Lecture 6:
Permutations: Products of 2-Cycles

Contents

To solve a permutation puzzle one must determine how the permutation representing the current position of the pieces can be decomposed into permutations representing the legal moves. It is this "decomposition problem" that will be the focus of our attention in many lectures to come.

In this lecture we will show every permutation can be decomposed as a product of 2-cycles. We will also see how this is connected to the solvability of the Swap puzzle.

It is standard terminology to refer to a 2-cycle as a **transposition**. So the title of this lecture could also be *Permutations: Products of Transpositions*.
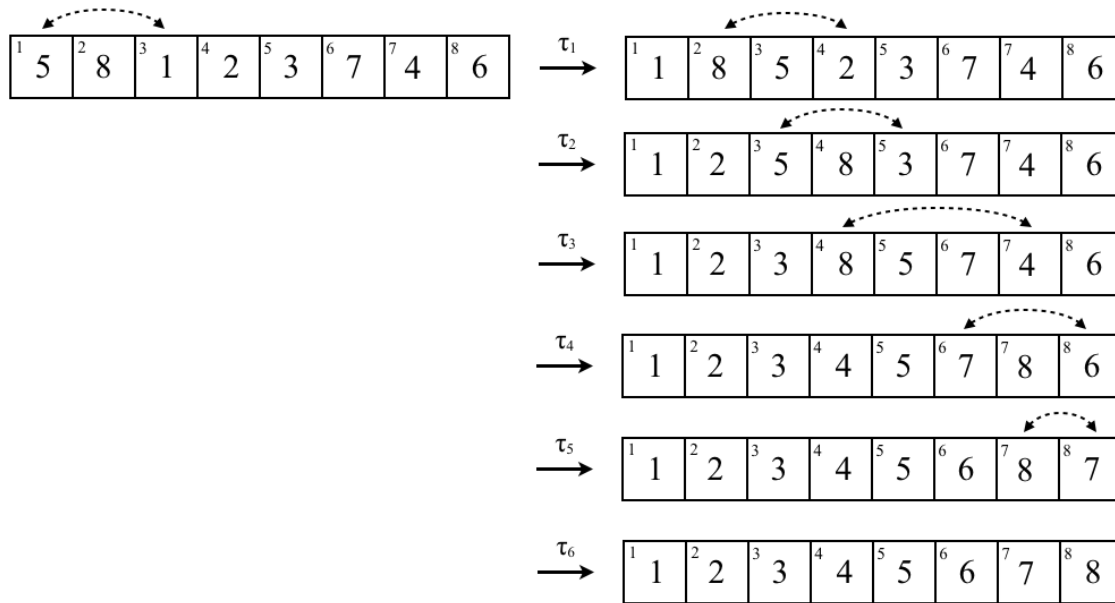
6.1   Introduction

Consider the permutation $\alpha = (1,3,5)(2,4,7,6,8)$. We would like to show it can be written as a product to 2-cycles.

To this permutation we consider the corresponding scramble of the Swap puzzle on 8 objects.

¹ 5	² 8	³ 1	⁴ 2	⁵ 3	⁶ 7	⁷ 4	⁸ 6

To solve the puzzle recall the objective is to restore all numbered tiles to their home positions where the only legal moves are to swap tiles from any two boxes (i.e. a 2-cycle). One possible play is as follows.

faculty of science

SFU department of mathematics

$$5 \; 8 \; 1 \; 2 \; 3 \; 7 \; 4 \; 6 \quad \xrightarrow{\tau_1} \quad 1 \; 8 \; 5 \; 2 \; 3 \; 7 \; 4 \; 6$$

$$\xrightarrow{\tau_2} \quad 1 \; 2 \; 5 \; 8 \; 3 \; 7 \; 4 \; 6$$

$$\xrightarrow{\tau_3} \quad 1 \; 2 \; 3 \; 8 \; 5 \; 7 \; 4 \; 6$$

$$\xrightarrow{\tau_4} \quad 1 \; 2 \; 3 \; 4 \; 5 \; 7 \; 8 \; 6$$

$$\xrightarrow{\tau_5} \quad 1 \; 2 \; 3 \; 4 \; 5 \; 6 \; 8 \; 7$$

$$\xrightarrow{\tau_6} \quad 1 \; 2 \; 3 \; 4 \; 5 \; 6 \; 7 \; 8$$

The dotted arrows indicate the two tile that are about to be swapped.

The permutations corresponding to the moves are:

$$\tau_1 = (1,3), \quad \tau_2 = (2,4), \quad \tau_3 = (3,5), \quad \tau_4 = (4,7), \quad \tau_5 = (6,8), \quad \tau_6 = (7,8)$$

and so the game-play corresponds to the composition: $\alpha \tau_1 \tau_2 \tau_3 \tau_4 \tau_5 \tau_6 = \varepsilon$. It follows that:

$$\alpha = \tau_6^{-1} \tau_5^{-1} \tau_4^{-1} \tau_3^{-1} \tau_2^{-1} \tau_1^{-1} \tag{1}$$
$$= (7,8)(6,8)(4,7)(3,5)(2,4)(1,3) \tag{2}$$

This is precisely what we wanted, α is written as a product of 2-cycles.

Exercise 6.1 *Write the permutation $\beta = (1,5,3,4,2)$ as a product of 2-cycles. Do this by using β as the starting scramble of the Swap puzzle, then solving the puzzle and keeping track of your moves as 2-cycles.*

6.2   Product of 2-Cycles

There doesn't seem to be anything special about the particular permutation α that we used in the last example. Our strategy was to just move the numbers, one at a time, to their home positions, and we chose to do this in increasing order, though we could have done it an any order we wanted.

This means we should be able to write *any* permutation as a product of 2-cycles. This is such an important observation that will state it as a theorem (a complete proof is given below).

Theorem 6.1 (Product of 2-Cycles) *Every permutation in $S_n, n > 1$, can be expressed as a product of 2-cycles.*

Playing with the Swap puzzle showed us intuitively why the theorem is true, it also gave us a method for finding such a decomposition into 2-cycles. As quick as it was to find a decomposition, we will

require a much quicker method: a way to "eyeball" the decomposition. Having to draw a Swap game each time we want to compute a decomposition into 2-cycles would be too time consuming. So how can we do this even more quickly?

Well, consider a 5-cycle: $\beta = (1, 5, 3, 4, 2)$. By direct computation we can check

$$(1, 5, 3, 4, 2) = (1, 5)(1, 3)(1, 4)(1, 2).$$

Check the product for yourself!

In general we have the following "quick" method for decomposing cycles.

Decomposition of a k-cycle into 2-cycles:
A k-cycle $(a_1, a_2, a_3, \ldots, a_{k-1}, a_k)$ in S_n can be decomposed into 2-cycles as follows:

$$(a_1, a_2, a_3, \ldots, a_{k-1}, a_k) = (a_1, a_2)(a_1, a_3) \cdots (a_1, a_{k-1})(a_1, a_k)$$

Using this method of decomposing k-cycles we can easily decompose any permutation by first writing the permutation as a product of disjoint cycles, and then decomposing each cycle into 2-cycles. For example, consider $\alpha = (1, 3, 5)(2, 4, 7, 6, 8)$ again:

$$\alpha = (1, 3, 5)(2, 4, 7, 6, 8) = (1, 3)(1, 5)(2, 4)(2, 7)(2, 6)(2, 8).$$

We now give a formal proof of Theorem 6.1.

Proof: First note that the identity can be expressed as $(1, 2)(1, 2)$, and so it is a product of 2-cycles. (This is why we needed $n > 1$ in the statement of the theorem.) Now consider any permutation $\alpha \in S_n$. We already know we can write α as a product of disjoint 2-cycles:

$$\alpha = (a_1, a_2, \ldots, a_r)(b_1, b_2, \ldots, b_s) \cdots (c_1, c_2, \ldots, c_t)$$

and each cycle can be decomposed into 2-cycles as we observed above:

$$\alpha = (a_1, a_2)(a_1, a_3) \cdots (a_1, a_r)(b_1, b_2)(b_1, b_3) \cdots (b_1, b_s) \cdots (c_1, c_2)(c_1, c_3) \cdots (c_1, c_t).$$

This completes the proof. $\square$

6.3   Solvability of Swap

A permutation α is obtainable as a puzzle position of Swap if and only if it can be expressed as a product of legal moves (2-cycles):

$$\alpha = \tau_k^{-1} \cdots \tau_2^{-1} \tau_1^{-1}.$$

See Equation 1 for example. In other words, if α is the current position then the moves required to solve the puzzle are $\tau_1, \tau_2, \ldots, \tau_k$.

Since every permutation is a product of 2-cycles (Theorem 6.1), then as a consequence we have the following:

Corollary 6.1 *The Swap puzzle, where the legal moves consist of swapping contents of any two boxes, is solvable from any configuration. In other words, all permutations in S_n can be obtained in the Swap puzzle on n-objects.*

This is the first in a series of solvability results we wish to obtain for all the puzzles.

Notice, the result only applies to Swap when the legal moves are swapping contents of any two boxes. We could consider other variations of Swap, for example:

Variation 1: Legal moves consist of swapping the contents of any other box with the object in box 1.

For this variation, a permutation α is obtainable as a position if and only if it can be written as a product of 2-cycles of the form: $(1, a)$ for $a \in \mathbb{Z}_n$. See Exercises 4 and 5.

Variation 2: Legal moves consist of picking any 3 boxes and cycling their contents either to the left or right (i.e. 3-cycles).

For this variation, a permutation α is obtainable as a position if and only if it can be written as a product of 3-cycles. See Exercises 6 and 7.

6.4   Exercises

1. For the permutation $\alpha = (1, 8, 4)(2, 3, 7)(5, 6)$ write it as a product of 2-cycles, first by: (1) Thinking of it as a scrambling of the Swap puzzle, and solving the puzzle as we did in the example in section 6.1, then by (2) Using the method developed in Section 6.2. Which method was the quickest to use?

2. Write the 3-cycle $(1, 2, 3)$ as a product of two 2-cycles.

3. For each of the following permutations, in cycle form, write it as a product of 2-cycles.

   (a) $(1, 6, 4, 3)$

   (b) $(2, 4, 7)(3, 9, 5, 8)$

   (c) $(1, 9, 4, 5)(3, 11, 4)(6, 7)$

   (d) $(1, 2, 3, 4, 5)(6, 7, 8, 9, 10)$

4. Using only the legal moves in Variation 1 of Swap described in Section 6.3, solve the puzzle with initial scrambling $\alpha = (1, 3, 5)(2, 4, 7, 6, 8)$.

5. Show for Variation 1 of Swap described in Section 6.3 that every permutation in S_n is obtainable as a puzzle position. This is equivalent to showing that every permutation in S_n can be written as a product of 2-cycle of the form $(1, a)$ where $a \in \mathbb{Z}_n$. (Hint: First show every cycle can be writing as a product of such transpositions.)

6. Consider only the legal moves in Variation 2 of Swap described in Section 6.3. Determine which of the following scramblings are solvable.
   (a) $\alpha = (1, 3, 5)(2, 4, 7, 6, 8)$
   (b) $\beta = (1, 6, 2)(3, 4, 8)(5, 7)$
   (Hint: Play the game of Swap with these configurations and see if you can solve it.)

7. Discover a solvability condition for Variation 2 of Swap described in Section 6.3. That is, determine the conditions a permutation α must satisfy in order for it to be obtainable as a puzzle configuration.

   (Given the current tools we have developed so far, this still may be a difficult problem. We'll soon develop the tools needed to completely solve this problem. However, for now see if you can discover a solvability condition.)