# Lecture 4: Permutations: Cycle Notation

## Contents



In this section we introduce a simple, yet extremely powerful, notation for permutations: *cycle form* We'll revisit the concepts of products (composition), order, and inverses, and see how our new notation simplifies calculations.

This lecture corresponds to Section 3.3 of Joyner's text.

### 4.1 Permutations: Cycle Notation

Consider the $5$-cycle permutation $\alpha$ defined as follows:

$$\alpha(1) = 2, \quad \alpha(2) = 3, \quad \alpha(3) = 4, \quad \alpha(4) = 5, \quad \alpha(5) = 1.$$

The *array form* of $\alpha$ is shown in Figure 1a, and the *arrow diagram* is shown in Figure 1b.

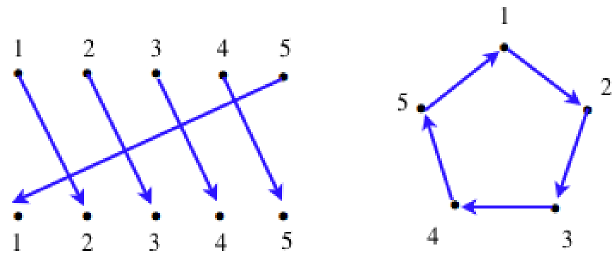
Another *arrow diagram* which provides a more visual display of the structure of the permutation is shown in Figure 1c. This is called the *cycle-arrow form*.

In this diagram all the information for $\alpha$ is still present. What is $\alpha(3)$? To determine this, look at the diagram and find $3$, then see where the arrow takes it. In this case it takes it to $4$, so $\alpha(3) = 4$.

There are a couple of nice things about cycle arrow form: (1) it displays more visually the cycle structure (i.e. we can see the $5$ numbers cycling around the circle, which is why we called it a $5$-cycle), and (2) it uses only one set of numbered dots, making the diagram more compact than our original arrow form.

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

(a) array form



(b) arrow form (c) cycle-arrow form

$$\alpha = (1, 2, 3, 4, 5)$$

(d) cycle form

Figure 1: Different representations for a 5-cycle.

Though mathematically satisfactory, the cycle arrow form is cumbersome to draw. However, leaving out the arrows we can simply write the 5-cycle as:

$$\alpha = (1, 2, 3, 4, 5)$$

This represents that fact that $\alpha$ maps each number to the next one in the list, and maps $5$ back around to the start of the list, which is $1$. This representation is shown in Figure 1d.

All representations in Figure 1 have their own benefits, but it is the *cycle notation* that is the most compact, and this will be the notation we primarily use in this course.

When working with cycle notation, $\alpha = (1, 2, 3, 4, 5)$, you should read it as follows:

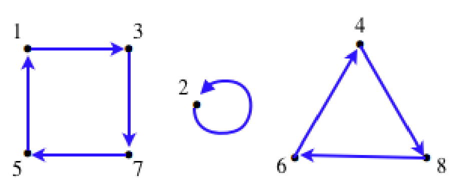
"$1$ goes to $2$, $2$ goes to $3$, $3$ goes to $4$, $4$ goes to $5$, and $5$ goes to $1$."

We don't need to start at $1$ when writing down the cycle form, if we started at $3$, for instance, and constructed the list of numbers we visit by traveling around Figure 1c then we get $(3, 4, 5, 1, 2)$. This is another perfectly acceptable representation of $\alpha$: reading this cycle notation as described above will tell us exactly how $\alpha$ acts as a function. In particular, we can represent $\alpha$ by any of the equivalent cycle forms:

$$\alpha = (1, 2, 3, 4, 5) = (2, 3, 4, 5, 1) = (3, 4, 5, 1, 2) = (4, 5, 1, 2, 3) = (5, 1, 2, 3, 4).$$

Despite this notation allowing for non-unique representations of permutations, there is an easy fix. Just writing the cycle so that the first number is the smallest number in the cycle. In this case we would then write $\alpha = (1, 2, 3, 4, 5)$ since $1$ is the smallest number in this cycle.

Let's look at another permutation: $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 7 & 8 & 1 & 4 & 5 & 6 \end{pmatrix}$. The cycle arrow form is:



This reveals so much about the permutation, especially when you imagine taking powers of it: $\beta^n$. For instance, 1,3,5,7 only get permuted amongst themselves, so there is no $k$ such that $\beta^k(1) = 4$. Also, since a $4$-cycle has order $4$, then $\beta^4$ would leave 1,3,5,7 untouched: $\beta^4(x) = x$ when $x = 1, 3, 5, 7$.

To construct the cycle form of $\beta$ we we look at the arrow form above and notice that $1$ goes to $3$, $3$ goes to $7$, $7$ goes to $5$ and $5$ goes back to $1$. This can simply be written as $(1, 3, 7, 5)$. Similarly, $2$ goes to $2$ so we write this as $(2)$, and the $4, 6, 8$ triangle can be written as $(4, 8, 6)$. This means we can write $\beta$ as:

$$\beta = (1, 3, 7, 5)(2)(4, 8, 6).$$

This is a compact way to represent the permutation $\beta$, and we haven't lost any information. For example, we can use the cycle form determine $\beta(3)$ by noticing in $(1, 3, 7, 5)(2)(4, 8, 6)$ the number $3$ is followed by $7$, so $\beta(3) = 7$. Similarly, $\beta(5) = 1$ since from $5$ we wrap around in the cycle and get back to $1$.

If we make one further convention, to leave off any number that gets mapped to itself, then $\beta$ can be written in a further compact form:

$$\beta = (1, 3, 7, 5)(4, 8, 6).$$

In this convention, any number not present in the cycle form is assumed to map back to itself.

An expression of the form $(a_1, a_2, \ldots, a_m)$ is called an $m$- *cycle*.

We say $\beta$ is the product of a $3$-cycle and a $4$-cycle.

**Example 4.1** *To determine the cycle form of the permutation*

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 1 & 6 & 8 & 4 & 10 & 7 & 2 & 9 & 3 \end{pmatrix}$$

*start with the smallest number in the set, in this case it is* $1$. *Since* $\alpha(1) = 5$ *we begin the cycle by writing*

$$(1, 5, \ldots) \ldots.$$

*Next,* $5$ *maps to* $4$*, so we continue building the cycle*

$$(1, 5, 4, \ldots) \ldots.$$

*Continuing in this way we construct* $(1, 5, 4, 8, 2, \ldots) \ldots$*, and since* $2$ *maps back to* $1$ *then we close off the cycle:*

$$(1, 5, 4, 8, 2) \ldots.$$

*Next, we pick the smallest number that doesn't appear in any previously constructed cycle. This is the number* $3$ *in this case. We now repeat what we just did and construct the cycle involving* $3$*:*

$$(1, 5, 4, 8, 2)(3, 6, 10) \ldots.$$

*We now pick the smallest number that doesn't appear in any previously constructed cycle, which is* $7$*, and construct the cycle to which it belongs. In this case* $7$ *just maps to itself:*

$$(1, 5, 4, 8, 2)(3, 6, 10)(7) \ldots.$$

*Finally, the only number remaining is* $9$ *and it maps back to itself so the cycle for of* $\alpha$ *is*

$$(1,5,4,8,2)(3,6,10)(7)(9)$$

*which simplifies to*

$$\alpha = (1,5,4,8,2)(3,6,10)$$

*since our convention is omit* $1$*-cycles. Therefore,* $\alpha$ *is the product of a* $3$*-cycle and a* $5$*-cycle.*

**Exercise 4.1** ***Converting from array to cycle form.*** *Convert the permutation given in array form:* $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ *to cycle form.*

**Exercise 4.2** ***Converting from cycle to array form.*** *For the permutation given in cycle form by* $(1,3,5,2)(4,7) \in S_8$*, express it in array form.*

## 4.2 Products of Permutations: Revisited

It is not efficient to convert permutations from cycle form to array from, then compose the permutations in array form, only to convert back to cycle form. Instead, we will work entirely with the cycle form but we do so by *thinking* of their representation in array form.

For example, consider the permutations $\alpha = (1,5,2,3)$ and $\beta = (1,5,4)(2,3)$ in $S_5$. What is the cycle for of $\alpha\beta$? Of course, we could just stick the two permutations together, end-to-end, and write

$$\alpha\beta = (1,5,2,3)(1,5,4)(2,3)$$

but it will be more convenient to express the permutation in *disjoint cycle form*, that is where the various cycles have no numbers in common.

We determine the cycle form of $\alpha\beta$ by determining exactly how it maps each number, beginning with $1$. Keep in mind that permutation composition is done from left-to-right, and each cycle that does not contain a number fixes that number. We have that: $(1,5,2,3)$ sends $1$ to $5$, $(1,5,4)$ sends $5$ to $4$, and $(2,3)$ fixes $4$. So the effect of $\alpha\beta$ is it sends $1$ to $4$.

$$1 \dashrightarrow (1,5,2,3)(1,5,4)(2,3) \dashrightarrow 4$$

Thus we begin writing the disjoint cycle form as $\alpha\beta = (1,4,\ldots)\ldots$.

Repeating this process with $4$, we have, cycle-by-cycle, left-to-right,

$$4 \xrightarrow{(1,5,2,3)} 4 \xrightarrow{(1,5,4)} 1 \xrightarrow{(2,3)} 1,$$

so that $\alpha\beta(4) = 1$, and the cycle form is now $\alpha\beta = (1,4)\ldots$.

Next we pick the smallest number that is not in any previously constructed cycle, this would be $2$. Repeating this process with $2$, cycle-by-cycle, left-to-right,

$$2 \xrightarrow{(1,5,2,3)} 3 \xrightarrow{(1,5,4)} 3 \xrightarrow{(2,3)} 2,$$

so that $\alpha\beta(2) = 2$, and the cycle for is now $\alpha\beta = (1, 4)(2)\dots$.

Continuing in this way we find that $\alpha\beta = (1, 4)(2)(3, 5) = (1, 4)(3, 5)$.

The important thing to keep in mind when multiplying cycles is to *keep moving* from one cycle to the next from left-to-right.

**Example 4.2** *Let* $\alpha = (1, 4, 6, 3, 7)(2, 8)$ *and* $\beta = (2, 5, 3)(4, 7, 8, 1)$ *be permutations in* $S_8$. *Then*

$$\alpha\beta = (1, 4, 6, 3, 7)(2, 8)(2, 5, 3)(4, 7, 8, 1) = (1, 7, 4, 6, 2)(3, 8, 5)$$

*and*

$$\beta\alpha = (2, 5, 3)(4, 7, 8, 1)(1, 4, 6, 3, 7)(2, 8) = (1, 6, 3, 8, 4)(2, 5, 7).$$

*Check this yourself. To start you off, lets consider what happens to* $1$ *under* $\alpha\beta$*:*

$$1 \xrightarrow{(1,4,6,3,7)} 4 \xrightarrow{(2,8)} 4 \xrightarrow{(2,5,3)} 4 \xrightarrow{(4,7,8,1)} 7,$$

*so* $(\alpha\beta)(1) = 7$.

## 4.3 Properties of Cycle Form

Two basic properties of permutations are: (a) **every permutation can be written as a product of disjoint cycles**, and (b) **disjoint cycles commute**.

The first property was implicit in our discussion of how to construct the cycle form of a permutation. In particular, when we finished constructing a cycle, the first thing we did was look for a number that did not appear in an previously constructed cycles. This guarantees that our cycles will be disjoint.

The second property: *disjoint cycles commute*, is also fairly straightforward consequence of the disjoint cycle notation. For example, consider the disjoint cycles $\alpha = (1, 3, 2)$ and $\beta = (4, 5)$. When multiplying these cycles it doesn't matter which order the product is taken: $\alpha\beta = (1, 3, 2)(4, 5) = (4, 5)(1, 3, 2) = \beta\alpha$. Both of these products represent the same permutation: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$. As a former student of mine once said, *it is kind of like two games of musical chairs going on in two different rooms, neither one has any influence on the other.*

Even though this property straightforward, it is very important, so we will state it as a theorem.

**Theorem 4.1 (Disjoint Permutations Commute)** *If* $\alpha, \beta \in S_n$ *and have no numbers in* $\mathbb{Z}_n$ *that are moved by both* $\alpha$ *and* $\beta$ *then* $\alpha\beta = \beta\alpha$. *In other words, if the disjoint cycle form of* $\alpha$ *has no number in common with the disjoint cycle form of* $\beta$ *then* $\alpha$ *and* $\beta$ *commute.*

As a more physical example of disjoint cycles commuting, consider the moves R and L of Rubik's cube. These moves are disjoint in the sense that their is no common piece that is moved by both R and L. Notice that RL and LR result in exactly the same position of the cube, so in this sense RL = LR, and so R and L commute.

## 4.4 Order of a Permutation: Revisited

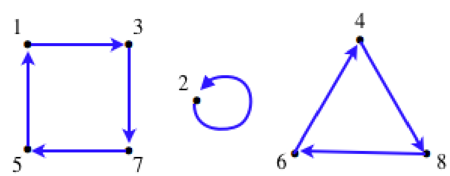
Recall the **order** of a permutation $\alpha \in \mathbb{Z}_n$ is the smallest positive integer $m$ such that $\alpha^m = \varepsilon$. To determine the order of a given permutation our only technique so far was to just continue computing powers until we hit the identity. This is a very inefficient way to compute orders.

The disjoint cycle form has the enormous advantage of allowing us to "eyeball" the order of a permutation.

For example the 5-cycle $(1, 2, 3, 4, 5)$ has order 5. In general, an $m$-cycle has order $m$. (You are asked to show this in Exercise 9.) The order of a product of disjoint cycles is given by the next theorem.

**Theorem 4.2 (Order of a Permutation)** *The order of a permutation written in disjoint cycle form is the least common multiple of the lengths of the cycles.*

Before we prove this theorem lets see why it should be true. Consider the permutation $\beta = (1, 3, 7, 5)(4, 8, 6)$, which is the product of a cycle of length 3 and a cycle of length 4. The arrow diagram is as follows.




We want to determine the smallest power $k$ so that $\beta^k$ is the identity. Every application of $\beta$ moves the numbers around the square (4-cycle) one position, so in order to have numbers return to their original position $\beta$ must be applied 4, or a multiple of 4, times. This means $4 \mid k$. [1] Similarly, considering the triangle (3-cycle) $\beta$ would need to be applied a multiple of 3 times to move numbers back to their original positions. This means $3 \mid k$. Since we require both 3 and 4 to divide $k$, and we want $k$ to be as small as possible, this means $k$ is the *least common multiple* or 3 and 4, that is $\text{ord}(\beta) = k = \text{lcm}(3, 4) = 12$. Sure enough, if we check we can see $\beta^{12} = \varepsilon$.

An easy way to see $\beta^{12} = \varepsilon$ is to do the following:

$$\beta^{12} = [(1, 3, 7, 5)(4, 8, 6)]^{12} = (1, 3, 7, 5)^{12}(4, 8, 6)^{12} = [(1, 3, 7, 5)^4]^3[(4, 8, 6)^3]^4 = \varepsilon^3\varepsilon^4 = \varepsilon.$$

Here we used the fact that an $m$-cycle has order $m$, and $(\sigma_1\sigma_2)^k = \sigma_1^k\sigma_2^k$, for *disjoint* cycles $\sigma_1$ and $\sigma_2$ (recall that disjoint cycles commute by Theorem 4.1).

This is precisely the idea that we use to give a general proof of the theorem.

**Proof: (Theorem 4.2)**

**One cycle:** As we noted above, a cycle of length $m$ has order $m$. (See Exercise 9.)

**Two disjoint cycles:** Now suppose $\alpha$ and $\beta$ are disjoint cycles of lengths $a$ and $b$. Let $k$ be the least common multiple of $a$ and $b$, that is, $k$ is the smallest positive integer which is divisible by both $a$ and $b$. Since $\alpha$ and $\beta$ commute then $(\alpha\beta)^k = \alpha^k\beta^k = \varepsilon$ (here we used that fact that $a|k$ implies $\alpha^k = \varepsilon$ and $b|k$ implies $\beta^k = \varepsilon$). It follows from Theorem 3.4 that the order of $\alpha\beta$, call it $t$, divides $k$. We now wish to show $t = k$. From $\varepsilon = (\alpha\beta)^t = \alpha^t\beta^t$ it follows that $\alpha^{-t} = \beta^t$. However, $\alpha$ and $\beta$ have no symbol in

[1] For integers, the vertical bar $\mid$ means "divides", so $a \mid b$ is read "$a$ divides $b$" and means $b = ak$ for some integer $k$.

common, and since raising a cycle to a power does not introduce new symbols, $\alpha^{-t}$ and $\beta^t$ also have no symbol in common. Since $\alpha^{-t} = \beta^t$ and have no commons symbols then they both must be the identity: $\alpha^{-t} = \beta^t = \varepsilon$. If follows from Theorem 3.4 that $t$ is divisible by $a$ and $b$. This means that $k = \text{lcm}(a, b)$ must also divide $t$. Therefore $t = k$, as desired.

**More than two disjoint cycle:** The general case involving more than two cycles is handled in an analogous way. □

**Example 4.3** *(a) The order of $\alpha = (1, 3, 4)(2, 5)$ is lcm$(3, 2) = 6$. Observe that*

$$\alpha^6 = [(1, 3, 4)(2, 5)]^6 = (1, 3, 4)^6(2, 5)^6 = \varepsilon.$$

*(b) The permutation $\beta = (1, 7, 4, 10, 3)(2, 5, 6, 9)(8, 11)$ has order lcm$(5, 4, 2) = 20$. Notice how quickly we were able to compute this order. If we tried to do it by successively computing powers of $\beta$ we would need to compute $20$ powers, and this assumes we didn't make any mistakes in the tedious calculations. This shows the power of Theorem 4.2.*

**Exercise 4.3** *Find the order of each of the following permutations:*
*(a)* $(1, 3)$ *(b)* $(1, 5, 2, 3)$ *(c)* $(1, 5, 3, 7)(2, 6, 8)$

## 4.5 Inverse of a Permutation: Revisited

Every permutation can be written as a product of disjoint cycles: $\alpha = \sigma_1\sigma_2\cdots\sigma_k$. We have already seen that the inverse of a product is the product of the inverses in the reverse order, so

$$\alpha^{-1} = \sigma_k^{-1}\cdots\sigma_2^{-1}\sigma_1^{-1}.$$

This means, in order to determine $\alpha^{-1}$ directly from its cycle form we just need to know how to find the inverse of a cycle.

Consider the $5$-cycle $\alpha = (1, 2, 3, 4, 5)$. We'd like to come up with a simple method for determining the inverse $\alpha^{-1}$ directly from the cycle form, and without having to change representation to array form, or arrow form.

We already know that if we have $\alpha$ in array form: $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$ then it is easy to write down the inverse: $\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 1 \end{pmatrix}$. If we express this back in cycle form we have $\alpha^{-1} = (1, 5, 4, 3, 2)$. An alternative way to write this cycle is $(5, 4, 3, 2, 1)$. This gives us a very simple method for computing an inverse of a cycle: just write the cycle backwards!

$$\alpha^{-1} = (1, 2, 3, 4, 5)^{-1} = (5, 4, 3, 2, 1) = (1, 5, 4, 3, 2)$$

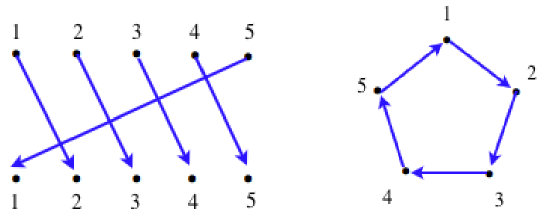
The last equality follows from our convention that we start the cycle with the smallest number in the cycle. See Figure 2 on page 8 for the various representation of $\alpha$ and $\alpha^{-1}$.

To make sure we nail this down, consider another example. The inverse of the permutation $\beta = (1, 5, 3)(2, 4)$ is $\beta^{-1} = (2, 4)^{-1}(1, 5, 3)^{-1} = (4, 2)(3, 5, 1) = (2, 4)(1, 3, 5)$.

To summarize:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

(a) array notation



(b) arrow notation

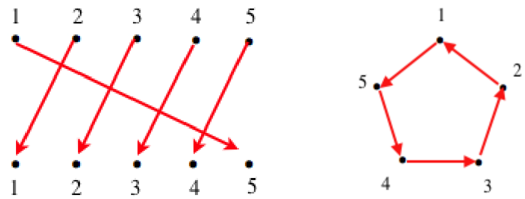
(c) cycle arrow notation for $\alpha$

$\alpha = (1, 2, 3, 4, 5)$

(d) cycle notation for $\alpha$

$$\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$$

(e) array notation



(f) arrow notation

(g) cycle arrow notation

$\alpha^{-1} = (1, 5, 4, 3, 2)$

(h) cycle notation

Figure 2: Different representations for $\alpha$ and $\alpha^{-1}$.

> To get from the cycle form of $\alpha$ to the cycle form of $\alpha^{-1}$, just write the representation for $\alpha$ down in the reverse order.

This means, reverse the order in which the numbers are written in each individual cycle, as well as reverse the order in which the cycles are written.

**Example 4.4** *(a) The inverse of the permutation* $\alpha = (1, 6, 3, 4, 5)$ *is* $\alpha^{-1} = (5, 4, 3, 6, 1) = (1, 5, 4, 3, 6)$.

*(b) The inverse of a* $2$*-cycle is itself. For example,* $(1, 2)^{-1} = (2, 1) = (1, 2)$.

*(c) The inverse of the permutation* $\beta = (1, 4, 3, 5)(3, 7, 6)(2, 5, 7, 3, 1)(6, 4)(2, 3, 5, 4)(4, 5, 3)$ *is*

$$\begin{aligned} \beta^{-1} &= [(1, 4, 3, 5)(3, 7, 6)(2, 5, 7, 3, 1)(6, 4)(2, 3, 5, 4)(4, 5, 3)]^{-1} \\ &= (4, 5, 3)^{-1}(2, 3, 5, 4)^{-1}(6, 4)^{-1}(2, 5, 7, 3, 1)^{-1}(3, 7, 6)^{-1}(1, 4, 3, 5)^{-1} \\ &= (4, 3, 5)(2, 4, 5, 3)(6, 4)(2, 1, 3, 7, 5)(3, 6, 7)(1, 5, 3, 4) \end{aligned}$$

*Since* $\beta^{-1}$ *is not in disjoint cycle form (due to the fact that* $\beta$ *itself was not), then we should probably put it in this form.*

$$\beta^{-1} = (1, 6)(2, 7, 3, 4, 5).$$

**Exercise 4.4** *Let* $\alpha = (1, 2)(4, 5)$ *and* $\beta = (1, 6, 5, 3, 2)$*. Compute (a)* $\alpha^{-1}$*, (b)* $\beta^{-1}$*, (c)* $(\beta\alpha)^{-1}$.

## 4.6 Summary of Permutations

Let's continue with our summary of what we know about $S_n$.

- $S_n$, the symmetric group of degree $n$, is the set of all permutation of $\mathbb{Z}_n = \{1, 2, \ldots, n\}$:
$$S_n = \{\alpha \mid \alpha : \mathbb{Z}_n \to \mathbb{Z}_n \text{ and } \alpha \text{ is a bijection } \}.$$
- $|S_n| = n!$
- Two elements $\alpha, \beta \in S_n$ can be composed (multiplied) to give another element $\alpha\beta \in \mathbb{Z}_n$.[2]
- The *identity* permutation $\varepsilon = (1)(2)(3)\cdots(n)$ has the property that $\varepsilon\alpha = \varepsilon\alpha = \alpha$ for all $\alpha \in S_n$. If we follow our convention of omitting 1-cycles, then when writing the cycle form for $\varepsilon$ we cannot omit all of them! In this case, we usually write just one 1-cycle. For example, $\varepsilon = (1)$. Just remember missing elements are mapped to themselves.
- Every $\alpha \in S_n$ has an inverse denoted by $\alpha^{-1}$. The defining property of an inverse is $\alpha\alpha^{-1} = \alpha^{-1}\alpha = \varepsilon$.
- Inverse of a product: $(\alpha_1\alpha_2\cdots\alpha_k)^{-1} = \alpha_k^{-1}\cdots\alpha_2^{-1}\alpha_1^{-1}$.
- Inverse of an $m$-cycle: $(a_1, a_2, \ldots, a_{m-1}, a_m)^{-1} = (a_m, a_{m-1}, \ldots, a_2, a_1)$.
- Permutation composition (multiplication) is associative: $(\alpha\beta)\gamma = \alpha(\beta\gamma) = \alpha\beta\gamma$.
- Permutation composition (multiplication) is not necessarily commutative. However, disjoint permutations commute.
- Cancelation Property: $\alpha\beta = \alpha\gamma$ implies $\beta = \gamma$, and $\beta\alpha = \gamma\alpha$ implies $\beta = \gamma$.
- For every $\alpha \in S_n$ the is a smallest number $m$, called the order of $\alpha$, denoted by ord$(\alpha)$, such that $\alpha^m = \varepsilon$. If a permutation is written in disjoint cycle form then ord$(\alpha)$ is the least common multiple of the lengths of the cycles.
- We've seen 5 ways to represent a permutation: (1) listing out all the values, (2) array form, (3) arrow form, (4) cycle-arrow form, and (5) cycle form. We will most frequently use cycle form since it is not only the most compact form, it also allows for easy calculations of products, inverses, and orders. We will see very soon that there are many more benefits to this notation.

## 4.7 Working with Permutations in SAGE

SAGE uses disjoint cycle notation for permutations, and permutation composition occurs left-to-right, which agrees with our convention. There are two ways to write the permutation $\alpha = (13)(254)$:
1. As a text string (include quotes): "(1,3)(2,5,4)"
2. As a list of tuples: [(1,3), (2,5,4)]

SAGE
```
sage: S5=SymmetricGroup(5)    # symmetric group on 5 objects, and names it S5
sage: a=S5("(2,3)(1,4)")      # constructs the permutation (2,3)(1,4) in S5
sage: b=S5("")                # constructs the identity permutation in S5
sage: c=S5("(2,5,3)")         # constructs the 3-cycle (2,5,3) in S5
sage: print a, b, c,
(1,4)(2,3)
()
```

[2]the convention of these notes is to compose permutations from left-to-right,

```
(2,5,3)
sage: a*c               # compose permutations by using multiplication sign
(1,4)(3,5)
sage: c.inverse()       # computes inverse
(2,3,5)
sage: c.order()         # computes order
3
```

Try these examples in SAGE, then change things and see what happens. Don't be afraid to experiment, this is how you learn. You won't break anything (at least it is unlikely you will).

## 4.8 Exercises

1. **Converting from array to cycle notation.** Convert each of the following permutations given in array form to cycle form

   (a) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 8 & 5 & 4 & 7 & 1 & 3 & 6 & 2 & 10 & 9 \end{pmatrix}$

   (c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 10 & 11 & 9 & 4 & 8 & 15 & 5 & 2 & 7 & 3 & 6 & 1 & 12 & 13 & 14 \end{pmatrix}$

2. **Converting from cycle to array notation.** For each of the following permutation in $S_8$ convert from cycle form to array form.

   (a) $(1, 5, 2)(3, 4)(7, 8)$

   (b) $(1, 7, 4, 6)(3, 5, 8)$

3. **Reducing cycle notation to disjoint cycles.**
   When multiplying permutations we will most likely end up with a product of cycles which are not necessarily disjoint, and our goal will be to find a representation in disjoint cycle form. To practice this, write the following permutations in disjoint cycle form.

   (a) $\alpha = (1, 4, 3, 5)(3, 7, 6)(2, 5, 7, 3, 1)(6, 4)(2, 3, 5, 4)(4, 5, 3)$

   (b) $\beta = (1, 2, 3)(1, 4, 5)(1, 6, 7)(1, 8, 9)$

   (c) $\gamma = (9, 3, 5, 6)(4, 5, 2, 3, 7)(3, 7, 8, 2)(1, 4)(7, 4)$

4. **Products and Inverses of permutations.**
   Consider the following permutations in $S_{10}$:

$$\alpha = (1, 5, 2, 7)(3, 4)(8, 10, 9), \quad \beta = (1, 10, 9, 7, 6, 5, 2, 4, 8),$$

$$\gamma = (1, 2, 3, 4)(6, 10, 8, 7, 9), \quad \delta = (1, 5, 8, 4)(2, 9, 10, 7)(3, 6).$$

   Compute the disjoint cycle form of each of the following:

(a) $\alpha\beta$
(b) $\beta\delta$
(c) $\gamma\alpha$
(d) $\delta^4$
(e) $\alpha\gamma\delta$
(f) $\alpha^1$
(g) $\delta^{-1}\beta^{-1}$
(h) $(\alpha\delta)^{-1}$

5. For each of the permutations below, determine its order.

   (a) $\sigma = (3, 7, 4)$

   (b) $\alpha = (1, 5, 8, 4)(2, 9, 10, 7)(3, 6)$

   (c) $\beta = (2, 6, 8, 3, 10, 9, 7, 4)$

   (d) $\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 7 & 1 & 5 & 4 & 3 & 6 \end{pmatrix}$

   (e) $\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 4 & 1 & 5 & 10 & 9 & 7 & 6 & 8 \end{pmatrix}$

6. For each of the permutations below, express the inverse in disjoint cycle form.

   (a) $\alpha = (1, 5, 8, 4)(2, 9, 10, 7)(3, 6)$

   (b) $\beta = (2, 6, 8, 3, 10, 9, 7, 4)$

   (c) $\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 7 & 1 & 5 & 4 & 3 & 6 \end{pmatrix}$

   (d) $\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 4 & 1 & 5 & 10 & 9 & 7 & 6 & 8 \end{pmatrix}$

7. Let $\alpha = (1, 3, 6)(2, 4)$ and $\beta = (1, 4, 5, 2)$. Compute each of the following.
   (a) $\alpha^{-1}$ (b) $\beta^{-1}$ (c) $\alpha\beta$ (d) $\beta\alpha$

8. Let $\alpha = (1, 2)(4, 5)$ and $\beta = (1, 6, 5, 3, 2)$. Compute $\beta^{-1}\alpha\beta$.

9. Show that the order of a $m$-cycle $(a_1, a_2, \ldots, a_m)$ is $m$?

10. What is the order of a pair of disjoint cycles of length $5$ and $3$? $4$ and $6$? $22$ and $18$?

11. What is the order of the product of three disjoint cycles of lengths $3$, $5$, and $7$? $6$, $12$ and $26$?

12. Show $S_5$ contains no element of order $7$?

13. What is the maximum order of any element in $S_{10}$?

14. Let $\alpha, \beta \in S_n$, show that $\alpha$ and $\beta^{-1}\alpha\beta$ have the same order.

15. Let $\beta = (1, 3, 5, 7, 9, 8, 6)(2, 4, 10)$. What is the smallest positive integer $n$ for which $\beta^n = \beta^{-7}$?

16. Let $\alpha = (1, 7, 4, 5, 9)(3, 8)(10, 6, 2)$. If $\alpha^m$ is a $5$-cycle, what can you say about $m$?

17. In $S_3$, find permutations $\alpha$ and $\beta$ so that ord$(\alpha) = 2$, ord$(\beta) = 2$, and ord$(\alpha\beta) = 3$.

18. Find permutations $\alpha$ and $\beta$ so that ord$(\alpha) = 3$, ord$(\beta) = 3$, and ord$(\alpha\beta) = 5$.

19. (a) If $\alpha \in S_n$ has order $k$, show that $\alpha^{-1} = \alpha^{k-1}$.
    (b) Use part (a) to find $\alpha^{11}$ for $\alpha = (1, 3, 6, 2)(4, 7, 5)$.

20. How many permutations of order $5$ are there in $S_6$?

21. Suppose $\alpha$ is a $10$ cycle. For which integers $i$ between $2$ and $10$ is $\alpha^i$ also a $10$-cycle?

22. **Splicing and dicing cycles.**[3] What happens to the cycle structure of a permutation $\alpha$ when you follow $\alpha$ by a transposition? The answer is you either splice two of the cycles of $\alpha$ into one bigger cycle, you cut one of the cycles of $\alpha$ into two smaller cycles, you extend one cycle by on element, or you add a new transposition to the cycle structure. Verify the special cases of this statement below, and then make an argument that the claim follows in general from these special cases.

    (a) If $\alpha = (a_1, a_2, \ldots, a_r)(b_1, b_2, \ldots, b_s)$ where these two cycles are disjoint, then

    $$\alpha(a_1, b_1) = (a_1, \ldots, a_r, b_1, \ldots, b_s).$$

    (b) If $\beta = (a_1, a_2, \ldots, a_r)$ and $1 \leq i < j \leq r$, then

    $$\beta(a_i, a_j) = (a_1, \ldots, a_{i-1}, a_j, a_{j+1}, \ldots, a_r)(a_i, a_{i+1}, \ldots, a_{j-1}).$$

    (c) If $\gamma = (a_1, a_2, \ldots, a_r)$ and $b \neq a_i$ for all $i$, then

    $$\gamma(a_1, b) = (a_1, a_2, \ldots, a_r, b).$$

    (d) If $\delta = (a_1, a_2, \ldots, a_r)$ and if $(b_1, b_2)$ is disjoint from $\delta$, then

    $$\delta(b_1, b_2) = (a_1, a_2, \ldots, a_r)(b_1, b_2).$$

[3] This exercise is from J. Kiltinen's book *Oval Track and Other Permutation Puzzles*.