Lecture 7: Permutations: The Parity Theorem

Contents

In this lecture we introduce one of the most important theorems about permutations: **The Parity Theorem**.

We already know every permutation can be expressed using 2-cycles. We now explore the question of how many 2-cycles are needed.

In Section 3.1 of Joyner's text he discusses parity, but his approach may look a little different than the one we take below. In fact, they are simiilar, where he talks about the *swapping number*, we are actually playing the Swap puzzle and physically swapping pieces. The only difference is we prove the Parity Theorem in a different way than he does.

7.1 Introduction

In Lecture 6 we saw that the permutation $\alpha = (1,3,5)(2,4,7,6,8)$ can be written as a product of 2-cycles in two different ways:

$$\begin{aligned}\alpha &= (7,8)(6,8)(4,7)(3,5)(2,4)(1,3)\\ &= (1,3)(1,5)(2,4)(2,7)(2,6)(2,8).\end{aligned}$$

The first decomposition we obtained by considering the permutation as an initial scrambling of the tiles of Swap, then solving the puzzle by restoring each tiles to its home position in increasing order, beginning with tile 1. The second decomposition was obtained using our "quick" method for decomposing permutations. There are many more possible decompositions of α, here are two more:

$$\begin{aligned}\alpha &= (1,6)(6,7)(1,4)(1,7)(2,8)(4,8)(1,5)(3,5)\\ &= (1,3)(1,2)(1,4)(1,2)(1,5)(1,2)(1,7)(1,6)(1,8)(1,2)\end{aligned}$$

The number of 2 cycles used in the decompositions are not always the same. In the four decompositions we have, two use 6 , one uses 8, and one uses 10. Even though the number of 2-cycles isn't constant, it always seems to be of the same parity, in this case it is *even*.

This observation is true in general. Before we state the general result we first better explain the term *parity*. We say for an integer m that its **parity is even** if m is a multiply of 2. If m is not a multiple of 2 then its **parity is odd**. Perhaps the term "parity" is not familiar, but certainly the distinction between an odd number and an even number is. Much in the same way that integers come in one of two types, based on parity: odd or even, permutations also come in one of two types, based on the parity of a permutation. This is what the next theorem says.

Theorem 7.1 (The Parity Theorem) *If a permutation α can be expressed as a product of an even number of 2-cycles, then every decomposition of α into 2-cycles must have and even number. On the other hand, if α can be expressed as a product of an odd number of 2-cycles, then every decomposition of α into 2-cycles must have an odd number. In symbols, if*

$$\alpha = \tau_1\tau_2\cdots\tau_r = \sigma_1\sigma_2\cdots\sigma_s$$

where the τ_i's and σ_i's are 2-cycles, then r and s are both even or both odd.

We will give two different proofs of this theorem in the next few sections. But for now lets look at some of the consequences of this theorem.

The Parity Theorem tells us that we can define what we mean by the parity of a permutation. If we decompose it as a product of 2-cycles in any way, then the parity of the number of 2-cycles that we used is either odd or even, and this is the parity we assign to the permutation. Here is the formal definition.

Definition 7.1 (Even and Odd Permutation) *A permutation that can be expressed as a product of an even number of 2-cycles is called an **even permutation**. A permutation that can be expressed as a product of an odd number of 2-cycles is called an **odd permutation**.*

This definition of parity may not seem to exciting, but just wait. This will allow us to answer important question about the puzzle, and sometimes allow us to abandon quests that are impossible.

Definition 7.2 (Sign of a Permutation) *The **sign** of a permutation α is defined to be 1 if α is even, or -1 if α is odd.*

$$\mathit{sign}(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is an even permutation,} \\ -1 & \text{if } \alpha \text{ is an odd permutation.} \end{cases}$$

Parity of the Identity:
The identity permutation ε is an even permutation.

This follows from $\varepsilon = (1,2)(1,2)$, which is a decomposition into and even number of 2-cycles.

SAGE
```
sage: S5=SymmetricGroup(5)
sage: a=S5("()")    #the identity permutation in cycle form.
sage: a.sign()
1
```

It is useful to be aware of the parity of cycles.

Parity of a Cycle:
An m-cycle, $(a_1, a_2, \ldots, a_m)$ is an even permutation if m is odd, and it is an odd permutation if m is even. (Confusing, I know.)

This follows from the fact that an m-=cycle can be expressed as a product of $m-1$ transpositions:

$$(a_1, a_2, \ldots, a_m) = (a_1, a_2)(a_1, a_3) \cdots (a_1, a_m).$$

If m is even then $m-1$ is odd, and vice-versa. This is why the parity of the permutation is opposite to the parity of the length of the cycle.

SAGE
```
sage: S5=SymmetricGroup(5)
sage: a=S6("(1,2,3,4)")
sage: a.sign()
-1
sage: b=S5("(1,2,3,4,5)")
sage: b.sign()
1
```

Example 7.1 *Determine whether the following permutations are odd or even.*

(a) $(1, 5, 11, 6, 7, 3)$

(b) $(1, 4, 12)(3, 8, 5, 9)(7, 10)$

(a) This is a 6*-cycle and therefore an odd permutation since it can be written as a product of* 5 *transpositions:*

$$(1, 5, 11, 6, 7, 3) = (1, 5)(1, 11)(1, 6)(1, 7)(1, 3).$$

SAGE
```
sage: S11=SymmetricGroup(11);
sage: a=S11("(1,5,11,6,7,3)");
sage: a.sign()
-1
```

(b) Writing each cycle as a product of transpositions we have:

$$(1, 4, 12)(3, 8, 5, 9)(7, 10) = (1, 4)(1, 12)(3, 8)(3, 5)(3, 9)(7, 10).$$

Since $(1, 4, 12)(3, 8, 5, 9)(7, 10)$ can be written as the product of 6 transpositions, it follows that it is even.

SAGE
```
sage: S12=SymmetricGroup(12);
sage: b=S12("(1,4,12)(3,8,5,9)(7,10)");
sage: b.sign()
1
```

7.2 Variation of Swap

in Lecture 6 we considered the following variation on the legal moves of Swap.

Variation: Legal move is to pick any 3 boxes and cycle their contents either to the left or right.

For this variation, a permutation corresponding to a scrambling of Swap is solvable if and only if it can be expressed as a product of 3-cycles (i.e. the legal moves). Since 3-cycles are even permutations, and products of even permutations are even (see Exercise 5) then any product of 3-cycles must be an even permutation. This means an odd permutation of Swap is not solvable under this variation of the legal moves.

For example, the scrambling

1	2	3	4	5	6	7	8
2	8	1	5	3	7	4	6

is not solvable. This corresponds to the permutation $\alpha = (1,3,5,4,7,6,8,2)$ which is an odd permutation.

Rather than trying many repeated failed attempts at solving the puzzle, the characterization of α as an odd permutation lead us to abandon any such question to solve the puzzle. This provides a glimpse into how we will be using the Parity Theorem to investigate the solvability of puzzles.

7.3 Proof of the Parity Theorem

In this section we provide a two different proofs of the Parity Theorem. However, rather than proving the Parity Theorem directly we will prove another result (Claim 7.2), from which the Parity Theorem follows. While reading this section, keep in mind we cannot assume that the Parity Theorem is true (yet), since this is what we are trying to prove.

Consider the following Claim.

Claim 7.1 *Any expression for the identity permutation ε as a product of transpositions uses an even number of them. That is, if*

$$\varepsilon = \tau_1\tau_2\cdots\tau_m$$

where the τ_i's are transpositions, then m is an even integer.

Before considering *why* this is true, let's see how the Parity Theorem is a consequence of this claim. Suppose $\tau_1\tau_2\cdots\tau_r$ and $\sigma_1\sigma_2\cdots\sigma_s$ are two decompositions of a permutation α into 2-cycles. Then

$$\varepsilon = \alpha\alpha^{-1} = (\tau_1\tau_2\cdots\tau_r)(\sigma_1\sigma_2\cdots\sigma_s)^{-1} = \tau_1\tau_2\cdots\tau_r\sigma_s^{-1}\cdots\sigma_2^{-1}\sigma_1^{-1}$$

is a decomposition of ε into $r+s$ transpositions. If Claim 7.1 is true, then $r+s$ must be even, from which it follows that r and s have the same parity. Therefore the Parity Theorem 7.1 is true.

Therefore, in order to prove the Parity Theorem it is sufficient to prove Claim 7.1. But how do we know Claim 7.1 is true? Well, one way is to prove that:

Claim 7.2 *If there is an expression $\tau_1\tau_2\cdots\tau_m$ for the identity permutation ε that uses m transpositions, then there is an expression for ε that uses $m-2$ transpositions.*

Again, before considering *why* Claim 7.2 is true, let's see how we can use it to prove Claim 7.1. Let's assume to the contrary that it was possible to have an expression $\tau_1\tau_2\cdots\tau_m$ for ε where m is odd. Then, assuming Claim 7.2 is true, we could get an expression using $m-2$ transpositions (which is still an odd number of transpositions). We could keep applying Claim 7.2, reducing the number of transpositions by 2 each time, until we end up with an expression for ε using only one transposition. But this is impossible since a single transposition is not equal to the identity (the two numbers in the cycle would not be fixed by the permutation). The fact that we get something impossible from the assumption that an expression for ε exists that uses an odd number of transpositions forces us to conclude that Claim 7.1 is true.

To summarize we have

$$\text{Claim 7.2} \Rightarrow \text{Claim 7.1} \Rightarrow \text{Theorem 7.1}$$

So it suffices to prove Claim 7.2. This is the proof will focus on here. We will provide two completely different proofs, one will be algebraic in nature and will involve playing around with the cycle decomposition of permutations (this is the classic proof), the other will be a little more tactile and has a game-like feel to it (this proof is due John O. Kiltinen).

7.3.1 Proof 1 of Claim 7.2

The rough idea of what we will do is: first we will pick the right-most occurrence of any number appearing in decomposition into transpositions. Then we will push this number to the left through the transpositions, while transforming the transpositions at the same time, until we eventually get two transpositions that cancel.

Before giving a formal proof let's look at an example. The product of the following 12 transpositions is the identity. Check for yourself!

$$\varepsilon = (1,2)(1,3)(1,4)(1,6)(1,5)(3,4)(3,5)(2,5)(2,3)(1,5)(2,6)(2,4) \tag{1}$$

We will transform this product to a product of only 10 transpositions, which still represents the identity. Choose a number appearing in any transpositions. We'll choose 3. Find the right-most transposition containing this number. In this case it would be the transposition $(2,3)$ (the ninth one in the list). We now want to push 3 to the left, so in this product we replace $(2,5)(2,3)$ with the equivalent permutation $(3,5)(2,5)$ (Check for yourself that $(2,5)(2,3) = (3,5)(2,5)$.):

$$\varepsilon = (1,2)(1,3)(1,4)(1,6)(1,5)(3,4)(3,5)\mathbf{(3,5)(2,5)}(1,5)(2,6)(2,4).$$

Now we can replace $(3,5)(3,5)$ with ε:

$$\varepsilon = (1,2)(1,3)(1,4)(1,6)(1,5)(3,4)(2,5)(1,5)(2,6)(2,4),$$

which is an expression using 2 fewer transpositions than we started with.

Sometime it may take a few more steps, for example if we decided to use 5 instead of 3, we would have proceeded as follows: Find the right-most transposition containing this number in Equation (1). In this case it would be the transposition $(1,5)$. We now want to push 5 to the left, so in this product we replace $(2,3)(1,5)$ with the equivalent permutation $(1,5)(2,3)$, since disjoint cycle commute.

$$\varepsilon = (1,2)(1,3)(1,4)(1,6)(1,5)(3,4)(3,5)(2,5)(1,5)(2,3)(2,6)(2,4)$$

Next replace $(2,5)(1,5)$ with $(1,5)(1,2)$:

$$\varepsilon = (1,2)(1,3)(1,4)(1,6)(1,5)(3,4)(3,5)(1,5)(1,2)(2,3)(2,6)(2,4),$$

then replace $(3,5)(1,5)$ with $(1,5)(1,3)$

$$\varepsilon = (1,2)(1,3)(1,4)(1,6)(1,5)(3,4)(1,5)(1,3)(1,2)(2,3)(2,6)(2,4).$$

Since $(3,4)$ and $(1,5)$ commute, the two $(1,5)$'s would cancel and we get:

$$\varepsilon = (1,2)(1,3)(1,4)(1,6)(3,4)(1,3)(1,2)(2,3)(2,6)(2,4),$$

which is an expression using 2 fewer transpositions than we started with.

With these two examples behind us, we now give the formal proof.

Proof of Claim 7.2:
Choose a number a that appears in the transposition τ_m. Since $(i,j) = (j,i)$ for any transposition (i,j), the product $\tau_{m-1}\tau_m$ can be expressed in one of the following ways as shown on the left:

$$\begin{aligned}
(a,b)(a,b) &= \varepsilon \\
(a,c)(a,b) &= (a,b)(b,c) \\
(c,d)(a,b) &= (a,b)(c,d) \\
(b,c)(a,b) &= (a.c)(c,b)
\end{aligned}$$

If the first case occurs we may delete $\tau_{m-1}\tau_m$ in the original product and obtain a product for ε using $m-2$ transpositions. In the other three cases we replace the form $\tau_{m-1}\tau_m$ with what appears on the right to obtain a new product of m transpositions that is still the identity, but where the right-most occurrence of a has now moved one 2-cycle to the left. We now repeat the process, where at each stage either we cancel two 2-cycles (and we're done), or we form a new product where a has moved another 2-cycle to the left. This process must terminate with a product of $(m-2)$ transpositions equal to the identity, because otherwise we have a product of m transpositions equal to the identity in which the only occurrence of a is in the left-most 2-cycle, and such a product does not fix a whereas the identity does. $\square$

This completes the proof of Claim 7.2, and therefore the proof of the Parity Theorem too.

7.3.2 Proof 2 of Claim 7.2

We now present a more tactile proof of Claim 7.2 which is due to John O. Kiltinen.

Let's reinterpret what the claim says in terms of the Swap puzzle. Suppose that you started with the Swap puzzle as the identity permutation. Now imagine you did some swaps at random, not paying particular attention to what you were doing. After doing this for a while, you then decide to put everything back in its proper place. In other words you have produced a sequence of transpositions that equals ε. If you count the number of total transpositions you have used, this number will be even. This is precisely what Claim 7.1 says.

Now imagine your friend showed you a sequence of transpositions that they used to produce ε. Is it possible for you to best your friend and produce another such sequence that uses two few transpositions? Claim 7.2 says the answer is yes, and what we'll do here is describe a method to produce the shorter sequence, which can be done in real-time.

Let's call your friend Alice. Imagine two copies of Swap stacked on top of each other, the top is Alice's set and the bottom is yours. We'll colour Alice's tiles green and yours blue, just to make it clear whose is whose.

	1	2	3	4	5	6	7	8
Alice	1	2	3	4	5	6	7	8
You	1	2	3	4	5	6	7	8

As Alice applies her transpositions, we will match/modify her moves, but in the end we will use two fewer. Here is how we'll do that.

Let's call box containing the tile she touches first the *First Box*. We will call the tile that Alice takes from the First Box *Alice's First Green Tile*, and call the box to which it went the *Tagged Box*. We will call the corresponding blue tile, *Our First Blue Tile*. To aid our memory, let us put markers into these boxes. The marker we place in the First Box (shaded background) will remain throughout the process, but the one in the Tagged Box (solid square in lower right corner) may move, depending on what Alice does.

After Alice makes her first move, we will not make a move of our own. This is the first transposition we omit, the second one is essentially the one she does when she needs to undo this move. Instead of matching her first move, we simply put markers on the boxes. From this point on, however, we will make a move in such a way that the following four conditions are always satisfied, up until she returns Alice's First Green Tile to the First Box, at that point we will just mirror her moves until she finishes.

Conditions to be satisfied after every turn up until she returns Alice's First Green Tile to the First Box:

(i) Alice's First Green Tile is always in the Tagged Box.

(ii) Our First Blue Tile is always at home in the First Box.

(iii) Whatever green tile number that Alice has in the First Box, we have the blue tile with that number in the Tagged Box.

(iv) All boxes other than the Tagged Box and the First Box contain green tiles and blue tiles with the same numbers.

Let's get our hands dirty and do an example. It may look a little confusing at first, but the idea is really straightforward. Suppose Alice's move sequence is

$$(3,7)(4,5)(1,3)(2,7)(3,8)(1,5)(2,6)(3,7)(2,6)(2,3)(1,4)(1,5)(1,7)(1,8).$$

The means Alice's first move is $(3,7)$, and so the First Box is box 3 (shaded background), Alice's First Green Tile is tile 3, and the Tagged Box is box 7 (square box in bottom right corner). So after her first move, we don't make a move, and we have.

	1	2	3	4	5	6	7	8
Alice	1	2	7	4	5	6	3	8
You	1	2	3	4	5	6	7	8

Alice's next move is $(4, 5)$. Since she doesn't touch tiles in either the First Box or the Tagged box then conditions (i)-(iv) will remain satisfied if we do the same move: $\tau_1 = (4, 5)$.

	1	2	3	4	5	6	7	8
Alice	1	2	7	5	4	6	3	8
You	1	2	3	5	4	6	7	8

Alice's next move is $(1, 3)$, and since this involves the First Box we move the contents of the Tagged Box instead, to satisfy condition (iii): $\tau_2 = (1, 7)$.

	1	2	3	4	5	6	7	8
Alice	7	2	1	5	4	6	3	8
You	7	2	3	5	4	6	1	8

Alice's next move is $(2, 7)$, and since this involves the Tagged Box 7 we move the Tagged Box marker as well as the tile, to satisfy condition (i). We also perform the same move on the blue tiles: $\tau_3 = (2, 7)$.

	1	2	3	4	5	6	7	8
Alice	7	3	1	5	4	6	2	8
You	7	1	3	5	4	6	2	8

Alice's next move is $(3, 8)$, and so we swap the blue tiles in the Tagged Box 2 and box 8. This will keep conditions (i)-(iv) satisfied. $\tau_4 = (2, 8)$.

	1	2	3	4	5	6	7	8
Alice	7	3	8	5	4	6	2	1
You	7	8	3	5	4	6	2	1

Alice's next move is $(1, 5)$ mirror this move: $\tau_5 = (1, 5)$.

	1	2	3	4	5	6	7	8
Alice	4	3	8	5	7	6	2	1
You	4	8	3	5	7	6	2	1

Alice's next move is $(2, 6)$ and we mirror it: $\tau_6 = (2, 6)$, while at the same time be move the tag to box 6.

	1	2	3	4	5	6	7	8
Alice	4	6	8	5	7	3	2	1
You	4	6	3	5	7	8	2	1

Alice's next move is $(3,7)$ and we do $\tau_7 = (6,7)$.

	1	2	3	4	5	6	7	8
Alice	4	6	2	5	7	3	8	1
You	4	6	3	5	7	2	8	1

Alice's next move is $(2,6)$ and we mirror it $\tau_8 = (2,6)$, while at the same time be move the tag to box 2.

	1	2	3	4	5	6	7	8
Alice	4	3	2	5	7	6	8	1
You	4	2	3	5	7	6	8	1

Alice's next move is $(2,3)$, and since this involves both the First Box and the Tagged Box this the the other transposition that we skip.

	1	2	3	4	5	6	7	8
Alice	4	2	3	5	7	6	8	1
You	4	2	3	5	7	6	8	1

Now both the green and blue tiles are in the same positions so we mirror the remaining moves Alice does: $\tau_9 = (1,4), \tau_{10} = (1,5), \tau_{11} = (1,7), \tau_{12} = (1,8)$. We have now produced as sequence permutations which is the identity:

$$\varepsilon = \prod_{i=1}^{12} \tau_i = (4,5)(1,7)(2,7)(2,8)(1,5)(2,6)(6,7)(2,6)(1,4)(1,5)(1,7)(1,8),$$

but uses 2 fewer permutations than Alice used.

This example illustrates the procedure, but how can we be sure it works in general. That is, how do we know there is always a move that we can do which keeps conditions (i)-(iv) satisfied. Well, the following rules provide us with these moves.

Rules to follow to ensure conditions (i)-(iv) are satisfied:

(a) If Alice does a transposition on her green tiles between boxes other that the First Box or the Tagged Box, then we do the same transposition on our blue tiles.

(b) If Alice does a transposition on her green tiles between the First Box and a box other than the Tagged Box, then we respond with a transposition on our blue tiles between the Tagged Box and the other box, not the First, that she used.

(c) If Alice does a transposition on her green tiles between the Tagged Box and a box other than the First Box, then we respond with a transposition on our blue tiles between the same two boxes. However, we also move the tag so that this other box now becomes the Tagged Box.

(d) If Alice does a transposition on her green tiles between the Tagged Box and the First Box, then we do not do this one.

(e) Once Alice has done a transposition of the type described in (d), which she must, then for every transposition of hers thereafter, we do the same transposition.

If we follow these rules when playing Alice, then we can be certain that conditions (i)-(iv) remain satisfied up until she moves Alice's First Green Tile back to the First Box. She must eventually have to make such a move since Alice's First Green Tile must return home (since the permutation is the identity) and the home position is precisely the First Box. Since we omit the first move and the move where she returns Alice's First Green Tile to the First Box, we have effectively reduced her sequence of transpositions by 2 moves. Thus completing the proof or Claim 7.2.

This may seem like a rather long-winded proof of Claim 7.2, and in fact it is. But this approach is designed to build on the tactile experience that you have developed from playing with the Swap puzzle and other permutation puzzles.

7.4 Exercises

1. Determine whether the following permutations are odd or even.

   (a) $(1,3,2)$
   (b) $(1,3,5,7,9)$
   (c) $(1,6,4,3)$
   (d) $(2,4,7)(3,9,5,8)$
   (e) $(1,9,4,5)(3,11,4)(6,7)$
   (f) $(1,2,3,4,5)(6,7,8,9,10)$

2. Determine whether the following permutations are odd or even.

   (a) $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 7 & 8 & 1 & 4 & 5 & 6 \end{pmatrix}$ (b) $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 2 & 1 & 6 & 7 & 8 & 3 & 4 \end{pmatrix}$

3. **The parity of 15-puzzle scrambles.** For each of the following arrangements of the 15-puzzle determine the parity of the corresponding permutation.

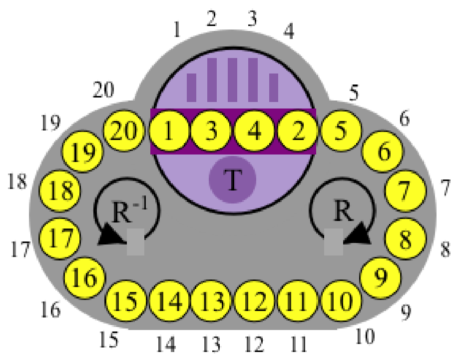
^{1} 1	^{2} 2	^{3} 3	^{4} 4
^{5} 5	^{6} 6	^{7} 7	^{8} 8
^{9} 9	^{10} 13	^{11} 15	^{12} 11
^{13} 14	^{14} 10	^{15} 12	^{16} empty

(a)

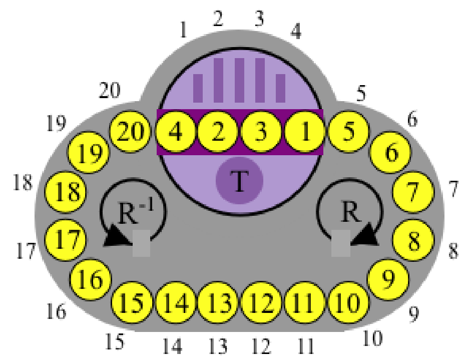
^{1} 13	^{2} empty	^{3} 5	^{4} 3
^{5} 2	^{6} 9	^{7} 7	^{8} 10
^{9} 15	^{10} 1	^{11} 14	^{12} 8
^{13} 12	^{14} 11	^{15} 6	^{16} 4

(b)

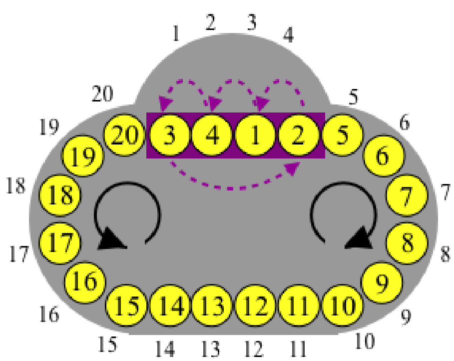
4. **The parity of some Oval Track end-game scrambles.** For each of the end-game arrangements of the Oval Track puzzle variations, determine its parity.



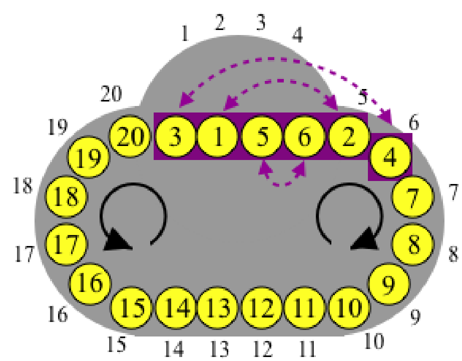
(a)



(b)



(c)



(d)

5. Show each of the following.
   (a) The product of two even permutations is an even permutation.
   (b) The product of two odd permutations is and even permutation.
   (c) The product of one even permutation and one odd permutation is and odd permutation.

6. In Definition 7.2 we defined the *sign* of an even permutation to be $+1$ and an odd permutation to be -1. Draw and analogy between the result of multiplying two permutations and the result of multiplying their corresponding signs: $+1$ and -1.
   (Hint: Use the results of the previous exercise.)

7. If α is even, prove that α^{-1} is even. If α is odd, prove that α^{-1} is odd.

8. Let $\alpha, \beta \in S_n$. Prove that $\alpha^{-1}\beta^{-1}\alpha\beta$ is and even permutation.

9. Let $\alpha, \beta \in S_n$. Prove that α and $\beta^{-1}\alpha\beta$ have the same parity.

10. Show that exactly half of the permutations in S_n are even.

11. Show that a permutation with odd order must be an even permutation.

12. Give and example of a permutation with even order, that is an even permutation. Also given an example of a permutation with even order, that is an odd permutation.

13. **The least number of transpositions to express a permutation.** Let $\alpha \in S_n$ with disjoint cycle form $\alpha = \sigma_1\sigma_2\cdots\sigma_r$, where σ_i is a k_i-cycle and they are arranged in such a way that

$k_1 \leq k_2 \leq \cdots \leq k_r$. In this situation we say α has cycle structure $(k_1, k_2, \ldots, k_r)$. If we express each cycle as a product of transpositions then we get an expression for α that uses $k - r$ transpositions, where $k = \sum_{i=1}^{r} k_i$. Show that this is the fewest transpositions that there can be in any expression for α in terms of transpositions.
(See the paper: J.O. Kiltinen, "How Few Transposiitons Suffice? ... You Already Know!", *Mathematics Magazine* 67 (1994) 45-47. for one such proof.)