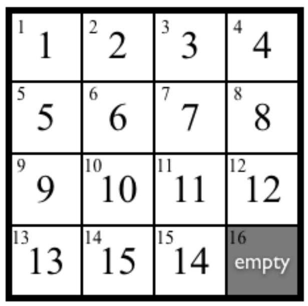


Figure 7: The 13-14-15 Problem. Can the puzzle be solved by starting from this position?

We will investigate whether this arrangement of the puzzle is solvable, as well as come up with a strategy for solving the puzzle in general.

Software: This puzzle is widely available as a free download for various operating system (mac/ win/ linux/ iphone/ android). Most versions have some sort of picture as the background, instead of the numbers 1 through 15. Find out more in the software section of our course webpage.

1.2.3 The Oval Track Puzzle (or TopSpin$^{\text{TM}}$)

The TopSpin puzzle was manufacture by Binary Arts (now called ThinkFun). It was invented by Ferdinand Lammertink, and patented on 3 Oct 1989, US 4,871,173. The puzzle consists of 20 numbered round pieces in one long looped track (see Figure 8). You can slide all the pieces around the loop. There is also a turntable in the loop (this is the purple circle which contains disks 1 through 4 in Figure 8), which can rotate any four adjacent pieces so that they will be in reverse order. This in effect swaps two adjacent pieces and the two pieces on either side of them. The aim is of course to mix up the ordering of the pieces, and then place the pieces back in numerical order (as show in Figure 8).

This puzzle became a North American classic with over a million copies sold. Nowadays though, the only place that this puzzle is available in its physical form is on ebay.

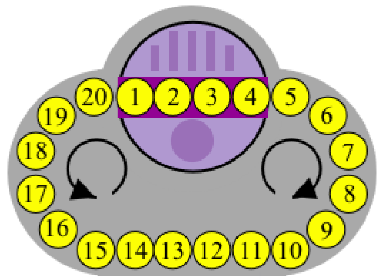
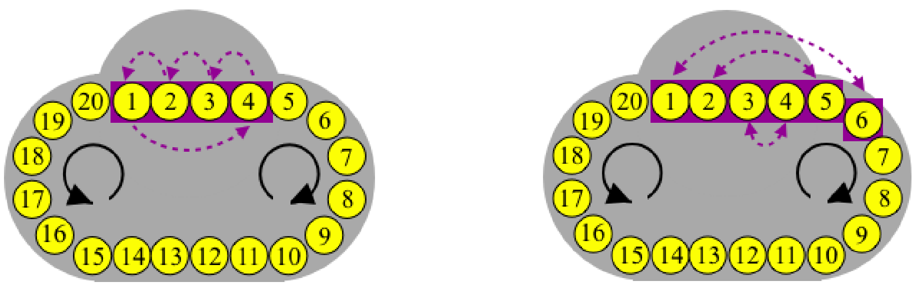


Figure 8: The TopSpin Puzzle in its solved state.

Variations of Oval Track: The name *Oval Track* has been given to a more general version of the puzzle, one that is updated for the digital era. When one moves away from having to construct a physical puzzle, and instead creates a virtual puzzle, then a whole new world of possible moves is available. For example, Figure 9 shows two variations of the puzzle. The *turntable move* in the original TopSpin puzzle is now replaced with the move indicated by the purple dashed lines. For instance, the new *turntable move* for the puzzle in Figure 9a moves the disk in spot 4 to spot 3, the disk in spot 3 to spot 2, the disk in spot 2 to spot 1, and takes the disk in spot 1 to spot 4. Another version of the *turntable move* involving 6 disks is given in Figure 9b. Variations of this puzzle are now limited only by your imagination.



(a) One variation of the turntable move

(b) Another variation of the turntable move

Figure 9: Variations on the Oval Track Puzzle

As usual, it will be convenient to indicate the home positions of the disks. So we put little numbers around the track indicating the number of the disk that should be in that position for the puzzle to be solved. See Figure 10.

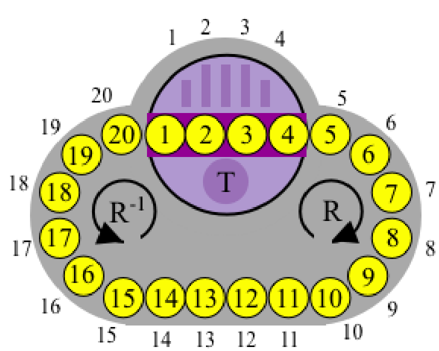


Figure 10: The TopSpin Puzzle with home positions labeled, and move notation.

Oval Track Notation: A clockwise rotation of numbers around the track, where each number moves one space, is denoted by R, a counterclockwise rotation is denoted by R^{-1}. A rotation of the turntable is (in general, an application of the *follow the arrows* move) is denoted by T. Figure 10 provides a visual summary of this notation.

Software: This puzzle is available in a virtual form on the web. Links are provided in the software section of our course webpage.

1.2.4 Hungarian Rings

The Hungarian Rings puzzle consists of two intersecting rings made up of a number of coloured balls. The rings of balls intersect at two places, so they share two of the balls. Each ring of balls can be rotated, so the balls can be mixed. The aim is to mix up the balls, and then place the balls back together so the colour form a continuous sequence (as show in Figure 11).

There are 38 balls of four colours: two colours have 9 balls (yellow and blue) and two colours have 10 balls (black, red). There are 4 balls between the intersections of the rings.

In the Rubik's Cubic Compendium [page 212], there is a picture of the Hungarian Rings and the following text by David Singmaster:

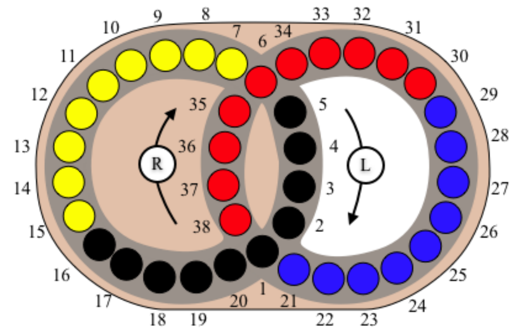
> Closer to Rubik's Magic Cube are 'interlocking cycle' puzzles where several rings of pieces cross each other. Endre Pap, a Hungarian engineer, invented a flat version with two rings which was marketed as the Hungarian Rings. The idea was not entirely new, as there is an 1893 patent for it.

The patent that Singmaster is referring to is US 507,215 by William Churchill, filed on May 28 1891, granted on October 24, 1893. (See Jaap's Puzzle page - link on course webpage - for a copy of the patent.)

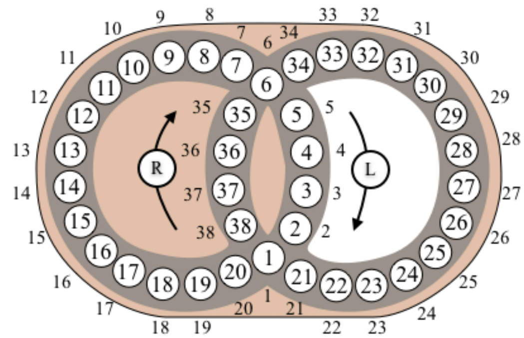
Figure 11: Hungarian Rings in its solved state. (manufactured 1982)

To study this puzzle we will temporarily ignore colours, and instead assign a number to each ball. See Figure 12b. We'll also indicate the home position of each ball by putting little numbers along the outside of the track. In effect we will study the puzzle of rearranging the numbers 1 through 38 on the two rings. In some sense this is a more difficult puzzle than the colour version of the puzzle simply because in the colour version there are really only 4 distinct balls, whereas in the number version there are 38 distinct balls and each one has only one home position. However, as we'll see, the added complexity inherited by using numbers, rather than colours, is not so bad, and the benefits to understanding the puzzle are numerous.

Hungarian Rings Notation: A clockwise rotation of the balls in the **right-hand ring**, where each ball moves one space around the track, is denoted by R, a counterclockwise rotation is denoted by R^{-1}. A clockwise rotation of the balls in the **left-hand ring**, where each ball moves one space around the track, is denoted by L, a counterclockwise rotation is denoted by L^{-1}. Figure 12 provides a visual summary of this notation.



(a) Hungarian Rings with home positions labeled by numbers.



(b) Hungarian Rings with numbers instead of colours.

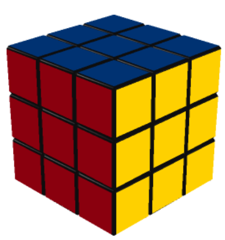
Figure 12: Hungarian Rings with disks labeled by numbers

Software: This puzzle is available in a virtual form on the web. Links are provided in the software section of our course webpage.

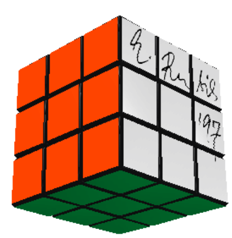
1.2.5 Rubik's Cube

This is probably the most well known mechanical puzzle. It was invented by Ern Rubik in Hungary around 1974, and the patent was filed 30 January 1975, HU 170062. Eventually it was produced and marketed by Ideal Toys in the early 1980s. It is quite possibly the most popular toy to have ever been manufactured, and many copycat cubes were made. It is estimated that there have been over 350 million cubes manufactured since 1980, and it is still being manufactured today.

The Rubik's Cube is a cube which is built from smaller cubes where there are 3 cubes along an edge, i.e. a $3 \times 3 \times 3$ cube. The 9 pieces on each face can rotate, which rearranges the small cubes at that face. The six sides of the puzzle are coloured, so every corner piece shows three colours, every edge piece shows 2 colours, and every face centre only one. See Figure 13.



(a) View of front (red), right (yellow) and up (blue) faces.



(b) View of back (orange), left (white) and down (green) faces.

Figure 13: The $3 \times 3 \times 3$ Rubik's Cube with classic colouring scheme: blue opposite green, red opposite orange, white opposite yellow.

Turning a face does not change the face centres (this is because twisting the face centres is not a visible change of pattern, if however, there was an image rather than a solid colour on the face then this would not be the case anymore) so these can be considered already solved. The other pieces have to be placed correctly around them. This is a particularly important observation because it implies the following:

> The colour of the centre piece of any face defines the only colour to which that face of the cube can be restored.

Cube Terminology & Notation: When playing with the cube the pieces begin to move all around. Since there are so many moving parts of the cube, it will be convenient to have some terminology to describe each piece, and its placement in the cube. It will also be convenient to have some notation for basic movements to aid in communication with one another. Of course, a good choice of notation can bring mathematics into the picture as well, as we will soon see. The notation we use was first introduced by David Singmaster in the early 1980's, and is the most popular notation in use today.

Fix an orientation of the cube in space. We may label the 6 sides as $\boldsymbol{f}$, $\boldsymbol{b}$, $\boldsymbol{r}$, $\boldsymbol{l}$, $\boldsymbol{u}$, $\boldsymbol{d}$ for *front*, *back*, *right*, *left*, *up*, and *down*.

The cube is made up of 26 smaller cubes called **cubies**. These are the ones that are visible, there actually isn't a 27^{th} cube in the middle, but instead a mechanism that allows things to twist and turn. The cube has 6 sides, or **faces**, each of which has $3 \cdot 3 = 9$ **facets**. Think of a facet as just one of the little coloured stickers. There are 54 facets in total for the $3 \times 3 \times 3$ Rubik's cube. The cubes split up into three types: **centre cubies** (having only one facet), **edge cubies** (having two facets), **corner cubies** (having three facets). See Figure 14.

(a) 6 centre cubies and mechanism

(b) 12 edge cubies

(c) 8 corner cubies

Figure 14: A disassembled Rubik's Cube showing the cubies.

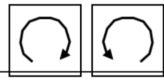
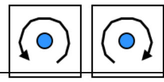
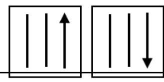
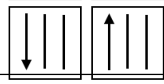
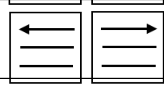
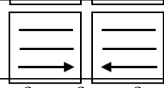
Each face of the cube is made up of a slice of 9 cubies that share a facet with the face. The face, along with all of the 9 cubies in the slice, can be rotated by 90 degrees clockwise (viewing the the face straight-on). We denote this move by uppercase letter of the name of the face. For example, F denote the move which rotates the front face by 90 degrees clockwise. See Table 1 for a complete description of cube moves and notation.

We call the space which a cubie can occupy a **cubicle**. As the pieces move around, the cubies move from cubicle to cubicle, and the facets move to the locations previously occupied by other facets. In order to solve the puzzle each cubie must get restored to its original cubicle, we call this its **home location**, and the facets must also be correctly positioned, we call this the cubies **home orientation**. Once *all* cubies are in their home positions and home orientations the puzzle will be solved.

Table 2 summarizes the terminology introduced here.

Since the center facets are fixed by the basic moves there are only $54 - 6 = 48$ facets that move. If we label the facets as in Figure 15 then we see that each basic move corresponds to a rearrangement, or *permutation* of the numbers 1 through 48. In this way we see that the Rubik's cube is much like the puzzle Swap, in the sense that we have a set of 48 numbers and a set of legal moves $\mathcal{M}$ (the 6 basic cube moves) which allow us to rearrange these numbers in some way.

Variations of Rubik's Cube: The Rubik's Cube is the puzzle that started the whole *Twisty Puzzle* craze. Since its invention hundreds of different types of twisty puzzle of all shapes and sizes have been designed. Puzzles of this type are often called *Rubik's Cube-like puzzles*, or *Twisty Puzzles*, or *Permutation Puzzles*. Here we will just briefly describe a few other Rubik's Cube-like puzzles.

notation (Singmaster)	**pictorial (view from front)**	**description of basic move (clockwise/counterclockwise refers to viewing the face straight-on)**
F , F^{-1}		F = quarter turn of **front** face in the **clockwise** direction. F^{-1}= quarter turn of **front** face in the **counterclockwise** direction.
B , B^{-1}		B = quarter turn of **back** face in the **clockwise** direction. B^{-1}= quarter turn of **back** face in the **counterclockwise** direction.
R , R^{-1}		R = quarter turn of **right** face in the **clockwise** direction. R^{-1}= quarter turn of **right** face in the **counterclockwise** direction.
L , L^{-1}		L = quarter turn of **left** face in the **clockwise** direction. L^{-1}= quarter turn of **left** face in the **counterclockwise** direction.
U , U^{-1}		U = quarter turn of **up** face in the **clockwise** direction. U^{-1}= quarter turn of **up** face in the **counterclockwise** direction.
D , D^{-1}		D = quarter turn of **down** face in the **clockwise** direction. D^{-1}= quarter turn of **down** face in the **counterclockwise** direction.

F^2, B^2. R^2, L^2, U^2, D^2 denote the corresponding *half-turn* of the face.
Since a clockwise half-turn is equivalent to a counterclockwise half-turn then
$F^2 = F^{-2}$, $B^2 = B^{-2}$, $R^2 = R^{-2}$, $L^2 = L^{-2}$. $U^2 = U^{-2}$, $D^2 = D^{-2}$

Table 1: Summary of cube move notation

Terminology	**Definition or Abbreviation**
cubies	The small cube pieces which make up the whole cube.
cubicles	The spaces occupied by the cubies.
facets	The faces of a cubie.
types of cubies: **corner**, **edge**, and **centre**:	A corner cubie has three facets. An edge cubie has two facets. A centre cubie has one facet
home location - of a cubie	The cubicle to which a cubie should be restored.
home position - of a cubie	The orientation in the home location to which a cubie should be restored.
positional names for cube faces	Up (u) Down (d) Right (r) Left (l) Front (f) Back (b)
Notation for cubicles - shown in *italics*	Lower case initials. For example, *uf* denotes the Up-Front edge cubicle
Notation for cubies - shown in *italics*	Upper case initials. For example, *URF* denotes cubie whose home position is in the the Up-Right-Front corner

Table 2: Summary of terminology and notation

$2 \times 2 \times 2$ Rubik's cube: The *Pocket* Rubik's Cube is a cube which is built from smaller cubes where there are 2 cubes along an edge, i.e. a $2 \times 2 \times 2$ cube. The 4 pieces on each face can rotate, which rearranges the small cubes at that face. The six sides of the puzzle are coloured, so every piece shows three colours. There are no face centres (or centre cubies), and there are no edge cubies, unlike its $3 \times 3 \times 3$ counterpart.

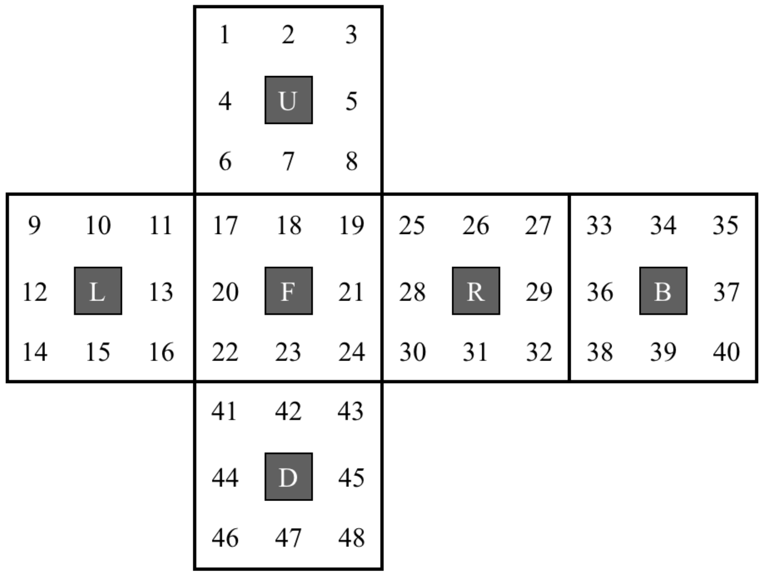
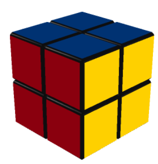


Figure 15: Facet labeling on the $3 \times 3 \times 3$ Rubik's cube.



$n \times n \times n$ **Rubik's cube:** Still sticking with the overall shape of a cube, one can increase the number of smaller cubes along each edge. Having 4 smaller cubes along and edge results in a $4 \times 4 \times 4$ cube called *Rubik's Revenge*, having 5 smaller cubes along an edge gives a $5 \times 5 \times 5$ cube called *The Professors Cube*. See Figure 16.

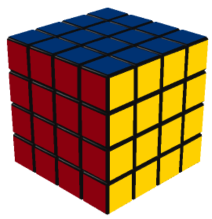
Being able to create larger versions of the cube turned out to be a substantial design problem, since the resulting object must twist smoothly and be robust enough that it doesn't fall apart easily. Greek inventor Panagiotis Verdes came up with a design that allowed for production of a physical $6 \times 6 \times 6$ cube and a $7 \times 7 \times 7$ cube. These versions of the cubes are known as *V-Cubes* after the inventor. See Figure 16.

1.3 Which brings us to the Definition of a Permutation Puzzle

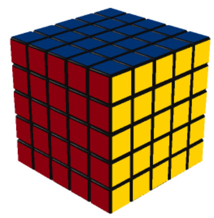
The 15-Puzzle, the Oval Track puzzle, Hungarian Rings and Rubik's Cube are all variations of the same theme. Each one consisted of pieces that were rearranged, or permuted, in some way, and the goal is to try to restore the pieces to their original positions. The legal moves that one is allowed to use is forced by the design or construction of the puzzle. Puzzles of this type known as *permutation puzzles*. Since these type of puzzles are the main focus of this course, we shall give a precise definition for this term.

A **one person game** is a sequence of moves following certain rules which satisfy:

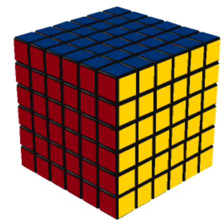
- there are finitely many moves at each stage,
- there is a finite sequence of moves which yields a solution,
- there are no "chance" or "random" moves (such as rolling a dice to determine what to do next),
- there is complete information about each move,



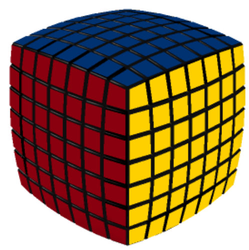
(a) Rubik's Revenge $4 \times 4 \times 4$



(b) Professor's Cube $5 \times 5 \times 5$



(c) V-Cube $6 \times 6 \times 6$



(d) V-Cube $7 \times 7 \times 7$

Figure 16: $n \times n \times n$ Cubes

- each move depends only on the present position, not on the existence or non-existence of a certain previous move (such as chess, where castling is made illegal if the king has been moved previously).

A **permutation puzzle** is a one person game (solitaire) with a finite set $T = \{1, 2, \ldots, n\}$ of puzzle pieces satisfying the following four properties:

1. For some $n > 1$ depending only on the puzzle's construction, each move of the puzzle corresponds to a unique permutation of the numbers in T,
2. If the permutation of T in (1) corresponds to more than one puzzle move then the two positions reached by those two respective moves must be indistinguishable,
3. Each move, say M, must be "invertible" in the sense that there must exist another move, say M^{-1}, which restores the puzzle to the position it was at before M was performed, In this sense, we must be able to "undo" moves.
4. If M_1 is a move corresponding to a permutation τ_1 of T and if M_2 is a move corresponding to a permutation τ_2 of T then $M_1 \cdot M_2$ (the move M_1 followed by the move M_2) is either
   - not a legal move, or
   - corresponds to the permutation $\tau_1\tau_2$.

Notation: We will always write successive puzzle moves from *left to right*, as we did in step (4) above.

1.4 Corresponding Sections of Joyner's textbook.

The textbook we will be referring to in this course is David Joyner's *Adventures in Group Theory.* I will usually state the sections of the text which parallel our lecture notes so you may investigate these topics further. This Lecture has been about introducing permutation puzzles, and giving a formal definition of the term *permutation puzzle*. See

Lecture 1 $\Rightarrow$ Chapter 4 of Joyner

However, in Joyner's textbook he has already introduced permutations and their notation by this point, so there will be some things that you won't understand yet. We will come back to these points in later lectures.

1.5 Exercises

1. Get your own Rubik's Cube. Whether you buy or borrow, make sure you have access to a Rubik's Cube throughout the entire semester, and bring it to class. If you don't know where to buy one, then check the course website, some suggestions are posted.

2. Get familiar with Rubik's Cube, and all the other puzzles we have just discussed. The course website has links to virtual versions of the puzzles. Download your own copy of the ones that are available, and bookmark the ones that are "online only" versions. Play with these puzzles. Don't worry if you can't solve them, this will come. But for now just get familiar with the puzzles, and the movements of the pieces.

3. Solve the Swap puzzle given in Figure 17, using the original set of legal move: *swap the contents of any two boxes*.

1	2	3	4	5	6
2	6	4	1	3	5

Figure 17: Swap position for Exercises 3, 4, 5.

4. Solve the Swap puzzle in Figure 17, using only legal moves of the form: *the contents of any box can only be swapped with box* 1.

5. Can you solve the Swap puzzle given in Figure 17, using only legal moves consisting of 3-cycles: *pick three boxes and cycle the contents either to the right (clockwise) or to the left (counterclockwise)*?

6. Consider the starting arrangement of tiles for the Swap puzzle in Figure 18.

   (a) Solve the puzzle using only legal moves of the form: *the contents of any box can only be swapped with box* 1.

   (b) Solve the puzzle using only legal moves consisting of 3-cycles: *pick three boxes and cycle the contents either to the right (clockwise) or to the left (counterclockwise)*.

   (c) Solve the puzzle using only legal moves consisting of pairs of swaps: *pick four boxes, swap the contents of two boxes, and swap the contents of the other two boxes.*

1	2	3	4	5	6
1	6	4	2	3	5

Figure 18: Swap position for Exercise 6.

7. Can you solve the Swap puzzle given in Figure 17, using only legal moves consisting pairs of swaps: *pick four boxes, swap the contents of two boxes, and swap the contents of the other two boxes*?

8. Verify that each of the puzzles we've encountered: Swap, 15-Puzzle, Oval Track, Hungarian Rings and Rubik's Cube, are permutation puzzles. That is, show that the definition of the term "permutation puzzle" is satisfied by these puzzles.