Lecture 16: Mastering the Hungarian Rings Puzzle

Contents

In this lecture we give a thorough analysis of the Hungarian Rings puzzle, both the coloured and the numbered versions.

The interested reader may wish to see the book *Oval Track and other Permutation Puzzles* by J.O. Kiltinen for further reading. A link in the software section of our website will take you to his page where you can download a demo version of the Hungarian Rings puzzle - both numbered and coloured versions.

16.1 Hungarian Rings - Numbered version

In this section we focus on the Hungarian Rings puzzle as shown in Figure 1. It seems reasonable that the numbered version is more difficult to solve than the coloured version. This is because in the coloured version has only 4 distinct disks, but the numbered version has 38 distinct disks. Even though it is more difficult we will start with the numbered version. In Section 16.4 we will apply our new-found knowledge to the coloured version and describe a simple and elegant solution.

The two basic moves of the Hungarian Rings puzzle are L, and R, where L denotes a clockwise rotation of disks around left ring, where each disk moves one space, and R denotes a clockwise rotation of numbers around the right ring.

The permutation corresponding to the legal moves R and L are as follows:

$$L = (1, 20, 19, 18, 17, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2)$$
$$R = (1, 38, 37, 36, 35, 6, 34, 33, 32, 31, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21)$$

and the Hungarian Rings puzzle group is $HR = \langle L, R\rangle$.

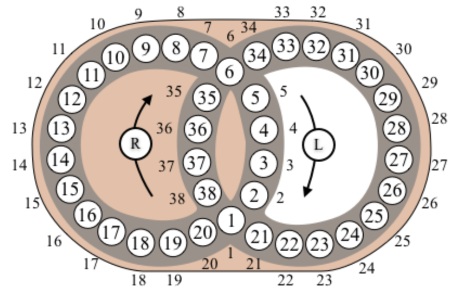


Figure 1: Hungarian Rings puzzle - numbered version.

Note that L^{-1} and R^{-1} represents a counterclockwise rotation of the disks around the respective rings.

Let's get right down to business and find out which permutations of the 38 disks are possible. We can set-up the corresponding puzzle group HR in SAGE and compute its order. Since the maximum possible number of permutation is 38! we'll ask if the order of HR is this value.

SAGE

```
sage: S38=SymmetricGroup(38)
sage: L=S38("(1,20,19,18,17,16,15,14,13,12,11,10,9,8,7,6,5,4,3,2)")
sage: R=S38("(1,38,37,36,35,6,34,33,32,31,30,29,28,27,26,25,24,23,22,21)")
sage: HR=S38.subgroup([L,R])
sage: HR.order()==factorial(38)
True
```

Therefore, *all* possible permutations of the puzzle pieces are possible. We could have instead asked if HR is the symmetric group S_{38} to achieve the same result.

SAGE

```
sage: HR==SymmetricGroup(38)
True
```

Theorem 16.1 (Solvability Criteria for Hungarian Rings puzzle) *For the Hungarian Rings puzzle every permutation of the* 38 *pieces is possible. In other words,* $HR = S_{38}$.

Much like the Oval Track puzzle we can see theoretically why $HR = S_{38}$. In Lecture 14 we saw that we could produce a 3-cycle as a compound conjugate:

$$[[L^5, R^5], R^{-1}LR] = (6, 11, 12).$$

There is enough "wiggle room" on the puzzle to bring any three disks into spots $6, 11, 12$, so we can perform any 3-cycle by conjugating this one. Therefore, we can perform any even permutation of the puzzle pieces. The move L is a 20-cycle, which is odd. This means HR contains A_{38} and at least one odd permutation. Therefore it must contain all of S_{38}. This is similar to the argument used to show $OT = S_n$ when the number of disks $n \geq 6$ is even.

Knowing that all permutations in S_{38} are obtainable is a start, but we actually would like to know how to solve the puzzle from any arrangement of the disks. As with the Oval Track puzzle, moving the first few disks home is straightforward, it is the end-game where we need theory-based strategies.

16.1.1 Start-game: Solve the first 20 disks

There is enough flexibility in the puzzle to solve disks 7 through 16 on the left ring, and disks 21 through 30 on the right ring. You may be able to get a couple more disks in place, such as 17 and 31.

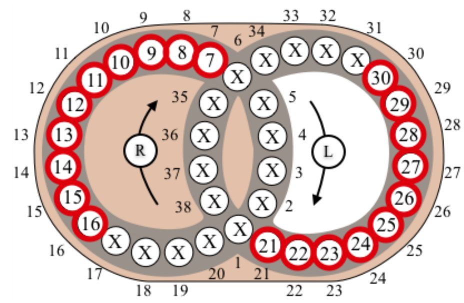


Figure 2: Start-game: begin by putting disks $7-16$ and $21-30$ in place.

Using general heuristics you may be able to get a few more disks into place. Once you are at the point where you think general heuristics cannot take you any further you are at the end-game. You will probably have 20 to 23 disks in their proper position. This leaves 15 to 18 disks we still need to solve.

16.1.2 End-game: A strategy

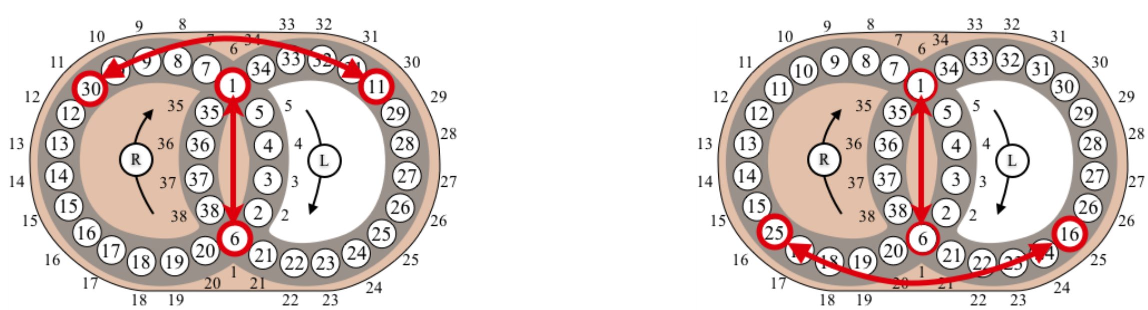
This puzzle has a rather large end-game, as compared to the Oval Track puzzle. Once we have made it to this point we express the remaining permutation in disjoint cycle form. It will possibly involve 2-cycles, 3-cycles, 4-cycles, 5-cycles, and perhaps longer cycles.

If we know some fundamental cycles of these lengths then we have a strategy to solve: just solve one cycle at a time. In the next section we go about building fundamental cycles.

Before doing this. let's recall the fundamental commutators (see Figure 3):

$$[L^5, R^{-5}] = (1,6)(11,30), \quad \text{and} \quad [L^{-5}, R^5] = (1,6)(16,25).$$

These will come in handy in the following sections. We'll come back to the end game strategy in Section 16.3.



(a) $L^5R^{-5}L^{-5}R^5 = (1,6)(11,30)$ (b) $L^{-5}R^5L^5R^{-5} = (1,6)(16,25)$

Figure 3: Basic commutators on the Hungarian Rings puzzle

16.2 Building Small Cycles: Tools for Our End-Game Toolbox

16.2.1 5-cycles

Starting with the intersection spots 1 and 6, there is a collection of 6 spots that are nicely spaced around the puzzle: each one five away from the next one. The locations of these spots are $1, 6, 11, 16, 24, 30$. With this observation, there is a nice 8-move sequence to creates a 5-cycle:

$$\sigma_5 = (L^5R^5)^4 = (1, 25, 30, 11, 16).$$

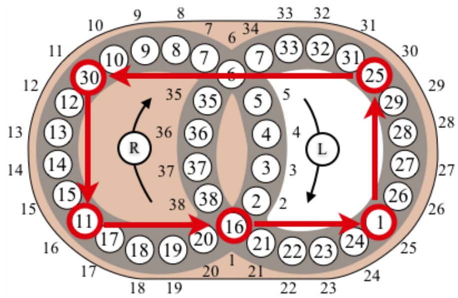


Figure 4: Fundamental 5-cycle: $\sigma_5 = (L^5R^5)^4 = (1, 25, 30, 11, 16)$.

Conjugation of this 5-cycle will come in useful when we need to deal with long cycles in the end-game.

16.2.2 4-cycles

Think back to the Oval Track puzzle and how we produced a 2-cycle. We had to send every disk through the turntable. It was theoretically impossible to produce a 2-cycle without doing this. The reason, as we discussed in Lecture 14, was that if one or more disks never passed through the turntable then it would be possible to do the same thing on the puzzle with 21 disks. But this puzzle doesn't have a 2-cycle since the basic moves are even. A similar argument would show that *every* odd permutation on the Oval Track puzzle must come from a sequence of moves that push every piece through the turntable.

There is a similar theoretical result for the Hungarian Rings puzzle.

Theorem 16.2 *On the Hungarian Rings puzzle, suppose there is a sequence of moves that produces an odd permutation β, which returns at least one disk on each ring to its home position. Then during the process, each piece r on the right ring where $\beta(r) = r$ must have been temporarily moved to the left ring, or each piece ℓ on the left ring where $\beta(\ell) = \ell$ must have been temporarily moved to the right ring. In other words, every piece on one of the rings that β keeps at home would have been temporarily sent to the other ring.*

To see why this is true, suppose β is an odd permutation that keeps a disk r on the right ring at home and keeps a disk ℓ on the left ring at home (i.e. $\beta(r) = r$ and $\beta(\ell) = \ell$), and suppose during the entire process it keeps r on the right ring, and ℓ on the left ring. Without loss of generality, we can assume $\beta = L^{m_1}R^{k_1}L^{m_2}R^{k_2}\cdots L^{m_\ell}R^{k_\ell}$, for integers m_i and k_j, in which some could be 0.

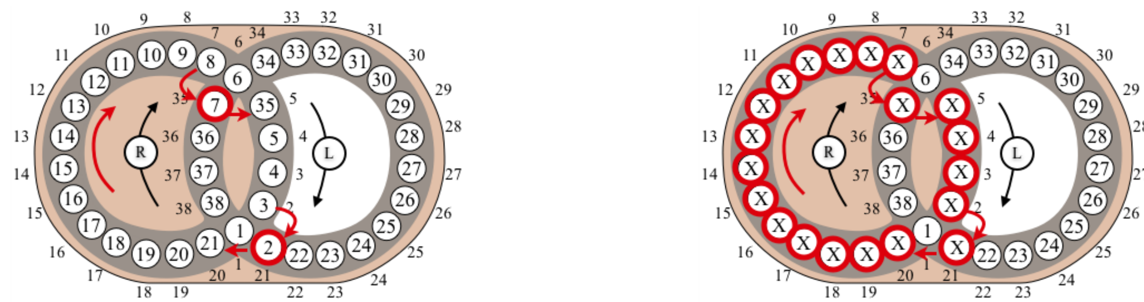
Since r never moves to the left ring, the only moves that affect it are the moves R^{k_i}, so the overall effect of β on r is the same as that of $R^{k_1}R^{k_2}\cdots R^{k_\ell} = R^{k_1+k_2+\cdots+k_\ell}$, which turns the right ring $k_1 + k_2 + \ldots + k_\ell$ positions. Since r is returned home then $k_1+k_2+\cdots+k_\ell$ must be divisible by 20, and hence the right ring moves contribute an even permutation to the process.

Similarly, by considering piece ℓ on the left ring, $m_1 + m_2 + \cdots m_\ell$ is divisible by 20, and so the left ring moves contribute an even permutation to the process. Therefore β must be even. A contradiction.

Theorem 16.2 gives us some insight into how we can construct a 4-cycle, or any odd permutation for that matter: for every disk on one ring that is to remain fixed by the permutation, we need to temporarily move it to the other ring. Let's try to do this with the disks on the left ring. The reason we use the left ring is purely aesthetic: the left ring consists of the numbers are 1 through 20 which are easy to remember.

First let's draw our attention to disks 35 and 21 in the solved state of the puzzle. See Figure 6a. We'd like to consider a move which only affects these disks in the right ring. Recall that our goal is to temporarily move every disk in the left ring to the right ring, as this is necessary if we wish to construct an odd permutation. To simplify what could potentially be a complicated set of moves, we would like to minimize the number of pieces that are moved on the right ring. We will only try to use move-sequences that affect positions 35 and 21 of the right ring, and these will be the positions where the left ring pieces temporarily visit.

The conjugate RLR^{-1} is a move that only affects positions 35 and 21 in the right ring, so let's begin with that move. It temporarily moves 1 and 6 off the left ring via move R, puts them in the holding spots (positions 38 and 34, respectively), after move L is applied then R^{-1} moves them back on the left ring to where they started. It also moves disks 2 and 7 off the left ring, and leaves them in our holding spots (positions 21 and 35, respectively) on the right ring. See Figure 5a. If we do this move again, it will put 2 back on the left ring, but it will be on the opposite side of 1. It also moves 1 and 6 off and on again. Repeated applications would keep moving 1 and 6 off and on the right ring, while at the same time moving another two disks off, then eventually back on. Figure 5b shows the result of repeated application of RLR^{-1}.



(a) The affect of applying RLR^{-1} once.

(b) Each application of RLR^{-1} moves the disks marked X one unit along the path.

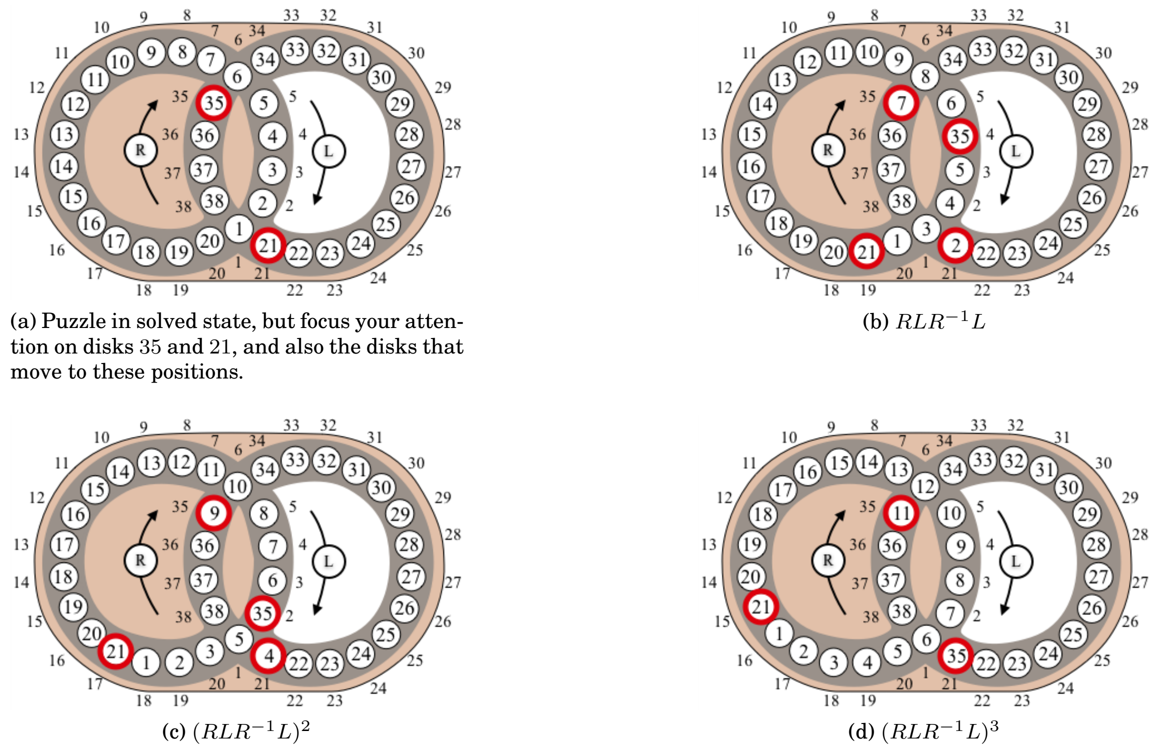
Figure 5: The result of applying move RLR^{-1} once, and repeatedly.

This would be a very slow process, to move every tile off the left ring then back on again, not to mention at some point we would need to do something to control which odd permutation we construct. Instead, it would be nice to move as many numbers off and on the left ring as possible, in a minimum number of moves, while at the same time keeping as many disks as we can in numerical order. To achieve this, we consider RLR^{-1} as the first move, then we advance the numbers on the left ring, before applying RLR^{-1} again, this would put two new numbers in positions 1 and 6 which would then be ready to be moved off and back on the left ring with the next application of RLR^{-1}. In other words, let's consider the move sequence $RLR^{-1}L$. The result of this move is shown in Figure 6b (in the figure we've drawn our attention to positions and disks 21 and 35).

There are a few things that we should note about the move $RLR^{-1}L$:

- All disks on the right ring were unaffected, except for disk 35 and 21.
- The disks in positions 7 and 2 were moved to storage on the right ring. And disks in positions 35 and 21 moved to take their place on the left ring.
- The disks in positions 1 and 6 were temporarily moved to positions 38 and 34 on the right ring, and then move back to the left ring, ending up on positions 20 and 5, respectively. That is, they moved only position clockwise around the ring.
- All other disks on the left ring advanced two positions clockwise around the ring.

Repeated application of $RLR^{-1}L$ is shown in Figure 6. A summary of which disks are moved off the left ring, then back on again, by repeated application of $RLR^{-1}L$ is given in Table 1. It is important to notice the change that was made by $(RLR^{-1}L)^3$, so compare Figure 6a to 6d. Disks that started in positions 1 to 10, are now back in their natural order after having been moved temporarily to the right ring. Disks that started in positions 12 through 20 are still in their natural order (they weren't moved to the right ring).



(a) Puzzle in solved state, but focus your attention on disks 35 and 21, and also the disks that move to these positions.

(b) $RLR^{-1}L$

(c) $(RLR^{-1}L)^2$

(d) $(RLR^{-1}L)^3$

Figure 6: The set-up moves for creating the 4-cycle $(1, 35, 11, 21)$.

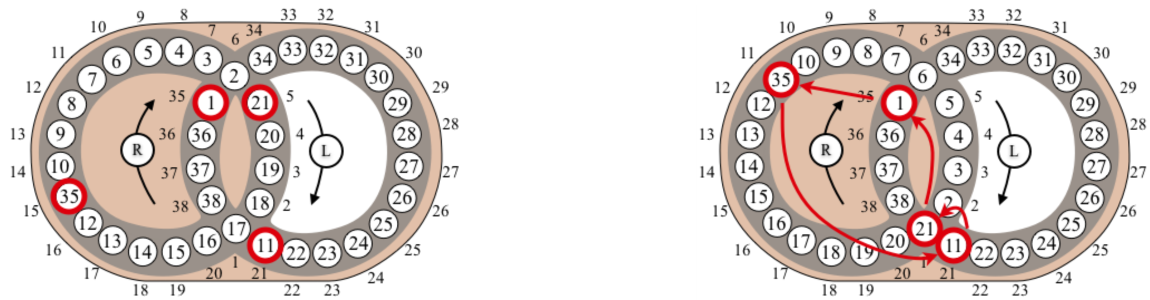
n	**prior to n^{th} move $RLR^{-1}L$:** all disks that have moved off/on the left ring	**during n^{th} move $(RLR^{-1}L)^n$:** disks that currently moved on and stayed on the left ring	disks that currently moved off/on the left ring	**after n^{th} move $(RLR^{-1}L)^n$:** disks that are currently off the left ring
1	$\emptyset$	35 (35) 21 (21)	6 (34) 1 (38)	7 (35) 2 (21)
2	$1, 6$	7 (35) 2 (21)	8 (34) 3 (38)	9 (35) 4 (21)
3	$1, 2, 3, 6, 7, 8$	9 (35) 4 (21)	10 (34) 5 (38)	11 (35) 35 (21)
2	$1, 2, 3, 4, 5, 6, 7, 8, 9, 10$			

Table 1: Summary of disks that moved off then back onto the left ring, and the positions affected, with first 3 application of $RLR^{-1}L$. The number in brackets next to the disk number represents the position the disk visited in the right ring.

If we continue to repeat the process then we would disturb the natural order of disk 1 to 10. Instead, we first rotate the left ring so that disks 1 through 10 are out of the way (apply L^5), then we apply the procedure $RLR^{-1}L$ three more times to move the other 10 disks off the left ring. The result is shown in Figure 7a. Disks 9 through 10 were far enough away that they weren't affect, but disk 1 got sent to position 35. Notice most disks on the left ring are back in their proper order, so rotating them back to their home position by L^4 results in the 4-cycle:

$$\sigma_4 = (RLR^{-1}L)^3 L^5 (RLR^{-1}L)^3 L^4 = (1, 35, 11, 21).$$

We have just shown how to get a 4-cycle (see Figure 7b) and Theorem 16.2 tells us that this is probably the best we could do.



(a) $(RLR^{-1}L)^2L^5(RLR^{-1}L)^3$

(b) The fundamental 4-cycle: $\sigma_4 = (RLR^{-1}L)^3L^5(RLR^{-1}L)^3L^4 = (1, 35, 11, 21)$

Figure 7: The set-up moves for creating the 4-cycle $(1, 35, 11, 21)$, continued.

16.2.3 3-cycles

The fundamental 3-cycle, which we call σ_3, was built using compound commutators:

$$\sigma_3 = [[L^5, R^5], R^{-1}LR] = (6, 11, 12).$$

See Lecture 14 for the discussion.

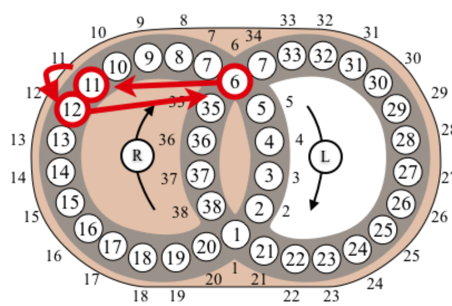


Figure 8: Fundamental 3-cycle: $\sigma_3 = [[L^5, R^5], R^{-1}LR] = (6, 11, 12)$.

16.2.4 2-cycles

Theorem 16.2 tells us that producing a 2-cycle is just as challenging as producing a 4-cycle since they are both odd. Luckily we already found a way to construct a 4-cycle:

$$\sigma_4 = (RLR^{-1}L)^3L^5(RLR^{-1}L)^3L^4 = (1, 35, 11, 21),$$

as shown in Figure 7b.

Using σ_4 we can construct the 2-cycle $(1, 11)$. Begin by applying σ_4. Now, if we can find a move sequence to swap disks 1 and 35, and swap disks 11 and 21 then we can produce the 2-cycle $(1, 11)$. To do this we can conjugate the pair of transpositions:

$$[L^5, R^5] = (1, 25)(6, 11)$$

by the four-step move sequence $\beta = RL^{-1}R^{-6}L$ which moves disks 1, 35, 11 and 21 to spots 6, 11, 1 and 25, respectively. Therefore,

$$\begin{aligned}\sigma_2 &= \sigma_4\beta[L^5, R^5]\beta^{-1} \\ &= ((RLR^{-1}L)^3L^5(RLR^{-1}L)^3L^4)(RL^{-1}R^{-6}L)(L^5R^5L^{-5}R^{-5})(L^{-1}R^6LR^{-1}) \\ &= (1, 11).\end{aligned}$$

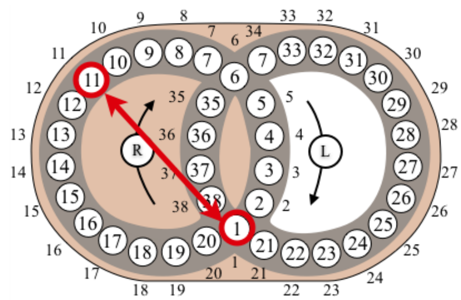


Figure 9: Fundamental 2-cycle: $\sigma_2 = (1, 11)$.

16.3 Solving the end-game

Theoretically, knowing how to perform 2-cycles is enough to solve the puzzle for any configuration. However, this would be very slow to perform manually. We now summarize a strategy for solving the puzzle.

(a) Starting with a scrambled puzzle put disks 7 through 16 on the left ring, and disks 21 through 30 on the right ring in proper numerical order.

(b) Still using general heuristics get a few more disks in their proper places if possible.

(c) Write down the remaining permutation in cycle form.

(d) Work on cycles that at length 5 or longer using conjugates of the fundamental 5-cycle $\sigma_5 = (1, 11, 35, 11, 16)$. If the cycle length is more than 5, you will be able to get 4 disks at a time into their right places. If the cycle is length 5 then you can solve all disks in the cycle this way.

(e) At this point all remaining cycles will be of length 5 or less. Using conjugates of the fundamental cycles: σ_5, σ_4, σ_3, σ_2 solve all disks in each cycle one cycle at a time.

16.4 Hungarian Rings - Coloured version

We now present a simple, and elegant strategy for solving the colour version of the puzzle shown in Figure 10.

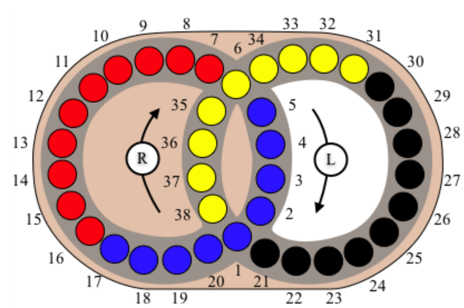


Figure 10: Hungarian Rings puzzle - coloured version.

There are 10 black disks and 10 red disks, but there are only 9 of each in blue and yellow. Solve the black and red disks first. There is enough room in the puzzle to do this using general heuristics. Once these are in their proper locations try to put as many blue and yellow disks in their home locations using general heuristics. To place the final remaining pieces (blue and yellow) you can swap two at a time, but if you're sneaky about how you do this then you actually don't need to use a 2-cycle. By placing the same coloured disks in spots 1 and 6, and the disks you want to swap in spots 11 and 30 (or 16 and 25), the commutators in Figure 3 can be used to swap pairs of disks. Since the intersection disks have the same colour this will go unnoticed, and this process will essentially allow you to swap any two blue and yellow disks. When using this method try to put either red or black disks in the intersection spots 1 and 6.

16.5 Exercises

1. Play with one of the virtual puzzles from the course website. The Hungarian Rings iphone app is pretty good. Try to solve each scrambling using the techniques developed in this section.

2. Show that for any cycle $\alpha = (a_1, a_2, a_3, a_4, a_5, \ldots, a_k)$ of length $k > 5$, there is a 5-cycle β so that $\alpha\beta$ has length $k - 4$. (This fact was used in the strategy for solving the end-game.)