Lecture 12:
Puzzle Groups

Contents

In this lecture we associate to each permutation puzzle a group, called the *puzzle group*. We then see that this group can be represented by a group of permutations, and we can use SAGE to investigate the puzzles.

12.1 Puzzle Groups

Let's first recall the definition of a *permutation puzzle*, since we would like to see how groups come into the picture. In Lecture 1 we defined what we mean by a *one person game*, and from that we gave the following definition of a permutation puzzle.

A **permutation puzzle** is a one person game (solitaire) with a finite set $T = \{1, 2, \dots, n\}$ of puzzle pieces satisfying the following four properties:

1. For some $n > 1$ depending only on the puzzle's construction, each move of the puzzle corresponds to a unique permutation of the numbers in T,
2. If the permutation of T in (1) corresponds to more than one puzzle move then the two positions reached by those two respective moves must be indistinguishable,
3. Each move, say M, must be "invertible" in the sense that there must exist another move, say M^{-1}, which restores the puzzle to the position it was at before M was performed, In this sense, we must be able to "undo" moves.
4. If M_1 is a move corresponding to a permutation τ_1 of T and if M_2 is a move corresponding to a permutation τ_2 of T then $M_1 \cdot M_2$ (the move M_1 followed by the move M_2) is either

- not a legal move, or
- corresponds to the permutation $\tau_1\tau_2$.

As indicated in part 4 it may happen that the composition of two moves is not legal. For example, this happens with the 15-Puzzle since legal moves change as the empty space moves around the board. See Section 12.5. This generally happens when dealing with a puzzle that contains a "gap". We won't consider such puzzles in this lecture, besides a remark in Section 12.5. Instead we will focus on puzzles for which two moves can always be composed. Typically these are the puzzles "without-gaps".

Let Puz be a permutation puzzle (where any two moves can be composed). For example Puz could be Rubik's cube, Oval Track, or Hungarian Rings. We consider two puzzle moves, m_1 and m_2, to be *equivalent* if the two positions reached by those two respective moves are indistinguishable.

Let $M(\mathrm{Puz})$ be the set of all inequivalent puzzle moves (what we typically refer to a move-sequences). We can think of $M(\mathrm{Puz})$ as just the set of all possible configurations, or positions of the puzzle pieces. We have a way to combine elements of $M(\mathrm{Puz})$: if $m_1, m_2 \in M$ then m_1m_2 represents the move-sequence m_1 followed by m_2, which is again in $M(\mathrm{Puz})$. (This is why we assume the puzzle does not have gaps.) It turns out that $M(\mathrm{Puz})$ is a group under this operation. The identity is the "do nothing" move, and inverses exist by part 3 of the definition above. Associativity follows from the fact that "moves" correspond to "permutations" and permutation composition is associative.

Definition 12.1 (Puzzle Group) *For a permutation puzzle* Puz*, the set of all inequivalent puzzle moves* $M(\mathrm{Puz})$ *is a group under move composition.* $M(\mathrm{Puz})$ *is called the* ***puzzle group*** *of* Puz*.*

Since puzzle moves and positions correspond to permutations we can represent $M(\mathrm{Puz})$ as a subgroup of a permutation group. To do this we just need to associate each basic legal move $m_i \in M(\mathrm{Puz})$, $1 \le i \le k$, to a permutation α_i. We then use the permutation group $\langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle$ to represent the puzzle. We've already done this with all of our puzzles, so here we are just emphasizing the fit within group theory.

12.2 Rubik's Cube

Let Puz be an $n \times n \times n$ Rubik's cube, then we call $M(\mathrm{Puz})$ the **n-cube group**. In the special case when $n = 3$ we call it the **Rubik's cube group**. We use the special notation RC_n to denote the n-cube group.

12.2.1 3-Cube Group

We will do a little investigation into the Rubik's cube group.

As we described in Lecture 1, we label the facets of the Rubik's Cube as shown Figure 1. Figure 2 shows the labeling on an actual cube.

The permutation corresponding to each of the basic moves of the Rubik's Cube are:

$$\begin{aligned}
R &= (25,27,32,30)(26,29,31,28)(3,38,43,19)(5,36,45,21)(8,33,48,24)\\
L &= (9,11,16,14)(10,13,15,12)(1,17,41,40)(4,20,44,37)(6,22,46,35)\\
U &= (1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)(11,35,27,19)\\
D &= (41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)(16,24,32,40)\\
F &= (17,19,24,22)(18,21,23,20)(6,25,43,16)(7,28,42,13)(8,30,41,11)\\
B &= (33,35,40,38)(34,37,39,36)(3,9,46,32)(2,12,47,29)(1,14,48,27)
\end{aligned}$$

$R^{-1}, L^{-1}, U^{-1}, D^{-1}, F^{-1}, B^{-1}$ correspond to the inverses of these permutations.

Since the centre's of the cube are fixed by these moves then any two of these moves are inequivalent. This means that RC_3 can be represented by the group subgroup of S_{48} generated by these permutations:

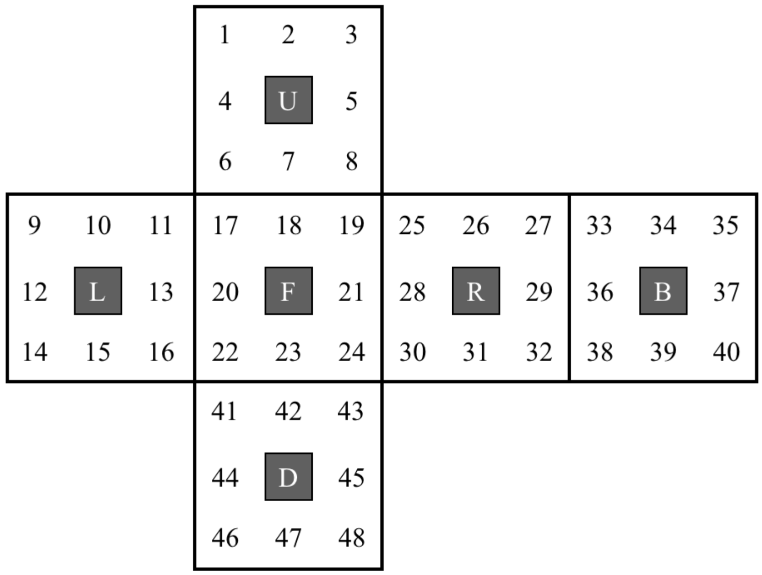
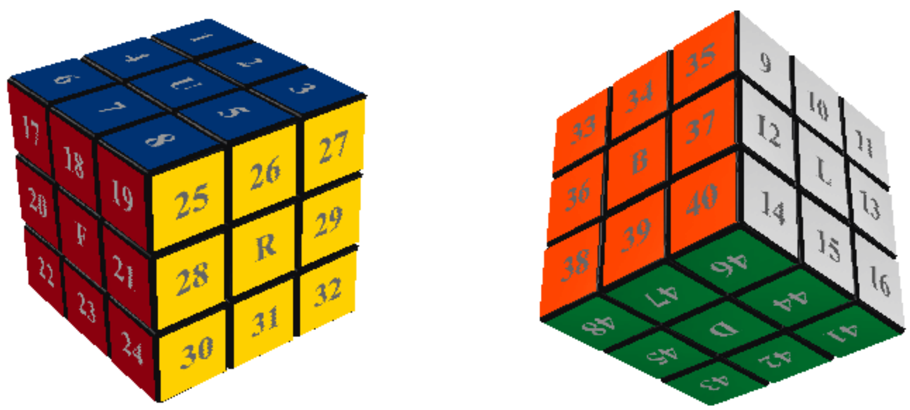


Figure 1: Facet labeling on the Rubik's cube.



(a) Labeling on Up, Right, Front faces

(b) Labeling on Down, Back, Left faces

Figure 2: The labeling of the facets of Rubik's Cube.

$$RC_3 = \langle \mathrm{R}, \mathrm{L}, \mathrm{U}, \mathrm{D}, \mathrm{F}, \mathrm{B} \rangle.$$

We can define RC_3 in SAGE as follows.

SAGE

```
sage: S48=SymmetricGroup(48)
sage: R=S48("(25,27,32,30)(26,29,31,28)(3,38,43,19)(5,36,45,21)(8,33,48,24)")
sage: L=S48("(9,11,16,14)(10,13,15,12)(1,17,41,40)(4,20,44,37)(6,22,46,35)")
sage: U=S48("(1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)(11,35,27,19)")
sage: D=S48("(41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)(16,24,32,40)")
sage: F=S48("(17,19,24,22)(18,21,23,20)(6,25,43,16)(7,28,42,13)(8,30,41,11)")
sage: B=S48("(33,35,40,38)(34,37,39,36)(3,9,46,32)(2,12,47,29)(1,14,48,27)")
sage: RC3=S48.subgroup([R,L,U,D,F,B])      # define Rubik's cube group to be RC3
```

Now that RC_3 is in SAGE we can calculate some facts about the Rubik's cube. For example, we can determine the size of RC_3. This is the number of different configurations there are of the cube.

SAGE
```
sage: RC3.order()
43252003274489856000
sage: factor(RC3.order())
2^27 * 3^14 * 5^3 * 7^2 * 11
```

Therefore there are approximately $4.3 \cdot 10^{19}$ configurations of the cube. And only one solution!

Theorem 12.1 *The Rubik's cube group RC_3 has order* $2^{27}3^{14}5^3 7^2 11 = 43,252,003,274,489,856,000$.

Since the order of an element in a group must divide the size of the group, then we immediately see from the factored form of $|RC_3|$ that there are no elements of prime order ≥ 13. Also, by Cauchy's theorem (see Lecture 11), there must be an element of order 11. Actually finding such an element is another story, all we know is one exists. In fact, 9 others must exist as well since it would generate a subgroup of order 11.

We can also check if it is possible to flip a single edge, while leaving everything else in place. Consider flipping the cubie in the *uf* cubical, the corresponding permutation is $(7, 18)$. The following calculation shows it is not in RC_3.

SAGE
```
sage: S48("(7,18)") in RC3
False
```

However, we can flip two edges, say for example the cubies in the *uf* and *ur* cubicals. This corresponds to the permutation $(7, 18)(5, 26)$.

SAGE
```
sage: S48("(7,18)(5,26)") in RC3
True
```

Notice this only tells us that it is possible to flip two edges using moves R, L, U, D, F, B, but it doesn't indicate what sequence of moves will do this. This is in fact a much harder problem. Basically what we are asking for is a method which can determine, for any element of RC_3, a way to write it as a product of the generators (or equivalently, as a word in R, L, U, D, F, B). This is known as the *word problem* in group theory and is very difficult in many situations.

However, there is an implementation in SAGE of an algorithm for solving the word problem in RC_3. It doesn't return the shortest possible move sequence, but it does a pretty good job nonetheless. For this we need to use the built-in `CubeGroup()` package.

SAGE
```
sage: rubik=CubeGroup();
sage: G=rubik.group();
sage: R=rubik.R();
sage: L=rubik.L();
sage: U=rubik.U();
sage: D=rubik.D();
sage: F=rubik.F();
sage: B=rubik.B();
sage: state = G("(7,18)(5,26)")
sage: rubik.solve(state)     # calls the solve algorithm
"F2 R2 B' F' D' F D B R2 F' R' F' R"
```

Therefore, one move-sequence for flipping edges uf and ur is

$$F^2R^2B^{-1}F^{-1}D^{-1}FDBR^2F^{-1}R^{-1}F^{-1}R.$$

12.2.2 2-Cube Group

We label the facets of the Pocket Cube as shown in Figure 3. Figure 9 shows the labeling on an actual cube.

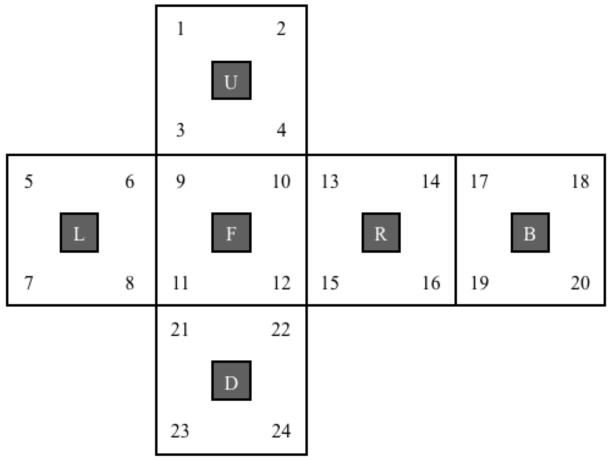
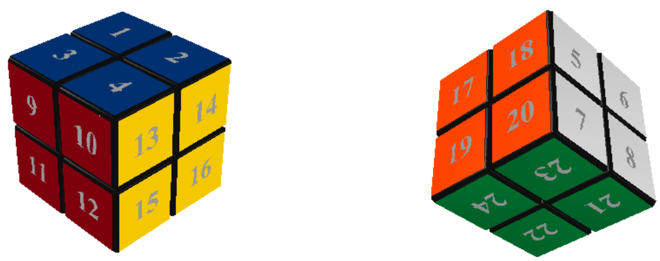


Figure 3: Facet labeling on the Pocket cube.



(a) Labeling on Up (blue), Right (yellow), Front (red) faces

(b) Labeling on Down (green), Back (orange), Left (white) faces

Figure 4: The labeling of the facets of the Pocket Cube.

The permutation corresponding to each of the basic moves of the Pocket Cube are:

$$\begin{aligned}
R &= (13,14,16,15)(10,2,19,22)(12,4,17,24)\\
L &= (5,6,8,7)(3,11,23,18)(1,9,21,20)\\
U &= (1,2,4,3)(9,5,17,13)(10,6,18,14)\\
D &= (21,22,24,23)(11,15,19,7)(12,16,20,8)\\
F &= (9,10,12,11)(3,13,22,8)(4,15,21,6)\\
B &= (17,18,20,19)(1,7,24,14)(2,5,23,16)
\end{aligned}$$

$R^{-1}, L^{-1}, U^{-1}, D^{-1}, F^{-1}, B^{-1}$ correspond to the inverses of these permutations.

There is one major difference between the Pocket cube and Rubik's cube: the Pocket cube does not have any fixed centres. Why does this matter? Consider the moves R and L. They are *equivalent*! Notice that applying R, leaves the cube in exactly the same position as L (the cube as a whole has just been rotated in space). Another way to say this is RL^{-1} is the identity in RC_2. Try it!

But if we were to use the permutations above to generate a group then this wouldn't be the group RC_2. Since the product of permutations associated with R and L don't have the property that $RL^{-1} = \varepsilon$. This means the permutations are picking up the fact that the cube rotated in space.

Again let's summarize the real difference between Rubik's cube and the Pocket cube: the Pocket cube can be rotated in space using only puzzle moves (which rotate faces), whereas Rubik's cube cannot be rotated in space using puzzle moves (since centres stay fixed under face rotations).

This means that RC_2 is smaller that the permutation group generated by the 6 permutations above. In fact, we really only need one of each of the following pairs of moves: $\{R, L\}, \{U, D\}, \{F, B\}$. We'll choose to only use R, D, F. This means the UBL cubie always remains in its home position. This is the piece we will keep fixed.

SAGE
```
sage: S24=SymmetricGroup(24)
sage: R=S24("(13,14,16,15)(10,2,19,22)(12,4,17,24)")
sage: D=S24("(21,22,24,23)(11,15,19,7)(12,16,20,8)")
sage: F=S24("(9,10,12,11)(3,13,22,8)(4,15,21,6)")
sage: RC2=S24.subgroup([R,D,F])      # define Pocket cube group to be RC2
```

We can determine the size of RC_2.

SAGE
```
sage: RC2.order()
3674160
sage: factor(RC2.order())
2^4 * 3^8 * 5 * 7
```

Therefore there are approximately 3.6 million configurations of the Pocket cube. And only one solution.

Theorem 12.2 *The Pocket cube group RC_2 has order $2^4 3^8 5 \cdot 7 = 3,674,160$.*

If we didn't realize that some moves are equivalent, and just constructed the group generated by all moves, what would happen?

SAGE
```
sage: S24=SymmetricGroup(24)
sage: R=S24("(13,14,16,15)(10,2,19,22)(12,4,17,24)")
sage: L=S24("(5,6,8,7)(3,11,23,18)(1,9,21,20)")
sage: U=S24("(1,2,4,3)(9,5,17,13)(10,6,18,14)")
sage: D=S24("(21,22,24,23)(11,15,19,7)(12,16,20,8)")
sage: F=S24("(9,10,12,11)(3,13,22,8)(4,15,21,6)")
sage: B=S24("(17,18,20,19)(1,7,24,14)(2,5,23,16)")
sage: S24.subgroup([R,L,U,D,F,B]).order()
88179840
sage: 88179840/3674160
24
```

We would have been off by a factor of 24. Why 24? This is precisely the number of different rotations there are for the whole cube. Since the permutation group was treating rotations of the cube as different states, but the cube group RC_2 should know these states really aren't different at all, then it is no surprise that we would be off by the number of rotations to the cube: 24.

This does illustrate, however, that we can't just assign a permutation to each move, and form the permutation group. Some thought needs to be taken as to whether the representation is faithful.

Swapping Corners on the Pocket Cube:

Are we able to swap two corners on the Pocket Cube, while keeping every other cubie in its home location (not necessarily with proper orientation)?

If we think about what a typical permutation would look like, well this would be quite tedious. Since corners can possibly twist and be returned to their home locations, it is not a simply matter of just asking if a 2-cycle is in RC_2. However, we aren't really interested in how the stickers move around, just the cubies themselves. So if we view RC_2 acting on the the 8 cubies, we just want to know if we can swap two cubies, and fix all other cubies in their current location.

If we number the cubicles as follows: 1 is the *ufr* cubical, 2 is the *urb* cubical, 3 is the *ubl* cubical, 4 is the *ulf* cubical, 5 is the *dfr* cubical, 6 is the *drb* cubical, 7 is the *dbl* cubical, 8 is the *dlf* cubical.

The action of each move on the cubies are then:

$$\begin{aligned} R &= (1,2,6,5) \\ L &= (3,4,8,7) \\ U &= (1,4,3,2) \\ D &= (5,6,7,8) \\ F &= (1,5,8,4) \\ B &= (2,3,7,6) \end{aligned}$$

We can the ask SAGE to compute whether it is possible to swap the 1 and 2 cubies.

SAGE

```
sage: S8=SymmetricGroup(8)
sage: R=S8("(1,2,6,5)")
sage: L=S8("(3,4,8,7)")
sage: U=S8("(1,4,3,2)")
sage: D=S8("(5,6,7,8)")
sage: F=S8("(1,5,8,4)")
sage: B=S8("(2,3,7,6)")
sage: H=S8.subgroup([R,L,U,D,F,B])
sage: S8("(1,2)") in H
True
sage: H.order()=factorial(8)
True
```

The computation shows that not only can we swap cubies 1 and 2, but in fact every permutation of the 8 cubies is possible. Remember though, the representation of RC_2 that we chose to work with here ignores any twisting of corners. So even though we can move the pieces anywhere we want, there may be limitations on how we can twist them.

12.3 Oval Track

Let Puz be the Oval Track puzzle (or one of its variations), then we call $M(\text{Puz})$ the **Oval Track group** and we use the special notation OT to denote this group.

We'll look at a few different variations of the puzzle, corresponding to different modifications of the turntable move T.

12.3.1 Oval Track - TopSpin: $T = (1,4)(2,3)$

The basic legal moves of the TopSpin version of the Oval Track puzzle are R, and T, where R denotes a clockwise rotation of numbers around the track, where each number moves one space, and T denotes a rotation of the turntable. See Figure 5.

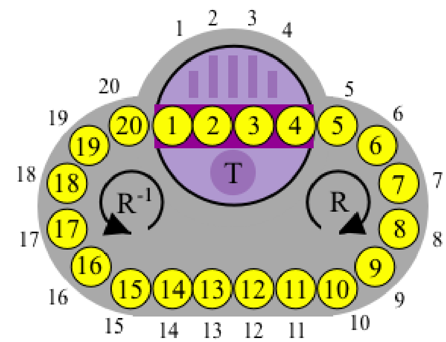


Figure 5: The Oval Track Puzzle.

The permutation corresponding to the legal moves R, and T are as follows:

$$R = (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20)$$
$$T = (1,4)(2,3)$$

Note that $T^{-1} = T$. This is due to the fact that spinning the turntable in either direction achieves the same result.

The basic moves R and T are not equivalent, so OT can be represented by the permutation group generated by these two permutations.

SAGE
```
sage: S20=SymmetricGroup(20)
sage: R=S20("(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20)")
sage: T=S20("(1,4)(2,3)")
sage: OT=S20.subgroup([R,T])        # define OT to be a permutation group
```

What is the size of OT. Since the puzzle consists of permuting 20 disks, we wonder if all permutations are possible. Since there are $20!$ permutations of 20 objects, we'd like to know if $|OT| = 20!$.

SAGE
```
sage: OT.order()==factorial(20)
True
```

This means OT is actually the symmetric group of degree 20: $OT = S_{20}$. Therefore, every permutation of the disks is possible. Of course, the key to solving this puzzle is to figure out how you can obtain each permutation using only moves R and T.

12.3.2 Oval Track - Variation 2: $T = (1,4,3,2)$

The *turntable move* in the original TopSpin puzzle is now replaced with the move indicated by the purple dashed lines. In this version, the new *turntable move* for the puzzle in Figure 6 moves the disk in spot 4 to spot 3, the disk in spot 3 to spot 2, the disk in spot 2 to spot 1, and takes the disk in spot 1 to spot 4.

The permutation corresponding to the legal moves R, and T are as follows:

$$R = (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20)$$
$$T = (1,4,3,2)$$

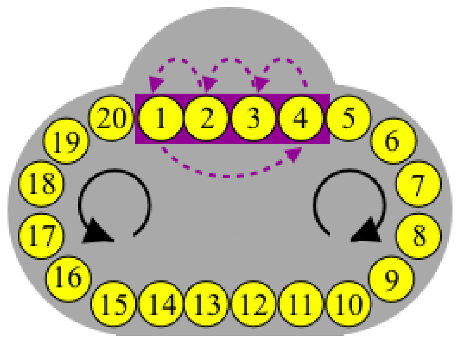


Figure 6: The Oval Track Puzzle.

SAGE

```
sage: S20=SymmetricGroup(20)
sage: R=S20("(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20)")
sage: T=S20("(1,4,3,2)")
sage: OT2=S20.subgroup([R,T])        # define OT2 to be a permutation group
sage: OT2.order()==factorial(20)
True
```

In this variation all possible permutations of the 20 disks are possible.

12.3.3 Oval Track - Variation 3: $T = (1,6)(2,5)(3,4)$

Another version of the *turntable move* involving 6 disks is given in Figure 7.

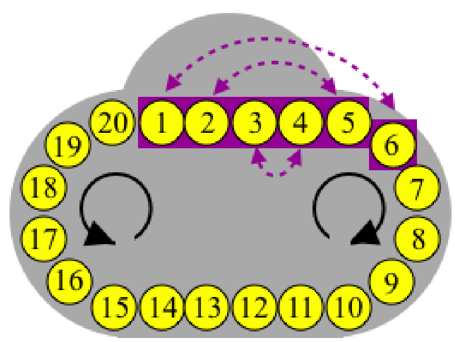


Figure 7: The Oval Track Puzzle.

SAGE

```
sage: S20=SymmetricGroup(20)
sage: R=S20("(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20)")
sage: T=S20("(1,6)(2,5)(3,4)")
sage: OT3=S20.subgroup([R,T])        # define OT3 to be a permutation group
sage: OT3.order()==factorial(20)
True
```

In this variation all possible permutations of the 20 disks are possible.

12.4 Hungarian Rings

Let Puz be the Hungarian Rings puzzle (numbered version), then we call $M(\text{Puz})$ the **Hungarian Rings group** and we use the special notation HR to denote this group.

The basic legal moves of the Hungarian Rings puzzle are R, and L, where R denotes a clockwise rotation of numbers around the right-hand ring (each number moves one space), and L denotes a clockwise rotation of numbers around the left-hand ring.

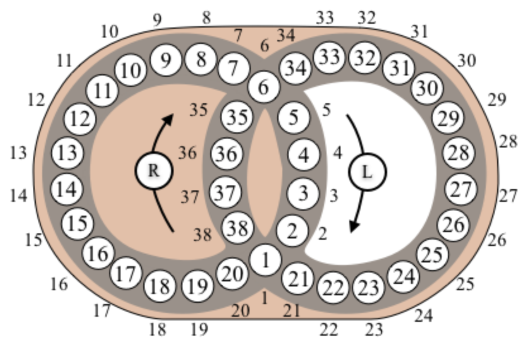


Figure 8: Hungarian Rings - numbered version.

The permutation corresponding to each of the legal moves R and L are:

$$R = (1, 38, 37, 36, 35, 6, 34, 33, 32, 31, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21)$$
$$L = (1, 20, 19, 18, 17, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2)$$

R^{-1} and L^{-1} correspond to the inverses of these permutations.

Since the moves R and L are inequivalent then HR can be represented by the group of permutations generated by R and L.

SAGE
```
sage: S38=SymmetricGroup(38)
sage: L=S38("(1,20,19,18,17,16,15,14,13,12,11,10,9,8,7,6,5,4,3,2)")
sage: R=S38("(1,38,37,36,35,6,34,33,32,31,30,29,28,27,26,25,24,23,22,21)")
sage: HR=S38.subgroup([R,L])==factorial(38)
True
```

Therefore, all possible permutations of the 38 balls are possible.

12.5 15-Puzzle

The 15-puzzle does not fit into group theory as neatly as our other puzzles do. The problem is that a move must involve the empty space, so the available legal moves at each stage changes depending on where the empty space is.

In what follows, we will describe a move of the pieces of the 15 puzzle by the first letter of the word *(u)p*, *(d)own*, *(l)eft*, *(r)ght*, which is to indicate the direction a tile is pushed into the empty space. For example, beginning with the empty space in spot 16, let m_1 be the sequence of moves:

$$m_1 = rrr.$$

Similarly, with the empty space in spot 16, let m_2 be the sequence of moves:

$$m_2 = rddd.$$

Move m_1 places the empty space in spot 13 by moving all tile on the bottom row to the right. Whereas, move m_2 places the empty space in spot 3. Therefore, it is impossible to perform the move sequence $m_1 m_2 = (rrr)(rddd)$ since once three r moves are applied there is no tile to the left of the empty space to apply another r move. The set of all legal moves is not closed under composition, therefore is not a group.

However, if we narrow our focus we can find a group lurking in there somewhere.

Represent each sequence of moves by its corresponding permutation, so the set of all such move sequence corresponds to a subset of the permutation group S_{16}. Let this subset be denoted by FP:

$$FP = \{\alpha \mid \alpha \text{ is the permutation corresponding to a legal position of the 15-puzzle}\}.$$

We already noted FP is not a group but the example gives us some insight into how we can fix this. If each moves starts with the empty space in box 16, then returns it to box 16, then the next move can be applied without any trouble. We let FP^* consist of the set of all moves that leaves the empty space in spot 16. In terms of permutations this means:

$$FP^* = \{\alpha \in FP \mid \alpha(16) = 16\}.$$

Now FP^* is a group. In fact we know it to be the group A_{15}.

In general when considering puzzles with gaps, we can look at the subset of legal moves where each move returns the space to its home position, this set will form a group.

12.6 Exercises

1. **Single Corner Twist.** Is it possible to rotate a single corner cubie of Rubik's cube, while leaving everything else in its home position?

(a) Figure for Exercise 1

(b) Figure for Exercise 2

(c) Figure for Exercise 3

Figure 9: Which corner twists are possible?

2. **Two Corner Twists.** For Rubik's cube, is it possible to rotate two corner cubies in the same direction, while leaving everything else in its home position?
3. **Another Two Corner Twists.** For Rubik's cube, is it possible to rotate two corner cubies in opposite directions, while leaving everything else in its home position?
4. **Swapping Corners on Rubik's cube.** Show that it is impossible to swap two corner cubies on Rubik's cube, while leaving all other cubies in their home locations (not necessarily with proper orientation)?
5. **Oval Track with 19 Disks.** Consider the Oval Track puzzle (TopSpin version) where only 19 disks are used. Are all permutations of the 19 disks possible? If not, can you describe exactly which permutation are possible?

6. **Varying the Number of Disks on Oval Track.** For the Oval Track puzzle with n disk, let OT_n denote the puzzle group, determine the size of OT_n, for $6 \leq n \leq 20$. In each case, describe exactly which permutations of the puzzle pieces are possible.

7. **Very Few Disks on Oval Track.** Consider OT_n for $n = 4, 5$. Investigate which permutations of the puzzle pieces are possible.

8. **Varying the turntable move T of the Oval Track puzzle.** In this exercise you will investigate, with the help of SAGE , some variations of the Oval Track puzzle. In all variations[1], we assume there are 20 disks, and the usual move consisting of rotating the pieces along the track is R. We will vary the turntable move T. We have already seen that if the turntable move is $T = (1,4)(2,3)$ or $T = (4,3,2,1)$ then we are still able to obtain *all* permutations of the 20 disks. Investigate the other variations of the move T given in Table 1. Under the column "permutation group", try to determine what groups of permutation of the 20 pieces is possible. The first two rows have been filled in already.

variation	**turntable move T**	**permutation group**
$OT\ 1$	$(1,4)(2,3)$	S_{20}
$OT\ 2$	$(4,3,2,1)$	S_{20}
$OT\ 3$	$(3,2,1)$	
$OT\ 4$	$(5,4,3,2,1)$	
$OT\ 5$	$(1,2)(3,4)$	
$OT\ 6$	$(1,11)(4,14)$	
$OT\ 7$	$(5,3,1)$	
$OT\ 8$	$(1,3)(2,4)$	

Table 1: Variations of the Oval Track puzzle

[1]Variation names are due to John O. Kiltinen who studies these in his book: *Oval Track and other Permutation Puzzles*.