Lecture 9:
Mastering the 15-Puzzle

Contents

We now have enough theory developed to give a full analysis of the 15-puzzle. We will present a solvability criteria which will allow us to easily see whether a given scrambling of the puzzle is solvable. We will also sketch a strategy for solving the puzzle.

This lecture corresponds to Section 7.4 in Joyner's text.

9.1 Solvability Criteria

Determining the solvability of a scrambling of the tiles on the 15-puzzle is a simple task as we will see. Let's first consider the case where a scrambling places the empty space back into its original box (box 16). This means the corresponding permutation α fixes 16: $\alpha(16) = 16$. We can think of such a permutation as an element of S_{15}. (Just think about the disjoint cycle form, 16 doesn't appear since it is mapped back to itself.)

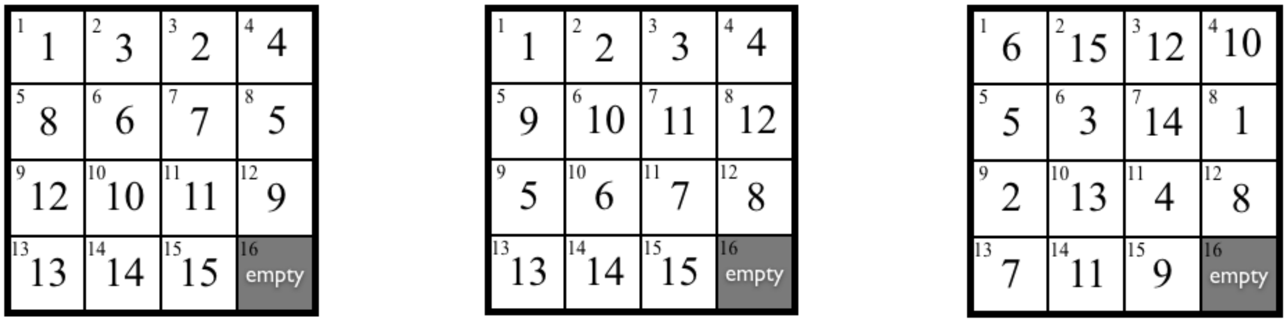
Figure 1 shows three different configurations of the 15-puzzle corresponding to permutations in S_{15}. The permutations are written below each puzzle. We'd like to be able to quickly determine which configurations are solvable.

The next theorem says any rearrangement of tiles in the 15-puzzle starting from the solved-state configuration which brings the empty space back to its original position must be an even permutation of the other 15 pieces. Moreover, it says that *every* even permutation of the 15 tiles can be obtained as a position on the 15-puzzle.

Theorem 9.1 (Solvability Criteria for 15-Puzzle - Part 1) *A permutation α of the 15-puzzle which fixes 16, is solvable if and only if it is even: i.e. $\alpha \in A_{15}$.*

It follows that the number of solvable positions of the 15-Puzzle, where the empty space is in its home position, is

$$|A_{15}| = \frac{15!}{2} = 653,837,184,000.$$



(2,3)(5,8)(6,7)(9,12)(10,11) (a)

(5,9)(6,10)(7,11)(8,12) (b)

(1,8,12,3,6)(2,9,15)(4,11,14,7,13,10) (c)

Figure 1: Which of the positions are solvable?

We immediately conclude from this theorem that the puzzles in Figures 1a and 1c are not solvable since the permutations are odd, whereas the puzzle in Figure 1b is solvable since the permutation is even.

We will provide a proof of this theorem in Section 9.2.

What about the case when the scrambling does not place the empty space back in box 16? We'll see that simply knowing the parity of the permutation and the position of the empty space is enough to determine solvability. But first it will be handy to talk about the *parity of a box*.

Colour the 15-puzzle like a checker board as in Figure 2. We will call the shaded boxes *even* and the white boxes *odd*. Under this definition boxes $1, 3, 6, 8, 9, 11, 14, 16$ are even, whereas the other boxes are odd.

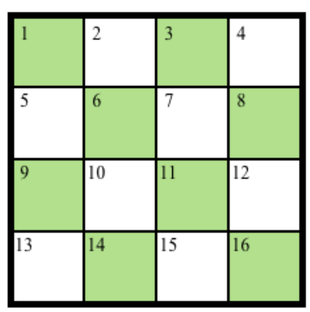


Figure 2: **Parity of box:** Define the shaded boxes to be *even* and the white boxes to be *odd*.

With this concept of odd and even boxes now defined, we can state the general solvability criteria for the 15-puzzle.

Theorem 9.2 (Solvability Criteria for 15-Puzzle - Part 2) *A permutation of the 15-puzzle is solvable if and only if the parity of the permutation is the same as the parity of the location of the empty space.*

When the empty space is in one particular box, there are

$$|A_{15}| = \frac{15!}{2} = 653,837,184,000$$

possible positions of the tiles. Since there are 16 different places to put the empty space, there is a total of

$$16\left(\frac{15!}{2}\right) = \frac{16!}{2} = 10,461,394,944,000$$

possible ways to rearrange the tiles on the board so that the puzzle is solvable. This means, of all $16!$ ways to arrange the tiles in the boxes, exactly half are solvable!

When the puzzle craze hit the world in the early 1880's people noticed that when they randomly placed the tiles in the box, the puzzle was solvable roughly half the time. This now explains why!

As an example, the permutation corresponding to the scrambling in Figure 3 is

$$(1,10,11,7,6)(2,3,4,8,12,16,5)(13,15)$$

which is odd (check this yourself), and the parity of the location of the empty space is odd, therefore the puzzle is solvable by the solvability criteria: Theorem 9.2.

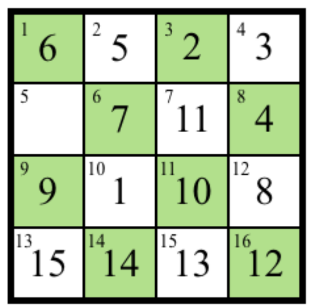


Figure 3: Is this position solvable?

9.2 Proof of Solvability Criteria

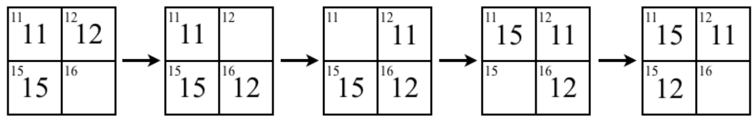
We will prove Theorem 9.1 and then show that Theorem 9.2 is a direct consequence of it.

Proof of Theorem 9.1: There are two directions we need to prove: (i) a solvable configuration is an even permutation, and (ii) every even permutation is a solvable configuration. The proof of (i) is very straightforward, but the proof of (ii) requires us to use the fact that even permutations can be expressed using 3-cycles.

(i) Suppose we have a solvable rearrangement of the 15 tiles, where the empty space is in its home position (box 16). Let $\alpha \in S_{15}$ be the corresponding permutation. Since puzzle moves consist of transpositions - the empty space is swapped with an adjacent tile - then let $\tau_1, \tau_2, \ldots, \tau_k$ be the moves (i.e. transpositions) which solve the puzzle (i.e. takes α to the identity permutation ε). As usual, this means $\alpha = \tau_k \ldots \tau_2\tau_1$. Since the empty space moves around the puzzle and then eventually returns home, the number of moves must be even. To see why this is true, refer to Figure 2, the empty space must start in shaded box 16, and after each move it alternates the colour of the box it is in, and so if it returns to a shaded box it must have moved an even number of times. This means k is even, and so α is expressible as a product of an even number of transpositions. Therefore α is even.

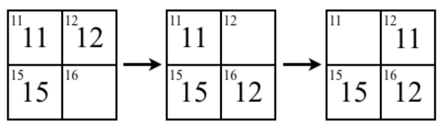
(ii) We wish to show *every* even permutation of the 15 tiles is obtainable through puzzle moves, starting from the solved-state. We will do this by showing we can obtain *any* 3-cycle of the tiles. This is enough to prove the theorem since any even permutation is expressible as a product of 3-cycles, and if we can produce any 3-cycle then we can produce any product of them, through sequential moves, and therefore we can produce any even permutation.

We begin by observing we can produce the 3-cycle $\sigma = (11, 12, 15)$, by focussing on the bottom right corner of the puzzle:

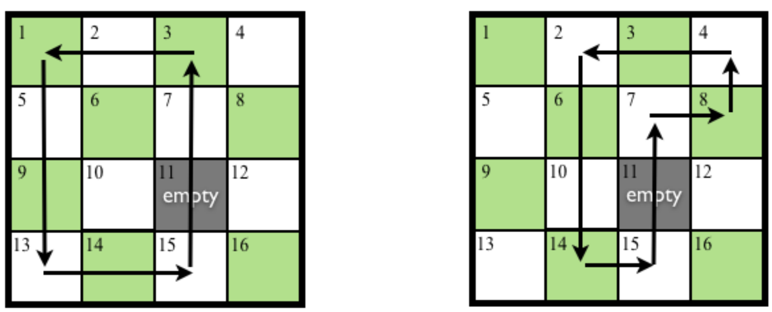


The sequence of moves is: $(12, 16)(11, 12)(11, 15)(15, 16)$

Now that we have one 3-cycle σ, we will show that we can use σ construct any other 3-cycle we want. From a solved puzzle, pick any tile, say $i \in \mathbb{Z}_{15}$. Move tiles 12 and 11 to boxes 16 and 12, respectively, by the move sequence $\alpha = (12, 16)(11, 12)$:.



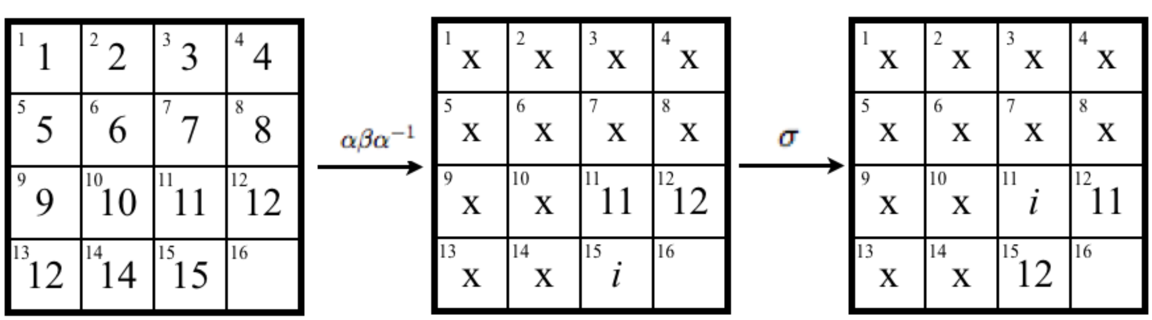
Then using one of the two tours in Figure 4 we can move tile i to box 15, without disturbing the contents of boxes 12 and 16. Call this move sequence β.



(a) (b)

Figure 4: Tours for producing 3-cycles.

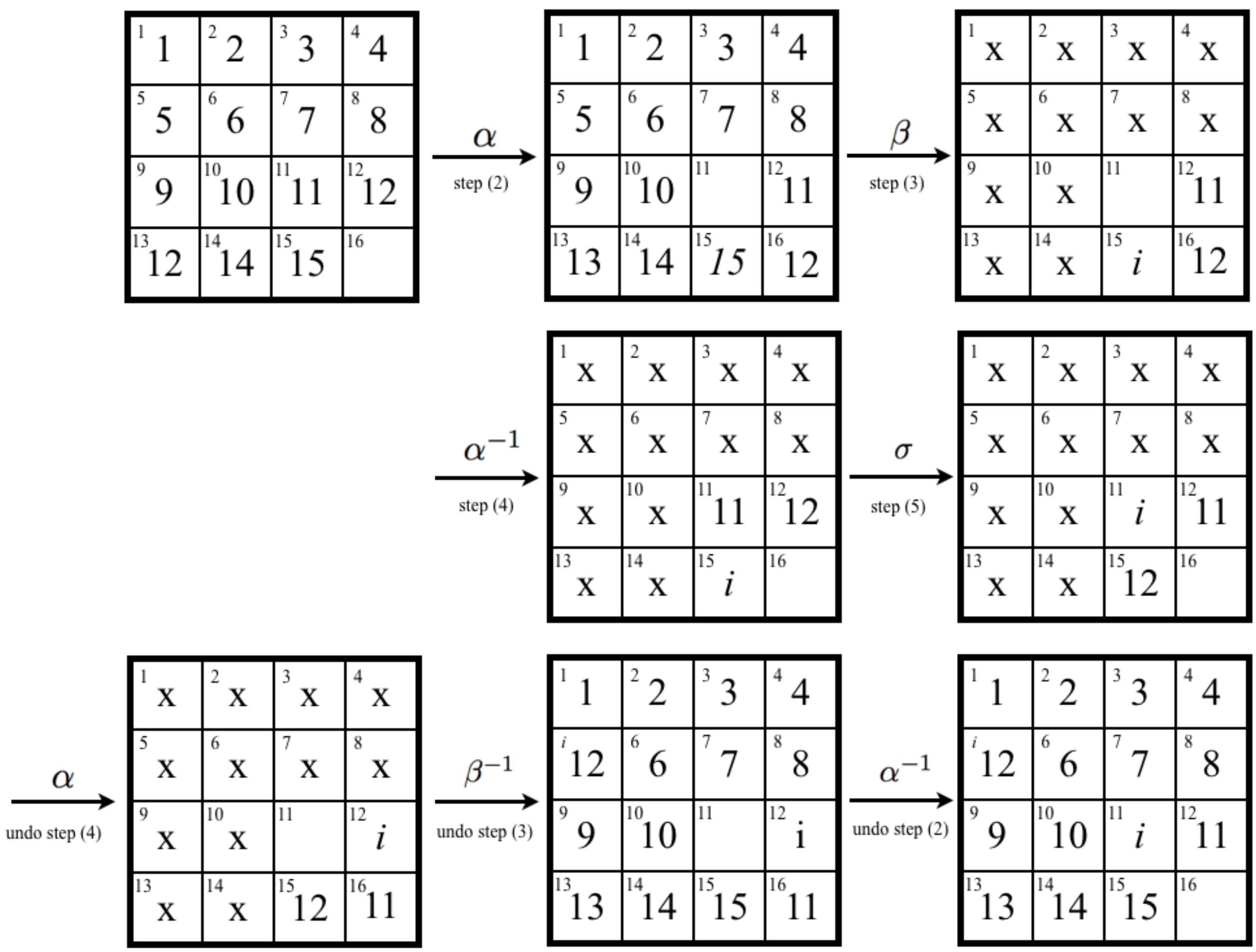
Applying α^{-1} then moves 11 and 12 back into their home positions. This puts the puzzle in the middle position in the following diagram, where the x's indicate these numbers may have moved around.



Applying the 3-cycle σ then moves tiles i, 11, and 12 around as indicated in the diagram. Now, we apply the inverse move sequence $(\alpha\beta\alpha^{-1})^{-1} = \alpha\beta^{-1}\alpha^{-1}$ and this takes everything back to where it was, except i stays in box 11, 11 stays in box 12, and 12 goes to box i. Therefore, we have created the 3-cycle $(11, 12, i)$, where i is any other tile we wish.

Let's summarize what we did:
(1) we chose some tile i,
(2) temporarily hide tiles 11 and 12 in boxes 12 and 16,
(3) used one of the tours in Figure 4 to bring tile i to box 15, and this didn't disturb tiles 11 and 12 hidden in boxes 12 and 16
(4) moved 11 and 12 back out to their original positions
(5) applied the 3-cycle $\sigma = (11, 12, 15)$
(6) then reversed all the steps (4) to (2), thus taking everything home except 11,12, and i have been cycled.

Here is an example (near the end we think of i as being 5 for concreteness).



So we now can construct any 3-cycle of the form $(11, 12, i)$, for any $i \neq 11, 12$.

Since $(11, 12, k)(11, 12, j) = (11, j)(12, k)$ we are able to put any tiles (j and k) in boxes 11 and 12, while leaving everything else in place. Moreover,

$$(11, j)(12, k)(11, 12, i)(11, j)(12, k) = (i, j, k),$$

where $i \neq j \neq k$ and $i, j, k \notin \{11, 12\}$. Therefore, we can produce any possible 3-cycle.

This completes the proof. □

The proof of the general solvability condition is a simple consequence of this specific case.

Proof of Theorem 9.2: Let α be the current permutation of the 15-puzzle. Move the empty space to box 16, then the new arrangement corresponds to the permutation

$$\alpha^* = \alpha\tau_1\tau_2\cdots\tau_k$$

where $\tau_1, \tau_2, \ldots, \tau_k$ were the transpositions used to move the empty space to box 16. Since the empty space is now in box 16 then, by Theorem 9.1, α^* is solvable if and only if it is an even permutation.

Let's think about when α^* is even. This really follows from the way we defined the parities of the boxes.

If the empty space was in an odd box, then it would have taken an odd number of transpositions to move it to box 16. That is, k would be odd. On the other hand, if the empty space was in an even box then k would be even, since it would have taken an even number of transpositions to move it to box 16. In either case, k is equal to the parity of the box the empty space was in. This is precisely the reason we defined the parity of a box as we did.

Now, putting it all together, α is solvable if and only if $\alpha^* = \alpha\tau_1\tau_2\cdots\tau_k$ is even, which is equivalent to α and k having the same parity, which is equivalent to α and the location of the empty space having the same parity. This completes the proof. □

9.3 Strategy for Solution

Of course, in proving Theorem 9.1 we've essentially presented a strategy for solution. First move the empty space into box 16, then the resulting permutation is even, so we may express it as a product of 3-cycles. In practice, our typical method for doing this is first to express it as a product of transpositions, then group pairs of transpositions and express them as 3-cycle or pairs of 3-cycles. We can now use the technique outlined in Section 9.2 to produce each 3-cycle, one-by-one, by moving the desired tiles into the $11, 12, 15$ boxes, performing the 3-cycle $(11, 12, 15)$, then moving everything back again.

Though theoretically possible, and a perfectly sound way to prove the theorem, this makes for a completely inelegant way to solve the puzzle. Not to mention you would need to remember, or write down, the move sequence $\alpha\beta\alpha^{-1}$ since you would need to apply the inverse. This move sequence could be very long. Instead we'll look for a more efficient solution, and one that doesn't require remembering any previously made moves.

Some hints to get you started:

Hint 1: Solve the puzzle by setting the tiles in their proper places, one-by-one, in numerical order. At some stages, it may be necessary to temporarily disturb placed pieces, but they shouldn't have too move to far out of place.

If you haven't tried this already, do so now.

Hint 2: There are some tricky parts. For instance if $1, 2, 3$ are all in place, but 4 is not, it will be necessary to disturb the previously placed pieces in order to get 4 in its proper place. Instead, before placing 3, join it with 4 to form a chain and bring the two of them into place together. Forming chains of tiles is a useful strategy.

If you haven't tried this already, do so now.

Hint 3: Getting the final few pieces in the proper places is of course tricky. But at this stage, making use of 3-cycles, as in the proof above, may be useful. After all, if you can use mathematics to shed some light on what to do, then do it!

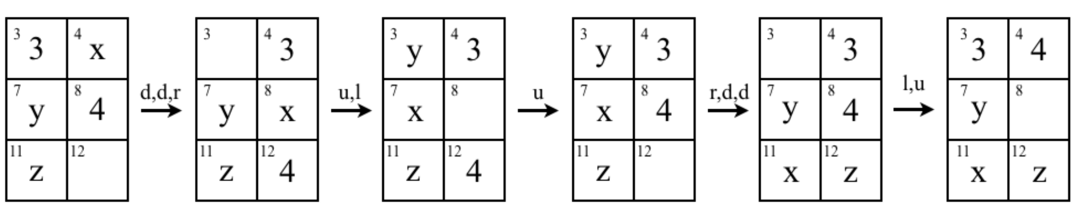
Most of all, just have some fun. Try some strategies of your own, if they are useful, then write them down so you won't forget them.

SPOILER ALERT: We now present a complete method for solving the 15-Puzzle, read only if you want to spoil the fun of discovering a solution yourself.

The following solution is due to Jaap Scherphuis (http://www.jaapsch.net/puzzles/). It is not an optimal solution, that is, it won't allow you to solve the puzzle in the minimum number of moves, but it does give a method that works on any solvable configuration, and it extends to puzzles of sizes other than 4-by-4. Using this method, with a smooth puzzle, or better yet a virtual version, solutions can take between 1 to 2 minutes, possibly faster.

Phase 1: Solve the top row from left to right.

1. Find the next tile you want to place in position in the top row.
2. If it is not the last tile of the row, it is fairly easy to place correctly, simply keep the following points in mind:
   (a) Never disturb any previously placed pieces.
   (b) To move the tile in a certain direction, move the other tiles around until the space is next to your tile on the side you want to move it to. Then you can move the tile.
3. If the last tile is not already in position, bring it to the position directly below its correct spot, with the space directly below that. Then move tiles in the following directions: *down, down, right, up, left, up, right, down, down, left, up*. This should place the piece in position. Note it does temporarily disturb the previously placed tile. See figure below.

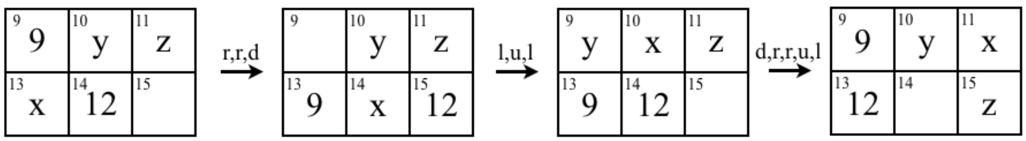


Phase 2: Solve the rest of the puzzle

1. Use the technique in phase 1 to solve each row in turn, until there are only two rows left.

2. Rotate the puzzle a quarter turn to the right. The left column of the two rows becomes the top row now.

3. Use the technique in phase 1 to solve each row in turn, until there are only two rows left. This means there is only a 2x2 square left to solve. For example, the next figure show how to get tile 12 in the correct place in the bottom left corner of the 4-by-4 version.

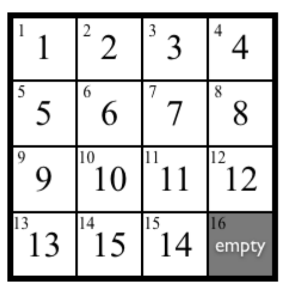


4. Move the pieces in the remaining 2x2 square around until one piece is positioned correctly, and the space is in the correct spot. The other two tiles should automatically be correctly positioned as well.

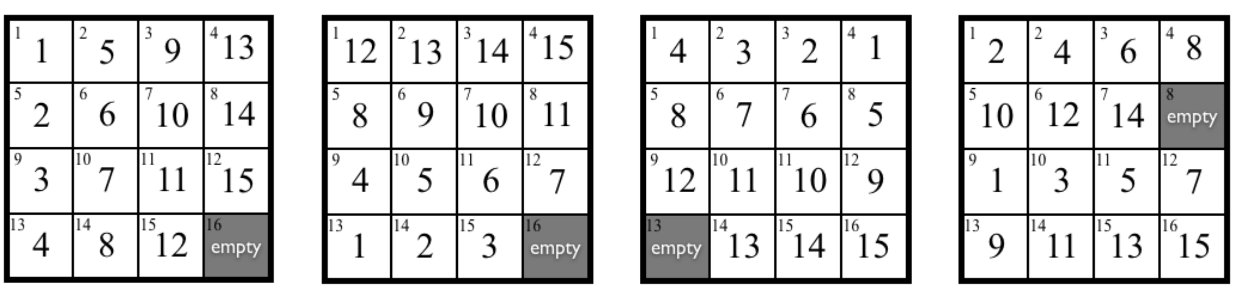
5. If there are two tiles that need to be swapped, then this cannot be done unless two other tiles are swapped as well. If there are two identical tiles somewhere in the puzzle, then you will have to swap them and solve the rest again. (This may happen if there are letters or pictures on the tiles instead of numbers.)

9.4 Exercises

1. In the early 1880's the world went crazy over trying to solve configuration of the 15-puzzle where the 14 and 15 were swapped. See the Figure below. Explain why no one was able to find a solution.



2. Show that each of the following scramblings of the 15-puzzle are solvable.



(a) (b) (c) (d)

3. Show that each of the following scramblings of the 15-puzzle are unsolvable.

(a)

^{1} 15	^{2} 14	^{3} 13	^{4} 12
^{5} 11	^{6} 10	^{7} 9	^{8} 8
^{9} 7	^{10} 6	^{11} 5	^{12} 4
^{13} 3	^{14} 2	^{15} 1	^{16} empty

(b)

^{1} 1	^{2} 8	^{3} 9	^{4} empty
^{5} 2	^{6} 7	^{7} 10	^{8} 15
^{9} 3	^{10} 6	^{11} 11	^{12} 14
^{13} 4	^{14} 5	^{15} 12	^{16} 13

(c)

^{1} 1	^{2} 2	^{3} 4	^{4} 7
^{5} 3	^{6} 5	^{7} 8	^{8} 11
^{9} 6	^{10} 9	^{11} 12	^{12} 14
^{13} 10	^{14} 13	^{15} 15	^{16} empty

(d)

^{1} 4	^{2} 3	^{3} 2	^{4} 1
^{5} 5	^{6} 14	^{7} 13	^{8} 12
^{9} 6	^{10} 15	^{11} empty	^{12} 11
^{13} 7	^{14} 8	^{15} 9	^{16} 10

4. Determine which of the following arrangements of the 15-puzzle are solvable and which are unsolvable.

(a)

^{1} 1	^{2} 2	^{3} 4	^{4} 3
^{5} empty	^{6} 9	^{7} 11	^{8} 6
^{9} 14	^{10} 13	^{11} 15	^{12} 8
^{13} 5	^{14} 12	^{15} 10	^{16} 7

(b)

^{1} 10	^{2} 9	^{3} 8	^{4} 7
^{5} 11	^{6} empty	^{7} 15	^{8} 6
^{9} 12	^{10} 13	^{11} 14	^{12} 5
^{13} 1	^{14} 2	^{15} 3	^{16} 4

(c)

^{1} empty	^{2} 1	^{3} 2	^{4} 3
^{5} 4	^{6} 5	^{7} 6	^{8} 7
^{9} 8	^{10} 9	^{11} 10	^{12} 11
^{13} 12	^{14} 13	^{15} 14	^{16} 15

(d)

^{1} 1	^{2} 2	^{3} 3	^{4} 4
^{5} 12	^{6} 13	^{7} 14	^{8} 15
^{9} 11	^{10} empty	^{11} 15	^{12} 6
^{13} 10	^{14} 9	^{15} 8	^{16} 7

5. This exercise is to help us understand the details of the proof in Section 9.2, and to get some practice with creating 3-cycles. Starting with the puzzle in the solved state write down a sequence of moves which will produce each of the 3-cycles:

   (a) $(11, 12, 13)$ (b) $(11, 12, 8)$ (c) $(11, 8, 13)$.

   Either write the moves using transpositions, or use the words “up”, “down”, “left”, “right”, to indicate the direction the tile adjacent to the empty space is moved. It may help to use a physical or virtual version of the puzzle. See the “software” section of the course webpage for links to virtual versions of the puzzle.

6. A 15-puzzle manufacturer wants to sell the puzzle with the tiles already mixed-up, and they want the pattern to be “pretty” so it catches the eye of the customer when sitting on a store shelf. This manufactured version of the puzzle does not allow the pieces to be removed, so the pattern needs to be solvable. They propose to use a pattern where all then even numbered tiles are in the first two rows, and the odd numbered tiles in the last two rows (see the Figure below). They also colour all the even tiles red, so that in the solved state the puzzle will have a pattern of vertical lines. If they manufacture the puzzle in this way, will it be solvable? Or will this result in angry customers wanting to return their puzzles?

[1] 2	[2] 4	[3] 6	[4] 8
[5] 10	[6] 12	[7] 14	[8] empty
[9] 1	[10] 3	[11] 5	[12] 7
[13] 9	[14] 11	[15] 13	[16] 15

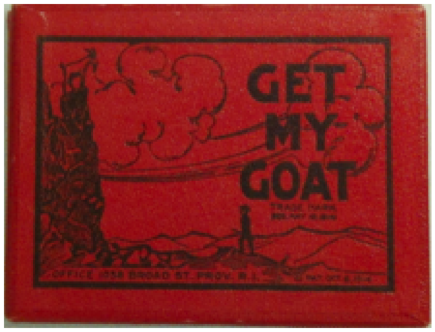
7. In 1959, the Plas-Trix Company in the USA produced a letter version of the 15-puzzle. The problem is to rearrange the blocks so they correctly spell RATE YOUR MIND PAL. They manufactured and sold the puzzle with the last two tiles switched. See the figure below. Explain why it is possible to solve this puzzle.
(Hint: At first glance it seems this is analogous to the 15-14 problem in Exercise 1, in which case it is not solvable. But this is not entirely equivalent, and the subtle differences are what allows this puzzle to be solved. Can you spot the reason this puzzle is solvable?)

R	A	T	E
Y	O	U	R
M	I	N	D
P	L	A	empty

8. The *Panama Canal Puzzle* dates back to 1915. The staring position has the red letter "P" and the black letter "C" swapped. The problem is to swap the two blocks back. Explain why it is possible to solve this puzzle.

9. The *Get My Goat Puzzle* was patented in 1914. The problem is to get the goat inside the fenced-in area, after removing the marked block. This basically requires a swap of the block with the picture of the goat's head and the block adjacent to it. Explain why this puzzle is solvable.

You can play an online version of this puzzle at "Nick Baxter's sliding puzzle page". See the link on our course website.



Conjugation:

Exercises 10 through 12 introduce the idea of *conjugation*.
First a definition:

If $\alpha, \beta \in S_n$, we call the permutation $\beta^{-1}\alpha\beta$ the **conjugate** of α by β.

Looking back at the proof of Theorem 9.1 we transformed the 3-cycle $\sigma = (11, 12, 15)$ into another 3-cycle $(11, 12, i)$ by:

$$\gamma\sigma\gamma^{-1} = (11, 12, i),$$

where $\gamma = \alpha\beta\alpha^{-1}$ was a sequence of moves that moved tile i to box 15, and left tiles 11 and 12 alone. We used conjugation twice in the proof: $\alpha\beta\alpha^{-1}$ and $\gamma\sigma\gamma^{-1}$. These types of products are used extensively when solving permutation puzzles. If you have some experience with permutation puzzles you will notice you frequently make moves of the form:

- do a move m_1,
- then do another move m_2,
- then undo the first move m_1^{-1}.

If you notice you do this, then you already have a working feel for conjugation. In the next few exercises we investigate conjugation, and show that $\beta^{-1}\alpha\beta$ and α have the same cycle structure. This general result is the reason why $\gamma\sigma\gamma$ is a 3-cycle.

10. For each of the following pairs of permutations $\alpha, \beta \in S_n$ calculate the conjugate of α by β. In other words, compute the product $\beta^{-1}\alpha\beta$.

    (a) $\alpha = (1, 2, 3, 4, 5)$, $\beta = (1, 5, 8)(2, 6)(3, 7, 4)$
    (b) $\alpha = (1, 5, 8)(2, 6)(3, 7, 4)$, $\beta = (1, 2, 3, 4, 5)$
    (c) $\alpha = (5, 7, 3, 6)(10, 11, 8, 12)$, $\beta = (1, 2)(4, 10, 5, 11, 7, 9, 12)$

    In each case notice, the cycle structure of $\beta^{-1}\alpha\beta$ is the same as α. For instance in (b), α is a product of two 2-cycles and one 3-cycle, and so is $\beta^{-1}\alpha\beta$.

11. For $\alpha = (1, 2, 3, 4)$ and $\beta = (1, 4)(3, 5, 2)$ do the following.

    (a) Calculate $\beta^{-1}\alpha\beta$.
    (b) Calculate the values of $\beta(1)$, $\beta(2)$, $\beta(3)$, $\beta(4)$, then write down the 4-cycle, $(\beta(1), \beta(2), \beta(3), \beta(4))$.
    (c) Observe that the 4-cycle in part (b) is the same as the answer to part (a). Coincidence? The next exercise says, this is no coincidence.

12. Show that for any $\alpha, \beta \in S_n$ the conjugate $\beta^{-1}\alpha\beta$ has the same cycle structure as α.

    Hint: express α in disjoint cycle form $\sigma_1\sigma_2\cdots\sigma_k$, where σ_i is a m_i-cycle, for $1 \leq i \leq k$. Then show

    $$\text{(i)} \quad \beta^{-1}\alpha\beta = (\beta^{-1}\sigma_1\beta)(\beta^{-1}\sigma_2\beta)\cdots(\beta^{-1}\sigma_k\beta).$$

    Then it suffices to only consider the case when α is a cycle. Which means, you just need to prove:

    $$\text{(ii)} \quad \beta^{-1}(a_1, a_2, \ldots, a_m)\beta = (\beta(a_1), \beta(a_2), \ldots, \beta(a_m)).$$

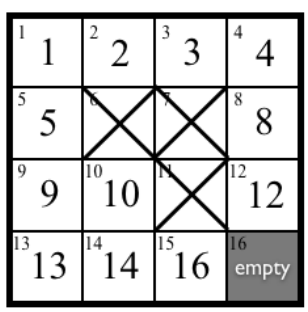
Other Board Sizes and Obstacles:
In Exercises 13 to 15 we investigate board sizes other than 4×4.

13. Consider 5 tiles and an empty space on a board consisting of 3 rows and 2 columns. Show, by using a similar argument to the one used for the 15 puzzle that a permutation $\alpha \in S_5$, where the empty space is in its home location, corresponds to a solvable configuration if and only if α is an even permutation.

    Hint: By using a two-by-two square of four boxes, show that a single 3-cycle can be obtained. Then show every 3-cycle can be obtained by conjugation, similar to the argument we used for the 15 puzzle..

14. The following board shows a variation of the 15 puzzle where boxes 6,7, and 11 are obstacles. That is, these boxes are "out-of-play" and cannot be used. We can still ask the question as to which permutations of the tiles are solvable. Show that, just like the original 15 puzzle, Theorems 9.1 and 9.2 remain true.

    Hint: For simplicity just focus on permutations leaving the empty space in its home location. Use the two-by-two square of four boxes to generate all 3-cycles: first show you can obtain one 3-cycle, then use conjugation to obtain all others.



15. **[Challenging]** The following is a general characterization of the solvability condition for rectangular boards, with obstacles. Verify it is true.

    Let α denote an arbitrary permutation of tiles on a rectangular $m \times n$ board such that the board

    (a) has one empty space,
    (b) has at least one two-by-two array of boxes all of which are in use, and
    (c) may have some obstacles (boxes that are out-of-play and cannot be used), but these obstacle do not trap tiles (in other words, any tile can be moved to any other location).

Then the permutation α is solvable if and only if the parity of α is the same as the parity of the location of the empty space.