# Lecture 23:
# Symmetry & Counting II:
# Burnside's Theorem

## Contents



In this lecture we continue our discussion of how to use group theory to *count like a professional*. We look at an application permutation groups to count the number of different designs there are of various objects.

## 23.1 A Motivating Example

Consider the task of colouring the six vertices of a regular hexagon so that there are three black and three white vertices. Figure 1 shows an example of one such colouring.

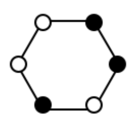
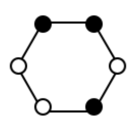
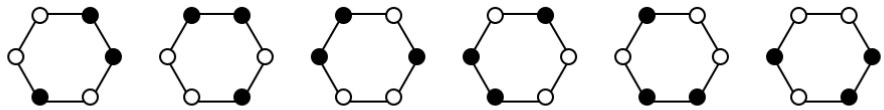


Figure 1: An example of a colouring of the vertices of the hexagon: three white, three black.

**Ceramic Tiles:**
If such a colouring appears on a ceramic tile, it wouldn't make sense to consider this different from the colouring



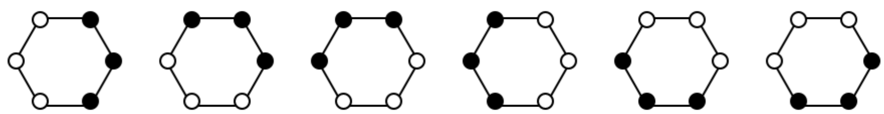
since this one can be obtained by rotating the one in Figure 1 counterclockwise by $60^\circ$. In this case, we should consider two colours equivalent if one can be obtained from the other by a rotation of the hexagon. In other words, a manufacturer would only need to make the tile in Figure 1, and simply by rotating the tile the following six colourings are equivalent under the group of rotations of a hexagon.



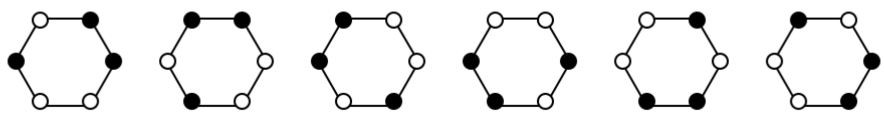
How many tiles would a manufacturer need to make in order to obtain all possible ways to colour three vertices black and three white (up to rotational equivalence)?

There are $\binom{6}{3} = 20$ ways to pick three vertices to colour black. As we observed above it would be nonsensical for a manufacturer to produce each of the $20$ designs, since up to rotation, the colouring in Figure 1 is equivalent to six different designs.

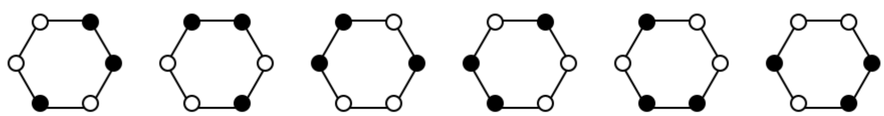
Figure 2 shows all $20$ possible colourings. They are organized into equivalence classes. For example, all colouring in 2a are equivalent under the rotational group of the hexagon. Similarly for the other three cases. This means, a manufacturer would only need to produce $4$ different tiles, say for example the first one in each collection of Figure 2.



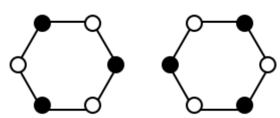
(a)



(b)



(c)



(d)

Figure 2: All the different ways to colour three vertices of a hexagon black, and the other three white.

**Beads on a Necklace:**
On the other hand, if we think of these colourings as representing beads on a necklace, then two colourings would be equivalent if one can be obtained from the other by an element of the dihedral group of the hexagon. In other words, colourings can also be reflected to obtain equivalent colourings. In this situation, the colourings in Figure 2b and 2c are equivalent. This means there are essentially $3$ different ways to make a necklace with three black beads and three white ones (up to rotational/reflexive symmetries of the hexagon).

In general, we say that two designs (arrangements) $A$ and $B$ are *equivalent under a group* $G$ of permutations if there is an element $\alpha \in G$ such that $\alpha(A) = B$. That is, two designs are equivalent under $G$ if they are in the same orbit of $G$. The set being permuted by $G$ is the set of designs or arrangements.

Therefore the number of inequivalent configurations is the number of orbit classes under $G$. In the next section we present a celebrated theorem which allows us to count the number of orbit classes.

In the ceramic tile example, the 20 designs in Figure 2 have been split up into 4 orbit classes ((a)-(d)) under the group of rotations of the hexagon. In the necklace example, there are only 3 orbit classes under the dihedral group $D_6$. Classes (b) and (c) merge to form one equivalence class in this case.

## 23.2 Burnside's Theorem

Let $X$ be a nonempty set, and $S_X$ the set of all permutations of $X$:

$$S_X = \{\alpha \mid \alpha : X \to X \text{ is a bijection}\}.$$

We first recall what we mean by the *fixed set* of a permutation in $S_X$.

For a permutation $\alpha \in S_X$, the **fixed set of** $\alpha$ is the set of all elements in $X$ that $\alpha$ doesn't move. We denote the set by $\mathrm{fix}(\alpha)$.

$$\mathrm{fix}(\alpha) = \{\mathrm{x} \in \mathrm{X} \mid \alpha(\mathrm{x}) = \mathrm{x}\}.$$

Note that

$$x \in \mathrm{fix}(\alpha) \iff \alpha \in \mathrm{stab}_{\mathrm{S_X}}(\mathrm{x}).$$

**Theorem 23.1 (Burnside's Theorem)** [1] *If $G$ is a finite group of permutations on a set $X$, then the number of distinct orbits of $G$ on $X$ is*

$$N = \frac{1}{|G|} \sum_{\alpha \in G} |\mathrm{fix}(\alpha)|.$$

**Proof:** Let the orbits be

$$\begin{aligned} \mathcal{O}_1 &= \{a_1, \ldots, a_m\} \\ \mathcal{O}_2 &= \{b_1, \ldots, b_n\} \\ &\vdots \\ \mathcal{O}_N &= \ldots \end{aligned}$$

Recall, the $\mathcal{O}_\rangle$'s partition $X$, so each element of $X$ appears in one and only one orbit. Then, by the orbit-stabilizer theorem,

$$|\mathrm{stab_G(a_1)}| + \cdots + |\mathrm{stab_G(a_1)}| = \underbrace{\frac{|G|}{m} + \cdots + \frac{|G|}{m}}_{m \text{ terms}} = |G|$$

$$|\mathrm{stab_G(b_1)}| + \cdots + |\mathrm{stab_G(b_1)}| = \underbrace{\frac{|G|}{n} + \cdots + \frac{|G|}{n}}_{n \text{ terms}} = |G|$$

$$\vdots$$

Summing all these equations, we obtain

$$\sum_{x \in X} |\mathrm{stab_G(x)}| = |\mathrm{G}| \cdot \mathrm{N}.$$

[1]This theorem is also commonly called the *Polya-Burnside Counting Theorem*.

On the other hand,

$$x \in \mathrm{fix}(\alpha) \iff \alpha \in \mathrm{stab_G(x)},$$

so

$$\begin{aligned}\sum_{x\in X} |\mathrm{stab_G(x)}| &= |\{(\alpha, x) \mid \alpha \in G, x \in X, \alpha(x) = x\}| \\ &= \sum_{\alpha\in G} |\mathrm{fix}(\alpha)|.\end{aligned}$$

Therefore,

$$\sum_{\alpha\in G} |\mathrm{fix}(\alpha)| = |\mathrm{G}| \cdot \mathrm{N}$$

so that

$$N = \frac{1}{|G|} \sum_{\alpha\in G} |\mathrm{fix}(\alpha)|.$$

□

## 23.3 Applications of Burnside's Theorem

**Example 23.1** *Let's return to the ceramic tile and necklace problems from Section 23.1 and see how to apply Burnside's theorem in this familiar context. It will be convenient to recall that the dihedral group $D_6$ consists of elements:*

$$D_6 = \{\varepsilon, r, r^2, r^3, r^4, r^5, f, rf, r^2f, r^3f, r^3f, r^5f\}$$

*where $r$ denotes a clockwise rotation through $60°$ and $f$ is a reflection about a line through opposite vertices. The groups of rotational symmetries is*

$$G = \langle r\rangle = \{\varepsilon, r, r^2, r^3, r^4, r^5\}.$$

*In the case of counting hexagonal tiles with three black vertices and three white vertices, the set of objects being permuted is the $20$ possible designs, whereas the group of permutations is $G$, the group of six rotational symmetries of a hexagon.*

*The identity fixes all $20$ designs in Figure 2. Rotations through $60°$, $180°$, or $300°$ fix none of the designs. That is, $|\mathrm{fix(r)}| = |\mathrm{fix(r^3)}| = |\mathrm{fix(r^5)}| = 0$. Rotations through $120°$ and $240°$ fix the two designs in Figure 2d, so $|\mathrm{fix(r^2)}| = |\mathrm{fix(r^4)}| = 2$. We summarize these results in Table 1.*

| **element:** $\alpha$ | **Number of arrangements fixed by this type of element:** $|\mathit{fix}(\alpha)|$ |
|---|---|
| $\varepsilon$ | 20 |
| $r$ | 0 |
| $r^2$ | 2 |
| $r^3$ | 0 |
| $r^4$ | 2 |
| $r^5$ | 0 |

Table 1: $|\mathrm{fix}(\alpha)|$ for each $\alpha \in \langle r\rangle < D_6$.

*By Burnside's Theorem, we have that*

$$\begin{aligned} \textit{number of orbits } &= N = \frac{1}{|G|}\sum_{\alpha\in G}|\text{fix}(\alpha)| \\ &= \frac{1}{6}(20+0+2+0+2+0) \\ &= \frac{24}{6} = 4. \end{aligned}$$

*Now let's use Burnside's Theorem to count the number of necklace arrangements. In this case we want to count the number of orbits under $D_6$. Table 2 summarizes the sizes of the fixed sets for each $\alpha \in D_6$.*

| **type of element** | **number of elements of this type** | **Number of arrangements fixed by this type of element** |
|---|---|---|
| identity | 1 | 20 |
| rotation of order 2 (180°) | 1 | 0 |
| rotation of order 3 (120° or 240°) | 2 | 2 |
| rotation of order 6 (60° or 300°) | 2 | 0 |
| reflection across diagonal | 3 | 4 |
| reflection across bisector | 3 | 0 |

Table 2: $|\text{fix}(\alpha)|$ for each type of $\alpha \in D_6$.

*By Burnside's Theorem, we have that*

$$\begin{aligned} \textit{number of orbits } &= N = \frac{1}{|D_6|}\sum_{\alpha\in D_6}|\text{fix}(\alpha)| \\ &= \frac{1}{12}(20+0+2\cdot 2+0+3\cdot 4+0) \\ &= \frac{36}{12} = 3. \end{aligned}$$

**Example 23.2** *Consider the number of ways to colour the faces of a regular tetrahedron with 4 different colours.*

*How should we decide when two colourings of the tetrahedron are nonequivalent? Certainly, if we were to pick up a tetrahedron coloured in a certain manner, rotate it, and put it back down, we would think of the tetrahedron as being positioned differently rather than being coloured differently. So our permutation group for this problem is just the group of 12 rotation of the tetrahedron, which we denoted by $G_T$. This group consists of the identity; eight elements of order 3, each which fix one vertex; and three elements of order 2, each which fix no vertices (but fix exactly two edges).*

*The total number of colourings, without regard to equivalence, is $4!$. Therefore*

$$\text{fix}(\varepsilon) = 4!$$

*while, for any $\alpha \in G_T$, $\alpha \neq \varepsilon$,*

$$\text{fix}(\alpha) = 0.$$

*Table 3 summarizes the results.*

*By Burnside's Theorem, we have that*

$$\begin{aligned} \textit{number of orbits } &= N = \frac{1}{|G_T|}\sum_{\alpha\in G_T}|\text{fix}(\alpha)| \\ &= \frac{1}{12}(4!+0+0+\cdots 0) \\ &= \frac{4!}{12} = 2. \end{aligned}$$

| type of element | number of elements of this type | Number of arrangements fixed by this type of element |
|---|---|---|
| identity | 1 | 4! |
| rotation of order 2 | 3 | 0 |
| rotation of order 3 | 8 | 0 |

Table 3: $|\text{fix}(\alpha)|$ for various types of $\alpha \in G_T$.

*Representative for the two orbit classes are showing in Figure 3.*

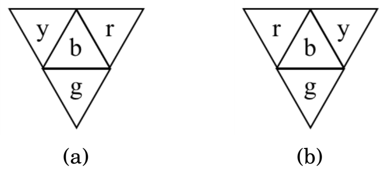



Figure 3: The two inequivalent colourings of the faces of a tetrahedron (tetrahedron is unfolded to see all sides).

**Example 23.3** *Suppose that we have the colours red (R), green (G), and blue (B), and we wish to colour the edges of a regular tetrahedron. First observe there are* $3^6 = 729$ *colourings without regard to equivalence. As with the previous example, we consider how the group of rotations of the tetrahedron,* $G_T$ *acts on these colourings. Two colourings are equivalent if they are in the same* $G_T$ *orbit. Every rotation permutes the* 729 *colourings, and to apply Burnside's theorem we must determine the size of* $\text{fix}(\alpha)$ *for each of the* 12 *rotations.*

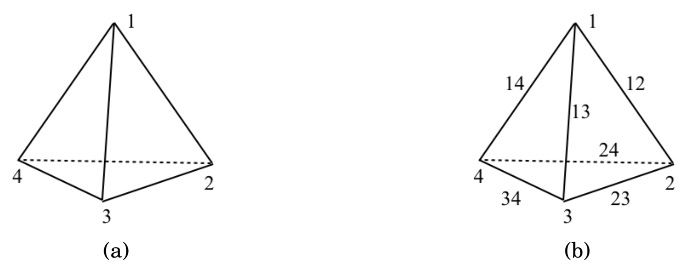



Figure 4: regular tetrahedron, with vertices and edges labeled.

*The identity fixes all* 729 *colourings:*

$$|\textbf{fix}(\varepsilon)| = 729.$$

*Now consider the rotation* $(2,3,4)$ *or order* 3*. (Here we are describing a rotation by the permutation it induces on the vertices.) Suppose that a specific colouring is fixed by this element (that is, the tetrahedron appears to be coloured the same before and after this rotation). Since* $(2,3,4)$ *takes edge* 12 *to edge* 13*, edge* 13 *to* 14*, and edge* 14 *to edge* 12*, these three edges must be coloured the same. The same argument shows that edges* 23*,* 24 *and* 34 *must be coloured the same. Since there are three choices of colour for each of these two sets, there are* $3^2 = 9$ *colourings of the tetrahedron in total that are fixed by the rotation* $(2,3,4)$*. Table 4 lists the* 9 *different colourings. Therefore,*

$$|\textbf{fix}((2,3,4))| = 9.$$

*For each of the other* 7 *rotations of order* 3 *a similar argument shows the fixed set has sized* 9*.*

*Now consider the rotation* $(1,2)(3,4)$ *of order* 2*. Since edges* 12 *and* 34 *are fixed they may be coloured in any way and will appear the same after the rotation* $(1,2)(3,4)$ *(since the rotation fixes these edges). This gives* $3 \cdot 3 = 9$

| **colouring** | 12 | 13 | 14 **egde colours** | 23 | 24 | 34 |
|---|---|---|---|---|---|---|
| scheme 1 | R | R | R | R | R | R |
| scheme 2 | R | R | R | G | G | G |
| scheme 3 | R | R | R | B | B | B |
| scheme 4 | G | G | G | G | G | G |
| scheme 5 | G | G | G | R | R | R |
| scheme 6 | G | G | G | B | B | B |
| scheme 7 | B | B | B | B | B | B |
| scheme 8 | B | B | B | R | R | R |
| scheme 9 | B | B | B | G | G | G |

Table 4: Nine colourings fixed by $(2, 3, 4)$.

*choices for these edges. Edges* $14$ *and* $23$ *are swapped by the rotation* $(1, 2)(3, 4)$ *and so must be coloured the same. Similarly, edges* $13$ *and* $24$ *are swapped and must be coloured the same. There are* $3$ *choices to colour each of these sets, so there are* $9$ *ways to colour these two sets altogether. Therefore, there are* $9 \cdot 9 = 81$ *ways to colour all the edges in such a way that the colouring remains fixed under the rotation* $(1, 2)(3, 4)$*. Table 5 lists the* $81$ *different colourings. Therefore,*

$$|\text{fix}((1, 2)(3, 4))| = 81.$$

*For each of the other* $2$ *rotations of order* $2$ *a similar argument shows the fixed set has sized* $81$*.*

| **colouring** | 12 | 13 | 14 **egde colours** | 23 | 24 | 34 |
|---|---|---|---|---|---|---|
| scheme 1 | X | Y | R | R | R | R |
| scheme 2 | X | Y | R | R | G | G |
| scheme 3 | X | Y | R | R | B | B |
| scheme 4 | X | Y | G | G | G | G |
| scheme 5 | X | Y | G | G | R | R |
| scheme 6 | X | Y | G | G | B | B |
| scheme 7 | X | Y | B | B | B | B |
| scheme 8 | X | Y | B | B | R | R |
| scheme 9 | X | Y | B | B | G | G |

Table 5: Eighty-one colourings fixed by $(1, 2)(3, 4)$. $X$ and $Y$ can be any of $R$, $G$, $B$.

*The results are summarized in Table 6.*

| **type of element** | **number of elements of this type** | **Number of arrangements fixed by this type of element** |
|---|---|---|
| identity | 1 | 729 |
| rotation of order $2$ | 3 | 81 |
| rotation of order $3$ | 8 | 9 |

Table 6: $|\text{fix}(\alpha)|$ for various types of $\alpha \in G_T$.

*By Burnside's Theorem, we have that*

$$\begin{aligned} \textit{number of orbits} = N &= \frac{1}{|G_T|} \sum_{\alpha \in G_T} |\mathbf{fix}(\alpha)| \\ &= \frac{1}{12}(729 + 3(81) + 8(9)) \\ &= \frac{1044}{12} = 87. \end{aligned}$$

*It would be a difficult task to solve this problem without Burnside's Theorem.*

At this point you may be wondering who besides mathematicians would be interested in counting problems such as these. Chemists for one, are interested in these types of counting problems. Though, their interests lie more in counting configurations of molecules. We'll now look at an example.

**Example 23.4** *Benzene is a chemical compound, each molecule of shich is made up of six carbon (C) atoms, and six hydrogen (H) atoms. The carbon atoms are arranged in a hexagon with alternating single and double bonds. Each carbon atom must have four bonds and each hydrogen atom must have one bond. See Figure 5.*

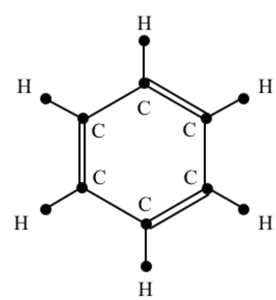


Figure 5: A benzene molecule.

*By replacing three of the hydrogen atoms by $CH_3$ clusters (see Figure 6) we can create a chemical derivative from benzene. Let's determine the number of such derivatives.*

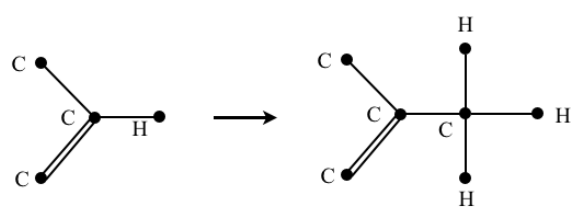


Figure 6: An example of how a hydrogen atom is replaced with a $CH_3$ cluster.

*Taking into account orientation, the number of possibilities would just be the number of ways of choosing three hydrogen atoms for replacement from six possibilities, in other words $\binom{6}{3} = 20$. However, those that are related by rotational/reflective symmetry clearly correspond to the same derivative chemical. So we wish to determine the number of derivatives up to equivalence under the symmetries of the molecule.*

*Let $r$ denote the rotation in the clockwise direction through an angle of $120°$, and let $f$ be a reflection about the "$x - axis$". The group of symmetries of the molecule (respecting the single/double bonds) is:*

$$G = \{\varepsilon, r, r^2, f, rf, r^2 f\}.$$

*A reflection ($f$, $rf$, or $r^2f$) swaps pairs of vertices so it would not fix any molecule. An order $3$ rotation would fix a molecule if all clusters lie on the same side of a double bond. There are two such molecules. Table 7 summarizes the number of arrangements, where three hydrogen atoms are replaced by $CH_4$ clusters, which are fixed by each element of $G$.*

| type of element | number of elements of this type | Number of arrangements fixed by this type of element |
|---|---|---|
| identity | 1 | 20 |
| rotation of order 2: $f, rf, r^2f$, | 3 | 0 |
| rotation of order 3: $r, r^2$ | 2 | 2 |

Table 7: $|\text{fix}(\alpha)|$ for various types of $\alpha \in G$.

*By Burnside's Theorem, we have that*

$$\begin{aligned} \textit{number of orbits } = N &= \frac{1}{|G|}\sum_{\alpha\in G}|\text{fix}(\alpha)| \\ &= \frac{1}{6}(20+3(0)+2(2)) \\ &= \frac{24}{6} = 4. \end{aligned}$$

*Therefore, there are $4$ such derivatives. You should try listing them.*

Another kind of molecule that chemists consider is visualized as a regular tetrahedron with a carbon atom at the centre and any of the four radicals $HOCH_2$ (hydroxymethyl), $C_2H_2$ (ethyl), $Cl$ (chlorine) or $H$ (hydrogen) at the four vertices. The number of such molecules can be easily counted using Burnside's Theorem.

## 23.4 Exercises

1. Determine the number of different ways there are of arranging $6$ keys on a key ring.
2. Determine the number of ways of colouring the vertices of a square so that two are red and two are green.
3. Determine the number of ways of colouring the vertices of a pentagon in each of the following ways:
   (a) with five distinct colours;
   (b) so that two are black and three are white;
   (c) so that two are black, two are white, and one is blue.
4. Determine the number of ways of colouring a regular $n$-gon with $n$ different colours.
5. Determine the number of ways of seating $n$ diplomats around a table.
6. Determine the number of (inequivalent) ways to colour the $6$ faces of a cube with $6$ distinct colours. Consider two colourings equivalent if one can be obtained from the other by a rotation of the cube.
7. Determine the number of (inequivalent) ways to colour the $6$ faces of the cube so that **three** faces are white and **three** faces are black. Consider two colourings equivalent if one can be obtained from the other by a rotation of the cube.

8. Determine the number of (inequivalent) ways to colour the $6$ faces of the cube so that **two** faces are white and **four** faces are black. Consider two colourings equivalent if one can be obtained from the other by a rotation of the cube.

9. Determine the number of (inequivalent) ways to colour the $12$ edges of the cube so that **six** edges are white and **six** edges are black. Consider two colourings equivalent if one can be obtained from the other by a rotation of the cube.

10. Determine the number of (inequivalent) ways to colour the $12$ pentagonal faces of a regular dodecahedron with $12$ distinct colours. Consider two colourings equivalent if one can be obtained from the other by a rotation of the dodecahedron.

11. Determine the number of (inequivalent) ways to colour the $12$ pentagonal faces of a regular dodecahedron so that $6$ faces are white and $6$ faces are black. Consider two colourings equivalent if one can be obtained from the other by a rotation of the dodecahedron.

12. Determine the number of (inequivalent) ways to colour the $20$ triangular faces of a regular icosahedron with $20$ different colours. Consider two colourings equivalent if one can be obtained from the other by a rotation of the icosahedron.

13. A benzene molecule can be viewed as six carbon atoms arranged in a regular hexagon. See Figure 7 (ignore double vs. single bonds). At each carbon atom, one of three radicals ($NH_2$, $COOH$, or $OH$). How many such compounds are possible?

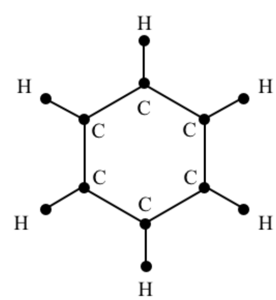



Figure 7: Diagram for Exercise 13.