Lecture 4 (hints/answers)

1. (a) $(1,2,4)$ (b) $(1,8,2,5)(3,4,7,6)(9,10)$

(c) $(1,10,3,9,7,5,8,2,11,6,15,14,13,12)$

2. (a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 1 & 4 & 3 & 2 & 6 & 8 & 7 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 2 & 5 & 6 & 8 & 1 & 4 & 3 \end{pmatrix}$

3.(a) $\alpha = (1,6)(2,5,4,3,7)$

4. (f) Should have been α^{-1}. In this case,

$\alpha^{-1} = (8,9,10)(3,4)(1,7,2,5)$.

5 (a) $\mathrm{ord}(\sigma) = 3$
(b) $\mathrm{ord}(\alpha) = \mathrm{lcm}(4,4,2) = 4$
(c) $\mathrm{ord}(\beta) = 8$
(d) $\gamma = (1,2,7,6,3)(4,5)$, $\mathrm{ord}(\gamma) = \mathrm{lcm}(5,2) = 10$
(e) $\delta = (1,2,3,4)(6,10,8,7,9)$, $\mathrm{ord}(\delta) = \mathrm{lcm}(4,5) = 20$.

6. (a) $\alpha^{-1} = (3,6)(2,7,10,9)(1,4,8,5)$.

8. $\beta^{-1}\alpha\beta = (1,2,3,5,6)(1,2)(4,5)(1,6,5,3,2)$
$= (1,6)(4,3)$

9. Let $\sigma = (a_1, \dots, a_m)$. Let $k = \mathrm{ord}(\sigma)$. We want to show $k = m$. Well, since
$\sigma^m(a_i) = a_{i+m}$ (assume indices "wrap around")
then $k \leq m$. Now,
$\sigma^k(a_1) = a_{1+k}$,
but if k is the order, then $\sigma^k(a_1) = a_1$. Therefore
$a_{1+k} = a_1$,
hence $k = m$.
$\therefore \mathrm{ord}(\sigma) = m$.

10. $\mathrm{lcm}(5,3) = 15$
$\mathrm{lcm}(4,6) = 12$
$\mathrm{lcm}(22,18) = 2 \cdot 11 \cdot 9 = 198$

11. $\mathrm{lcm}(3,5,7) = 105$
$\mathrm{lcm}(6,12,26) = 2^2 \cdot 3 \cdot 13 = 12 \cdot 13 = 156$

12. hint: would have to contain only 7-cycles, since 7 prime.

13. List all possible cycle structures and compute orders.

14. Let $m = \text{ord}(\alpha)$ and $k = \text{ord}(\beta^{-1}\alpha\beta)$.
Then
$$(\beta^{-1}\alpha\beta)^m = \beta^{-1}\alpha^m\beta = \beta^{-1}\varepsilon\beta = \beta^{-1}\beta = \varepsilon, \text{ so } \boxed{k \le m.}$$
On the other hand,
$$\varepsilon = (\beta^{-1}\alpha\beta)^k = \beta^{-1}\alpha^k\beta \Rightarrow \beta\varepsilon\beta^{-1} = \alpha^k$$
$$\Rightarrow \varepsilon = \alpha^k$$
$$\Rightarrow \boxed{m \le k}, \text{ since } \text{ord}(\alpha) = m.$$
$$\therefore m = k.$$

15. If $\beta^n = \beta^{-7}$ then $\beta^{n+7} = \varepsilon$. The smallest n for which this happens satisfies
$$n + 7 = 21, \text{ since } \text{ord}(\beta) = 21$$
$$\therefore \boxed{n = 14}$$

16. Hint: m must kill the 2-cycle, 3-cycle, but leave the 5-cycle.

17. $(1,2)$ & $(1,3)$ do the trick.
$$(1,2)(1,3) = (1,2,3).$$

18. $(1,2,3)$, $(1,4,5)$ do the trick.
$$(1,2,3)(1,4,5) = (1,2,3,4,5)$$

19. (a) α has order k means $\alpha^k = \varepsilon$. Multiply both sides by α^{-1} gives,
$$\alpha^{k-1} = \alpha^{-1}.$$
(b) α has order 12, so $\alpha^{11} = \alpha^{-1} = (1,2,6,3)(4,5,7)$.

20. These would be the 5-cycles,
$$(\_,\_,\_,\_,\_)$$
\# choices 6 5 4 3 2

But each 5-cycle has 5 different representations
$$\therefore \# \text{ of 5-cycles is } \frac{6\cdot5\cdot4\cdot3\cdot2}{5} = \boxed{144}$$

21. See assignment 2 #6.

Lecture 6. (Hints/answers)

1. quick method : $(1,8)(1,4)(2,3)(2,7)(5,6)$

2. $(1,2,3) = (1,2)(1,3)$

3. See assignment 3 #· for similar questions

5. hint: $(a,b) = (1,a)(1,b)(1,a)$

6.
7. } we cover these in Lecture 8.

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Lecture 7 (hints/answers)

1. (a) even (b) even (c) odd (d) odd (e) even (f) even

2. $\alpha = (1,3,7,5)(4,8,6) \rightsquigarrow$ odd

   $\beta = (1,5,7,3)(4,6,8) \rightsquigarrow$ odd

3. (a) $(10,14,13)(11,12,15)$ even

   (b) $(1,10,8,12,13)(2,5,3,4,16)(6,15,9)(11,14)$ odd

4. (a) assignment 3 # 4a
   (b) $(14)(23)$ even
   (c) $(1,3)(2,4)$ even
   (d) assignment 3 # 4b

5. Assignment 3 # 5

7. Assignment 3 #6

8. If α can be written as a product of k transpositions, and β a product of m transpositions, then

   $\alpha^{-1}\beta^{-1}\alpha\beta$ can be written as $\underbrace{2k+2m}_{\text{even}}$ transpositions !

9. Assignment 3 # 7.

10. hint: the map $\varphi: A_n \to O_n$, $\alpha \mapsto (1,2)\alpha$ is a bijection.

11. Assignment 3 # 9.

12. Assignment 3 # 11

Lecture 8 (Hints/Answers)

1. Recall permutations in A_n are <u>even</u>.

$(1,2,3,4)(5,6) \in A_7$

$(1,2,3)(4,5,6,7)(8,9) \in A_{10}$

} needed the 2-cycles, otherwise they would be <u>odd</u>.

2. (a) $\alpha = (1,2)(1,3) = (1,2,3)$
(b) $\beta = (1,2)(3,4) = (1,2,3)(1,4,3)$
(c) ⋮ (e) } write as a product of 2-cycles, then pair up two 2-cyles and write as a single 3-cycle, or product of two 3-cycles as done in (a) & (b).

(See assignment 4 #2)

3. See assignment 4 #3.

4. Solvable $\Longleftrightarrow$ an even permutation.

5. See assignment 4 # 4(a).

6. assignment 4 # 5 is similar.

7. Order can be determined from cycle structure: it is is the least common multiple of the cycle lengths. Possible cycle structures and orders in A_{10} are

cycle structure	order
(9-cycle)	9
(8-cycle)(2-cycle)	8
(7-cycle)	7
(7-cycle)(3-cycle)	21
(6-cycle)(4-cycle)	12
(6-cycle)(2-cycle)	6
(5-cycle)	5
(5-cycle)(5-cycle)	5
(5-cycle)(3-cycle)	15
(5-cycle)(2-cycle)(2-cycle)	10
(4-cycle)(4-cycle)	4
(4-cycle)(3-cycle)(2-cycle)	12
(4-cycle)(2-cycle)(2-cycle)(2-cycle)	4
(4-cycle)(2-cycle)	4
(3-cycle)	3
(3-cycle)(3-cycle)	3
(3-cycle)(3-cycle)(3-cycle)	3
(3-cycle)(3-cycle)(2-cycle)(2-cycle)	6
(3-cycle) [even # of 2-cycles]	6
even # of 2-cycles	2

$\therefore$ possible orders : 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 21

8. Example 8.1 lists elements of A_4.

9. Assignment 4 # 4(b)

10. Assignment 4 # 6

11. one-to-one: $\varphi(\alpha) = \varphi(\beta) \iff (1,2)\alpha = (1,2)\beta$
$\iff \alpha = \beta$ by cancellation property.

onto: If $\gamma \in (1,2)A_n$ then there is an $\alpha \in A_n$ s.t.
$\gamma = (1,2)\alpha$.
Therefore $\varphi(\alpha) = \gamma$.

12. Assignment 4 # 12.

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Lecture 9 (Hints/Answers)

1. Position is give by permutation (14,15) which is odd, and empty space is in box 16 (an even box). Since parities don't match, it is not solvable

2. Show the parity of the permutation is <u>equal</u> to the parity of the box where the empty space is.

3. Show the parity of the permutation is <u>not equal</u> to the parity of the box where the empty space is.

4. Similar to (2) & (3), except you aren't told in advance whether solvable/unsolvable. As usual, use Thm 9.2.

5. see assignment solutions

6. Permutation is (1,9,13,15,16,8,4,2)(3,10,5,11,14,7,12,6) which is <u>even</u> and empty space is in an even box (only 2 moves away from box 16).

$\therefore$ Solvable.

7. Even permutations are possible, so swap the two occurences of the letter R, in addition to swapping L & A. This is a product of two transpositions and therefore possible.

8. The first and second occurance of the letter A have similar backgrounds, so the double swap

C $\boxed{A}$ N $\boxed{A}$ M A
P A N A L

is possible since it is an even permutation, and this solves the puzzle.

9. Similar to the previous puzzles, since only even permutations are possible, in order to swap the goat and the fence, another swap of identical pieces must be done. There are a few identical pairs to choose from.

13–15 : Focus on showing any 3-cycle is possible by using arguing that any 3-tiles can be moved to a 2-by-2 array, then cycled. Then moved back.