Lecture 26:
The Best Card Trick

Contents

You are about to pull off the most amazing card trick... The audience draws 5 cards from the deck and gives them to the volunteer who, in turn, shows you four of the five cards, say:

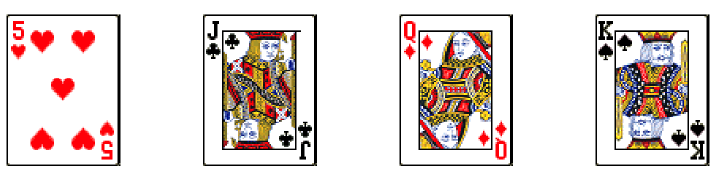
$$4\spadesuit, K\diamondsuit, 4\heartsuit, 5\clubsuit.$$

After a dramatic moment of "reading" the volunteers mind, and the minds of the audience members, you reveal that the fifth card, the one that was hidden from you, must be the $9\spadesuit$*. The audience gasps... How could this be?*[1]

26.1 How to Perform the *Best Card Trick*

You could come up with some story about how the first four cards were shown to you in order to allow you to tune into your volunteers mind, thus allowing you to read the fifth card. Or explain that you are having trouble reading his/her mind so you need the entire audience to concentrate on the fifth card. Of course all this is all just for show. The fact is, your volunteer is actually your accomplice and he/she passed you enough information to determine the final card. There is no element of chance here, your accomplice simply "encoded" the value of the last card in the first four cards. This begs the question: How did they do this?

Let's first describe the standard deck of playing cards. There are four *suits*: **clubs** $\clubsuit$, **diamonds** $\diamondsuit$, **hearts** $\heartsuit$, and **spades** $\spadesuit$. Within each suit there are 13 ranks: (**A**)ce, 2, 3, 4, 5, 6, 7, 8, 9, 10, (**J**)ack, (**Q**)ueen, (**K**)ing. It will be convenient for us to think of Ace $= 1$, Jack $= 11$, Queen $= 12$, and King $= 13$. In card jargon we are considering "Aces to be low", by which we mean the Ace is the lowest ranking card. There are $4 \cdot 13 = 52$ cards in all. See Figure 1 for some examples.



(a) 5 of Hearts. (b) Jack of clubs. (c) Queen of Diamonds. (d) King of Spades.

Figure 1: Examples of names of playing cards.

[1] See original article by Michael Kleber in Mathematical Intelligencer 24 #1, 2002

Why are five cards necessary for the trick? Given a collection of five cards from the deck there must be two of the same suit (this is known in mathematics as the *pigeonhole principle*). Let A and B be two cards of the same suit. One of these cards will be taken as the fifth card, i.e. the hidden card. The other card will be shown first, thus communicating the suit of the hidden card. All that remains now is to communicate the rank of the hidden card.

Once the suit is known there are 12 possibilities for the rank (since the hidden card is certainly not the one that was just shown to you). How can the remaining 3 cards be arranged to communicate the rank? Since there are 6 ways to arrange 3 objects we can communicate a number from 1 to 6 with the remaining 3 cards. This doesn't seem to be enough information.

Does it matter which of the two cards, A and B, of the same suit, is shown and which is hidden? Yes, it does (to do the trick in the way we describe). Let's define the *distance*, dist, between two cards of the same suit using the following diagram:

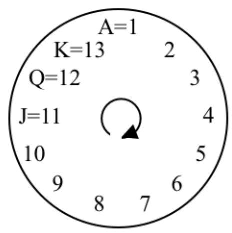


Figure 2: The distance between card ranks is the number of steps it takes to move from one rank to the next in the clockwise direction.

For example, $\text{dist}(3,6) = 3$, $\text{dist}(8, J) = 3$, $\text{dist}(Q, 5) = 6$, $\text{dist}(A, 8) = 7$. Notice that $\text{dist}(6, 3) = 10$, which is not the same as $\text{dist}(3, 6) = 3$. So the distance function is not symmetric in its arguments. However, we do have $\text{dist}(a, b) = 13 - dist(b, a)$, for any two ranks a, b. It follows that for any two ranks a, b (i.e. two cards of the same suit) that either $\text{dist}(a, b)$ or $\text{dist}(b, a)$ is less than or equal to 6. If $\text{dist}(a, b) \leq 6$ then we say a is *smaller than* b.

So, back to the two cards A, B. We will show the performer the smaller of the two cards and hide the larger. The larger card (i.e. the hidden card) will be at distance at most 6 from the smaller card and we can use the remaining 3 cards to communicate this distance.

All that remains now is to assign numbers 1 through 6 to arrangements of three objects. We will use the playing cards natural ordering (from smallest to largest):

$$A♣ < A♦ < A♥ < A♠ < 2♣ < 2♦ < 2♥ < 2♠ < 3♣ < 3♦ < 3♥ < 3♠ < 4♣ < 4♦ < 4♥ < 4♠ < \dots$$

$$\dots < 10♣ < 10♦ < 10♥ < 10♠ < J♣ < J♦ < J♥ < J♠ < Q♣ < Q♦ < Q♥ < Q♠ < K♣ < K♦ < K♥ < K♠.$$

That is, cards are first ordered by rank: $A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K,$ and ties are broken using the suit ordering: ♣ < ♦ < ♥ < ♠ (i.e. alphabetic in first letter of name: (C)lub, (D)iamond, (H)eart, (S)pade). Given three cards, say a, b, c, where $a < b < c$, we can list the six arrangements lexicographically (dictionary order) and assign numbers as follows:

$$abc = 1$$
$$acb = 2$$
$$bac = 3$$
$$bca = 4$$
$$cab = 5$$
$$cba = 6.$$

For example, the three cards 4♦, 5♣, K♦, have the following arrangements numbers (since $4\heartsuit < 5\clubsuit < K\diamondsuit$).

$$\begin{array}{lll} 4\heartsuit\ 5\clubsuit\ K\diamondsuit = 1, & 4\heartsuit\ K\diamondsuit\ 5\spadesuit = 2, & 5\spadesuit\ 4\heartsuit\ K\diamondsuit = 3, \\ 5\clubsuit\ K\diamondsuit\ 4\heartsuit = 4, & K\diamondsuit\ 4\heartsuit\ 5\clubsuit = 5, & K\diamondsuit\ 5\clubsuit\ 4\heartsuit = 6. \end{array}$$

Since the three cards had distinct ranks we don't need to look at the suit to break ties, since there will be no ties to break. For an example with ties see Exercise 2(c).

Now we can lay out the procedure for performing the trick. We use the term *Accomplice* to refer to the one who knows all five cards, and *Performer* for the one attempting to guess the hidden card.

Procedure:

Accomplice:

(1) The audience select five cards at random. Let s_1, s_2, r_1, r_2, r_3 be the five cards drawn from the deck, where s_1 and s_2 have the same suit.

(2) Picks two of the same suit: s_1, s_2. (Note: there could be more than two cards of the same suit, just pick any two for s_1 and s_2.)

(3) Picks one as the hidden card: after re-labeling if necessary assume that s_1 is smaller than s_2 (i.e. dist$(s_1, s_2) \le 6$). The hidden card will then be s_2.

(4) Arrange the remaining 3 cards r_1, r_2, r_3 to correspond to the number dist(s_1, s_2).

(5) Reveal cards one at a time. Reveal card s_1 first, then reveal the remaining cards in the order found in Step (4).

Performer:

(1) Determines the hidden card: The first card s_1 gives the suit of the hidden card and a place to start counting (namely its rank). Determine the number (between 1 and 6) to which the arrangement of the last 3 cards corresponds and add this to the rank of s_1, thus determining the hidden card s_2.

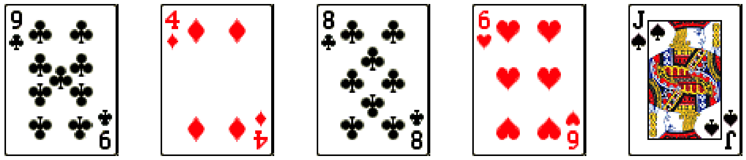
(2) Reveals the hidden card and waits for applause.

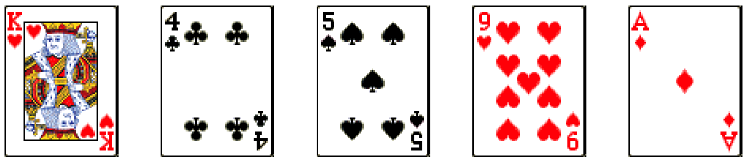
The Accomplice needs to be able to think fast since they need to arrange the cards rather quickly in order to start revealing them one at a time. The Performer, however, can stall for time by drawing large copies of the cards on the blackboard as they are revealed. This extra time will allow the Performer to work out the number corresponding to the arrangement of the final three cards.

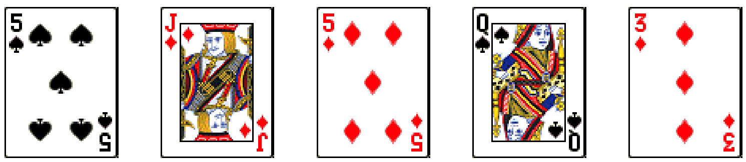
Warning: If the trick is performed a few times as outlined above the audience will pick up, rather quickly, that the first card is the same suit as the hidden one. So you may want to mix up the position of the suit card when performing the trick. It has been suggested to play the suit card in position $i \pmod 4$ when performing for the i^{th} time.

26.2 Exercises

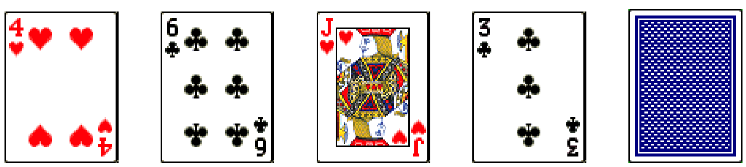
1. In each scenario below, five cards, which the audience has drawn, are given. Pick a card to hide and find an arrangement of the remaining four cards which determines the hidden card.

(a)


(b)


(c)


2. Your accomplice has presented you with the following arrangements of four cards. Determine the fifth card. You may assume the first card is the suit card.

(a)


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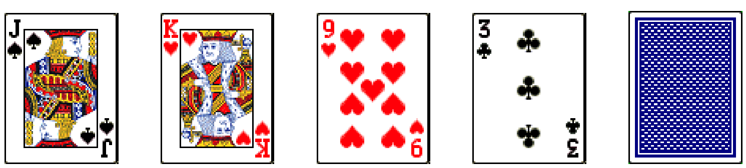

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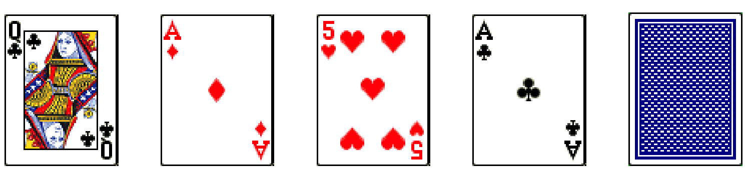
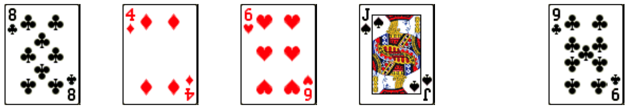
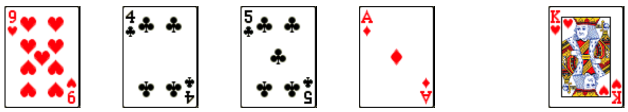


Solutions to Exercises:

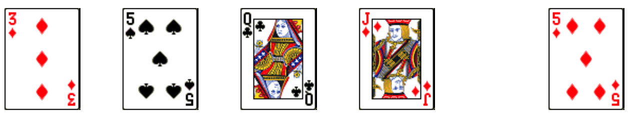
1. (a) Hide the 9♣ and present the arrangement:



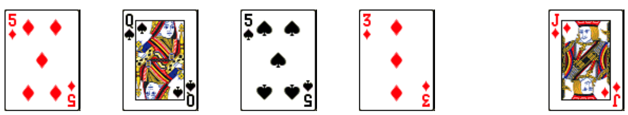
(b) Hide the K♥ and present the arrangement:



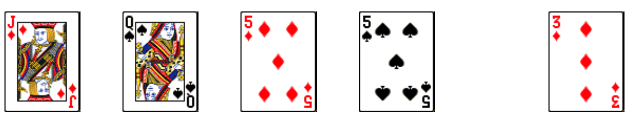
(c) There are three possible choices here:
(i) hide the 5♦ and present:



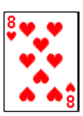
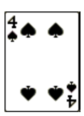
(ii) hide the J♦ and present:



(i) hide the 3♦ and present:



In this last case we need to use the suit ordering to break the tie between the two 5's: recall ♦ $<$ ♠.

2. (a)  (b)  (c)