Lecture 2:
A Bit of Set Theory

Contents

A Rubik's cube is made up a number of different pieces. There are corner cubies, edge cubies, and center cubies (see Lecture 1 for definitions of these terms). Each collection of these pieces forms a set. Part of understanding how these pieces move around will be to understand how the cube moves (F, B, R, L, U, D) act on these sets. In this lecture we recall some basic terminology and notation from set theory which will form the foundation for our mathematical investigations into Rubik's cube and other puzzles.

This lecture corresponds to material in Section 1.2 of Joyner's text.

2.1 Sets and Subsets

A **set** is a well-defined collection of objects. The objects of the set are called **elements**, and are said to be **members** of, or **belonging** to, the set.

By *well-defined* we mean that for any element we wish to consider, we are able to determine, under some scrutiny, whether or not it is a member of the set.

Typically we will use capital letters, such as A, B, C, $\ldots$ to represent sets and lower case letters to represent elements. For a set A we write $x \in A$ **if** x **is an element of** A, and $y \notin A$ **if** y **is not an element in** A.

Sets are usually defined in one of two ways:

(a) Listing all of its elements in braces: $A = \{a, b, c, \ldots\}$. For example $A = \{1, 2, 3, 4, 5\}$ is the set of integers from 1 to 5. Therefore, $3 \in A$, but $6 \notin A$.

(b) Using *set-builder* notation: $A = \{x \mid x \text{ has property } P\}$. For example we could define the previous set A as $\{x \mid x \text{ is an integer and } 1 \leq x \leq 5\}$. The vertical bar " | " is read "such that". The symbols $\{x \mid \ldots\}$ are read "the set of all x such that ...".

Some basic sets of numbers we should be familiar with are:

- $\mathbb{Z} =$ the set of *integers* $= \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$,
- $\mathbb{N} =$ the set of *nonnegative integers* or *natural numbers* $= \{0, 1, 2, 3, \ldots\} = \{x \in \mathbb{Z} \mid x \geq 0\}$,
- $\mathbb{Z}^+ =$ the set of *positive integers* $= \{1, 2, 3, \ldots\} = \{x \in \mathbb{Z} \mid x > 0\}$,
- $\mathbb{Q} =$ the set of *rational numbers* $= \left\{ \frac{a}{b} \,\middle|\, a, b \in \mathbb{Z}, b \neq 0 \right\}$,
- $\mathbb{Q}^+ =$ the set of *positive rational numbers* $= \{x \in \mathbb{Q} \mid x > 0\}$,
- $\mathbb{R}$ is the set of *real numbers*.
- $\mathbb{Z}_n = \{1, 2, \ldots, n\}$ = the set of integers from 1 to n, where $n \in \mathbb{Z}^+$. [1]

Let A and B be sets. If all the elements of A also belong to B then we say A is a **subset** of B and we write $\boldsymbol{A \subset B}$. For example, $\mathbb{Z}^+ \subset \mathbb{Z}$ since every positive integer is itself an integer, but $\mathbb{Q} \not\subset \mathbb{Z}$, since there are rational numbers that are not integers, consider $\frac{1}{2}$.

Two sets A and B are said to be **equal**, and we write $\boldsymbol{A = B}$, if $A \subset B$ and $B \subset A$.

If a set has a finite number of elements then we say it is an **finite set**. Otherwise it is an **infinite set**. For any finite set A, $|A|$ denotes the number of elements in A and is called the **cardinality**, or *size*, of A. For example, $|\mathbb{Z}_n| = n$, whereas $\mathbb{Z}$ in an infinite set.

The **empty set**, or **null set**, is the set that contains no elements. The empty set is denoted by $\emptyset$, or $\{\}$, and has that property that $|\emptyset| = 0$.

Let A and B be two sets. The set of all elements belonging to either A or B is denoted $\boldsymbol{A \cup B}$ and is called the **union** of A and B. The set of all elements belonging to both A and B is denoted $\boldsymbol{A \cap B}$ and is called the **intersection** of A and B. The set of all elements not belonging to A is denoted A^c or sometimes by $\overline{A}$, and is called the **complement** of A. [2] The set of all elements in A that are not in B is denoted $\boldsymbol{A - B}$ and is called the **difference of** A **with** B. We sometimes refer to this as A **take away** B.

The **Cartesian product** of A and B is the set of all ordered pairs (x, y) where $x \in A$ and $y \in B$ and is denoted by $\boldsymbol{A \times B}$.

The following summarizes the different operations we have on sets:

$$\begin{aligned} A \cup B &= \{x \mid x \in A \textbf{ or } x \in B\}, \\ A \cap B &= \{x \mid x \in A \textbf{ and } x \in B\}, \\ A^c = \overline{A} &= \{x \mid x \notin A\}, \\ A - B &= \{x \mid x \in A \textbf{ and } x \notin B\} = A \cap B^c \\ A \times B &= \{(x, y) \mid x \in A \text{ and } y \in B\}. \end{aligned}$$

We call two sets **disjoint** if they have not element in common: A and B are disjoint if $A \cap B = \emptyset$.

[1] Sometimes $\mathbb{Z}_n$ is defined to be the set $\{0, 1, 2, \ldots n - 1\}$. When reading the literature you should be aware of which definition the author is using.

[2] In defining the complement we need to specify the elements we are considering, that is we need to consider A as a subset of some larger set. To see why, just think about what could be meant by $\mathbb{Z}^c$? Does this mean all elements in $\mathbb{Q}$ not in $\mathbb{Z}$, or all elements in $\mathbb{R}$ not in $\mathbb{Z}$, or something else entirely. The larger set to which we consider A as a subset will be called the *universe* or *universe of discourse* denoted by $\mathcal{U}$. It will be clear, given the context, as to what universe we are working in. What this means though is that we should really write $A^c = \{x \mid x \in \mathcal{U} \text{ and } \notin A\}$.

2.2 Laws of Set Theory

Some of the major laws that govern set theory are the following.

For any sets A, B, and C taken from a universe $\mathcal{U}$

1)	$(A^c)^c = A$	Law of Double Negation
2)	$(A \cup B)^c = A^c \cap B^c$ $(A \cup B)^c = A^c \cap B^c$	DeMorgan's Laws
3)	$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Laws
4)	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative Laws
5)	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive Laws
6)	$A \cup A = A$ $A \cap A = A$	Idempotent Laws
7)	$A \cup \emptyset = A$ $A \cap \mathcal{U} = A$	Identity Laws
8)	$A \cup A^c = \mathcal{U}$ $A \cap A^c = \emptyset$	Inverse Laws
9)	$A \cup \mathcal{U} = A$ $A \cap \emptyset = \emptyset$	Domination Laws
10)	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorbtion Laws

These set theoretic laws are similar to the arithmetic properties of the real numbers, where "$\cup$" plays the role of "$+$", and "$\cap$" plays the role of "$\times$". However, there are several differences.

We will prove the first part of the Distributive Law, and leave the proof of all others to the reader. See Exercise 7 and 8 for the second part of the Distributive Law and DeMorgan's Law,

Proof: Let $x \in \mathcal{U}$. Then

$$
\begin{aligned}
x \in A \cup (B \cap C) &\Leftrightarrow x \in A \quad \text{or} \quad x \in B \cap C \\
&\Leftrightarrow x \in A \quad \text{or} \quad x \text{ is in both } B \text{ and } C \\
&\Leftrightarrow x \in A \cup B \quad \text{and} \quad x \in A \cup C \\
&\Leftrightarrow x \in (A \cup B) \cap (A \cup C)
\end{aligned}
$$

This completes the proof. □

We also state a result about the cardinality of a disjoint union of sets.

Theorem 2.1 *Let $A_1, A_2, \ldots, A_n$ be disjoint finite sets. Then*

$$|A_1 \cup \cdots \cup A_n| = |A_1| + \cdots + |A_n|.$$

2.3 Examples Using SAGE

Example 2.1 *In this example we show how to define a set, and compute cardinalities, unions, intersections, differences and cartesian products.* SAGE.

SAGE

```
sage: S1=Set([1,2,3,4,5]);
sage: S2=Set([3,4,5,6,7]);
sage: S1;S2;
{1, 2, 3, 4, 5}
{3, 4, 5, 6, 7}
sage: S1.cardinality()
5
sage: S1.union(S2)
{1, 2, 3, 4, 5, 6, 7}
sage: S1.intersection(S2)
{3, 4, 5}
sage: S1.difference(S2)
{1,2}
sage: S2.difference(S1)
{6,7}
sage: CartesianProduct(S1, S2)
Cartesian product of {1, 2, 3, 4, 5}, {3, 4, 5, 6, 7}
sage:  CartesianProduct(S1,S2).list()
[[1, 3], [1, 4], [1, 5], [1, 6], [1, 7], [2, 3], [2, 4], [2, 5], [2, 6],
[2, 7], [3, 3], [3, 4], [3, 5], [3, 6], [3, 7], [4, 3], [4, 4], [4, 5],
[4, 6], [4, 7], [5, 3], [5, 4], [5, 5], [5, 6], [5, 7]]
sage: CartesianProduct(S1,S2).cardinality()
25
sage: 2 in S1
True
sage: 1 in S2
False
```

Example 2.2 SAGE *has a number of commonly used sets already built in:* $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$. *The commands are ZZ, NN, QQ, and RR, respectively.*

SAGE

```
sage: ZZ
Integer Ring
sage: 1 in ZZ
True
sage: 1/2 in ZZ
False
sage: 0 in NN
True
sage: -1 in NN
False
```

Example 2.3 *We can build a set by using properties in* SAGE. *Here we use Python's modulo operator %:* $a\%b$ *returns the remainder of* a *when divided by* b.

SAGE
```
sage: Set(x for x in range(0,10) if x%2==0)
{0, 2, 4, 6, 8}
sage: Set(x for x in range(0,10) if x%2==1)
{1, 3, 9, 5, 7}
```

Example 2.4 *The* `is_prime()` *function returns True if the input is a prime integer, and False if not. Such functions are called* ***boolean valued functions****. We can use boolean valued function to create subsets as this example shows.*

SAGE
```
sage: is_prime(29)
True
sage: is_prime(4)
False
sage: Set(x for x in range(0,10) if is_prime(x))
{2, 3, 5, 7}
sage: Set(x for x in range(0,1000) if is_prime(x)).cardinality()
168
```

Alternatively, we could use the `filter()` *command in Python. You can get more information on* `filter()` *by typing* `filter?` *at the* SAGE *prompt.*

SAGE
```
sage: S=Set(1..20)  #constructs a set of all integers from 1 to 20
sage: filter(is_prime,S)
[2, 3, 5, 7, 11, 13, 17, 19]
```

2.4 Exercises

1. Which of the following sets are equal?

   (a) $\{1,2,3\}$ (b) $\{2,3,1,3\}$ (c) $\{3,2,1,1,2\}$ (d) $\{1,3,3,2,1,3\}$

2. Let $A = \{1, \{1\}, \{2\}\}$. Which of the following statements are true?

   (a) $1 \in A$ (c) $\{1\} \subset A$ (e) $\{2\} \in A$ (g) $\{\{2\}\} \subset A$
   (b) $\{1\} \in A$ (d) $\{\{1\}\} \subset A$ (f) $\{2\} \subset A$ (h) $\{1,2\} \subset A$

3. Determine all the elements of the following sets.

   (a) $\{1 + (-1)^n \mid n \in \mathbb{N}\}$
   (b) $\{n \in \mathbb{N} \mid n \leq 20$ and n is divisible by $3\}$
   (c) $\{n \in \mathbb{N} \mid n \leq 20,\ n$ is prime, and $2n+1$ is divisible by $3\}$

4. Determine the cardinality of the following sets.

   (a) The set of all cubies of the Rubik's cube which have a blue facet.
   (b) The set of all corner cubies of the Rubik's cube which have a blue facet.

5. Consider the set $A = \{1, 2, 3, 4, 5\}$.
   (a) How many subsets of cardinality 1 does A have?
   (b) How many subsets of cardinality 2 does A have?
   (c) How many subsets does A have in total?
   (Hint: don't forget the empty subset, and the set A itself, when counting subsets.)
6. For $\mathcal{U} = \mathbb{Z}_{10}$, let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 4, 8\}$ and $C = \{1, 2, 3, 5, 7\}$. Determine each of the following.
   (a) $(A \cup B) \cap C$
   (b) $(A \cup B) - C$
   (c) $A^c \cap B^c$
   (d) $|A \cup B|$
7. Prove the second *Distributive Law* stated in Section 2.2.
8. Prove *DeMorgan's laws* stated in Section 2.2..