Lecture 17:
Partitions & Equivalence Relations

Contents

The cubies of Rubik's cube come in three types: corner cubies, edge cubies, and center cubies. In some sense we can think of any two edge cubies as equivalent since, using cube moves, we can take any edge cubie to the location of any other edge cubie (at the cost of possibly moving other pieces around). Similarly any two corner cubies are equivalent. Grouping similar elements together when trying to understand a large complicated set is a very powerful idea.

In this lecture we recall the concept of a *partition* of a set, and discuss its connection with the concept of an *equivalence relation* on a set.

17.1 Partitions of a Set

Consider the set of integers $\mathbb{Z}$. There are two well known subsets: the set of odd integers and the set of even integers. Every integer is a member of one of these subsets, and no integer is a member of both, so this gives a *partition* of $\mathbb{Z}$:

$$\mathbb{Z} = \{\ldots -5, -3, -1, 1, 3, 5, \ldots\} \cup \{\ldots -4, -2, 0, 2, 4, \ldots\}.$$

Definition 17.1 *A* ***partition*** *of a set A is a finite collection of non-empty subsets $A_1, A_2, \ldots, A_n$ satisfying the following properties.*

(a) A is the union of all the A_i's: $A = A_1 \cup A_2 \cup \cdots \cup A_n$,

(b) the A_i's are disjoint: $A_i \cap A_j = \emptyset$ for all $i \neq j$, $1 \leq i, j \leq n$.

Example 17.1 *Let E be the set of edge cubies of Rubik's cube, let V be the set of corner cubies, and let C be the set of centre cubies. E, V and C are disjoint sets, and their union is the set of all cubies. Therefore $E \cup V \cup C$ is a partition of the set of all cubies.*

Example 17.2 *(a) The three sets*

$$\begin{aligned}
A_0 &= \{\ldots -9, -6, -3, 0, 3, 6, 9, \ldots\} = \{3k \mid k \in \mathbb{Z}\}, \\
A_1 &= \{\ldots -8, -5, -2, 1, 4, 7, 10 \ldots\} = \{3k+1 \mid k \in \mathbb{Z}\}, \\
A_2 &= \{\ldots -7, -4, -1, 2, 5, 8, 11 \ldots\} = \{3k+2 \mid k \in \mathbb{Z}\},
\end{aligned}$$

form a partition of the integers $\mathbb{Z}$. A_0 is all the integers which are divisible by 3, A_1 are those integers whose remainder is 1 when divided by 3, and A_2 are those whose remainder is 2 when divided by 3. These exhaust all the possibilities of the remainder, and so $A_0 \cup A_1 \cup A_3 = \mathbb{Z}$. Moreover, for any particular integer, the remainder (upon division by 3) is unique so these sets are disjoint.

SAGE

```
sage: A0=[x for x in range(-10,10) if x%3==0]; print A0
[-9, -6, -3, 0, 3, 6, 9]
sage: A1=[x for x in range(-10,10) if x%3==1]; print A1
[-8, -5, -2, 1, 4, 7]
sage: A2=[x for x in range(-10,10) if x%3==2]; print A2
[-10, -7, -4, -1, 2, 5, 8]
sage: Set(A0).union(Set(A1).union(Set(A2)))==Set(range(-10,10))
True
```

(b) *A partition of the positive integers $\mathbb{Z}_+$ into two sets is $P \cup \overline{P}$ where P is the set of prime numbers, and $\overline{P} = \mathbb{Z}^+ - P$ is the set of non-prime positive integers.*

(c) *The sets $\{1,2,3\}$ and $\{3,4,5\}$ do not form a partition of $\mathbb{Z}_5 = \{1,2,3,4,5\}$ since they are not disjoint. They have the element 3 in common.*

We partitioned $\mathbb{Z}$ in three different ways: (i) into odd and even sets, (ii) into sets where the remainder upon division by 3 were the same, and (iii) into the set of primes, and non-primes. This illustrates there is more than one way to partition a set. As for which one to use, this really depends on the problem you are trying to solve.

Partitioning a set gives us a nice way to group elements with similarities. This allows us to focus our attention on subsets rather than the whole set, and this comes in handy when dealing with permutation puzzles. Partitions are closely related to another concept known as an *equivalence relation*. We now introduce this concept and show its connection with partitions.

17.2 Relations

We are familiar with many types of relations: “parent”, “brother”, “sister”, “sibling”, “spouse”, $<$, $=$, $>$, $\subset$, and other types of comparisons. In essence what we are doing is comparing two objects from the same set.

Definition 17.2 *Let A be a set. A subset $\mathcal{R} \subset A \times A$ is called a* ***relation on*** *A. If $(x,y) \in \mathcal{R}$ then we say x and y are related (and we sometimes write $x\mathcal{R}y$ for simplicity).*

Notice this definition is quite basic. It just says that by a “relation” we just mean a subset of $A \times A$. Any such subset will be a relation.

Example 17.3 *Let $A = \{1,2,3,4,5\}$, then each of the following is a relation on A.*

(a) $\mathcal{R}_1 = \{(1,4),(3,2)\}$

(b) $\mathcal{R}_2 = \{(1,1),(2,2),(3,3),(4,4),(5,5)\} = \{(a,b) \in A \times A \mid a = b\}$

(c) $\mathcal{R}_3 = \{(1,2),(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,4),(3,5),(4,5)\} = \{(x,y) \in A \times A \mid x < y\}$

In relation $\mathcal{R}_1$ we say: 1 is related to 4 and 3 is related to 2. But 1 is not related to 2. Also, 4 is not related to 1 in this case since $(4,1) \notin \mathcal{R}_1$. Read this carefully, 1 IS related to 4, but 4 IS NOT related to 1. Order matters in a relation. For example, John is the father of Jack, but Jack is not the father of John. This subtlety won't bother us too much (we are more interested in equivalence relations, which are symmetric, as discussed in the next section).

Since, by definition, a relation is a subset of $A \times A$, and $|A \times A| = 5^2 = 25$ then there are 2^{25} possible relations on A (each element of $|A \times A|$ can either be included in the relation, or not, hence there are two choices for each element). Some relations, of course, are more interesting than others.

Example 17.4 *Let $A = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ (that is, A is the set of all subset of $\{1,2\}$). Consider the relation*

$$\mathcal{R} = \{(\emptyset,\ \emptyset),\ (\emptyset, \{1\}),\ (\emptyset, \{2\}),\ (\emptyset, \{1,2\}),\ (\{1\}, \{1\}),\ (\{1\}, \{1,2\}),\ (\{2\}, \{2\}),\ (\{2\}, \{1,2\}),\ (\{1,2\}, \{1,2\})\}.$$

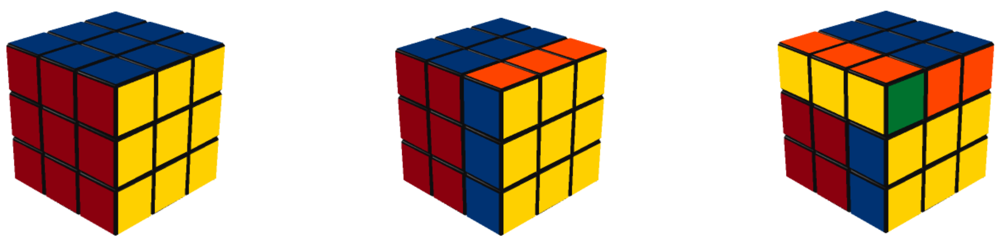
This is an example of the "subset" relation, since $(X, Y) \in \mathcal{R}$ precisely when $X \subset Y$.

Example 17.5 *Let $\mathcal{C}$ be the set of all the different configurations of Rubik's cube (that is, all the ways to mess up a cube). Let's say two configurations X and Y are related if there is a quarter turn of one of the 6 faces which takes configuration X to configuration Y:*

$$(X, Y) \in \mathcal{R} \quad \textit{if} \quad Y \textit{ can be obtained from } X \textit{ by a quarter turn of one face.}$$

This defines a relation on $\mathcal{C}$. In Figure 1 the cube in 1a and 1b are related (by a quarter turn of the r face), and the cubes in 1b and 1c are related (by a quarter turn of the u face). However, the cubes in 1a and 1c are not related, since it takes two face turns to get from one cube to the other.

Note that if $(X, Y) \in \mathcal{R}$ then $(Y, X) \in \mathcal{R}$, since each quarter turn has an inverse. In this case we would say $\mathcal{R}$ is a symmetric relation.



(a) (b) (c)

Figure 1: Three different configurations of Rubik's cube.

17.3 Equivalence Relation

For a given set, some relations are more useful than others. We saw in Example 17.3 that there are 2^{25} different possible ways to define a relation on $A = \{1, 2, 3, 4, 5\}$, but relations (b) and (c) seem much more useful (or should we say meaningful) than relation (a). Perhaps this is because we are so familiar with the relations "$=$" and "$<$". In this section we focus our attention on a special type of relation that is very useful in mathematics.

First a digression into relationships amongst people. For this let's just consider the set of all people who are currently alive, call this set $\mathcal{P}$. There are a number of relations we can consider on $\mathcal{P}$, for example if we are interested in who is whose child then the relation we would consider is: *$x\mathcal{R}y$ if x is a child of y*. Or maybe we want to consider the relationship of being a brother: *$x\mathcal{R}y$ if x is a brother of y*. Perhaps maybe we just want to know who is married, and to whom: *$x\mathcal{R}y$ if x is a spouse of y*. If your interest is in relationships on a more global scale then you can consider a proximity relation: *$x\mathcal{R}y$ if x lives in the same city as y*.

There are some differences in the behaviour of these relations. Consider the "brother of" relation. *Tim could be a brother of Alice*, but (assuming Alice is female) *Alice is not a brother of Tim*. We say that $\mathcal{R}$ is not *symmetric* in this case. However, the "spouse of" relation is *symmetric*: *if X is the spouse of Y then Y is the spouse of X*.

For the "proximity" relation, *if X lives in the same city as Y* and *Y lives in the same city as Z*, then it should follow that *X lives in the same city as Z*. We refer to this property as *transitivity*. Notice the "child relation" is not necessarily transitive, since *if Emma is a child of Karen*, and *Karen is a child of Henry*, then *Emma is not a child of Henry* (at least we hope not).

Another property that some relations may possess is the ability for an element to be related to itself. For example, *X lives in the same city as X* is certainly true. But, *X is a child of X* is impossible (though this would make a disturbing plot for some science fiction movie). A relation where all elements are related to themselves is known as *reflexive*.

An important, and very useful, class of relations are the relations that are *reflexive*, *symmetric* and *transitive*.

Definition 17.3 *Let $\mathcal{R}$ be a relation on a set A. We call $\mathcal{R}$ an* ***equivalence relation*** *on A if it satisfies the following properties:*

(a) *Each element is related to itself: $(a, a) \in \mathcal{R}$ for all $a \in A$ (reflexive property)*

(b) *If a is related to b then b is related to a: $(a, b) \in \mathcal{R}$ implies $(b, a) \in \mathcal{R}$ (symmetric property)*

(c) *If a is related to b, and b is related to c then a is related to c: $(a, b) \in \mathcal{R}$ and $(b, c) \in \mathcal{R}$ implies $(a, c) \in \mathcal{R}$ (transitive property).*

Notation: If $\mathcal{R}$ is an equivalence relation on A then we often write $x \equiv y$, or $x \sim y$ in place of $(x, y) \in \mathcal{R}$ for simplicity.

The "child of", "brother of", and "spouse of" relations are not equivalence relations. To see why we just need to observe that one of the three properties doesn't hold. In each case the reflexive property fails to hold. However, the "proximity" relation is an equivalence relation.

In Example 17.3 the relations $\mathcal{R}_1$ and $\mathcal{R}_3$ are not equivalence relations. For instance, neither one is symmetric. However, $\mathcal{R}_2$ is an equivalence relation.

The "proximity" relation $\sim$ on $\mathcal{P}$ is an equivalence relation. Pick some person, say person X from *Vancouver*. What does the set of all people related to X represent: $\{Y \in \mathcal{P} \mid Y \sim X\}$? Well, this would consist of all the people who live in Vancouver. Think about why? Sets of this type will be important for us, so we give them a special name.

Definition 17.4 *Let $\sim$ be an equivalence relation on a set A. For each $a \in A$ the set*

$$[a] = \{x \in A \mid x \sim a\}$$

is called the ***equivalence class of A containing*** *a. We call a a* ***representative*** *of the equivalence class $[a]$.* [1]

Lemma 17.1 *If $\sim$ is an equivalence relation on a set A and $x, y \in A$, then*

(a) *$x \in [x]$ (an equivalence class contains its representative)*

(b) *$x \sim y$ if and only if $[x] = [y]$ (if two elements are related then their equivalence classes are equal)*

(c) *$[x] = [y]$ or $[x] \cap [y] = \emptyset$ (equivalence classes are either equal or disjoint).*

Proof: (a) Since $\sim$ is reflexive $x \sim x$, therefore $x \in [x]$.

(b) Suppose $x \sim y$. We want to show that this implies $[x] = [y]$. To do this, let $z \in [x]$, then $z \sim x$ and since $x \sim y$ it follows that $x \sim y$, by the transitive property, and so $z \in [y]$. Therefore $[x] \subset [y]$. Moreover, $y \sim x$ by symmetry and a similar argument show $[y] \subset [x]$. Therefore $[x] = [y]$.

[1] The equivalence class of a is sometimes denoted by $[a]_{\mathcal{R}}$ or $[a]_{\sim}$.

Conversely, suppose $[x] = [y]$. By part (a), $x \in [x] = [y]$, and so $x \sim y$.

(c) If $[x] \cap [y] \neq \emptyset$ then let $z \in [x] \cap [y]$. It follows that $z \sim x$ and $z \sim y$, and so $x \sim y$ by transitivity. Now applying part (b) we have $[x] = [y]$.

□

Partitions and equivalence relations are related as the next result suggests.

Theorem 17.1 *(a) If A is a set and $\mathcal{R}$ is an equivalence relation on A then the set of equivalence classes form a partition of A.*

(b) If $A_1, \ldots, A_n$ is a partition of a set A (see Definition 17.1) then the relation $\mathcal{R}$ defined by

$$a\mathcal{R}b \quad \textit{if} \quad a, b \in A_i \textit{ for some } i,$$

is an equivalence relation on A. This relation can written as

$$\mathcal{R} = \bigcup_{i=1}^{n} A_i \times A_i.$$

The sets A_i are the equivalence classes of relation $\mathcal{R}$.

Proof: (a) This is a direct consequence of Lemma 17.1.

(b) By definition of $\mathcal{R} = \bigcup_{i=1}^{n} A_i \times A_i$ symmetric. Reflexivity follows from the fact that A is the union of the A_i's, and transitivity follows from the fact that the A_i's are disjoint.

□

Definition 17.5 *If $\sim$ is an equivalence relation on a set A, then a* ***set of class representatives*** *is a subset of A which contains exactly one element from each equivalence class. We denote the set of class representative by $A/\sim$.*

If $\sim$ is an equivalence relation on a set A , and $x \sim y$ then we say x and y are **equivalent**, rather than saying they are simply related.

Let's look at some examples to get a little more comfortable with these ideas.

Example 17.6 (Congruence relation on $\mathbb{Z}$) *Let n be a positive integer. Define an equivalence relation $\equiv$ on $\mathbb{Z}$ by*

$$a \equiv b \quad \textit{if } a - b \textit{ is divisible by } n.$$

We say a is ***congruent to*** *b* ***modulo*** *n and write $a \equiv b \pmod{n}$.*

For example, $26 \equiv 4 \pmod{11}$ since $26 - 4 = 22$ is divisible by 11. We say 26 is equivalent to 4 modulo 11. On the other hand, $7 \not\equiv 3 \pmod{5}$ since 5 does not divide $7 - 3 = 4$.

The equivalence class of x modulo n is often called the ***congruence class*** *of $x \pmod{n}$.*

The equivalence relation $\equiv \pmod{2}$ on $\mathbb{Z}$ has two equivalence (congruence) classes:

$$[0] = \{0, \pm 2, \pm 4, \ldots\} \quad \textit{and} \quad [1] = \{\pm 1, \pm 3, \pm 5, \ldots\}$$

A set of equivalence class representatives is $\{0, 1\}$.

The equivalence relation $\equiv \pmod{3}$ on $\mathbb{Z}$ has three equivalence (congruence) classes:

$$[0] = \{0, \pm 3, \pm 6, \ldots\}, \quad [1] = \{\pm 1, \pm 4, \pm 7, \ldots\} \quad \textit{and} \quad [2] = \{\pm 2, \pm 5, \pm 8, \ldots\}$$

A set of equivalence class representatives is $\{0, 1, 2\}$.

In general, for $n \in \mathbb{Z}^+$ *and* $a \in \mathbb{Z}$, *the class of* a *is*

$$[a] = \{a + kn \mid k \in \mathbb{Z}\}.$$

The set of equivalence class representatives (also called congruence class representatives modulo n*) is*

$$(\mathbb{Z}/\equiv) = \{0, 1, 2, \ldots, n-1\}.$$

Example 17.7 *Let* $\mathcal{C}$ *be the set of all the different configurations of Rubik's cube. The relation on* $\mathcal{C}$ *given in Example 17.5 is not transitive as we saw in that example.*

Instead, let's consider another relation on $\mathcal{C}$ *defined by* $X \equiv Y$ *if there is a sequence of moves involving only* U *and* R *that takes configuration* X *to configuration* Y. *This is an equivalence relation. Check for yourself that the three properties hold.*

The 3 *configurations shown in Figure 1 are equivalent, and therefore are elements of the same equivalence class. A representative for this class is the solved cube 1a. How many other configurations are equivalent to the solved cube? It turns out that there are a whopping* $73,483,200$ *configurations all equivalent to the solved cube. This means that by only twisting the* R *and* U *faces of the cube, you can generate over* 73 *million different configurations of the cube.*

SAGE

```
 sage: S48=SymmetricGroup(48)
 sage: R=S48("(25,27,32,30)(26,29,31,28)(3,38,43,19)(5,36,45,21)(8,33,48,24)")
 sage: L=S48("(9,11,16,14)(10,13,15,12)(1,17,41,40)(4,20,44,37)(6,22,46,35)")
 sage: U=S48("(1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)(11,35,27,19)")
 sage: D=S48("(41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)(16,24,32,40)")
 sage: F=S48("(17,19,24,22)(18,21,23,20)(6,25,43,16)(7,28,42,13)(8,30,41,11)")
 sage: B=S48("(33,35,40,38)(34,37,39,36)(3,9,46,32)(2,12,47,29)(1,14,48,27)")
 sage: H=S48.subgroup([R,U])
 sage: H.order()
73483200
```

Example 17.8 *Let* $\mathcal{A}$ *denote the set of all possible ways to reassemble Rubik's cube. That is, first you take it apart, then put it back together in the shape of a cube again. Define a relation* $\sim$ *on* $\mathcal{A}$ *as follows:*

$X \sim Y$ *if through a sequence of legal cube moves (i.e. twists of the* 6 *faces),* X *can be taken to* Y.

All this means is we consider two cubes equivalent if one can be twisted into the other.

What is the equivalence class of the solved cube?

This is really asking, what configurations are equivalent to the solved state configuration? In other words, what are all the possible configurations one can achieve from the solved cube by twisting faces. In this context, where we are considering all assembled cubes $\mathcal{A}$, *this is an interesting question, since if the equivalence class is not all of* $\mathcal{A}$ *it means there are ways to reassemble the cube which are not solvable. In other words, you can mess with your friends cube by taking it apart and reassembling it into an unsolvable cube.*

Using the notation introduced in this section, and letting X_0 *denote the cube in the solved state, then what we want to know is* $[X_0]$. *Moreover, if there is more than one equivalence class then it would be interesting to know how many there are and a set of equivalence class representative, i.e.* $\mathcal{A}/\sim$.

We will investigate this question later. But for now we'll note that $|\mathcal{A}/\sim| \geq 5$ *since Figure 2 shows five assemblies of Rubik's cube which are not equivalent under legal cube moves.*

We also know that a corner swap, see Figure 3, is not equivalent to X_0.

However, it is equivalent to the "edge swap" in Figure 2b. We'll see this when we study the Fundamental Theorem of Cubology.

(a) solved (b) edge flip (c) edge swap (d) clockwise corner twist (e) counterclockwise corner twist

Figure 2: Five different equivalence class representatives of $\mathcal{A}$. How many more are there?

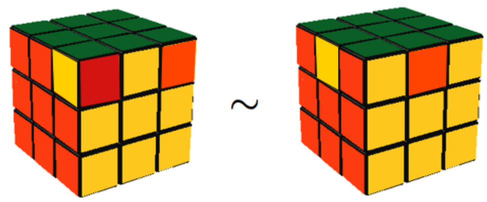


Figure 3: A corner swap is equivalent to an edge swap, but not equivalent to the solved state.

17.4 Exercises

1. Consider the "cousin of" relation:
$$x\mathcal{R}y \quad \text{if } x \text{ is a cousin of } y.$$
Is $\mathcal{R}$ symmetric? Is it transitive?

2. In Example 17.6 it was stated that $\equiv \pmod{n}$ is an equivalence relation on $\mathbb{Z}$. Prove this statement. That is, show it is reflexive, symmetric and transitive.

3. For each the following relations defined on the set X determine whether or not the relation is reflexive, symmetric, or transitive.
   (a) $X = \mathbb{Z}$, $a\mathcal{R}b$ if $a \mid b$ (i.e. a divides b)
   (b) $X = \mathbb{Z}$, $a\mathcal{R}b$ if $a + b = 10$
   (c) $X = \mathbb{Z}$, $a\mathcal{R}b$ if $a - b > 0$
   (d) $X = \mathbb{Z}$, $a\mathcal{R}b$ if $a + b$ is even
   (e) $X = \mathbb{Z}$, $a\mathcal{R}b$ if $a - b$ is even
   (f) $X = \mathbb{Z}$, $a\mathcal{R}b$ if $3 \mid a + b$
   (g) $X = \mathbb{Z}$, $a\mathcal{R}b$ if $\gcd(a, b) = 1$
   (h) $X = \mathbb{Z} \times (\mathbb{Z} - \{0\})$, $(a, b)\mathcal{R}(c, d)$ if $ad = bc$
   (i) $X = \mathbb{R} \times \mathbb{R}$, $(a, b)\mathcal{R}(c, d)$ if $\sqrt{(a-c)^2 + (b-d)^2} \leq 1$
   (j) $X = \mathbb{R} \times \mathbb{R}$, $(a, b)\mathcal{R}(c, d)$ if $ac + bd = 0$

4. Define the relation $\mathcal{R}$ on $\mathbb{R} \times \mathbb{R}$ by
$$(a, b)\mathcal{R}(c, d) \quad \text{if} \quad b - a = d - c.$$
Show that $\mathcal{R}$ is an equivalence relation and describe the set $\mathcal{R}$ geometrically.

5. Define the relation $\mathcal{R}$ on $\mathbb{R} \times \mathbb{R}$ by
$$(a, b)\mathcal{R}(c, d) \quad \text{if} \quad a^2 + b^2 = c^2 + d^2.$$
Show that $\mathcal{R}$ is an equivalence relation and describe the set $\mathcal{R}$ geometrically.

6. Define the relation $\mathcal{R}$ on $X = \{1, 2, 3, \ldots, 20\}$ by
$$a\mathcal{R}b \quad \text{if} \quad 3 \mid a - b.$$
Show that $\mathcal{R}$ is an equivalence relation. Describe the equivalence classes of the corresponding partition of X.

7. Define the relation $\mathcal{R}$ on $X = \{1, 2, 3, \ldots, 20\}$ by
$$a\mathcal{R}b \quad \text{if} \quad a \text{ and } b \text{ have the same prime divisors.}$$
Show that $\mathcal{R}$ is an equivalence relation. Describe the equivalence classes of the corresponding partition of X.

8. For each of the following statements about relations on a set A, where $|A| = n$, determine whether the statement is true or false. If it is false, give a counterexample.
   (a) If $\mathcal{R}$ is a reflexive relation on A, then $|\mathcal{R}| \geq n$.
   (b) If $\mathcal{R}$ is a relation on A and $|\mathcal{R}| \geq n$, then $\mathcal{R}$ is reflexive.
   (c) If $\mathcal{R}_1$, $\mathcal{R}_2$ are relations on A and $\mathcal{R}_1 \subset \mathcal{R}_2$, then $\mathcal{R}_1$ reflexive (symmetric, transitive) $\Rightarrow$ $\mathcal{R}_2$ reflexive (symmetric, transitive).
   (d) If $\mathcal{R}_1, \mathcal{R}_2$ are relations on A and $\mathcal{R}_1 \subset \mathcal{R}_2$, then $\mathcal{R}_2$ reflexive (symmetric, transitive) $\Rightarrow$ $\mathcal{R}_1$ reflexive (symmetric, transitive).
   (e) If $\mathcal{R}$ is and equivalence relation on A, then $n \leq |\mathcal{R}| \leq n^2$.

9. If $A = \{a, b, c, d\}$, determine the number of relations on A that are (i) reflexive, (ii) symmetric, (iii) reflexive and symmetric, (iv) reflexive and contains (a, b), (v) symmetric and contains (a, b).

10. If $A = \{1, 2, 3, 4\}$, give and example of a relation $\mathcal{R}$ on A that is
    (a) reflexive and symmetric, but not transitive.
    (b) reflexive and transitive, but not symmetric.
    (c) symmetric and transitive, but not reflexive

11. Describe a partition of the set of all prime numbers into four classes.

12. What is wrong with the following argument?
    Let A be a set and $\mathcal{R}$ a relation on A. If $\mathcal{R}$ is symmetric and transitive, then $\mathcal{R}$ is reflexive.
    Proof: Let $(x, y) \in \mathcal{R}$. By the symmetric property $(y, x) \in \mathcal{R}$. Then with $(x, y), (y, x) \in \mathcal{R}$, it follows by the transitive property that $(x, x) \in \mathcal{R}$. Consequently $\mathcal{R}$ is reflexive. □

13. Let A be a set with $|A| = n$, and let $\mathcal{R}$ be an equivalence relation on A with $|\mathcal{R}| = r$. Why is $r - n$ always even?

14. **Conjugation is an equivalence relation.** Let G be a group, show that the relation
$$g\mathcal{R}h \quad \Longleftrightarrow \quad g \text{ is a conjugate of } h,$$
is and equivalence relation.

15. Let G be a group and H a subgroup of G. Define a relation $\mathcal{R}$ on G by
$$a\mathcal{R}b \quad \text{if} \quad b^{-1}a \in H.$$
    (a) Show $\mathcal{R}$ is an equivalence relation.
    (b) Show that each equivalence class $[a]$ has the form $aH = \{ah \mid h \in H\}$ for some a. The is called the *left coset of H in G containing a*.

(c) Show that each equivalence class has the same cardinality. That is, show $|aH| = |bH|$, for any $a, b \in H$.

(d) Conclude from Theorem 17.1 that $|H|$ divides $|G|$. This proves Lagrange's Theorem: the order of a subgroup divides the order of a group.

16. Consider the set of all 2×2 matrices with real entries:

$$M_{2,2}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}.$$

Define a relation $\mathcal{R}$ on $M_{2,2}(\mathbb{R})$ by

$$A\mathcal{R}B \quad \text{if} \quad A \text{ is row equivalent to } B.$$

(By *row equivalent* we mean A can be converted to B through elementary row operations: (i) multiply a row by a scalar, (ii) swap two rows, (iii) add a multiple of another row to an existing row.)
Show $\mathcal{R}$ is an equivalence relation. How many equivalence classes are there? Determine a set of class representatives.

17. Define a relation $\mathcal{R}$ on $M_{2,2}(\mathbb{R})$ by

$$A\mathcal{R}B \quad \text{if} \quad \text{there exists and invertible matrix } C \text{ such that } B = CA.$$

Show $\mathcal{R}$ is an equivalence relation. How does this relation compare to the one in Exercise 16.