Lecture 13: Commutators

Contents

> *fanwuq:* Just solve one corner at a time like LBL until you get to last layer. Then, you can just use commutators to solve the rest of the corners.
>
> *JBogwith:* I'm sorry, I don't understand. I can get to the last layer, it is then where I get stuck. **What are commutators?**
>
> www.speedsolving.com forum discussion. Dec. 2007

In this lecture we look at a product known as a *commutator*. These types of move sequences are useful for *creating* moves on permutation puzzles.

Commutators are discussed in Section 5.8 of Joyner's text.

13.1 Commutators

When playing with permutation puzzles, certain move sequences can occur more often than others. For instance, a move sequence of the form "move 1, then move 2, then inverse of move 1, then inverse of move 2" turns out to be quite useful. This type of move is called a "commutator". As you read through this lecture, you will find it useful to have a puzzle on hand to try things out for yourself.

Definition 13.1 *If g, h are two elements of a group G, then we call the element*

$$[g,h] = ghg^{-1}h^{-1}$$

the ***commutator*** *of g and h.*

Note that if g and h commute then $[g,h]=e$. To see this observe,

$$[g,h] = ghg^{-1}h^{-1} = (gh)(g^{-1}h^{-1}) = (hg)(g^{-1}h^{-1}) = h(gg^{-1})h^{-1} = heh^{-1} = hh^{-1} = e.$$

Conversely, if $[g,h]=e$ then g and h commute. See Exercise 2. Commutators are useful in mathematics wherever non-commutative operations occur.

The commutator $[g, h]$ provides a measure of how much g and h fail to commute with each other. In particular. if g and h are permutations and they fail to commute with each other by "just a little bit" then $[g, h]$ will be close to the identity, i.e. it will only permute a few numbers. This is why commutators will be of interest to us in solving permutation puzzles, they will help us to "create" good moves. You may have just realized that you frequently use "commutator moves" when solving puzzles, if this is the case then you already have a working understanding of commutators.

Example 13.1 *Consider the symmetric group S_3 and the elements $s_1 = (1, 2)$, $s_2 = (1, 3, 2)$. Then the commutator $[s_1, s_2]$ is*

$$[s_1, s_2] = s_1 s_2 s_1^{-1} s_2^{-1} = (1, 2)(1, 3, 2)(1, 2)(1, 2, 3) = (1, 3, 2),$$

and the commutator $[s_2, s_1]$ is

$$[s_2, s_1] = s_2 s_1 s_2^{-1} s_1^{-1} = (1, 3, 2)(1, 2)(1, 2, 3)(1, 2) = (1, 2, 3).$$

It is not a coincidence that $[s_2, s_1] = [s_1, s_2]^{-1}$, see Exercise 3.

13.2 Creating Puzzle moves with Commutators

We will explore some properties of commutators of permutations and then see how we can apply what we learn to our standard collection of puzzles.

For a permutation $\alpha \in S_n$ define the **fixed set of** α to be the set of all numbers in $\mathbb{Z}_n = \{1, 2, 3, \dots, n\}$ that α doesn't move:

$$\text{fix}(\alpha) = \{m \in \mathbb{Z}_n \mid \alpha(m) = m\}.$$

The set of numbers that are not fixed by α, the ones that are moved, is the compliment of this set, which we denote by M_α:

$$M_\alpha = \overline{\text{fix}(\alpha)} = \{m \in \mathbb{Z}_n \mid \alpha(m) \neq m\}.$$

By way of contrast we will refer to this as the **moved set of** α. fix(α) is precisely the set of numbers that would appear as 1-cycles in the disjoint cycle form of α, and M_α are those numbers that appear in cycles of length ≥ 2. Since α and α^{-1} fix precisely the same objects It follows that fix$(\alpha) =$ fix(α^{-1}) and $M_\alpha = M_{\alpha^{-1}}$.

In terms of permutation puzzles, M_α is the list of all positions where the pieces are moved when α is applied, and fix(α) are those positions where the pieces are left alone.

We'll need one more bit of notation to simplify things to come. For a subset $A \subset \mathbb{Z}_n$ and a permutation $\alpha \in S_n$, we denote the set of all images of the elements of A under α as αA: [1]

$$\alpha A = \{\alpha(m) \mid m \in A\}.$$

Since α is injective then $|\alpha A| = |A|$.

Example 13.2 *For $\alpha = (1, 7, 3, 4, 12)(5, 9) \in S_{13}$, the set of objects that are moved is $M_\alpha = \{1, 3, 4, 5, 7, 9, 12\}$ and the set of objects that are fixed is fix$(\alpha) = \{2, 6, 8, 10, 11, 13\}$. For $A = \{2, 4, 6, 8, 10, 12\}$ and $B = \{3, 7, 11\}$, $\alpha A = \{\alpha(2), \alpha(4), \alpha(6), \alpha(8), \alpha(10), \alpha(12)\} = \{2, 12, 6, 8, 10, 1\}$, and $\alpha B = \{\alpha(3), \alpha(7), \alpha(11)\} = \{4, 3, 11\}$. This can be done in SAGE by using the `map` function: `map(f,L)` applies function `f` to each element of a list/set `L`.*

SAGE

```
sage: S13=SymmetricGroup(13)
sage: a=S13("(1,7,3,4,12)(5,9)")
sage: map(a,Set([2,4,6,8,10,12]))
[2,12,6,8,10,1]
sage: map(a,Set([3,7,11]))
[4,3,11]
```

[1] This type of set is sometimes denoted by $\alpha(A)$.

Now we are ready to investigate why the commutator $[\alpha, \beta]$ is likely to be "close" to the identity.

Let $\alpha, \beta \in S_n$, and m a number in $\mathbb{Z}_n$. Then m is moved by the commutator $[\alpha, \beta]$, i.e. $m \in M_{[\alpha,\beta]}$ if both:

(a) $m \in M_\alpha$ or $\beta(m) \in M_\alpha$, and

(b) $m \in M_\beta$ or $\alpha(m) \in M_\beta$.

In set notation, we can write this as:

$$M_{[\alpha,\beta]} \subset (M_\beta \cup \alpha^{-1}(M_\beta)) \cap (M_\alpha \cup \beta^{-1}(M_\alpha)). \tag{1}$$

To see why (b) is true assume that $m, \alpha(m) \notin M_\beta$, then $[\alpha, \beta]$ must leave m fixed:

$$[\alpha, \beta](m) = (\alpha\beta\alpha^{-1}\beta^{-1})(m) = \beta^{-1}(\alpha^{-1}(\beta(\alpha(m))) = \beta^{-1}(\alpha^{-1}(\alpha(m))) = \beta^{-1}(m) = m,$$

so $m \notin M_{[\alpha,\beta]}$. This proves (b). The proof of (a) is analogous.

We can describe the set of pieces that are moved in a more verbal way. First we need an alternate expression for (1). An equivalent way to write the set on the right in (1) is $(M_\alpha \cap M_\beta) \cup \alpha^{-1}(M_\alpha \cap M_\beta) \cup \beta^{-1}(M_\alpha \cap M_\beta)$. This follows from the facts that $\gamma(M_\delta \cap M_\sigma) = \gamma M_\delta \cap \gamma M_\sigma$ and $\gamma^{-1} M_\gamma = M_\gamma$ (See Exercises 7 and 8). Therefore,

$$M_{[\alpha,\beta]} \subset (M_\alpha \cap M_\beta) \cup \alpha^{-1}(M_\alpha \cap M_\beta) \cup \beta^{-1}(M_\alpha \cap M_\beta). \tag{2}$$

Notice $M_\alpha \cap M_\beta$ is the set of pieces affected by both α and β, and $\alpha^{-1}(M_\alpha \cap M_\beta)$ is the set of pieces that are moved to $M_\alpha \cap M_\beta$ by α, and $\beta^{-1}(M_\alpha \cap M_\beta)$ is the set of pieces moved to $M_\alpha \cap M_\beta$ by β. In words (2) says the following:

Remark 13.1 *If α and β are puzzle moves, the permutation produced by $[\alpha, \beta]$ only affects pieces that are in, or moved to, locations that are moved by both α and β.*

This remark will guide our choices for α and β. We want very little overlap in these two moves, and we want very few new pieces moved into this overlap. It can be challenging to find two moves with this property, but we can state some weaker conditions as to when $[\alpha, \beta]$ may still be a good move.

Since $|\alpha^{-1}(M_\beta)| = |M_\beta|$ and $|\beta^{-1}(M_\alpha)| = |M_\alpha|$ then (1) tells us that $|M_{[\alpha,\beta]}|$ is at most twice the size of the smaller of the sets M_α and M_β:

$$|M_{[\alpha,\beta]}| \leq 2 \min\{|M_\alpha|, |M_\beta|\}. \tag{3}$$

So if one of $|M_\alpha|$ and $|M_\beta|$ is small, then so is $|M_{[\alpha,\beta]}|$. Which means $[\alpha, \beta]$ may be a good puzzle move. We can actually say something more here.

Remark 13.2 *If the commutator $[\alpha, \beta]$ is to move the fewest possible pieces then α should bring as few new pieces into the locations where they will be moved by β. In other words, $\alpha^{-1}(M_\beta) \cap M_\alpha$ should be small.*

This remark is weaker than Remark 13.1 but its conditions are sometimes easier to check in practice. With that little bit of theory behind us, let's put it into practice on a number of our favourite puzzles.

13.2.1 Rubik's Cube

Here we consider the Rubik's cube group RC_3 generated by permutations R, L, U, D, F, B. It is best if you have your Rubik's cube handy as your read this part of the lecture.

Consider the move sequence $URU^{-1}R^{-1}$. Although it is not the identity (apply it to your cube to see this), it is a lot less complex than UR alone.

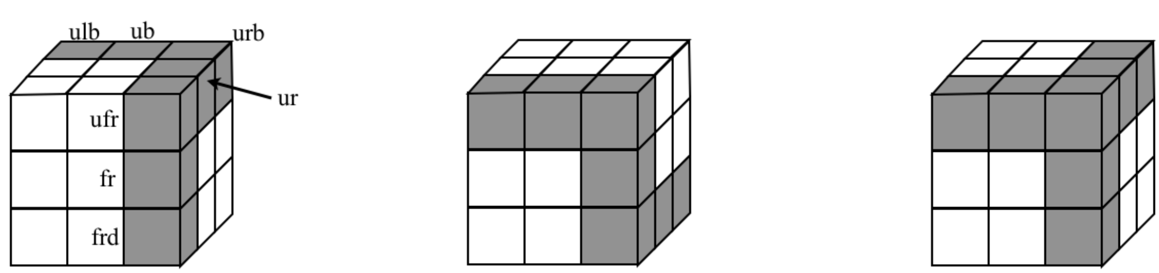
SAGE
```
sage: S48=SymmetricGroup(48)
sage: R=S48("(25,27,32,30)(26,29,31,28)(3,38,43,19)(5,36,45,21)(8,33,48,24)")
sage: L=S48("(9,11,16,14)(10,13,15,12)(1,17,41,40)(4,20,44,37)(6,22,46,35)")
sage: U=S48("(1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)(11,35,27,19)")
sage: D=S48("(41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)(16,24,32,40)")
sage: F=S48("(17,19,24,22)(18,21,23,20)(6,25,43,16)(7,28,42,13)(8,30,41,11)")
sage: B=S48("(33,35,40,38)(34,37,39,36)(3,9,46,32)(2,12,47,29)(1,14,48,27)")
sage: RC3=S48.subgroup([R,L,U,D,F,B])      # define Rubik's cube group to be RC3
```

SAGE
```
sage: commutator = lambda x,y: x*y*x^(-1)*y^(-1)      # define a function called commutator
sage: commutator(U,R)
(1,3,9,33,35,27)(2,5,21)(8,24,19,43,25,30)(26,28,34)
sage: commutator(U,R).order()
6
sage: U*R
(1,38,43,19,11,35,32,30,25,17,9,48,24,8,6)(2,36,45,21,5,7,4)(3,33,27)(10\
,34,29,31,28,26,18)
sage: (U*R).order()
105
```

In the above code we defined a function called `commutator` which takes two arguments x and y and returns the product $xyx^{-1}y^{-1}$. We use a Python `lambda` function to do this, which is just a quick way to define a function in one line where no complicated decision making has to be done. Of course, we really didn't need to define the function, we could have just typed in `U*R*U^-1*R^-1`, but with this function now defined we can quickly work out other commutators with less typing (just cut-and-paste).

Why should we have expected $URU^{-1}R^{-1}$ to be less complicated than UR? Many of the pieces that are moved by UR are returned to where they started by $U^{-1}R^{-1}$. For instance, consider the cubie in the ufl cubicle. The move U sends it to the ubl cubicle which is untouched by the move R, then it is moved back to the ufl cubicle by move U^{-1}, and finally move R^{-1} leave it where it is. This means the move sequence $URU^{-1}R^{-1}$ leaves the ufl cubicle untouched.

In general, if a piece is moved by U to a place that is not moved by R, then it will be moved back by U^{-1} to where it started. If the place where it started is not moved by R^{-1} – or equivalently, is not moved by R – then $URU^{-1}R^{-1}$ ends up leaving the piece where it started. Only where there is an overlap of the moves U and R are the pieces affected. The permutation produced by $URU^{-1}R^{-1}$ only affects pieces that are in, or moved to, locations common to both the *up* and *right* faces. This is precisely what (2) (and thus Remark 13.1) says. See Figure 1a.



(a) Possible cubies moved by $URU^{-1}R^{-1}$.

(b) **Z-commutator**: Shading indicates locations changed by $FRF^{-1}R^{-1}$

(c) **Y-commutator**: Shading indicates locations changed by $FR^{-1}F^{-1}R$

Figure 1: Y- and Z- commutators

In terms of the notation introduced, since the 3 pieces moved by both U and R are $M_U \cap M_R = \{urf, ur, urb\}$, and the pieces moved to these positions by U and R are:

$$U^{-1}(M_U \cap M_R) = \{ubr, ub, ubl\} \quad \text{and} \quad R^{-1}(M_U \cap M_R) = \{lrd, fr, frb\},$$

then $URU^{-1}R^{-1}$ moves at most the 7 pieces shaded in Figure 1a.

Commutators of two faces which share an edge occur so frequently that they have been given special names: the **Z-commutator** is $[F, R] = FRF^{-1}R^{-1}$, and the **Y-commutator** is $[F, R^{-1}] = FR^{-1}F^{-1}R$. The names, Z-commutator and Y-commutator are used regardless of which two adjacent faces are used, all that matters is both faces are turned in the same direction (Z-commutator), or turned in opposite directions (Y-commutator). See Figure 1.

The cycle structure of a commutator may be such that taking powers of it will kill-off some cycles, and therefore reduce the number of pieces moved even further. This is illustrated in the next exercise.

Exercise 13.1 *If x and y are basic moves of Rubik's cube associated with faces that share an edge, verify that*

(a) $[x, y]^2$ *permutes exactly* 3 *edges and does not permute any corners;*

(b) $[x, y]^3$ *permutes exactly* 2 *pairs of corners and does not permute any edges.*

Let's try to create a move-sequence, using commutators, that moves only a few pieces of the cube around. Looking back at (2) (and Remark 13.1) we keep in mind that for any move sequences x and y, the commutator only affects pieces that are in, or moved to, locations that are moved by both x and y. For example, consider the move

$$x = LD^2L^{-1}.$$

Amongst other things, this move sequence takes rbd to ufl, and leaves all other cubies in the *up* face in their original positions. If we then consider the move

$$y = U,$$

there is only one cubie that both x and y move: the ufl cubie. Since y only moves ufr to ulf, and x only moves rbd to luf, then the only cubies that are possibly affected by $[x, y]$ are: ufl, ufr, and rbd. Trying this new move sequence out we see it moves all 3 of these cubies: the ones shaded in the Figure 2.The order of $[x, y] = [LD^2L^{-1}, U]$ is 3.

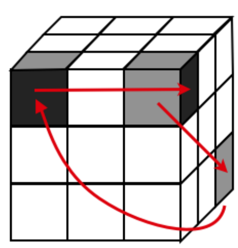


Figure 2: cubies moved by $[LD^2L^{-1}, U]$.

SAGE

```
sage: commutator(L*D^2*L^(-1),U)
(6,8,38)(11,19,32)(17,25,48)
sage: commutator(L*D^2*L^(-1),U).order()
3
```

As another example, let's construct a move to untwist two corner pieces. Consider the two moves

$$x = L^{-1}D^2LBD^2B^{-1}, \quad \text{and} \quad y = U.$$

The first move may look a little complicated, but try it out for yourself. It is actually quite simple: it moves ulb to the bottom layer, then brings it back into its home location, but twisted into position blu. The only location that is affected by both x and y is ulb, but x does not move it to another location, it only twists it in place. Once x is applied, then applying y followed by x^{-1} restores the *down* and *middle* layers of the cube, and will untwist the piece that moved from ulf to ubl by y. Finally y^{-1} moves the piece that started in ufl back home, but now twisted. The result is that $[x, y]$ twists the corner piece in ulf clockwise, and the corner piece in ulb counter-clockwise as shown in Figure 3. When we write the move sequence for $[x, y] = [L^{-1}D^2LBD^2B^{-1}, U]$ it is an impressive 14 moves long:

$$[x, y] = L^{-1}D^2LBD^2B^{-1}UBD^2B^{-1}LD^2L^{-1}U^{-1}.$$

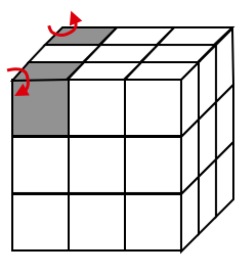


Figure 3: cubies moved by $[L^{-1}D^2LBD^2B^{-1}, U]$.

The move notation that we are using doesn't take into account that we can twist the whole cube around in our hands. This may make it difficult to see that a move sequence is a commutator. For example, the move sequence

$$x = U^2LR^{-1}F^2L^{-1}R$$

doesn't look like it has the form of a commutator. However, if we let $\boldsymbol{\mathcal{R}}$ denote a clockwise rotation of the whole cube around an axis through the right face, then F^2 can be written as $\boldsymbol{\mathcal{R}}U^2\boldsymbol{\mathcal{R}}^{-1}$ and so x can be seen to be the move sequence:

$$\begin{aligned} x &= U^2LR^{-1}\boldsymbol{\mathcal{R}}U^2\boldsymbol{\mathcal{R}}^{-1}RL^{-1} \\ &= [U^2, LR^{-1}\boldsymbol{\mathcal{R}}], \end{aligned}$$

which is a commutator. This move sequence is order 3 and permutes 3 edge cubies as shown in Figure 4.

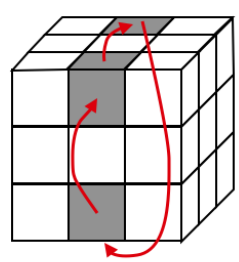


Figure 4: cubies moved by $[U^2, LR^{-1}\boldsymbol{\mathcal{R}}] = U^2LR^{-1}F^2L^{-1}R$.

13.2.2 Hungarian Rings

We now consider the Hungarian Rings group HR generated by permutations R and L. It is best if you have your puzzle handy (virtual or physical) as your read through this part.

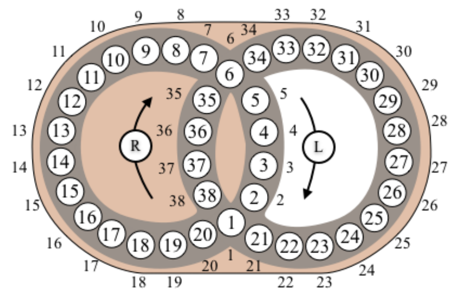


Figure 5: Hungarian rings puzzle.

Since each move affects over half the pieces of the puzzle then (3) isn't very helpful. It says a commutator moves at most 40 pieces, but this is more than the number of pieces on the puzzle. However, using Remark 13.2 as a guide will help us create moves that affect only a few pieces.

This puzzle has the feature that the two rings intersect at only two locations (1 and 6), so the two moves L and R have very little overlap. Specifically, $M_L = \{1, 2, 3, \ldots, 20\}$ and $M_R = \{1, 6\} \cup \{21, 22, \ldots, 38\}$, and the intersection is $M_L \cap M_R = \{1, 6\}$. Consequently, from (2) a commutator $[R^i, L^j]$, $1 \leq i, j \leq 19$, moves at most 6 disks:

$$
\begin{aligned}
M_{[R^i,L^j]} &= \{1, 6\} \cup R^{-i}\{1, 6\} \cup L^{-j}\{1, 6\} && (4)\\
&= \{1, 6, R^{-i}(1), R^{-i}(6), L^{-j}(1), L^{-j}(6)\}. && (5)
\end{aligned}
$$

Remark 13.3 *On the Hungarian Rings puzzle, any commutator of the form $[L^i, R^j]$ moves at most 6 disks.*

This maximum number can be reached, for example the commutator $[L, R^{-1}]$ moves 6 disks: $1, 2, 6, 7, 34, 38$.

SAGE

```
sage: S38=SymmetricGroup(38)
sage: L=S38("(1,20,19,18,17,16,15,14,13,12,11,10,9,8,7,6,5,4,3,2)")
sage: R=S38("(1,38,37,36,35,6,34,33,32,31,30,29,28,27,26,25,24,23,22,21)")
sage: commutator(L,R^(-1))  # this is our user defined function - see a previous code block
(1,38,2)(6,34,7)
```

For $[L^i, R^j]$ to move fewer than 6 disks we would need some elements in (4) to be the same. Remark 13.2 tells us we should look for a move L^j which moves as few new disks into spots 1 and 6 as possible. The values of j that do this are 5 and 15 (or equivalently -5). If we take $i, j \in \{5, 15\}$ then one of $L^{-i}(1) = 6$ or $L^{-i}(6) = 1$ is true, and one of $R^{-j}(1) = 6$ or $R^{-j}(6) = 1$ is true, which means $M_{[R^i,L^j]}$ has 4 elements. This gives the following.

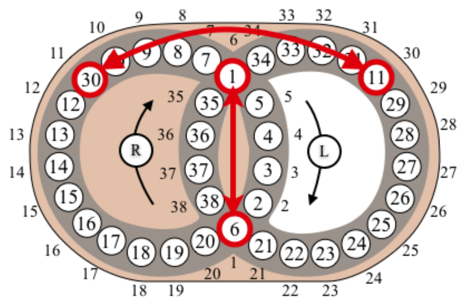
Remark 13.4 *On the Hungarian Rings puzzle, any commutator of the form $[L^i, R^j]$ where $i, j \in \{5, 15\}$ moves exactly 4 disks.*

As an example,

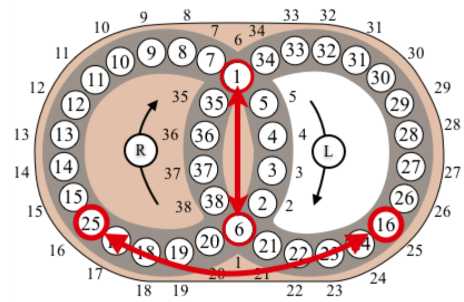
$$[L^5, R^{-5}] = (1, 6)(11, 30), \quad \text{and} \quad [L^{-5}, R^5] = (1, 6)(16, 25).$$

Knowing these commutators is enough to solve the colour version of this puzzle. We'll pick this up in a later lecture.

We could have SAGE determine all the powers i and j for which $|M_{[L^i,R^j]}| = 4$. The first line of code below defines a function `M` whose input is a permutation `a` and whose output is the set of all numbers between 1 and n which `a` moves. The command `len(a.tuple())` just gets the value of n from the permutation in cycle form by first converting the permutation to a list, then computing its length.



(a) $L^5R^{-5}L^{-5}R^5$



(b) $L^{-5}R^5L^5R^{-5}$

Figure 6: Basic commutators on the Hungarian Rings puzzle

SAGE
```
sage: M= lambda a: Set([ m for m in (1..len(a.tuple())) if a(m)!=m])
sage: for i in range(1,20):
sage:     for j in range(1,20):
sage:         if M(commutator(L^(i),R^(j))).cardinality()==4:
sage:             print i, j
5 5
5 15
15 5
15 15
```

13.2.3 Oval Track Puzzle

The Oval Track group OT generated by permutations R and T. As with the other sections, it is best if you have your puzzle handy (virtual or physical) as you read through this part.

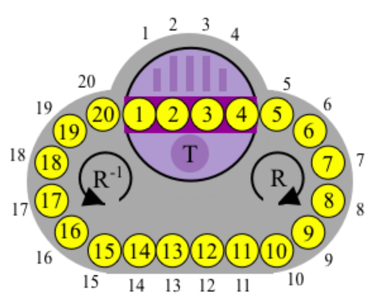


Figure 7: Oval Track puzzle.

A natural type of commutator to consider for this puzzle is $[R^i, T]$ where R^i is a rotation of the disks around the track by i positions, and T is a rotation of the turntable. In this case $M_{R^i} = \{1, 2, 3, \ldots, 20\}$ and $M_T = \{1, 2, 3, 4\}$, and so by (3) a commutator of this type will move at most $2\min\{20, 4\} = 8$ disks.

This maximum can sometimes be reached, for example the commutator $[R^{-4}, T] = (1, 4)(2, 3)(5, 8)(6, 7)$ moves 8 disks: $1, 2, 3, 4, 5, 6, 7, 8$.

For the commutator $[R^{-1}, T]$ the numbers of disks moved is less. This is because R^{-1} moves only one new disk into the turntable, namely disk number 5. As a result $[R^{-1}, T] = (1, 4, 2, 5, 3)$ only moves 5 disks: $1, 2, 3, 4, 5$.

SAGE
```
sage: S20=SymmetricGroup(20)
sage: R=S20("(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20)")
```

```
sage: T=S20("(1,4)(2,3)")
sage: OT=S20.subgroup([R,T])
sage: commutator(R^(-4),T)   # this is our user defined function - see a previous code block
(1,4)(2,3)(5,8)(6,7)
sage: commutator(R^(-1),T)
(1,4,2,5,3)
```

We will look for a commutator of the form $[R^{-i}, T]$ with a useful cycle structure. We can run a simple loop in SAGE to see this quite quickly.

SAGE

```
sage: for i in (1..19):
sage:     print i, commutator(R^(-i),T)
1 (1,4,2,5,3)
2 (1,4,5)(2,3,6)
3 (1,4,7)(2,3)(5,6)
4 (1,4)(2,3)(5,8)(6,7)
5 (1,4)(2,3)(6,9)(7,8)
6 (1,4)(2,3)(7,10)(8,9)
7 (1,4)(2,3)(8,11)(9,10)
8 (1,4)(2,3)(9,12)(10,11)
9 (1,4)(2,3)(10,13)(11,12)
10 (1,4)(2,3)(11,14)(12,13)
11 (1,4)(2,3)(12,15)(13,14)
12 (1,4)(2,3)(13,16)(14,15)
13 (1,4)(2,3)(14,17)(15,16)
14 (1,4)(2,3)(15,18)(16,17)
15 (1,4)(2,3)(16,19)(17,18)
16 (1,4)(2,3)(17,20)(18,19)
17 (1,18,4)(2,3)(19,20)
18 (1,20,4)(2,19,3)
19 (1,3,20,2,4)
```

For $4 \leq i \leq 16$ it is no surprise the cycle structure is a product of four disjoint 2-cycles. The commutator $[R^{-i}, T]$ brings four *new* disks: disks $i+1, i+2, i+3, i+4$, into the turntable, permutes them, then sends them back, and finally it permutes the original four disks: $1, 2, 3, 4$. The resulting permutation is:

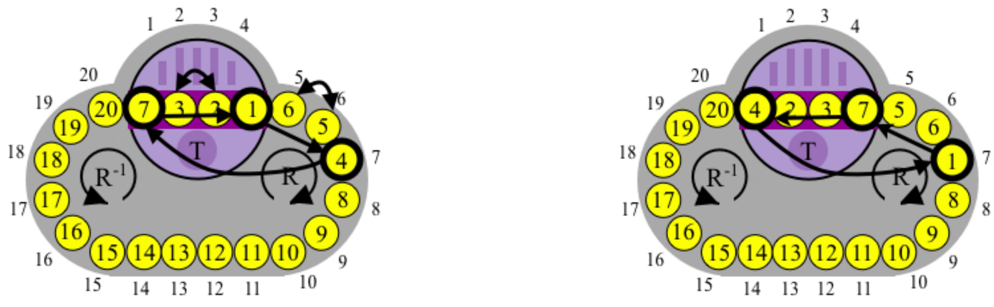
$$[R^{-i}, T] = (1,4)(2,3)(i+1, i+4)(i+2, i+3) \quad \text{for } 4 \leq i \leq 16.$$

Consider the case when $i = 1, 2, 3$. The cases when $i = 17, 18, 19$ are similar, only the rotation move R^{-i} is clockwise $20 - i$ spots. Perhaps at this point we should mention why we are considering a negative exponent on R. This is really just because for $i = 1, 2, 3$, $[R^{-i}, T]$ only brings other small numbered disks into the turntable. If we were to rotate clockwise first, then some high numbered disks (i.e. $20, 19$, etc) would enter the turntable. Eventually we would like to consider variations of the puzzle where the number of disks is changed, so it would be nice to have our results expressed in such a way that does not depend on the total number of disks.

The commutator $[R^{-3}, T]$ has a particularly advantageous cycle structure, it consists of one 3-cycle and two 2-cycles. We can kill-off the 2-cycles by applying the commutator twice:

$$\begin{aligned}[R^{-3}, T]^2 &= ((1,4,7)(2,3)(5,6))^2 = (1,4,7)^2(2,3)^2(5,6)^2 \\ &= (1,7,4).\end{aligned}$$

This should be a useful move to know in solving end-game problems on this puzzle. Also, since commutators are even (see Exercise 1) this is the smallest permutation we could get using products of commutators.



(a) $R^{-3}TR^3T = (1,4,7)(2,3)(5,6)$ (b) $(R^{-3}TR^3T)^2 = (1,7,4)$

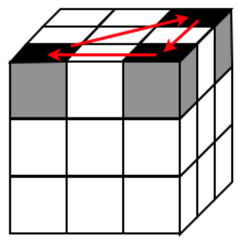
Figure 8: Basic commutators on the Oval Track puzzle

13.3 Exercises

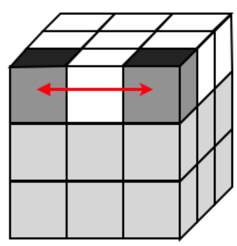
1. Let $\alpha, \beta \in S_n$. Show that the commutator $[\alpha, \beta]$ is an even permutation.
2. Show that if $[g, h] = e$ then g and h commute.
3. Let G be a group and $g, h \in G$, show that $[g, h]^{-1} = [h, g]$.
4. Prove each of the following.
   (a) A permutation commutes with the commutator $[\alpha, \beta]$ if and only if $[\alpha, \beta] = [\beta, \alpha^{-1}]$.
   (b) A permutation commutes with the commutator $[\alpha, \beta]$ if and only if $[\beta, \alpha] = [\alpha^{-1}, \beta]$.
   (c) Both α and β commute with $[\alpha, \beta]$ if and only if $[\alpha, \beta] = [\beta, \alpha^{-1}] = [\beta^{-1}, \alpha]$.
5. We have already seen that if α and β commute then $(\alpha\beta)^n = \alpha^n\beta^n$. But this can fail if α and β do not commute. Show that if α and β satisfy the weaker hypothesis that both commute with $[\alpha, \beta]$, then for every positive integer n, $(\alpha\beta)^n = \alpha^n\beta^n[\beta, \alpha]^{n(n-1)/2}$.
6. Let $\alpha, \beta \in S_n$.
   (a) If M_α and M_β have no locations (elements) in common (i.e. $M_\alpha \cap M_\beta = \emptyset$), what is the permutation of $[\alpha, \beta]$?
   (b) If M_α and M_β have two locations (elements) in common (i.e. $|M_\alpha \cap M_\beta| = 2$), what is the largest number of locations which can be in $M_{[\alpha,\beta]}$.
   (c) If M_α and M_β have two locations (elements) in common, what are the possibilities for $|M_{[\alpha,\beta]}|$.
7. Let $\gamma, \delta, \sigma \in S_n$. Prove the following.
   (a) $M_\gamma = M_{\gamma^{-1}}$
   (b) $\gamma^{-1}M_\gamma = M_\gamma$
   (c) $\gamma(M_\delta \cap M_\sigma) = \gamma M_\delta \cap \gamma M_\sigma$
8. Prove that for permutations α and β,
$$(M_\beta \cup \alpha^{-1}(M_\beta)) \cap (M_\alpha \cup \beta^{-1}(M_\alpha)) = (M_\alpha \cap M_\beta) \cup \alpha^{-1}(M_\alpha \cap M_\beta) \cup \beta^{-1}(M_\alpha \cap M_\beta).$$
   (Hint: Use the Distributive Law: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, and the results of Exercise 7.)

Rubik's Cube:

9. Find the order of the Y-commutator $[F, R^{-1}] = FR^{-1}F^{-1}R$ and of the Z-commutator $[F, R] = FRF^{-1}R^{-1}$.
10. Find the order of $[R, [F, U]]$.
11. What is the permutation produced by $[F, R^{-1}][R, U^{-1}][U, F^{-1}]$?
12. Show that
    (a) $[F, R^{-1}]^5 = R^{-1}[F, R]R$
    (b) $[F^{-1}, R^{-1}] = R^{-1}F^{-1}[F, R]FR$.
13. What is the permutation produced by $[(R^2U^2F^2)^3, U^2]$?
14. 3**-cycle of corners.** In this exercise you will build as a commutator a move which cycles 3 corner cubies as shown in the diagram.



    (a) To begin with, consider the move sequence $\alpha = F^{-1}D^{-1}FR^{-1}D^2RF^{-1}DF$. Verify that this move swaps the two corner cubes in the *up* layer, keeping their orientation (i.e. the *up* colour remains in the *up* face, which in the diagram is indicated by black). The lightly shaded cubies in the *middle* and *down* layer in the diagram move around, but the unshaded cubies remain fixed.



    You may wonder how this move was constructed. The idea is to basically take one or two cubies from the *up* layer, move them to the bottom layer, do some various moves, then bring them back to the *up* layer. Since we don't require pieces in the *middle* and *down* layers to be returned home, coming up with these moves isn't so difficult.

    (b) Since α only affects two cubies in the *up* layer, let $\beta = U$ and consider the commutator $[\alpha, \beta]$. Can you predict the effect of this move on the cubies? Hint: Remark 13.1 tells us which cubies can be affected. And with a little more thought you should be able to see how they are affected.
    (c) Perform the move $[\alpha, \beta] = F^{-1}D^{-1}FR^{-1}D^2RF^{-1}DFU(F^{-1}D^{-1}FR^{-1}D^2RF^{-1}DF)^{-1}U^{-1}$ and verify your prediction from the previous part.
15. **Flip** 2 **adjacent edges**. Let $S\ell_R$ denote the "slice move" which consists of rotating the middle slice, parallel to the R face, in the clockwise direction, from the perspective of the R face. Consider the move sequence
$$\alpha = S\ell_R U S\ell_R^{-1} U^{-1} S\ell_R U^2 S\ell_R^{-1}.$$
    (a) Verify α flips the edge in the fd position, and fixes everything else in the *down* layer.

(b) Since α only affects one cubies in the *down* layer, let $\beta = D$ and consider the commutator $[\alpha, \beta]$. Can you predict the effect of this move on the cubies?

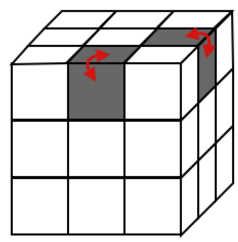
(c) Perform the move $[\alpha, \beta]$ and verify your prediction from the previous part.

16. **Another, flip** 2 **adjacent edges**. If we instead would like to flip 2-edges in the *up* layer. We could consider the move sequence
$$\alpha = S\ell_R^{-1} D S\ell_R D^{-1} S\ell_R^{-1} D^2 S\ell_R.$$

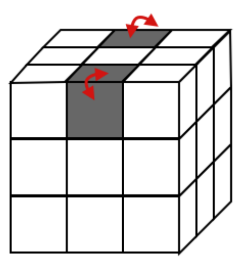
(a) Verify α flips the edge in the uf position, and fixes everything else in the *up* layer.

(b) Since α only affects one cubies in the *up* layer, let $\beta = U$ and consider the commutator $[\alpha, \beta]$. Can you predict the effect of this move on the cubies?

(c) Perform the move $[\alpha, \beta]$ and verify your prediction from the previous part. The move sequence should produce the double edge flip as shown in the figure below.



17. **Flip** 2 **opposite edges**. Find moves α and β so that the commutator $[\alpha, \beta]$ flips two opposite edges (as shown in the diagram below), and fixes everything else. (Hint: Modify the moves in the previous exercise.)



18. Investigate the commutators $[\alpha, \beta]$ for each of the following choices of α and β.

(a) $\alpha = RUR^{-1}$ and $\beta = D^{-1}$

(b) $\alpha = F^{-1}D^{-1}FR^{-1}D^2RF^{-1}DF$ and $\beta = U^2$

(c) $\alpha = RUR^{-1}U^{-1}RUR^{-1}$ and $\beta = D^{-1}$

(d) $\alpha = S\ell_R^{-1}$ and $\beta = S\ell_U^{-1}$. ($S\ell_X$ denotes a slice move of the middle slice parallel to face X.)

19. Create some of your own moves using commutators. Start by creating a move α which affects very few cubies in the *up* layer. Then take the commutator with $\beta = U$. Try to predict what your move with do before you even apply it.

Hungarian Rings:

20. Exploring the following commutators on the Hungarian Rings puzzle. Express the resulting permutation in disjoint cycle form.

(a) $[L,R]$ (d) $[R,L^{-1}]$ (g) $[L^5,R^{-1}]$ (j) $[R^5,L^5]$
(b) $[L,R^{-1}]$ (e) $[L^5,R]$ (h) $[L,R^{-5}]$ (k) $[L^{-5},R^{-5}]$
(c) $[R,L]$ (f) $[R,L^5]$ (i) $[L^5,R^5]$ (l) $[L^5,R^{-5}]$

21. **Getting a 3-cycle with compound commutators.** In this exercise we investigate the compound commutator: $[[L^5,R^5],R^{-1}LR]$. It may look pretty complicated at first glance, but its construction has been well controlled. Let $\alpha=[L^5,R^5]$ and $\beta=R^{-1}LR$, so the compound commutator is $[\alpha,\beta]$. The overlap of pieces moved by both α and β consists of a single disk as we'll see below. This indicates that the commutator $[\alpha,\beta]$ will likely be a good move to know.

   (a) Show that the permutation corresponding to the commutator $\alpha=[L^5,R^5]$ is $(1,25)(6,11)$. Conclude that $M_\alpha=\{1,6,11,25\}$.
   (b) Show that the only pieces of the right ring that β affects are the pieces in positions 34 and 38. Note that β affects all pieces in the left ring, except for 1 and 6. Conclude that
   $$M_\beta=(M_L-\{1,6\})\cup\{34,38\}=\{38\}\cup\{20,19,18,\ldots,8,7\}\cup\{34\}\cup\{5,4,3,2\}.$$
   (c) If you didn't already do so in the previous part, determine the cycle form of β.
   (d) Show that $M_\alpha\cap M_\beta=\{11\}$.
   (e) Show that $\alpha^{-1}(M_\alpha\cap M_\beta)=\{6\}$ and $\beta^{-1}(M_\alpha\cap M_\beta)=\{12\}$.
   (f) Conclude from formula (2) that $[\alpha,\beta]$ moves only $6, 11$, and 12, and verify that $[\alpha,\beta]=(6,11,12)$.

Note: One could just use SAGE to compute $[\alpha,\beta]=[[L^5,R^5],R^{-1}LR]$, however this wouldn't help to understand how to "build" this useful commutator in the first place. The exercises above are to get you to investigate how the commutator was constructed, so you may discover how to build your own commutators in the future.

Oval Track:

22. Determine the permutation corresponding to the commutator $[R^{-1}TR,T]$ on the Oval Track puzzle.

23. Consider the variation of the Oval Track puzzle where the turntable move T corresponds to the permutation $T=(4,3,2,1)$. See Figure 9.

   (a) Show that $[R^{-1},T]=(1,2,5)$.
   (b) Show that $[T^{-1},R^{-1}]=(1,5,4)$.
   (c) Show the product $[R^{-1},T][T^{-1},R^{-1}]$ is $(1,2,4)$.
   (d) Since commutators are even, so is any product of commutators. This means that 3-cycles are the best we can do. However, the turntable move T is odd, so combining this move with a commutator may allow us to produce a 2-cycle. See what the product $[R^{-1},T][T^{-1},R^{-1}]T$ gives you.

24. **Varying the turntable move T of the Oval Track puzzle.** In this exercise you will investigate, with the help of SAGE, some variations of the Oval Track puzzle. In all variations[2], we assume there are 20 disks, and the usual move consisting of rotating the pieces along the track is R. We will vary the turntable move T. We have already investigated commutators on OT 1, the original Oval Track puzzle. In the previous exercise we investigated commutators on OT 2 where the turntable move is $T=(4,3,2,1)$. In each case below, write out the permutation resulting from the commutator in cycle form.

[2]Variation names are due to John O. Kiltinen who studies these in his book: *Oval Track and other Permutation Puzzles*.

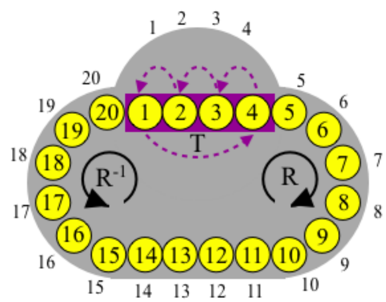


Figure 9: Oval Track puzzle variation for Exercise 23.

(a) $[R^{-1}, T]$ on OT 2 where $T = (4, 3, 2, 1)$

(b) $[R^{-2}, T]$ on OT 2 where $T = (4, 3, 2, 1)$

(c) $[T^2, R^{-2}]$ on OT 2 where $T = (4, 3, 2, 1)$

(d) $[R^{-1}, T^2]$ on OT 2 where $T = (4, 3, 2, 1)$

(e) $[R^{-1}, T]$ on OT 4 where $T = (5, 4, 3, 2, 1)$

(f) $[R^{-1}, T^2]$ on OT 4 where $T = (5, 4, 3, 2, 1)$

(g) $[R^{-1}, T]$ on OT 5 where $T = (1, 2)(3, 4)$

(h) $[R^{-2}, T]$ on OT 5 where $T = (1, 2)(3, 4)$

(i) $[R^{-3}, T]$ on OT 5 where $T = (1, 2)(3, 4)$

(j) $[R^{-5}, T]$ on OT 17 where $T = (1, 6)(2, 5)(3, 4)$