Lecture 19: Rubik's Cube: Beginnings

Contents

In this lecture, we summarize Rubik's cube terminology and notation that we have been using so far, as well as introduce Singmaster notation for each piece of, and each position on, the cube. There is no "one size fits all" notation when modelling Rubik's cube, we'll see that each notation has its benefits depending on what you are trying to do with it.

19.1 Rubik's Cube terminology and notation

The notation we use was first introduced by David Singmaster in the early 1980's, and is the most popular notation in use today.

19.1.1 Move Notation:

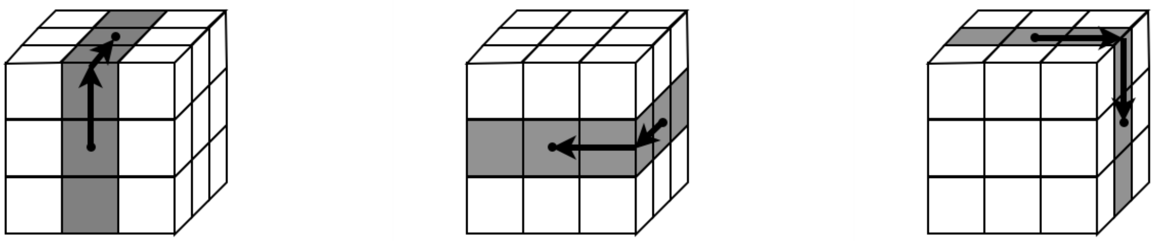
Fix an orientation of the cube in space. We may label the 6 sides as $\boldsymbol{f}$, $\boldsymbol{b}$, $\boldsymbol{r}$, $\boldsymbol{l}$, $\boldsymbol{u}$, $\boldsymbol{d}$ for *front*, *back*, *right*, *left*, *up*, and *down*.

Face moves:
A quarter twist of a face by 90 degrees in the clockwise direction (looking at the face straight on) is denoted by the uppercase letter corresponding to the name of the face. For example, F denote the move which rotates the front face by 90 degrees clockwise. See Table 1 for a complete description of cube moves and notation.

Slice moves:
We also indicate the names of some **slice** moves. These are moves in which one of the three middle slices is rotated. For example, if the slice between the l and r face is rotated upwards, that is, in the clockwise direction when viewed from the right face, then we denote this move by $S\ell_R$. We could also view this move from the left side as a counterclockwise rotation, so we could denote it by $S\ell_L^{-1}$. Similarly, we have slice moves for the slice parallel to the u and d face, and for the slice parallel to the f and b face. These moves are denoted by:

$$S\ell_R = S\ell_L^{-1}, \quad S\ell_U = S\ell_D^{-1}, \quad S\ell_F = S\ell_B^{-1}.$$

We can also square these moves. See Figure 1.



(a) Slice move $S\ell_R$. (b) Slice move $S\ell_U$. (c) Slice move $S\ell_F$.

Figure 1: Three basic *slice moves* of Rubik's Cube.

Whole cube moves:
The whole cube, as a single object, can be rotated in space. For example, we can rotate the cube about an axis through the centres of the left and right faces. If the rotation is in the clockwise direction as viewed from the right face then we denote the move by $\mathcal{R}$. This could also be viewed as a counterclockwise rotation from the left face perspective, so we could also denote it by $\mathcal{L}^{-1}$.

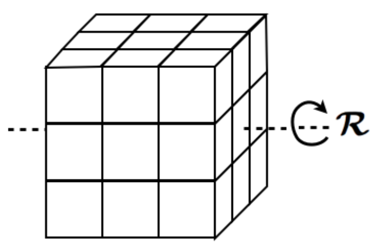


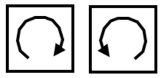
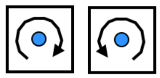
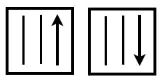
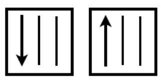
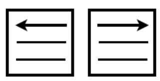
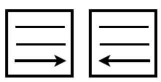
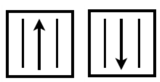
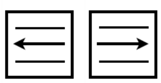
Figure 2: Whole cube rotation $\mathcal{R}$. Also denoted by $\mathcal{L}^{-1}$.

19.1.2 Position and Piece Notation:

The 26 pieces of the cube, called **cubies**, split up into three distinct types: **centre cubies** (having only coloured sticker), **edge cubies** (having two coloured stickers), **corner cubies** (having three coloured stickers).

We call the space which a cubie can occupy a **cubicle**, and we call the space a sticker can occupy a **facet**. We can also describe a facet as the face of a cubicle. As the pieces move around, the cubies move from cubicle to cubicle, and the stickers move from facet to facet. In the 15-puzzle, Oval Track, and Hungarian Rings puzzles, we called the location a piece could occupy a *position* or *spot*, the term *cubicle* is customary to use when talking about the Rubik's cube.

To solve the puzzle each cubie must get restored to its original cubicle, we call this the cubies **home location**, and each sticker must get returned to its original facet (i.e. the facets must also be correctly positioned), we

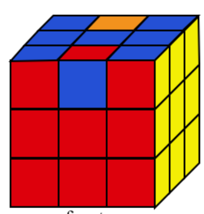
notation (Singmaster)	**pictorial** (view from front)	**description of basic move** (clockwise/counterclockwise refers to viewing the face straight-on)
$F\,,\quad F^{-1}$		F = quarter turn of **front** face in the **clockwise** direction. F^{-1} = quarter turn of **front** face in the **counterclockwise** direction.
$B\,,\quad B^{-1}$		B = quarter turn of **back** face in the **clockwise** direction. B^{-1} = quarter turn of **back** face in the **counterclockwise** direction.
$R\,,\quad R^{-1}$		R = quarter turn of **right** face in the **clockwise** direction. R^{-1} = quarter turn of **right** face in the **counterclockwise** direction.
$L\,,\quad L^{-1}$		L = quarter turn of **left** face in the **clockwise** direction. L^{-1} = quarter turn of **left** face in the **counterclockwise** direction.
$U\,,\quad U^{-1}$		U = quarter turn of **up** face in the **clockwise** direction. U^{-1} = quarter turn of **up** face in the **counterclockwise** direction.
$D\,,\quad D^{-1}$		D = quarter turn of **down** face in the **clockwise** direction. D^{-1} = quarter turn of **down** face in the **counterclockwise** direction.
$S\ell_R\,,\quad S\ell_R^{-1}$		$S\ell_R$ = quarter turn of **vertical** slice in the **clockwise** direction. $S\ell_R^{-1}$ = quarter turn of **vertical** slice in the **counterclockwise** direction.
$S\ell_U\,,\quad S\ell_U^{-1}$		$S\ell_R$ = quarter turn of **horizontal** slice in the **clockwise** direction. $S\ell_R^{-1}$ = quarter turn of **horizontal** slice in the **counterclockwise** direction.

$F^2, B^2, R^2, L^2, U^2, D^2$ denote the corresponding *half-turn* of the face.
Since a clockwise half-turn is equivalent to a counterclockwise half-turn then
$F^2 = F^{-2}, \quad B = B^{-2}, \quad R^2 = R^{-2}, \quad L^2 = L^{-2}, \quad U^2 = U^{-2}, \quad D = D^{-2}$

$\mathcal{F}, \mathcal{B}, \mathcal{R}, \mathcal{L}, \mathcal{U}, \mathcal{D}$ denote clockwise rotations of the whole cube behind the indicated face.

Table 1: Summary of cube move notation

call this the cubies **home orientation**.[1] See Figure 3 for an example of this distinction. Once *all* cubies are in their home locations and home orientations the puzzle will be solved.



front

Figure 3: Cubie UF is in its home location, but not in its home orientation since it is flipped. Similarly for cubie UB.

We will describe a labeling of facets and cubicles below. It is important to keep in mind that facets and cubicles don't move, only the pieces (cubies and stickers) move. So when describing a labeling of the cubies and facets it is best to think of this label as appearing on a fictitious layer of skin surrounding the puzzle. The pieces can move around under the skin but the skin remains in place.

[1]This can also be called its **home position.**

Facet Notation

Figures 4 and 5 shows a labeling of the facets of the cube. This labeling is due to mathematician, and puzzle enthusiast, David Singmaster. Our typical labeling uses numbers (see Lecture 1), but this labeling uses strings of symbols. That advantage to this labeling is that it allows us to easily determine where a facet position is located. For example, thinking back to our numerical labeling, if asked where facet position 41 is, you likely don't know without looking at a diagram. However, with this new labeling, facet 41 is facet dlf, which you know is on the dlf cubicle. As for which of the three sides it is, this is denoted by the first letter in the name: d for *down*. So facet dlf is the *down* side of the dlf cubicle.

If you are wondering how the order of the other two letters were chosen (i.e. why didn't we call it dfl?), the answer is simple: we wrote them in the order the faces appear when moving around the corner in the clockwise direction. You can check all the labelings in Figure 4 to verify this is the convention.

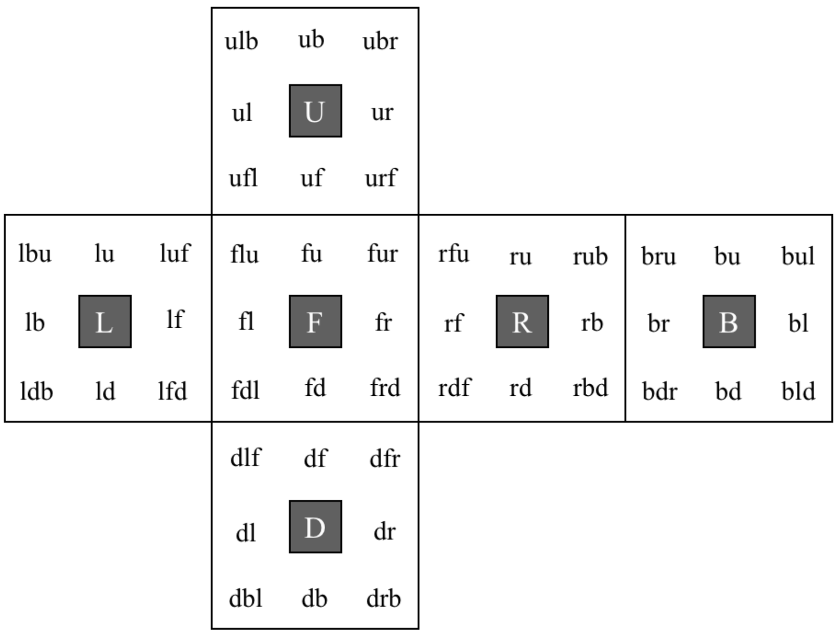


Figure 4: Facet labeling on the $3 \times 3 \times 3$ Rubik's cube.

(a) View of front (red), right (yellow) and up (blue) faces, labelled with Singmaster notation.

(b) View of back (orange), left (white) and down (green) faces, labeled with Singmaster notation.

Figure 5: Rubik's Cube with classic colouring scheme: blue opposite green, red opposite orange, white opposite yellow. Each cubicle is labeled using Singmaster notation.

Cubicle notation:
A cubicle can be identified by faces it touches. For example, the cubicle that touches the *up*, *right* and *front* faces can be denoted by urf. In particular, we can denote a cubicle by the labeling of any of the 3 facets that are on the cubicle, in addition to any of the other three orderings of the letters. For example the *front-up-right* cubicle can be denoted by any one of the 6 symbols: fur, urf, rfu, fru, rfu, or ufr.

Since a corner cubie has three facets we denote it by three letters. Similarly, edge cubies are denoted by two letters. Figures 4 and 5 shows a labelling of all the cubicles (use any one of the facet labelings to denote the cubicle to which the facet belongs).

There is a benefit to labeling cubicles and facets in a similar fashion. For the moment we focus our attention on cubies rather than cubicles/facets. For example consider the move R^{-1}. The cubie in position urf moves to cubicle dfr. However, there are three different ways a corner cubie can be placed in a cubicle, so just stating that urf moves to dfr doesn't indicate how it is oriented once it gets to dfr. Notice that the up face of the cubie is placed in the *front* face when it moves to the new cubicle. Similarly, the *right* face stays on the *right* face. It would be more descriptive to say that R^{-1} takes the cubie in position urf to position frd. We can write

$$ufr \xrightarrow{R^{-1}} fdr.$$

This indicates that cubie in cubicle ufr moves to cubicle fdr, and the stickers moved as follows: u-facet moves to f-facet, f-facet moves to d-facet, and r-facet moves to r-facet.

Cubie notation:
A cubie is identified by its home cubicle. We use capital letters to denote cubies, and lower case to denote cubicles. For example, URD denotes the cubie whose home location is the urd cubicle. It may seem that using the same notation to denote cubies as cube moves is a bad idea, however, we'll see that this doesn't cause any trouble at all. We just need to be aware as to whether we are talking about cube moves, or cube pieces.

Table 2 summarizes the terminology introduced here.

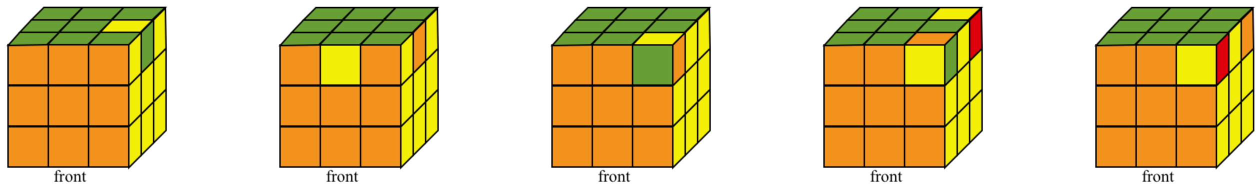
Terminology	**Definition or Abbreviation**
cubies	The small cube pieces which make up the whole cube.
cubicles	The spaces occupied by the cubies.
facets	The faces of a cubicle
types of cubies: **corner**, **edge**, and **centre**:	A corner cubie has three facets. An edge cubie has two facets. A centre cubie has one facet
home location - of a cubie	The cubicle to which a cubie should be restored.
home orientation - of a cubie	The orientation in the home location to which a cubie should be restored.
positional names for cube faces	Up (u) Down (d) Right (r) Left (l) Front (f) Back (b)
Notation for cubicles - shown in *italics*	Lower case initials. For example, *uf* denotes the Up-Front edge cubicle, *dbl* denotes the Down-Back-Left cubicle.
Notation for cubies - shown in *italics*	Upper case initials. For example, *URF* denotes cubie whose home position is in the the Up-Right-Front corner

Table 2: Summary of terminology and notation

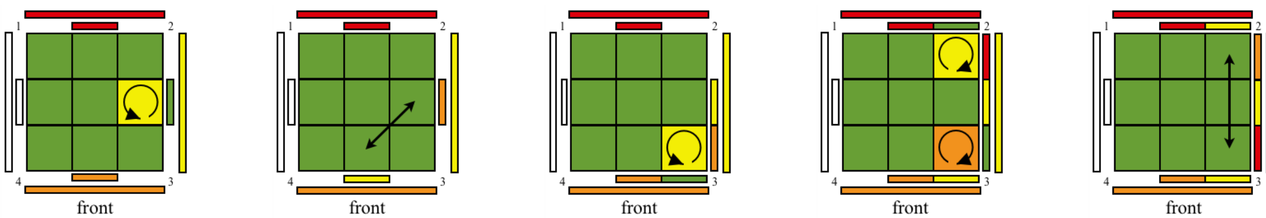
With all this notation now in our tool box, we are ready to investigate Rubik's cube.

19.2 Impossible Moves

Through previous investigations we've found that there are some moves that are impossible to do on the cube. Figure 6 shows five moves that are impossible. This will be helpful when coming up with a strategy to solve the cube since knowing what is impossible to do, will prevent us from going on a search we would never come back from.



(a) single edge flip (b) edge swap (c) single corner twist (d) double corner twist in same direction (e) corner swap



(f) single edge flip (g) edge swap (h) single corner twist (i) double corner twist in same direction (j) corner swap

Figure 6: Five different moves that are impossible to perform. The image in the bottom row is a face-on perspective of the top face of the corresponding cube in the top row. The thin rectangular boxes on the sides indicate the colour of the side facets, and the long rectangular box indicates the side face colour.

We've given permutation-parity arguments to show why it is impossible to (i) flip an edge, (ii) swap two edges, and (iii) swap two corners. The impossibility of the corner twist configurations were investigated using SAGE. In a later lecture we will come back to these configurations and give mathematical proofs that they are indeed impossible. Therefore confirming the computations done by SAGE. Since we were relying on group theoretic algorithms in SAGE, that we don't know/understand, providing an independent proof will provide us will some closure on this topic.

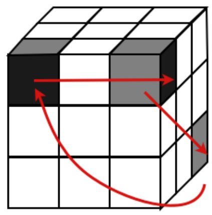
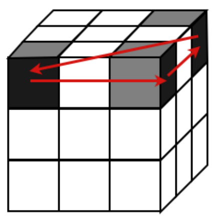
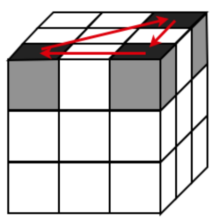
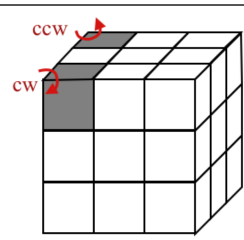
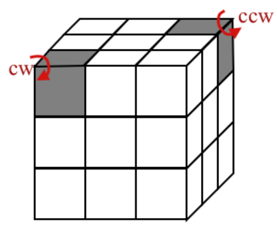
19.3 A Catalog of Useful Move Sequences

Over the previous few lectures we have built some useful moves using commutators. These were move sequences that affected only a few pieces, while returning everything other piece to the position it started. Using conjugation we are able to modify these move sequences to produce other useful moves of the same form. Below is a list of the moves we've created for convenient reference.

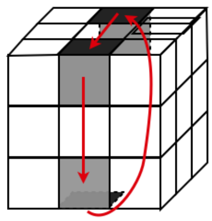
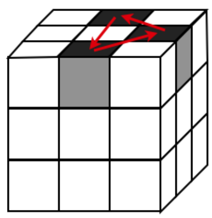
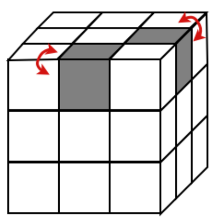
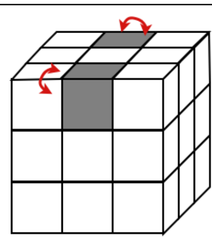
Notice that for each type of cubie (corners and edges) we can (i) 3-cycle any three cubies of the same type, and (b) twist/flip a pair of cubies of the same type. Knowledge of these moves is enough to solve the cube: first place cubies in their home locations (using 3-cycles), then orient the cubies in their home orientation (using twist/flip moves).

Reminder: $[x, y] = xyx^{-1}y^{-1}$ is the *commutator* of x and y and $y^{-1}xy$ is the *conjugate* of x by y. In the following tables, the move labeled C/E# is created using commutators, and the corresponding move denoted by C/E#' is the conjugate of it by the indicated move sequence y.

19.3.1 Corner Moves

name	effect	move-sequence
C1		$[LD^2L^{-1},U]$ $= LD^2L^{-1}ULD^2L^{-1}U^{-1}$
C1'		conjugate C1 by $y = B$: $B^{-1}[LD^2L^{-1},U]B$ $= B^{-1}LD^2L^{-1}ULD^2L^{-1}U^{-1}B$
C2		$[F^{-1}D^{-1}FR^{-1}D^2RF^{-1}DF,U]$ $= F^{-1}D^{-1}FR^{-1}D^2RF^{-1}DFUF^{-1}D^{-1}FR^{-1}D^2RF^{-1}DFU^{-1}$
C3		$[L^{-1}D^2LBD^2B^{-1},U]$ $= L^{-1}D^2LBD^2B^{-1}UBD^2B^{-1}L^{-1}D^2LU^{-1}$
C3'		conjugate C3 by $y = B^{-1}$: $B[L^{-1}D^2LBD^2B^{-1},U]B^{-1}$ $= BL^{-1}D^2LBD^2B^{-1}UBD^2B^{-1}L^{-1}D^2LU^{-1}B^{-1}$

19.3.2 Edge Moves

name	effect	move-sequence
E1		$[S\ell_R, U^2]$ $= S\ell_R U^2 S\ell_R^{-1} U^2$
E1'		conjugate E1 by $y = DR^2$: $R^2 D^{-1}[S\ell_R, U^2]DR^2$
E2		$[S\ell_R^{-1} D S\ell_R D^{-1} S\ell_R^{-1} D^2 S\ell_R, U]$
E2'		conjugate E2 by by $y = B^{-1}R^{-1}$: $RB[S\ell_R^{-1} D S\ell_R D^{-1} S\ell_R^{-1} D^2 S\ell_R, U]B^{-1}R^{-1}$

19.4 Strategy for Solution

Our primary goal is in understanding the cube. With that goal in mind we should come away with a strategy for solving the cube. We will not find an optimal strategy, nor will will look for a large collection of moves to tackle all sorts of configurations. Instead, we will be content with a method that systematically solves the cube and uses the tools we have developed in this course. Ideally the method should not involve lots of memorization, but should rely on a solid understanding of the mathematics of permutations (i.e. commutators and conjugates).

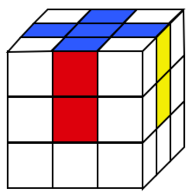
If you haven't already tried to use the moves listed in Section 19.3 to find a strategy yourself, try it now. The fun of discovering a solution on your own may be lost if you read the strategy described below.

More efficient methods than the ones described here, all of which require memorization, are left for the reader to find. A simple google search can keep you busy for weeks.

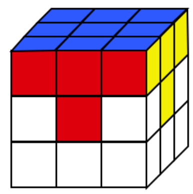
19.4.1 The Layer Method

The method we will use to solve the cube is known as the *layer method*. We begin by solving the top layer, followed by the middle layer, and finally the bottom layer. A sketch of the steps involved in implementing this strategy are shown in Figure 7.

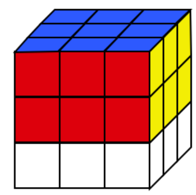
You may begin by solving any colour, and it is best to choose a colour that stands out to you from the rest. This way it is easy to find the pieces on the scrambled cube. In these notes we'll begin by solving the blue layer, in which case the bottom layer will be green.



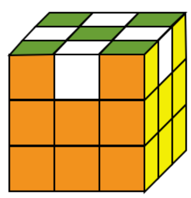
(a) **Step 1**: Solve edges in top layer.



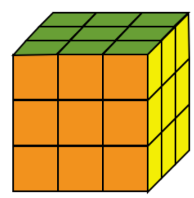
(b) **Step 2**: Solve corners in top layer.



(c) **Step 3**: Solve edges in middle layer



(d) **Step 4**: Flip over, solve remaining corners (first permute then orient).



(e) **Step 5**: Solve remaining edges (first permute then orient).

Figure 7: The Five-Step strategy for solution.

Solving the top and middle layers are pretty straightforward. You should be able to do this with a little practice and using general heuristics. A theory based strategy won't be needed until the end-game, which is when we reach the bottom layer.

19.4.2 Solving the Top Layer

Solving the top layer is a straightforward task. You can send pieces to the bottom layer, then bring them back to the top layer, to achieve desired twists. That is, make use of conjugation.

Step 1: Solve the edge cubies in the top layer.

Keep in mind that centres remain fixed, so there is only one proper home orientation for an edge cube. Use the centres as a guide. This is indicated in Figure 7a where the centres are shown, and the facets of the edge cubies must match the centres.

Step 2: Solve the corner cubies in the top layer.

Let α be any of the moves R, L, F, B. This will bring one corner cubie into the down layer. Rotating the down layer will then bring a new cubie into the cubicle whose contents are moved back up to the top layer by α^{-1}. In other words, $\alpha D \alpha^{-1}$ allows you to change a corner cubie in the top layer without affecting any other cubies in the top layer. This should help you finish the top layer completely.

19.4.3 Solving the Middle Layer

Step 3: Solve the edge cubies in the middle layer.

We could modify some of the move sequences in Section 19.3 to solve edges in the middle layer. However, this may be overkill, since at this stage there is plenty of "wiggle room" in the down layer so we should be able to find a general heuristic that works. Try to find one yourself.

One method that is pretty straightforward is described here.

If the cubie that is to be placed in the middle layer is currently in the bottom layer then rotate the bottom layer so one facet of the edge cubie is directly beneath the centre cubie of the same colour. For example, see Figure 8 where the cubie to be moved in the middle layer has a red facet on the side layer, so it is placed directly under the red center cubie. Whatever the colour of your cubie is, rotate the entire cube so the cubie is now in the fd cubicle, and the colour of the facet in the f face matches the centre cubie of the f face right above it.

Depending on whether the cubie is to be moved to the right of the left we can apply one of the two sequences:

$$\begin{aligned} \text{right:} \quad & [D^{-1}, R^{-1}][D, F] = D^{-1}R^{-1}DRDFD^{-1}F^{-1} \\ \text{left:} \quad & [D, L][D^{-1}, F^{-1}] = DLD^{-1}L^{-1}D^{-1}F^{-1}DF \end{aligned}$$

Notice that in either case the move sequence is a product of Y and Z commutators (see Lecture 13 for a discussion of these commutators).

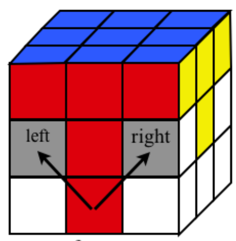


Figure 8: Moving an edge piece into the middle layer. To move right apply $[D^{-1}, R][D, F]$, to move left apply $[D, L][D^{-1}, F^{-1}]$.

19.4.4 Solving the Bottom Layer

We now have one layer left to solve. This is the end-game of Rubik's cube since it is here where things get a bit more difficult. Trying to place the remaining few pieces while leaving previously places pieces alone requires a collection of strategic moves: ones that move only a few pieces at a time. Luckily, the theory of commutators and conjugates has provided us with such moves (see Section 19.3).

Flip the cube over, so the bottom layer is now the top layer. This will allow us to see everything we need to solve.

Step 4: Solve the remaining corner cubies.

We'll do this in two steps:

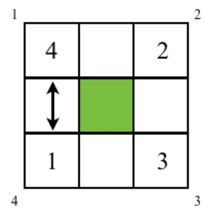
Step 4a: Place the remaining corner cubies in their home locations. Don't worry about twisting them into their home orientations just yet.

Look at the facets of each of the remaining corner cubies. The colours that appear will tell you exactly where its home location is. For example, the corner cubie with green, white and red facets belongs to the location which is the intersection of the green, white and red faces. Recall the colour of a face is given by the colour of the centre cubie.

Now that you know where each corner cubie must be moved, see if a simple rotation of the up face will restore all corners to their proper locations. If not, then we are in one of the following cases:

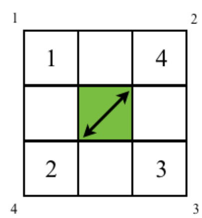
Case 1: It is possible to put exactly one corner cubie in its correct location, and have the other 3 out of position. Use the 3-cycle move sequence C1' in Section 19.3, or its inverse, to move the remaining 3 corner cubies into their correct positions.

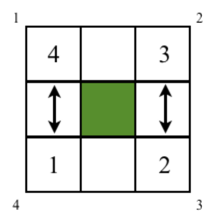
For example, if we need to swap two corner cubies as in the following diagram



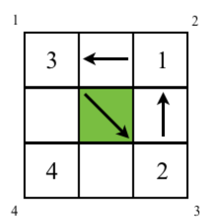
then we can first rotate the face so 1 is home, and 2, 3, 4 are out of position, then we just need to perform a 3-cycle $(2,4,3)$.

Case 2: Up to a physical rotation of the whole cube, we are in either one of the two following positions.:





The first case can be taken to the second case by rotating the face counterclockwise $90°$. So assume we are in the second case. Apply C1' to produce the 3-cycle $(1,4,2)$, which produces the following position.



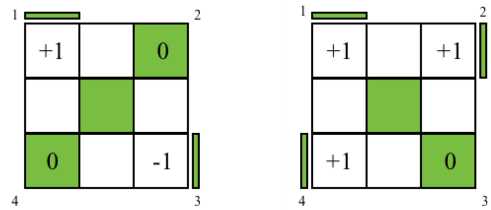
Now use C1' to produce a 3-cycle $(1,3,2)$.

Therefore, to restore the corner cubies to their correct locations at most two 3-cycles need to be applied.

Step 4b: Orient (twist) the remaining corner cubies into their home orientations.

Repeated applications of C3 will be enough to orient the corners. (Note, we already used SAGE to discover it is impossible to have exactly two corners twisted in the same direction.)

For example, denoting a corner cubie that must be rotated clockwise to be restored by $+1$, and one that must be rotated clockwise by -1, here are a couple of possible scenarios that we could be faced with:



In the first case, applying C3' will solve the corners. In the second case, we can apply C3 on corners 1 and 4 to solve corner 4, and take corner 1 to -1. Then applying C3 to corners 1 and 2 will solve the remaining two corners. Other scenarios are possible and can be dealt with similarly.

Step 5: Solve the remaining edge cubies.

We'll do this in two steps:

Step 5a: Place the remaining edge cubies in their home locations. Don't worry about flipping them into their home positions just yet.

Much like the corners, we can use 3-cycles E1' to restore all the edge cubies.

Step 5b: Orient (flip) the remaining edge cubies into their home orientations.

Using E2 and E2' we can flip any pair of edges to restore to their home orientation.

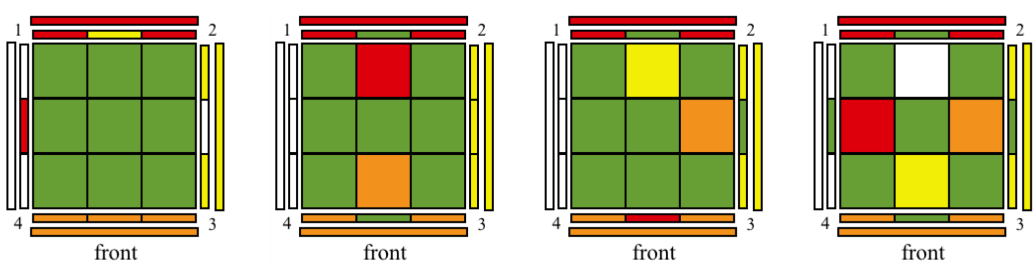
Note, it is impossible to have a single edge flipped as we've already discovered. Therefore, flipped edges occur in pairs and so E2, E2' are the only moves we will need.

Congratulations! Not only have we solved the cube, we built the moves to do it from scratch! Behold the power of the theory of permutations.

19.5 Exercises

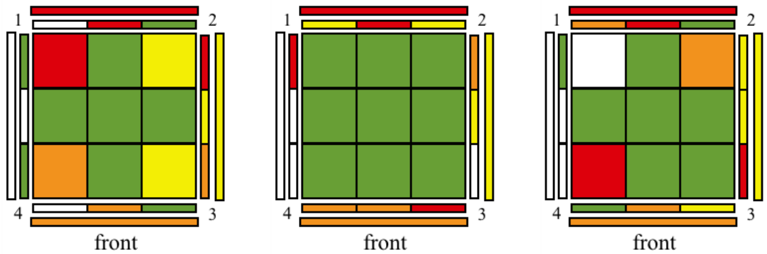
1. Practice solving the first two layers of your cube. Repeatedly scramble and solve until you are confident you can easily solve the first two layers.

2. **Practice with Step 5: solving edges in final layer.** In each part below, a configuration of the last layer is shown. The only pieces out of place are the indicated edge pieces. All other non-visible cubies are in their home orientations. Write down a strategy to solve the puzzle.



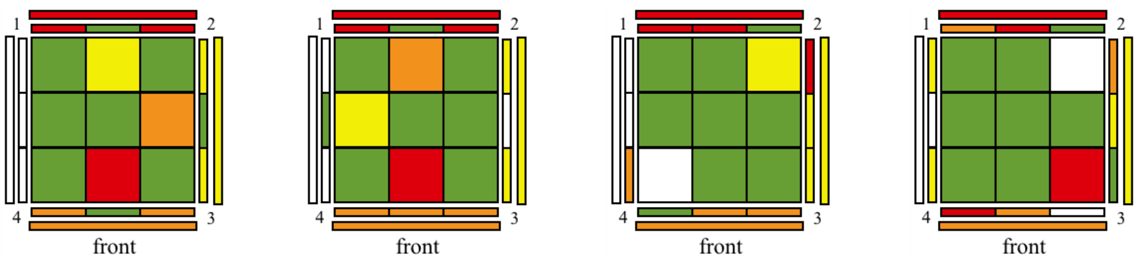
(a) (b) (c) (d)

3. **Practice with Step 4: solving corners in final layer.** In each part below, a configuration of the last layer is shown. The only pieces out of position are the indicated corner pieces. All other non-visible cubies are in their home orientations. Write down a strategy to solve the puzzle.



(a) (b) (c)

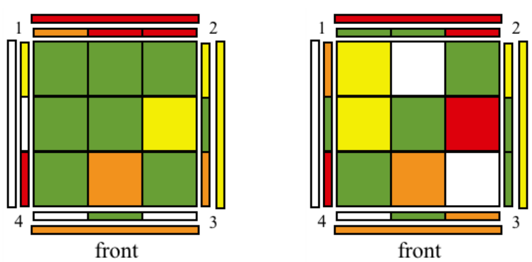
4. **Impossible Configurations.** In each part below, a configuration of the last layer is shown. All non-visible cubies are in their home orientations. Show that each configuration is impossible.



(a) (b) (c) (d)

(Hint: Try showing the configuration is equivalent to one shown in Section 19.2.)

5. **Practice with Steps 4 and 5: solving corners and edges in final layer.** In each part below, a configuration of the last layer is shown. Some edge and corner pieces are out of position. All other non-visible cubies are in their home orientations. Write down a strategy to solve the puzzle.



(a) (b)