Lecture 25:
Puzzle: Prisoners Names in Boxes

Contents

This Lecture is based on the puzzle "Names in Boxes" in the article *Seven Puzzles You Think You Must Not Have Heard Correctly* by Peter Winkler.

> *The names of* 100 *prisoners are places in* 100 *wooden boxes, one name to a box, and the boxes are lined up on a table in a room. One by one, the prisoners are led into the room, each may look in at most* 50 *boxes, but must leave the room exactly as he found it and is permitted no further communication with the others. The goal is for every prisoner to find his/her name in the one of the* 50 *boxes they examine - if they all do, then they will be sent free.*
>
> *The prisoners have a chance to plot their strategy in advance, and they are going to need it, because unless every single prisoner finds his own name all will subsequently be executed.*
>
> *Find a strategy for them which has probability of success exceeding* 30%.

Comment: If each prisoner opens a random selection of 50, the probability they will find their own name is $1/2$. So the probability of everyones survival is $1/2^{100} \approx 8 \cdot 10^{-31} = 0.0000000000000000000000000000008$. They could even do worse if they all looked in the same 50 boxes - their probability of survival would be 0 in this case. 30% seems ridiculously out of reach – but you read the problem correctly.

25.1 Solution to The Puzzle

The players first agree on a random labeling of themselves and the boxes. For example, they can label the boxes 1, 2, 3 etc, by their proximity to the door, and the prisoners could label themselves by the order in which they enter the room. See Figure 1.

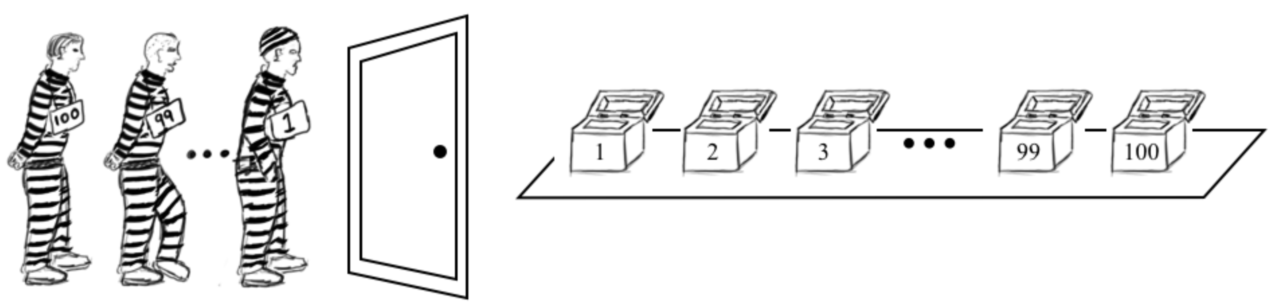

Figure 1: The boxes lined up on table. The names 1 through 100 are randomly placed inside the boxes. The lids are closed, and the prisoners enter one at a time to examine 50 boxes.

The strategy we describe could easily be foiled by the Warden if he (i) knows their strategy and (ii) knows their labeling, so perhaps a more randomly assigned order to the prisoners and boxes would be better. In any case, we assume every prisoner knows the labelings of all the prisoners and all the boxes.

Strategy:
When released into the room, each prisoner looks in his own box (the one which has the same label as himself). He then looks in the box of the name he just found, and then into the box belonging to the name he found in the second box, etc. until he either finds his own name, or haa opened 50 boxes.

Why the Strategy Works:
The process of placing 100 names in 100 boxes is a permutation of $\mathbb{Z}_{100} = \{1, 2, 3, \ldots, 100\}$. Let $\rho \in S_{100}$ be the permutation representing the arrangement of names in boxes:

$$\rho(i) = j, \quad \text{if box } i \text{ contains name } j.$$

If the Warden randomly assigns name to boxes, and the prisoners randomly assign labels to themselves, and the boxes, then ρ is a random element of S_{100}.

In the strategy described each prisoner is following a cycle in ρ. The prisoner begins in a cycle corresponding to the name found in his box, and since he will eventually have to return home (since a permutation is bijective) then the box that will send him home would be the one with his name in it. If every cycle in ρ has length ≤ 50 then this strategy will result in every prisoner finding their own name within the 50 box limit. Therefore, the probability of freedom is equal to the probability that a random permutation in S_{100} consists only of cycles of length ≤ 50 in its disjoint cycle form.

Let's determine how many permutations in S_{100} consists only of cycles of length ≤ 50. in fact, let's determine the number of permutations in S_{2n} consisting only of cycle of length $\leq n$.

For $k > n$, let a_k be the number of permutations containing a cycle of length exactly k. We start by determining a_k. There are $\binom{2n}{k} = \frac{(2n)!}{(2n-k)!k!}$ ways to pick the entries of the cycle, $(k-1)!$ ways to order them within the cycle, and $(2n-k)!$ ways to permute the rest. The product of these numbers is a_k:

$$a_k = \binom{2n}{k} \cdot (k-1)! \cdot (2n-k)! = \frac{(2n)!}{k}.$$

Since at most one cycle of length $> n$ can exist in a given permutation in S_{2n}, the number of permutations containing a cycle of length $> n$ is

$$\sum_{k=n+1}^{2n} a_k = \sum_{k=n+1}^{2n} \frac{(2n)!}{k}.$$

Therefore, the probability of choosing a permutation from S_{2n} which contains a cycle of length $> n$ is

$$\begin{aligned} \frac{1}{(2n)!} \sum_{k=n+1}^{2n} a_k &= \frac{1}{(2n)!} \sum_{k=n+1}^{2n} \frac{(2n)!}{k} \\ &= \sum_{k=n+1}^{2n} \frac{1}{k}. \end{aligned}$$

The probability of choosing a permutation from S_{2n} which contains a cycle of length $\leq n$ is

$$[\text{prob. of short cycle length in } S_{2n}] \quad = \quad 1 - \sum_{k=n+1}^{2n} \frac{1}{k} \quad = \quad 1 - b_{2n} + b_n. \tag{1}$$

where $b_m = \sum_{k=1}^{m} \frac{1}{k} \approx \ln m$. Therefore the probability is approximately

$$1 - \ln(2m) + \ln m = 1 - \ln 2 \approx 0.30685 \approx 30.7\%.$$

For any particular value of n we can use Equation 1 to calculate the probability more precisely. For example, in the case where $n = 50$,

$$[\text{prob. of short cycle length in } S_{100}] \quad = \quad 1 - \sum_{k=51}^{100} \frac{1}{k} \quad \approx \quad 31\%.$$

SAGE

```
sage: n=50
sage: (1-sum(1/k,k,n+1,2*n)).n()
0.311827820689805
```

Therefore, using this strategy the prisoner's have a 31% chance of obtaining freedom. And of course, a 69% chance of getting executed.

Peter Winkler states in his article that Eugene Curtain and Max Washauer have proved that this strategy cannot be improved.