Lecture 5:
From Puzzles to Permutations

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5.1 Introduction

The puzzles we have encountered so far all have a common theme: the pieces can be mixed up, and the goal is to restore the pieces back to some proper order. In other words, the pieces are permuted. We have already introduced, from a mathematical viewpoint, the notion of a permutation: *A permutation of a set A is defined to be a bijection $\alpha : A \to A$.* We will now see how to represent a puzzle by a permutation.

There are two types of permutations associated with a puzzle:

(a) the permutation describing the puzzles current position,

(b) the permutation corresponding to a move sequence applied to the puzzle.

Though the second one can really be thought of as a special case of the first, since the permutation we assign to a move sequence is just the one which represents the puzzle position after the move is applied to the solved state.

In this section we discuss how to write down these permutations for the standard set of puzzles we are studying. If we are thoughtful in how we do this, then the permutation describing the puzzles position is a composition of the permutations corresponding to the puzzle moves which takes the puzzle from the solved state to that position. That is, multiplying the permutations corresponding to the *moves*, should give us the permutation of the resulting *position*.

The way we do this will be the same for all puzzles where both the moving pieces and home positions have been labelled by numbers in $\mathbb{Z}_n$.

Definition 5.1 (Puzzle Position → Permutation) *For a given position (scrambling) of the puzzle, the* ***permutation corresponding to this position*** *is $\alpha : \mathbb{Z}_n \to \mathbb{Z}_n$ where*

$$\alpha(i) = j \quad \text{if piece labelled } i \text{ moved to position labelled } j.$$

This permutation describes precisely how the pieces in the home (or solved) state configuration were moved to produce the current configuration.

Definition 5.2 (Puzzle Move → Permutation) *For a given move sequence applied to the puzzle, the* ***permutation corresponding to this move sequence*** *is $\beta : \mathbb{Z}_n \to \mathbb{Z}_n$ where*

$$\beta(i) = j \quad \text{if the piece in position labelled } i \text{ moved to position labelled } j.$$

These definitions show how to construct the function α which corresponds to a position/move, but we should really say a few words about why this function is actually a permutation. That is, we want to observe α is one-to-one and onto. To see this, notice in any scrambling of the pieces no position has more than one piece occupying it, in other words, distinct pieces have gone into distinct positions. This means α is a one-to-one map from $\mathbb{Z}_n$ to $\mathbb{Z}_n$, which is then necessarily onto. This observation suggests why α is indeed a permutation.

Theorem 5.1 (Multiplying Moves) *Let α be the permutation corresponding to the current position of the puzzle, and β_1, β_2, $\ldots$ β_k be a move sequence applied to the puzzle which results in a final position γ. Then*

$$\alpha\beta_1\beta_2\cdots\beta_k = \gamma.$$

To see why this is true, consider any piece of the puzzle, say the piece labelled ℓ. Then, before the move sequence is applied, the piece ℓ starts in position $x_0 = \alpha(\ell)$. As the moves are applied one-by-one the ℓ piece moves to position x_1, then to position x_2, and so on, until it finally ends up in position x_k, where

$$\begin{aligned}
x_1 &= \beta_1(x_0) = \beta_1(\alpha(\ell)) = (\alpha\beta_1)(\ell) \\
x_2 &= \beta_2(x_1) = \beta_2((\alpha\beta_1)(\ell)) = (\alpha\beta_1\beta_2)(\ell) \\
&\ \vdots \\
x_k &= \beta_k(x_{k-1}) = \beta_k((\alpha\beta_1\beta_2\cdots\beta_{k-1})(\ell)) = (\alpha\beta_1\beta_2\cdots\beta_k)(\ell)
\end{aligned}$$

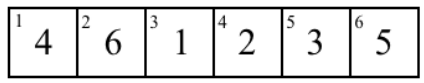
Therefore, $\gamma(\ell) = x_k = (\alpha\beta_1\beta_2\cdots\beta_k)(\ell)$ for every $\ell \in \mathbb{Z}_n$, and so $\gamma = \alpha\beta_1\beta_2\cdots\beta_k$. This proves the theorem.

Over the next few sections we will look at each puzzle individually.

This lecture corresponds to Chapter 4 of Joyner's text. Joyner discusses more 3-dimensional puzzles like Rubik's cube, but does not discuss the Oval Track puzzle or Hungarian Rings.

5.2 Swap

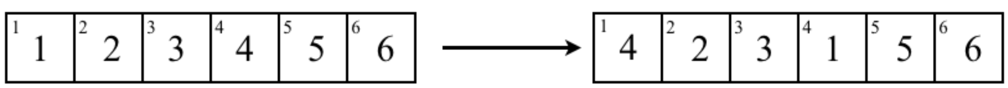
Each arrangement of the numbers in the Swap Puzzle, say with n numbers, is a permutation of the set $\mathbb{Z}_n = \{1, 2, 3, \dots, n\}$. For example, consider the following position of Swap with 6 numbers.



The permutation $\alpha : \mathbb{Z}_6 \to \mathbb{Z}_6$ we associate to this position is determined as follows. Since tile number 1 moved to box number 3, then $\alpha(1) = 3$. Since tile 2 moved to box 4, then $\alpha(2) = 4$. Continuing in this fashion we find α maps numbers 1 through 8 as follows.

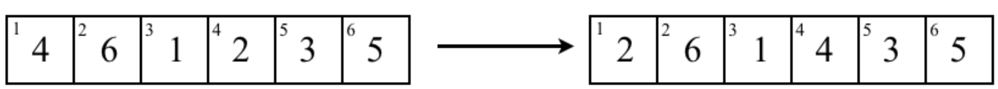
$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 1 & 6 & 2 \end{pmatrix} \qquad \text{or in cycle form} \qquad \alpha = (1, 3, 5, 6, 2, 4).$$

Consider the move obtained by swapping the tiles in boxes 1 and 4. What permutation should we use to represent this move? If we think of applying this move to the solved state of the puzzle, for example:



then we can represent this move by the permutation β corresponding to the position it leaves the puzzle in: $\beta = (1, 4)$.

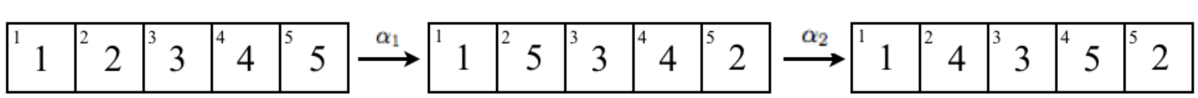
Now imagine the puzzle was not in the solved state, and we were applying the $1, 4$ swap. For example, we apply the move as follows:



How should we assign a permutation to this move? Well, the simplest way is to say it is just the same move as above, ignoring the actually objects in the boxes. All that matters is that the contents of box 1 and box 4 were switched. The permutation should then only depend on the boxes involved and how the contents move between boxes, but is shouldn't depend on what exactly is in the boxes. This is the essence of Definition 5.2.

From now on, when we wish to describe a move, we can just state it by giving the corresponding permutation. For example, the permutation $(3, 7)$ represents the move of switching the contents of boxes 3 and 7.

Example 5.1 *Consider the following sequence of moves in Swap.*



The first move consists of swapping the contents of boxes 2 *and* 5 *so it corresponds to the permutation* $\alpha_1 = (2,5)$*. The second move consists of swapping the contents of boxes* 2 *and* 4 *so it corresponds to the permutation* $\alpha_2 = (2,4)$*. The product* $\alpha_1\alpha_2 = (2,5)(2,4) = (2,5,4)$*, which is the permutation representing the move sequence as a whole, is precisely the permutation corresponding to the final position.*

Example 5.2 *Apply the move sequence* $\tau_1 = (3,5)$*,* $\tau_2 = (1,2)$*,* $\tau_3 = (2,5)$*,* $\tau_4 = (1,4)$ *to the game of Swap with* 6 *objects, and draw the final position of the game board, assuming you began with it in the solved state.*

The move sequence corresponds to the single maneuver: $\alpha = \tau_1\tau_2\tau_3\tau_4 = (3,5)(1,2)(2,5)(1,4) = (1,5,3,2,4)$*, (Theorem 5.1) which means the resulting game board position is as follows.*

1	2	3	4	5	6
4	3	5	2	1	6

We could have also applied the move sequences one-by-one to achieve the same result (here we simply write the numbered tiles, as they appear on the game board, separated by vertical bars $|$ *):*

$$1|2|3|4|5|6 \xrightarrow{\tau_1=(3,5)} 1|2|5|4|3|6 \xrightarrow{\tau_2=(1,2)} 2|1|5|4|3|6 \xrightarrow{\tau_3=(2,5)} 2|3|5|4|1|6 \xrightarrow{\tau_4=(1,4)} 4|3|5|2|1|6$$

Example 5.3 *Write the permutation* $\alpha = (1,5,3,7)(4,8,6)$ *as a product of* 2*-cycles. (Hint: Solve the corresponding Swap puzzle.)*

The permutation α *corresponds to the position*

1	2	3	4	5	6	7	8
7	2	5	6	1	8	3	4

which we will simply write as $7|2|5|6|1|8|3|4$*. To solve the puzzle from this state we may do the following:*

$$\begin{aligned} \alpha \Rightarrow 7|2|5|6|1|8|3|4 &\xrightarrow{\tau_1=(1,5)} 1|2|5|6|7|8|3|4 \xrightarrow{\tau_2=(3,7)} 1|2|3|6|7|8|5|4 \xrightarrow{\tau_3=(4,8)} 1|2|3|4|7|8|5|6 \\ &\xrightarrow{\tau_4=(5,7)} 1|2|3|4|5|8|7|6 \xrightarrow{\tau_5=(6,8)} 1|2|3|4|5|6|7|8 \Rightarrow \varepsilon. \end{aligned}$$

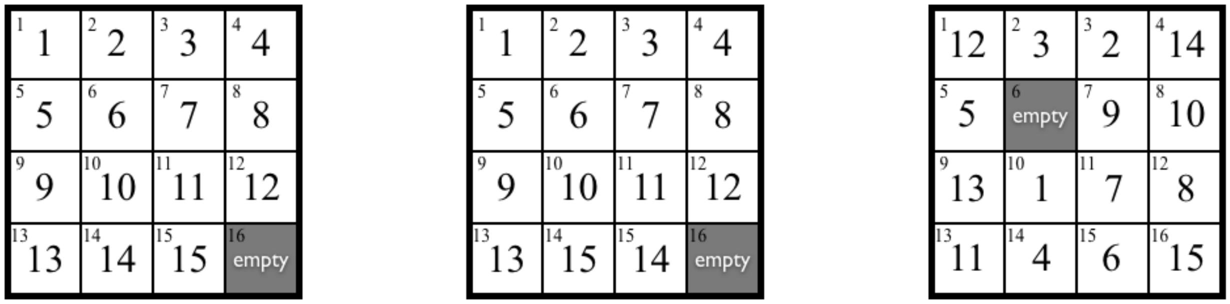
This means $\alpha\tau_1\tau_2\tau_3\tau_4\tau_5 = \varepsilon$*, or* $\alpha = \tau_5^{-1}\tau_4^{-1}\tau_3^{-1}\tau_2^{-1}\tau_1^{-1}$*. Therefore, we have found a decomposition of* α *into* 2*-cycles:*

$$\alpha = (1,5,3,7)(4,8,6) = (6,8)(5,7)(4,8)(3,7)(1,5).$$

5.3 15-Puzzle

Imagine the tiles in the 15 puzzle mixed-up. Consider Figure 1c for example. Each tile was moved from some numbered box (its home box) to some other numbered box: for example the tile in box 1 moved to box 10, but the tile in box 5 stayed in box 5. Here we think of the empty space as tile number 16, which we will often call the "empty tile".

We can write down the permutations describing each of the positions in 1 by using Definition 5.1.



(a) (b) (c)

Figure 1: The 15 Puzzle

(a) This puzzle is in the solved state, so no tiles have been moved. This corresponds to the identity permutation ε. The array form of this permutation is

$$\varepsilon = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \end{pmatrix}.$$

(b) In this puzzle the tiles in boxes 14 and 15 were switched. This corresponds to the permutation $(14, 15)$. The array form of this permutation is

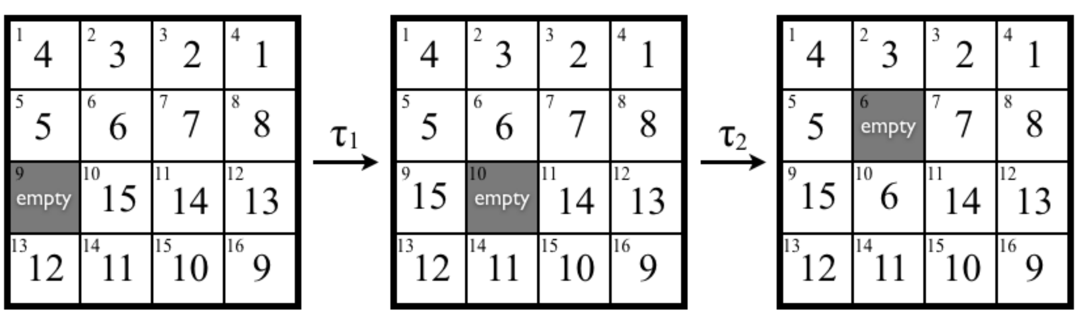
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 15 & 14 & 16 \end{pmatrix}.$$

(c) The tile originally in box 1 (that is, the tile labeled by 1) was moved to box 10, so $1 \mapsto 10$ for this permutation. The tile originally in box 10 (that is, the labeled by 10) was moved to box 8, so $10 \mapsto 8$. Continuing in this fashion we construct the cycle form of the corresponding permutation: $(1, 10, 8, 12)(2, 3)(4, 14)(6, 15, 16)(7, 11, 13, 9)$, where we omitted the 1-cycle (5). The array form of this permutation is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 10 & 3 & 2 & 14 & 5 & 15 & 11 & 12 & 7 & 8 & 13 & 1 & 9 & 4 & 16 & 6 \end{pmatrix}.$$

By the construction of the 15-Puzzle a legal move consists of swapping a tile with the empty tile, provided it is adjacent to the empty tile. This means legal moves are 2-cycles, and move sequences are products of 2-cycles.

Example 5.4 *Consider the following sequence of moves in the 15-Puzzle.*



The first move consists of moving the empty space from box 9 to 10 so it corresponds to the permutation $\tau_1 = (9, 10)$. The second move consists of moving the empty space form box 10 to 6 so it corresponds to the permutation $\tau_2 = (10, 6)$.

The first position is given by $\alpha = (1, 4)(2, 3)(9, 16)(15, 10)(11, 14)(12, 13)$, the last position is $\beta = (1, 4)(2, 3)(6, 10, 15, 9, 16)(11, 14)(12, 13)$, and we have

$$\begin{aligned} \alpha\tau_1\tau_2 &= (1, 4)(2, 3)(9, 16)(15, 10)(11, 14)(12, 13)(9, 10)(10, 6) \\ &= (1, 4)(2, 3)(6, 10, 15, 9, 16)(11, 14)(12, 13) \\ &= \beta. \end{aligned}$$

This provides and illustration of Theorem 5.1.

5.4 Oval Track Puzzle

Since there are 20 moving disks on the Oval Track puzzle (Figure 2), each position/move can be described as a permutation of $\mathbb{Z}_{20}$.

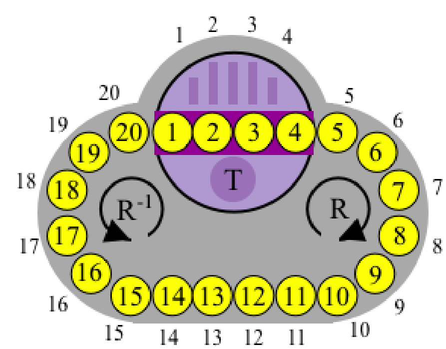


Figure 2: The Oval Track Puzzle.

Recall from Lecture 1, the basic legal moves of the oval track puzzle are R, R^{-1}, and T, where R denotes a clockwise rotation of numbers around the track, where each number moves one space, R^{-1}denotes a counterclockwise rotation of the numbers around the track, and T denotes a rotation of the turntable. See Figure 3.

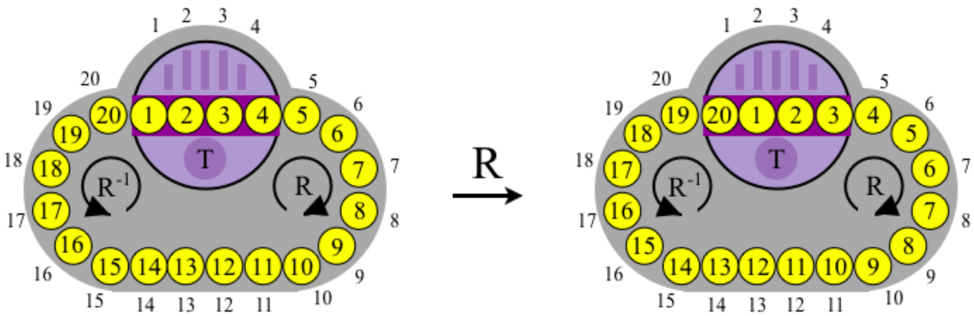
The permutation corresponding to the legal moves R, R^{-1}, and T are as follows:

$$\begin{aligned} &R = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20) \\ &R^{-1} = (1, 20, 19, 18, 17, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2) \\ &T = (1, 4)(2, 3) \end{aligned}$$

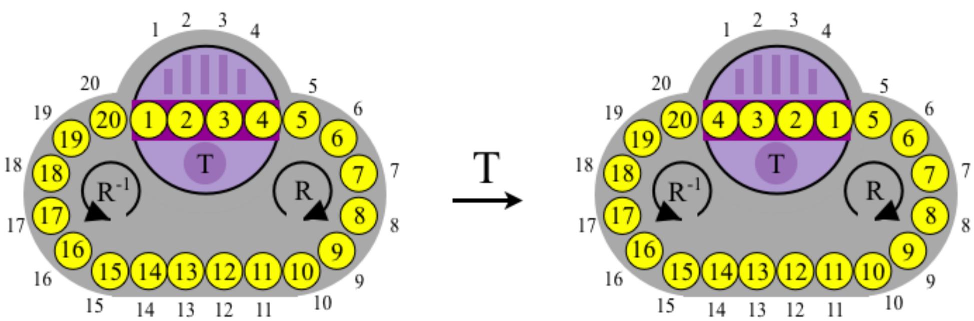
Note that $T^{-1} = T$. This is due to the fact that spinning the turntable in either direction achieves the same result.

Example 5.5 *Express, in cycle form, the permutations describing each of the positions in Figure 4.*

(a) Disk 1 was moved to slot 4, disk 4 was moved to slot 6, disk 6 was moved to slot 5, disk 5 was moved to slot 2, disk 2 was moved to slot 3, and disk 3 was moved to slot 1. All other disks are still in their home positions, so the corresponding permutation is $(1, 4, 6, 5, 2, 3)$.

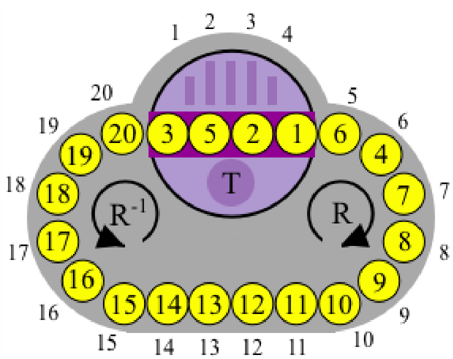


(a) R$= (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20)$

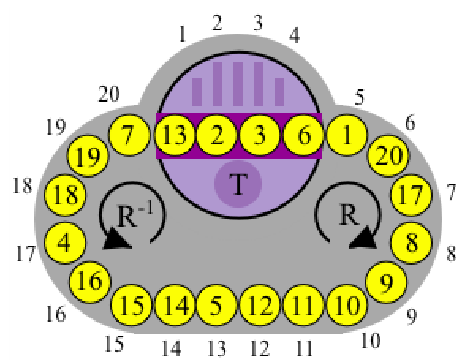


(b) T $= (1,4)(2,3)$

Figure 3: Basic Moves R and T of Oval Track.



(a)



(b)

Figure 4: Oval Track scramblings for Example 5.5.

(b) Similar to part (a) we just follow where each disk ended up. The corresponding permutation is $(1,5,13)(4,17,7,20,6)$.

Example 5.6 *For each of the following move sequences, which were applied to the solved-state Oval Track puzzle, draw the resulting configuration of the disks on the puzzle.*

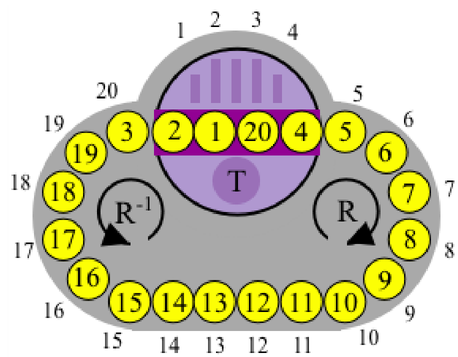
(a) RTR^{-1}

(b) $R^{-4}TR^2TR^{-1}$

(a) *If you have a physical puzzle, or one of the virtual ones linked to from the course website, then you can actually perform the move sequence and attain the resulting configuration. We can also do this using the permutation representations of the move sequence:*

$$\begin{aligned} RTR^{-1} &= (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20)(1,4)(2,3)R^{-1} \\ &= (1,3)(4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20)R^{-1} \\ &= (1,3)(4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20) \\ &\qquad (1,20,19,18,17,16,15,14,13,12,11,10,9,8,7,6,5,4,3,2) \\ &= (1,2)(3,20) \end{aligned}$$

The resulting position is drawn below.

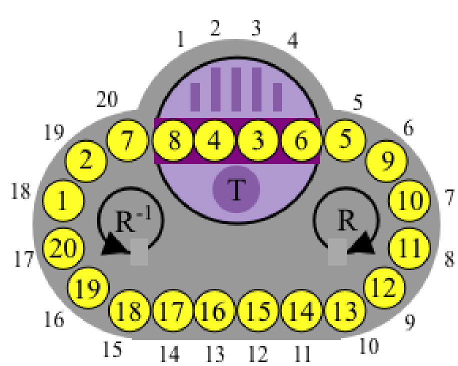


(b) *Multiplying two* 20*-cycles and a* 4*-cycle in part (a) was not technically difficult, it was just tedious. This product* $R^{-4}TR^2TR^{-1}$ *would be very tedious, and the actual calculation wouldn't be too enlightening. No mathematician would actually do the calculation by hand. In fact, most would not have done part (a) by hand either. It is really just the end result we are interested in, so we should do what any normal person would do, have a computer do the calculation. We'll use* SAGE *to do this.*

SAGE

```
sage: S20=SymmetricGroup(20)
sage: R=S20("(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20)")
sage: T=S20("(1,4)(2,3)")
sage: R^(-4)*T*R^(2)*T*R^(-1)
(1,18,15,12,9,6,4,2,19,16,13,10,7,20,17,14,11,8)
```

Later on we will discuss in detail the commands used above, and what each line of code does (you may be able to figure this out for yourself). For now, we have our answer to the question, $R^{-4}TR^2TR^{-1}$ *corresponds to the permutation returned by* SAGE *and the puzzle looks like this*



5.5 Hungarian Rings

There are 38 moving disks in the the (numbered) Hungarian Rings puzzle (Figure 5), so each position/move can be described as a permutation of $\mathbb{Z}_{38}$.

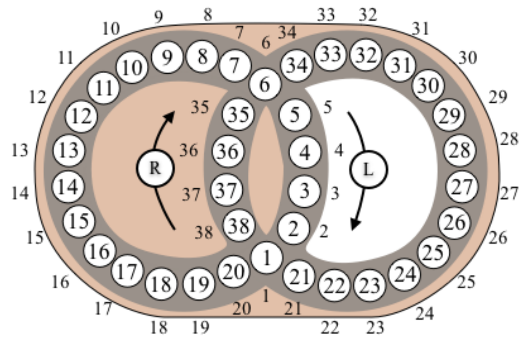


Figure 5: Hungarian Rings - numbered version.

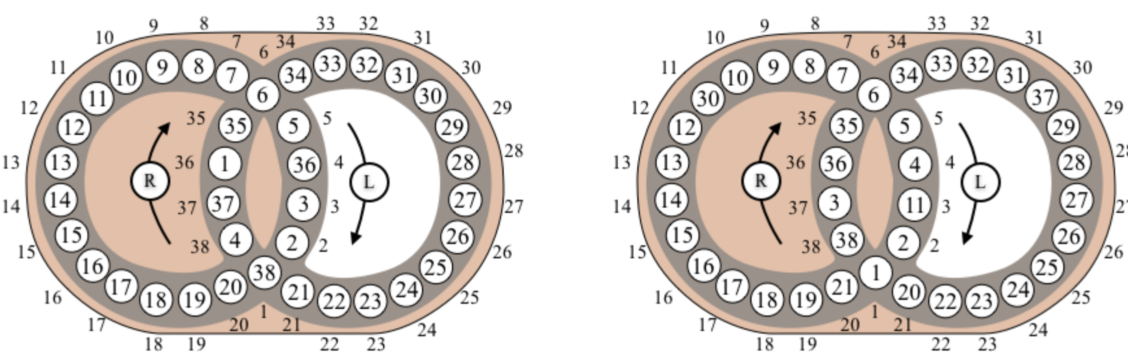
Recall from Lecture 1, the basic legal moves of the Hungarian Rings puzzle are R, R^{-1}, L, and L^{-1}, where R denotes a clockwise rotation of numbers around the right-hand ring (each number moves one space), R^{-1}denotes a counterclockwise rotation of the numbers around the right-hand ring, L denotes a clockwise rotation of numbers around the left-hand ring, and L^{-1}denotes a counterclockwise rotation of the numbers around the left-hand ring.

The permutation corresponding to each of the legal moves R and L are:

$$R = (1, 38, 37, 36, 35, 6, 34, 33, 32, 31, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21)$$
$$L = (1, 20, 19, 18, 17, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2)$$

$\mathrm{R}^{-1}$and L^{-1}correspond to the inverses of these permutations.

Example 5.7 *Express, in cycle form, the permutations describing each of the positions in Figure 6.*



(a) (b)

Figure 6: Oval Track scramblings.

(a) We simply follow where each disk has ended up. The corresponding permutation is $(1, 36, 4, 38)$.

(b) *Similar to part (a), following where each disk has ended up, the corresponding permutation is* $(3, 37, 30, 11)(20, 21)$.

Example 5.8 *For each of the following move sequences, which were applied to the solved-state Hungarian Rings puzzle, draw the resulting configuration of the disks on the puzzle.*

(a) $R^{-1}LR$

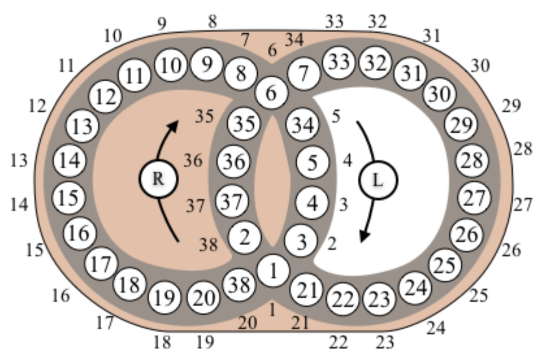
(b) $L^5R^5L^{-5}R^{-5}$

(a) *If you have a physical puzzle, or one of the virtual ones linked to from the course website, then you can actually perform the move sequence and attain the resulting configuration. We can also do this using the permutation representations of the move sequence, by multiplying the permutations. We'll use* SAGE *to do the computations.*

SAGE

```
sage: S38=SymmetricGroup(38)
sage: L=S38("(1,20,19,18,17,16,15,14,13,12,11,10,9,8,7,6,5,4,3,2)")
sage: R=S38("(1,38,37,36,35,6,34,33,32,31,30,29,28,27,26,25,24,23,22,21)")
sage: R^(-1)*L*R
(2,38,20,19,18,17,16,15,14,13,12,11,10,9,8,7,34,5,4,3)
```

The resulting position is drawn below.

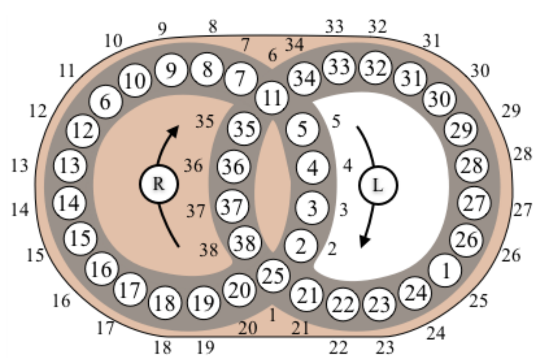


(b) *Again, we'll use* SAGE *to do the desired calculation.*

SAGE

```
sage:  L^5*R^5*L^(-5)*R^(-5)
(1,25)(6,11)
```

The resulting position is drawn below.



5.6 Rubik's Cube

To keep track of how the pieces of the cube move around, and to be able to describe movements and positions by permutations, we label each of the facets with numbers. For the $2 \times 2 \times 2$ cube there are 24 facets, whereas for the $3 \times 3 \times 3$ cube there are 54, but only 48 actually move. The 6 centres can be thought of as remaining fixed (though they can rotate, but this is only noticeable if the sticker has an image on it).

5.6.1 $2 \times 2 \times 2$ Cube

We label the facets of the Pocket Cube as shown in Figure 7. Figure 8 shows the labeling on an actual cube.

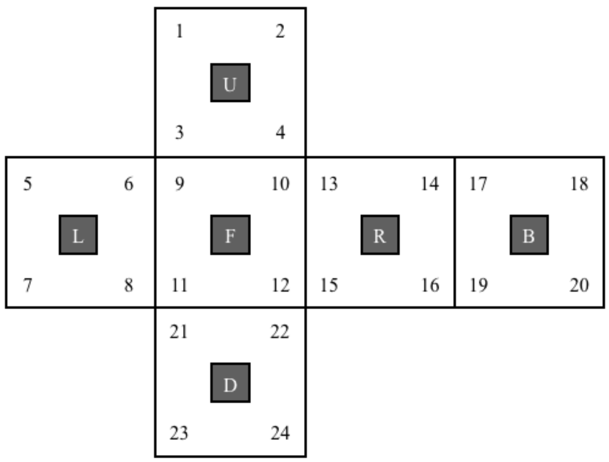
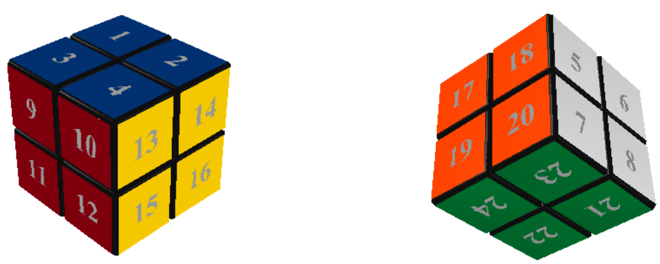


Figure 7: Facet labeling on the Pocket cube.



(a) Labeling on Up (blue), Right (yellow), Front (red) faces

(b) Labeling on Down (green), Back (orange), Left (white) faces

Figure 8: The labeling of the facets of the Pocket Cube.

We associate permutations to positions and moves in the usual way (Definitions 5.1 and 5.2). The basic moves of the Rubik's Cube are R, L, U, D, F, B, and their inverses. Each one denotes a clockwise quarter turn of the corresponding face. See Lecture 1 for a thorough discussion of this notation.

The permutation corresponding to each of the basic moves of the Pocket Cube are:

$$
\begin{aligned}
R &= (13,14,16,15)(10,2,19,22)(12,4,17,24)\\
L &= (5,6,8,7)(3,11,23,18)(1,9,21,20)\\
U &= (1,2,4,3)(9,5,17,13)(10,6,18,14)\\
D &= (21,22,24,23)(11,15,19,7)(12,16,20,8)\\
F &= (9,10,12,11),(3,13,22,8),(4,15,21,6)\\
B &= (17,18,20,19),(1,7,24,14),(2,5,23,16)
\end{aligned}
$$

R^{-1}, L^{-1}, U^{-1}, D^{-1}, F^{-1}, D^{-1}correspond to the inverses of these permutations.

5.6.2 $3 \times 3 \times 3$ Cube

As we described in Lecture 1, we label the facets of the Rubik's Cube as shown Figure 9. Figure 10 shows the labeling on an actual cube.

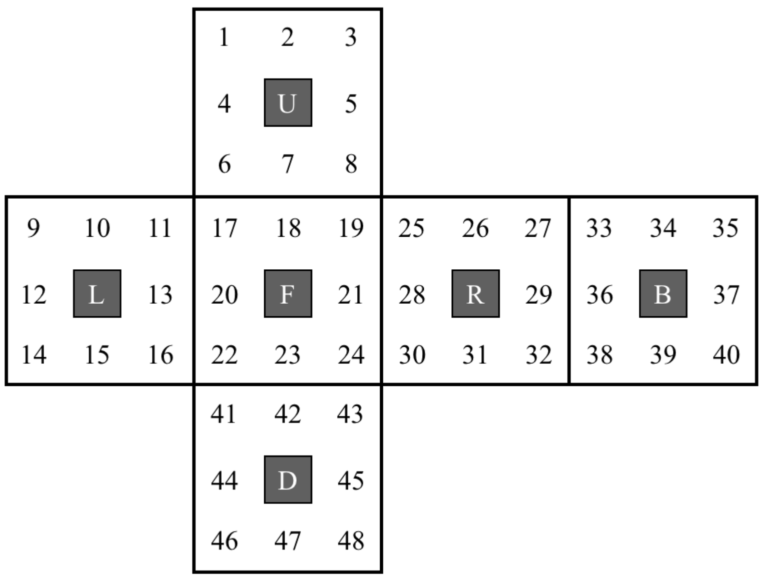


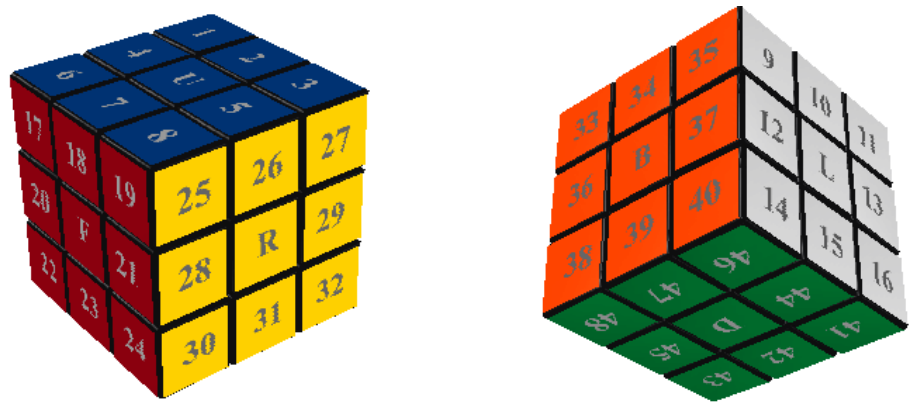
Figure 9: Facet labeling on the Rubik's cube.

The permutation corresponding to each of the basic moves of the Rubik's Cube are:

$$
\begin{aligned}
R &= (25,27,32,30)(26,29,31,28)(3,38,43,19)(5,36,45,21)(8,33,48,24)\\
L &= (9,11,16,14)(10,13,15,12)(1,17,41,40)(4,20,44,37)(6,22,46,35)\\
U &= (1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)(11,35,27,19)\\
D &= (41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)(16,24,32,40)\\
F &= (17,19,24,22)(18,21,23,20)(6,25,43,16)(7,28,42,13)(8,30,41,11)\\
B &= (33,35,40,38)(34,37,39,36)(3,9,46,32)(2,12,47,29)(1,14,48,27)
\end{aligned}
$$

R^{-1}, L^{-1}, U^{-1}, D^{-1}, F^{-1}, D^{-1}correspond to the inverses of these permutations.

Uncovering the secrets of the cube will involve playing around with these permutations, and our playground will be SAGE . Here is how we can input the permutations for the Rubik's Cube into SAGE.



(a) Labeling on Up, Right, Front faces

(b) Labeling on Down, Back, Left faces

Figure 10: The labeling of the facets of Rubik's Cube.

SAGE
```
sage: S48=SymmetricGroup(48)
sage: R=S48("(25,27,32,30)(26,29,31,28)(3,38,43,19)(5,36,45,21)(8,33,48,24)")
sage: L=S48("(9,11,16,14)(10,13,15,12)(1,17,41,40)(4,20,44,37)(6,22,46,35)")
sage: U=S48("(1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)(11,35,27,19)")
sage: D=S48("(41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)(16,24,32,40)")
sage: F=S48("(17,19,24,22)(18,21,23,20)(6,25,43,16)(7,28,42,13)(8,30,41,11)")
sage: B=S48("(33,35,40,38)(34,37,39,36)(3,9,46,32)(2,12,47,29)(1,14,48,27)")
```

We could then, for instance, see what the move sequence RU does to the cube.

SAGE
```
sage: R*U
(1,3,38,43,11,35,27,32,30,17,9,33,48,24,6)(2,5,36,45,21,7,4)(8,25,19)  \
(10,34,26,29,31,28,18)
```

We can easily "eyeball" the order of this permutation, it is $\mathrm{lcm}(15, 7, 3) = 105$. This gives us a glimpse into the kind of questions we can easily answer through computation.

Also, since RU consists of a 15-cycle, a 3-cycle and two 7-cycles, raising it to the power of 15 would get rid of the 15- and 3-cycles, and would leave us with some 7-cycles.

SAGE
```
sage: (R*U)^15
(2,5,36,45,21,7,4)(10,34,26,29,31,28,18)
```

This means we can move fewer pieces by taking powers of some move sequences. We'll later how this is an effective strategy for solving these puzzles.

5.7 Exercises

1. **Swap Puzzle arrangements into cycle notation.** For each of the following scramblings of the tiles in Swap, express them as permutations in S_n using cycle notation.

(a)

1	2	3
2	3	1

(b)

1	2	3	4	5
1	4	5	2	3

(c)

1	2	3	4	5	6	7	8	9	10
1	8	3	2	5	4	7	6	9	10

(d)

1	2	3	4	5	6	7	8	9	10
9	5	10	6	2	1	3	8	4	7

(e)

1	2	3	4	5	6	7	8	9	10	11	12
4	6	9	12	8	10	1	7	2	5	3	11

2. **Swap Puzzle arrangements from cycle notation.** For each of the following permutations, given in cycle form, draw the corresponding scrambling of the tiles on the Swap puzzle.

   (a) $(1,5,3,8)(2,4,7)$
   (b) $(3,7,4,10,6,5,8)$
   (c) $(1,12)(2,11)(3,10)(5,6,7)$

3. **Swap Puzzle arrangements and moves in cycle notation.** In each part (a) - (c) below, a sequence of moves has been applied to a scrambling of the tiles in Swap. Do the following:
(i) Express the starting position α as a permutation in cycle notation.
(ii) Express each move τ_i as a 2-cycle.
(iii) Express the whole move sequence as a permutation in cycle notation.
(iv) Express the final position β as a permutation in cycle notation and show that $\alpha\tau_1 \cdots \tau_n = \beta$.

(a)

1	2	3	4	5
3	4	5	2	1

$\xrightarrow{\tau_1}$

1	2	3	4	5
1	4	5	2	3

$\xrightarrow{\tau_2}$

1	2	3	4	5
1	4	3	2	5

(b) $5|4|2|8|1|3|6|7 \xrightarrow{\tau_1} 5|2|4|8|1|3|6|7 \xrightarrow{\tau_2} 8|2|4|5|1|3|6|7 \xrightarrow{\tau_3} 1|2|4|5|8|3|6|7 \xrightarrow{\tau_4} 1|2|4|5|8|6|3|7$

(c) $5|10|4|1|6|7|2|3|8|9 \xrightarrow{\tau_1} 5|10|4|1|7|6|2|3|8|9 \xrightarrow{\tau_2} 1|10|4|5|7|6|2|3|8|9 \xrightarrow{\tau_3} 1|9|4|5|7|6|2|3|8|10$
$\xrightarrow{\tau_4} 1|9|4|7|5|6|2|3|8|10 \xrightarrow{\tau_5} 1|2|4|7|5|6|9|3|8|10$

4. **Swap Puzzle move sequence in cycle notation.** For each move sequence α given below, express it as a permutation in cycle form.

(a)

1	2	3	4	5
4	1	5	3	2

$\xrightarrow{\alpha}$

1	2	3	4	5
1	5	3	4	2

(b)

1	2	3	4	5	6	7	8
3	8	1	2	5	7	4	6

$\xrightarrow{\alpha}$

1	2	3	4	5	6	7	8
1	2	3	4	5	8	7	6

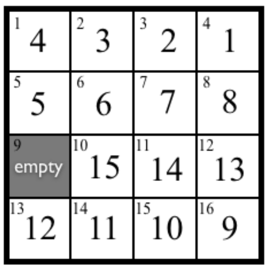
5. **Decomposing a permutation into 2-cycles.** Write the permutation $\alpha = (1,2,3)$ as a product of 2-cycles. (Hint: Solve the corresponding Swap puzzle.)

6. **Decomposing a permutation into 2-cycles.** Write the permutation $\alpha = (1,2,8,3,7)(4,5,6)$ as a product of 2-cycles. (Hint: Solve the corresponding Swap puzzle.)

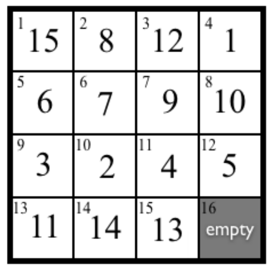
7. **Decomposing a permutation into 3-cycles.** Write the permutation $\alpha = (1,2)(3,4)$ as a product of 3-cycles. (Hint: Solve the corresponding Swap puzzle, under the variation where the legal moves are now 3-cycles.)

8. **Decomposing a permutation into 3-cycles.** Write the permutation $\alpha = (1,2,8,3,7)(4,5,6)$ as a product of 3-cycles. (Hint: Solve the corresponding Swap puzzle, under the variation where the legal moves are now 3-cycles.)

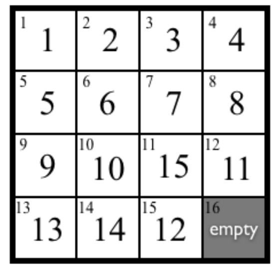
9. **15-Puzzle arrangements into cycle notation.** Express each of the following scramblings of the 15-puzzle as a permutation in cycle form.



(a)



(b)



(c)

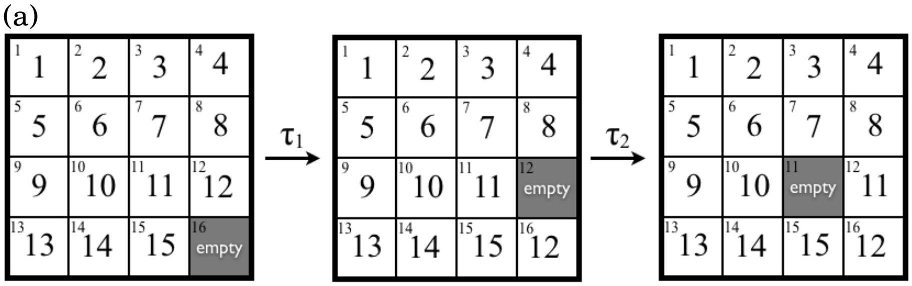
10. **15-Puzzle arrangements from cycle notation.** For each of the following permutations, given in cycle form, draw the corresponding scrambling of the tiles on the 15 puzzle.

    (a) $(6,7,11,10)$

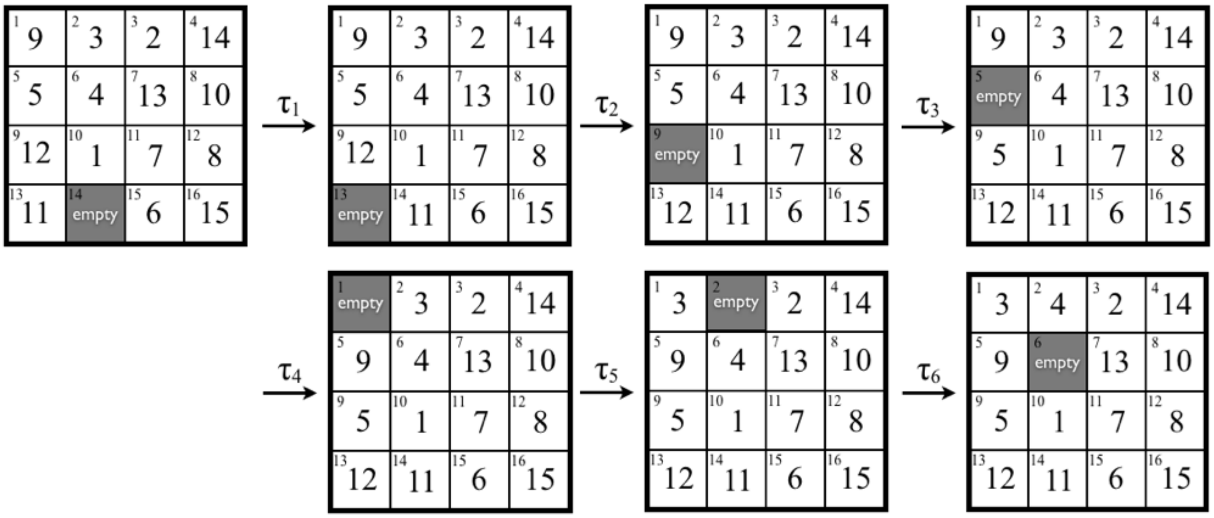
    (b) $(1,5,3,10,15,2,14,12,11,6,7,4)(9,16)$

    (c) $(2,10,13,5)(1,3)(7,8,9)$

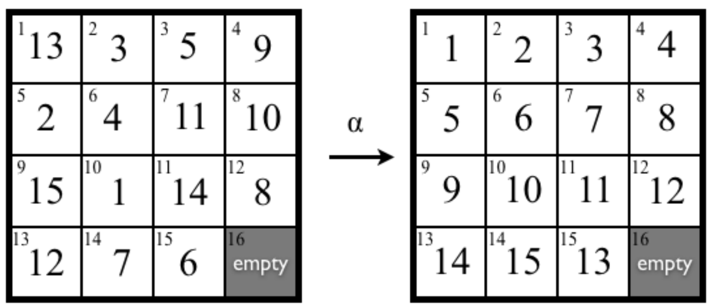
11. **15-Puzzle arrangements and moves in cycle notation.** In each part (a) - (c) below, a sequence of moves has been applied to a scrambling of the tiles in the 15-Puzzle. Do the following:
(i) Express the starting position α as a permutation in cycle notation.
(ii) Express each move τ_i as a 2-cycle.
(iii) Express the whole move sequence as a permutation in cycle notation.
(iv) Express the final position β as a permutation in cycle notation and show that $\alpha\tau_1 \cdots \tau_n = \beta$.
(a)



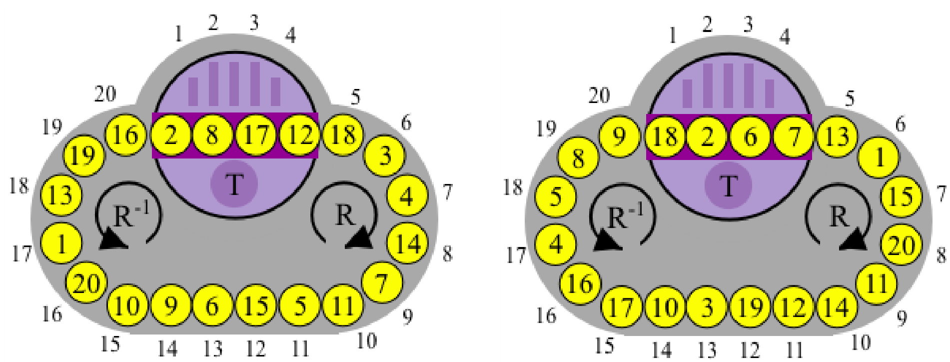
(b)



12. **15-Puzzle move sequence in cycle notation.** For the move sequence α given below, express it as a permutation in cycle form.

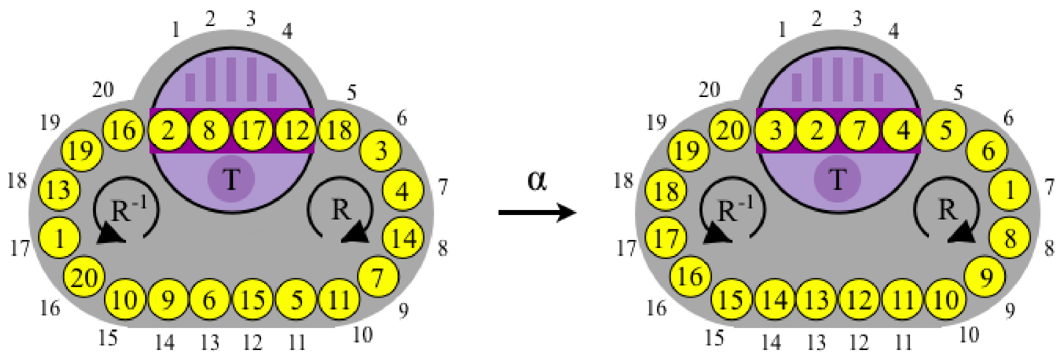


13. **Oval Track Puzzle arrangements into cycle notation.** Express, in cycle form, the permutation describing each of the positions of the Oval Track puzzle drawn below.



(a) (b)

14. **Oval Track Puzzle move sequence in cycle notation.** Express the move sequence α given in the diagram below as a permutation in cycle notation.



15. For each of the following move sequences, which are applied to the solved-state Oval Track puzzle, draw the resulting configuration of the disks on the puzzle.

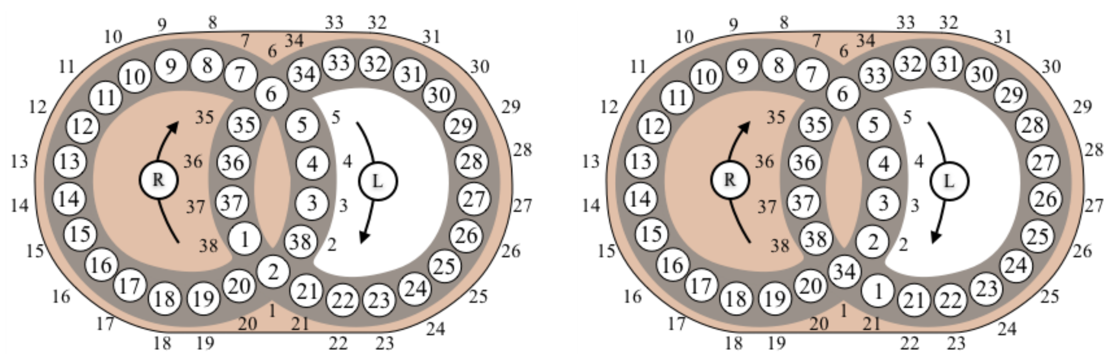
    (a) T^2

    (b) R^{19}

    (c) $R^{-1}TR$

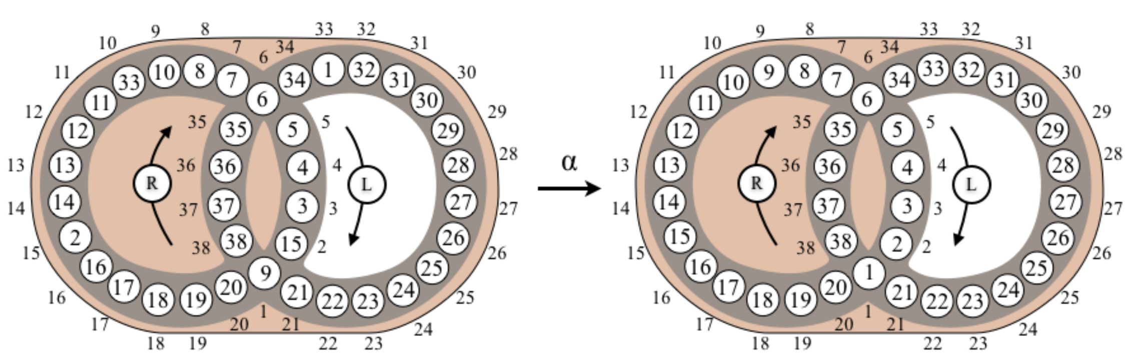
    (d) $TR^{-1}TR$

16. **Hungarian Rings arrangements into cycle notation.** Express, in cycle form, the permutation describing each of the positions of the Hungarian Rings puzzle drawn below.



(a) (b)

17. **Hungarian Rings move sequence in cycle notation.** Express the move sequence α given in the diagram below as a permutation in cycle notation.



18. For each of the following move sequences, which are applied to the solved-state Hungarian Rings puzzle, draw the resulting configuration of the disks on the puzzle.

    (a) R^2

    (b) RL

    (c) $L^5R^5L^{-5}R^{-6}LR^6L^5R^{-5}L^{-5}R^{-1}L^{-1}R$ (use SAGE to compute this)

19. **Rubik's Cube arrangements into cycle notation.** Express, in cycle form, the permutation corresponding to the position of the Rubik's Cube where the cubies have been moved and positioned as follows:

    - the UR cubie is in the bu cubicle (recall this means the U face of the UR cubie is in the B face of the bu cubicle)
    - the UB cubie is in the lu cubicle
    - the UL cubie is in the ur cubicle.

    (Look back at Lecture 1 where the terms "cubie" and "cubicle" are discussed.)