

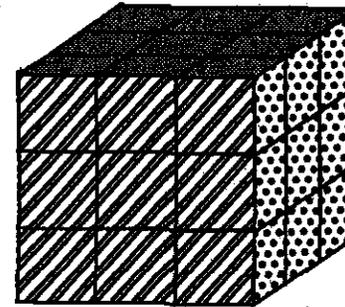
The Complete Cube Book
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THE COMPLETE CUBE BOOK

Illustrations by Philip Cary

**Regnery Gateway
Chicago**



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To the memory of
Unni Namboodiri, 1956-1981—
friend, mathematician, cubist

Acknowledgements

Some of the ideas in this book were obtained from conversation with other cubists, including Rick Miranda, Dan Goldstein, Nick Reingold, and Tom Hunter. I am also grateful to Dave Leep and Zal Usiskin for cubic advice.

The notation and several other ideas were drawn from "Notes on Rubik's Magic Cube" by David Singmaster. The alternative methods for obtaining Figs. 8.7 and 8.8 at the end of chapter 8 are credited to Richard Walker in the March 1981 *Scientific American*. The alternative method for Fig. 8.9 is derived from *Cube Games* by Don Taylor and Leanne Rylands.

Preface

If you are as fascinated by Rubik's Cube as I am, then this book is written for you. In my opinion, the Cube is the most intriguing puzzle ever invented.

There are several books available giving a solution to Rubik's Cube, as this book does, but most give solutions that require several minutes, even with practice.

Using the system presented in this book, you will soon be able to solve the cube in times consistently under one minute. I can solve the cube in 40 seconds. Teenagers seem to have better manual dexterity than the rest of us, so I have no doubt that many of you will be able to beat 40 seconds consistently.

Furthermore, this book does not require memorizing a lot of complicated algorithms, that is, methods. Most of the methods given here are fairly simple consequences of the conjugation principle and the commutator principle. These principles are explained in Chapter 2 and form the key to understanding Rubik's Cube.

This book is also the only book that clearly explains exactly which patterns are possible with Rubik's Cube and which are not. Previously, this information was available only to experts working in an esoteric branch of mathematics called "group theory." This book, however, is self-contained and does not presuppose any abstruse mathematical knowledge.

The complete solution to Rubik's Cube is given in Chapters 3 through 6. Chapter 7 gives some tips for solving the cube more quickly. Procedures for getting pleasing—even pretty—patterns are given in Chapter 8. Chapters 9 and 10 prove which patterns are possible and which are not. The number of possible positions is shown to be exactly 43,252,003,274,489,856,000. Some games and exercises are given in Chapter 11, and a glossary at the end of the book defines all of the relevant terms.

Chapter 1 Introduction

In this chapter I will describe Rubik's Cube, and define some notation and terminology that is used throughout this book. The basic problem, of course, is to restore the cube to its original position in which each face was one solid color. A complete, fast, and simple solution is given in chapters 2 through 6.

Rubik's Cube has six faces, which are abbreviated as follows:

- F* front
- B* back (*not* bottom)
- R* right
- L* left
- U* up
- D* down

Each face appears to be composed of nine small cubes, which I shall call "cubies" to distinguish them from the entire large cube. There are three types of cubies: *corners*, *edges*, and *centers*. Altogether the cube has 8 corners, 12 edges, and 6 centers, for a total of 26 cubies (see Fig. 1.1).

Each corner cubie has three colored stickers, each edge cubie has two stickers, and each center has one. No two cubies have the same colors.

A twist of a face moves each corner cubie to a location previously occupied by another corner cubie. In the same manner, edge cubies are sent to other edges.

In a sense, the centers never move. Twisting a face rotates the center of that face but leaves it in the same position as before. Of course, you could turn the entire cube, but no matter how the cube is scrambled, it is always possible to turn the entire cube (without twisting any faces) so that the center cubies remain in their original positions. Thus, the centers may be regarded as being fixed.

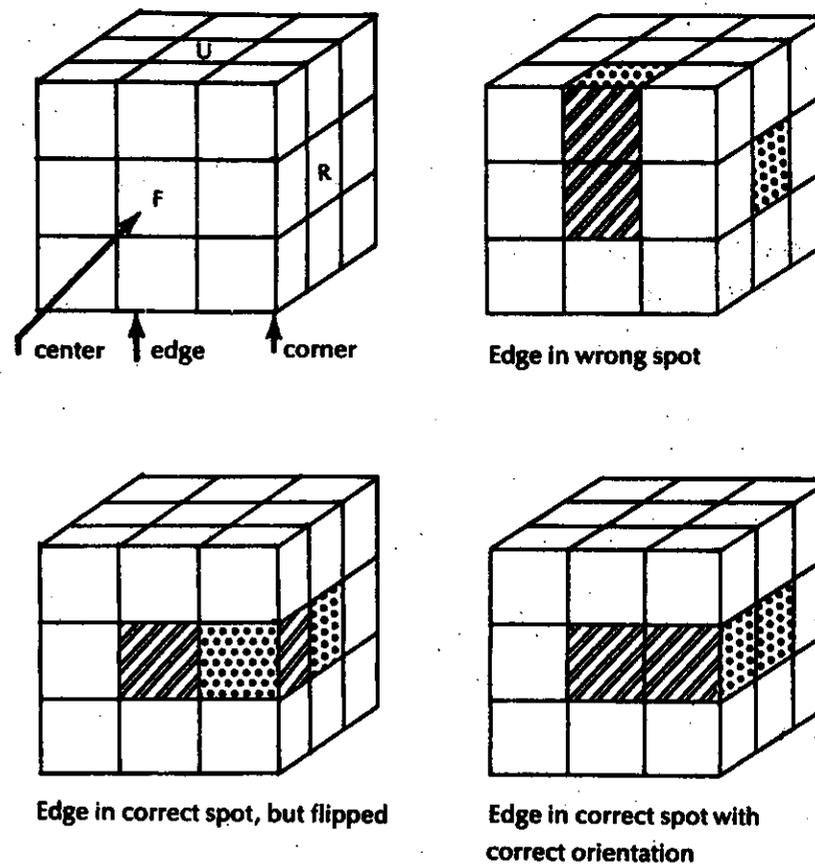


Fig. 1.1. The cube

Once we regard the centers as fixed, we can identify each face by the color of its center. To solve the cube it is necessary, for example, for the face with the red center eventually to become a solid red face.

As for each cubie, there is one and only one place where it belongs. To determine where a cubie belongs, look at its colors and find the matching centers.

For example, a red and green edge cubie must border on the red and green centers, and hence belongs between them.

To solve the cube you must not only get each cubie in its correct location, but you must orient it correctly. A cubie is oriented correctly if each sticker matches the color of the adjacent center. Each edge cubie has two

orientations and each corner cubie has three. If an edge does not match the adjacent centers, it must be flipped. If a corner is not oriented correctly, it must be rotated 120 degrees until its stickers match the colors of the centers they touch (see Fig. 1.2).

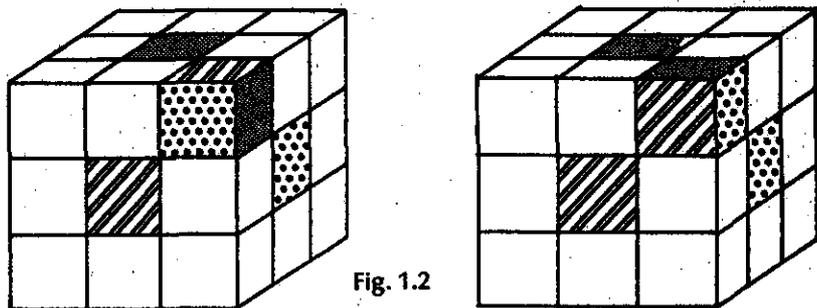
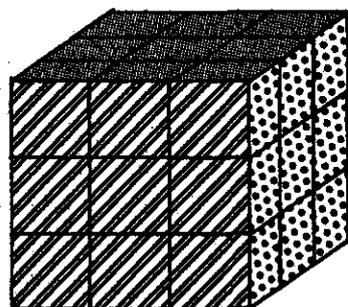


Fig. 1.2

Corner in place but with wrong orientation

Corner in place with correct orientation



Solved cube

Result of F applied to solved cube

Result of D applied to solved cube

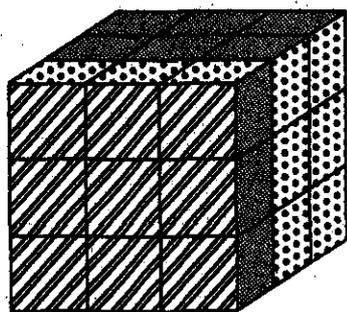
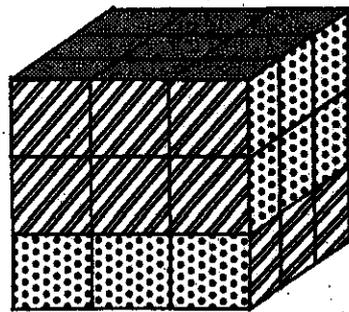


Fig. 1.3



Each face may be turned clockwise or counterclockwise. F means a 90-degree clockwise twist of the front face and F^{-1} refers to a 90-degree counterclockwise twist of the front face. R , L , U , D , and B refer to clockwise turns of the other faces. Turning a face clockwise always means turning that face in the direction which would be clockwise if that face were viewed directly, head-on. See Fig. 1.3.

The notation $LF^{-1}R$ means first turn the left face 90 degrees clockwise, then turn the front face 90 degrees counterclockwise, and lastly turn the right face 90 degrees clockwise. Having done $LF^{-1}R$, $R^{-1}FL^{-1}$ restores the cube.

R and R^{-1} are inverses to each other in the sense that RR^{-1} has no net effect on the cube. (This is the reason for the $^{-1}$ in the notation. R^{-1} should be read "R inverse.") $R^{-1}R$ also has no net effect. RR (which has the same result as $R^{-1}R$) rotates the right face by 180 degrees and can be written R^2 . (R^2 is read "R squared.")

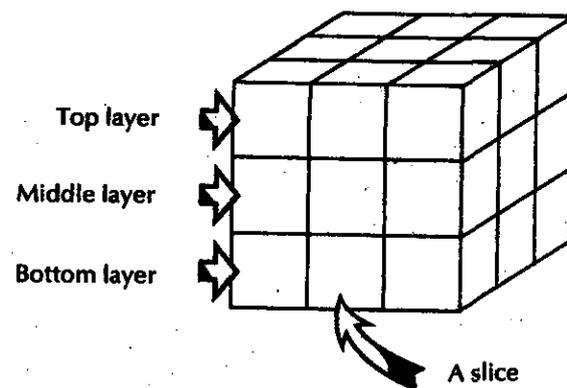


Fig. 1.4. The layers

The following five chapters give a solution to the cube by doing one layer at a time. The layers are shown in Fig. 1.4. The top layer is the same as the upper face. The middle layer is an example of a slice. A slice consists of those cubies which lie between opposite faces.

Rubik's Cube may be dismantled by rotating one face 45 degrees and prying one of the edge cubies out as in Fig. 1.5. You can then remove the other cubies easily to reveal the interior mechanism. Exactly how the

cube works will then be clear. When re-assembling the cube, be sure to put it in its original position. Otherwise, as explained in Chapter 10, the cube may be impossible to solve.

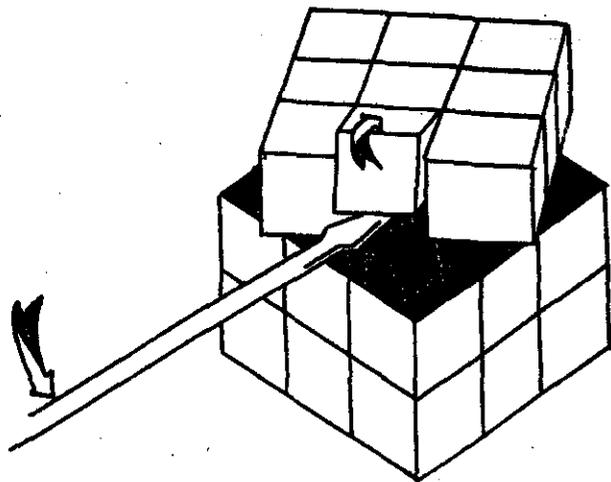


Fig. 1.5. Disassembling the cube

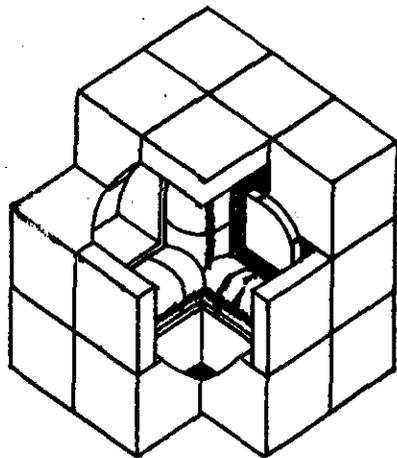


Fig. 1.6. Interior of cube

Chapter 2 Conjugates and Commutators

The one-minute solution to Rubik's Cube begins in this chapter and is completed in Chapter 6. The solution proceeds by first solving the top face, then the middle layer, then the bottom corners, and finally the remaining edges. At each stage you will use processes which move certain cubies around while only temporarily disrupting the work you have already done. Understanding these processes is the key to solving the cube.

A process is simply a sequence of face twists. For example, a particularly nice process is given in Fig. 2.1. It interchanges a pair of edges on the front face, and also interchanges a pair of edges on the right face, while leaving the rest of the cube unchanged.

There are several ways to generate new processes from old ones. The simplest way is just to turn the entire cube before performing a process. For example, the process shown in Fig. 2.2. could be obtained from Fig 2.1 by turning the entire cube.

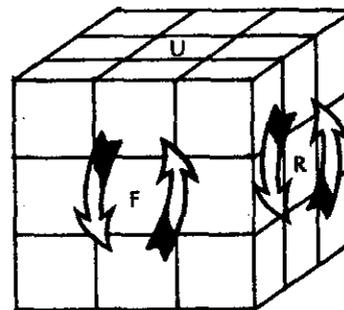


Fig. 2.1. $F^2R^2F^2R^2F^2R^2$

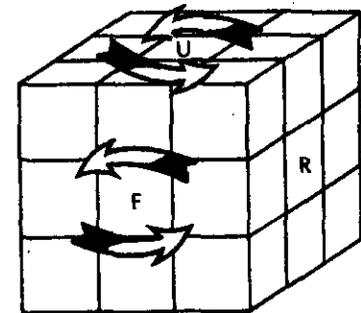


Fig. 2.2. $F^2U^2F^2U^2F^2U^2$

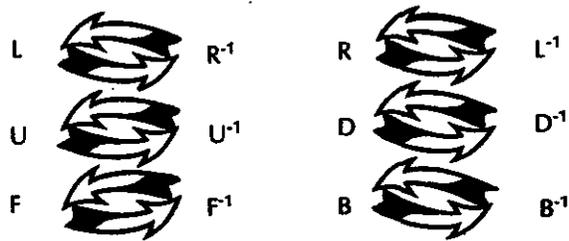


Fig. 2.3. The mirror image moves

A more interesting new process is the inverse process. This is obtained by performing all of the operations backwards and in reverse order. Any process followed by its inverse process will return the cube to its original position. For example, the inverse to $RUF^1U^1R^1D^1R^1DR$ is $R^1D^1RDRUFU^1R^1$. The process in Fig. 2.1 is its own inverse.

Another useful process is the "mirror image." This is obtained by proceeding as if you were imitating someone else working on the cube while observing him in a mirror. The operations are changed as shown in Fig. 2.3. For instance, the mirror image of $RUF^1U^1R^1D^1R^1DR$ is $L^1U^1FULDL^1L^1$.

A more subtle method is to use the conjugation principle. This says that you get a variant ("conjugate") of a process by applying a sequence of moves, then the process, and then the inverse to the first sequence of moves. The conjugate is essentially the same process as the original process, but moves different cubies.

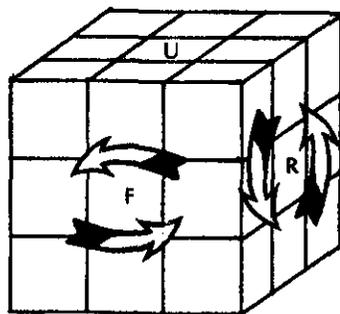


Fig. 2.4. $F^2R^2F^2R^2F^2R^2F^{-1}$

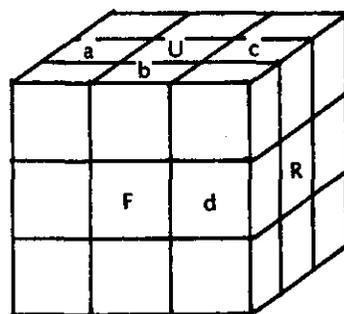


Fig. 2.5

To see how useful conjugation is, suppose you are looking for a process which will interchange two pairs of cubies as shown in Fig. 2.4. This figure is very similar to Fig. 2.1. In fact, if we rotate the front face of Fig. 2.4 by 90 degrees, it will be exactly the same as Fig. 2.1. Then an application of the Fig. 2.1 process will interchange the desired edges. The front face can then be rotated back into position. We would refer to this as the conjugate of the Fig. 2.1 process by F .

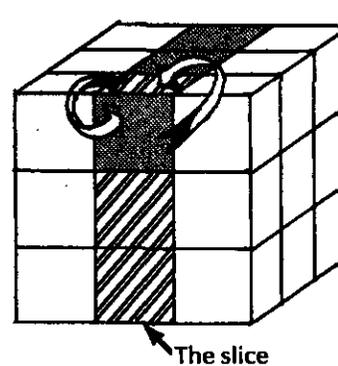


Fig. 2.6. $U^1FR^1UF^{-1}$

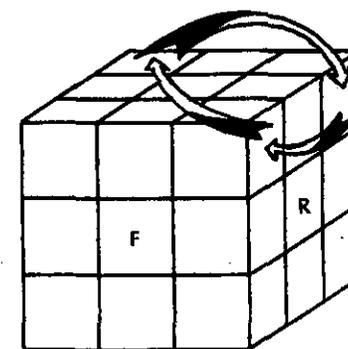


Fig. 2.7. $U^1L^1URU^{-1}LU^{-1}R^{-1}$

For another example of conjugation, see Fig. 2.5. Chapter 6 gives processes which permute (that is, rearrange) a , b , and c in Fig. 2.5 without changing the other cubies. R followed by this process followed by R^{-1} permutes a , b , and d without changing any other cubies.

The conjugation principle is very powerful. Once you learn a process which permutes three edges without changing other cubies, then any three edges may be permuted by an appropriate conjugate. This will be illustrated in detail in Chapter 6.

Another useful method is the commutator principle. The commutator of two processes is obtained by doing the two processes, and then doing the inverses to the two processes. For example, the commutator of L and D is LDL^1D^{-1} .

For the typical application of the commutator principle, the objective is to do some operation on one portion of the cube while doing the inverse operation to another portion of the cube. To accomplish this, you first find a process which does the operation while possibly disrupting part of the rest of the cube. The effect on some cubies can be saved by moving

them over to the unchanged portion of the cube and then performing the inverse process. Finally, move what you have saved back to where it belongs.

As an example of the commutator principle, consider the process $U^1FR^1UF^1$, which flips one edge on the slice between the right and left faces, as indicated in Fig. 2.6. The rest of the slice is unchanged, although the right and left faces become disrupted. Now rotate the slice, apply the inverse process (FU^1RF^1U), and rotate the slice back. The result is that two edges on the slice will be flipped without changing the rest of the cube.

To take another example, the conjugate U^1L^1U moves the far upper right corner off the right face without disturbing the rest of the right face. R followed by the inverse conjugate U^1LU puts it back in a different spot. Thus the commutator $U^1L^1URU^1LU^1R^1$ permutes three upper corners while leaving the rest of the cube unchanged, as shown in Fig. 2.7. This process is called a "corner 3-cycle."

Chapter 3 The Top Face

To solve the cube in stages, we start by solving the top face. Chapter 4 explains how to solve the middle layer and the solution is completed in chapters 5 and 6.

To solve the top face, we not only want the top face to be one solid color, but we also want the colors on the perimeter of the top face to match the colors of the centers of the sides, as in Fig. 3.1. (In this figure and in the figures to follow, a blank sticker means that its color is irrelevant for present considerations.)

First solve the top edges. This can be done quite quickly and easily, as

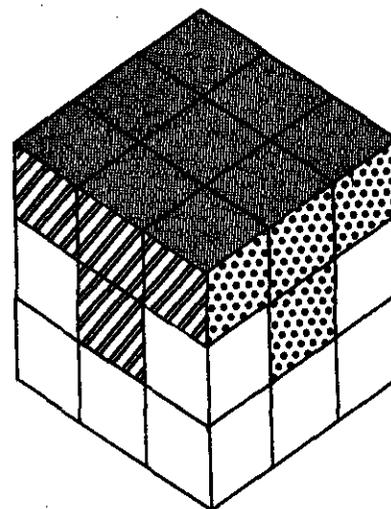


Fig. 3.1. The top layer is solved

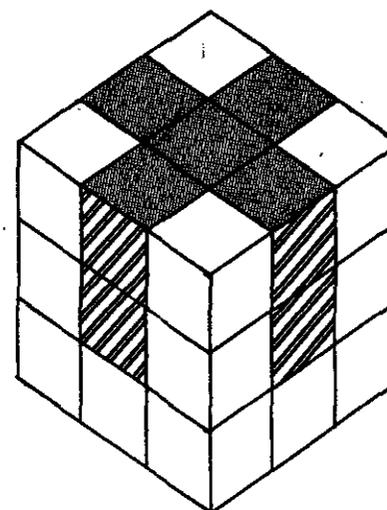


Fig. 3.2. Top edges have been solved

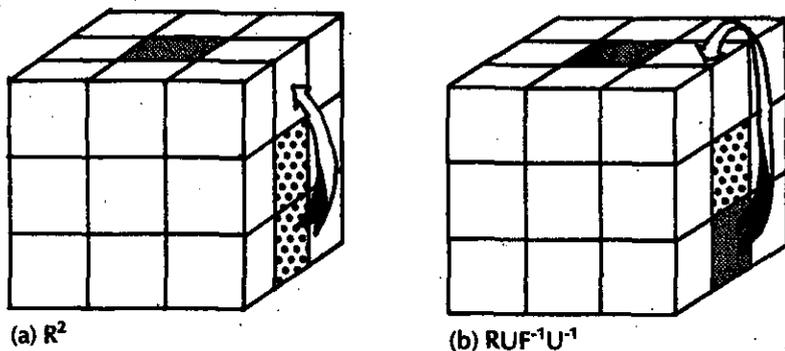


Fig. 3.3. Putting a top edge in place

I will explain in a moment. When the top edges are completed, the cube will appear as in Fig. 3.2.

To solve an edge, locate the edge as well as the place where it belongs. (Recall that there is only one correct position for each of the cubies.) If the edge is on the bottom layer, turn the bottom layer until the edge is directly below the spot where it belongs and hold the cube as in Fig. 3.3 (a) or (b). For case (a), R^2 will put the edge in place. For case (b), apply $RUF^{-1}U^{-1}$.

If the edge is on the middle layer, hold the cube so that the sticker which matches the color of the top center is in front, as in Fig. 3.4 (a) or (b). For case (a), rotate the top face until the spot where the edge belongs is on the right. Then apply R and rotate the top face back. Case (b) is the mirror image, so apply the mirror image process. (That is, rotate the top

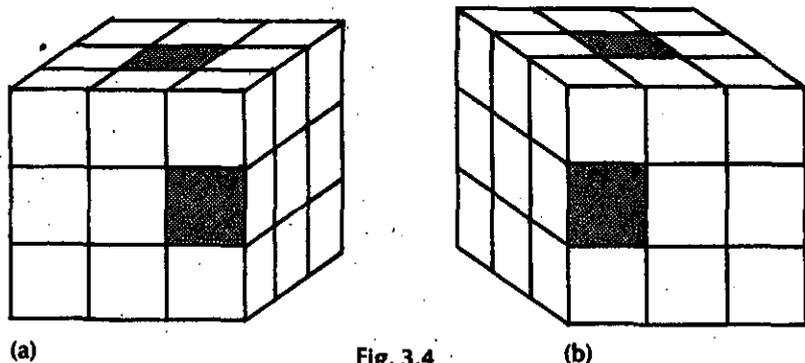


Fig. 3.4

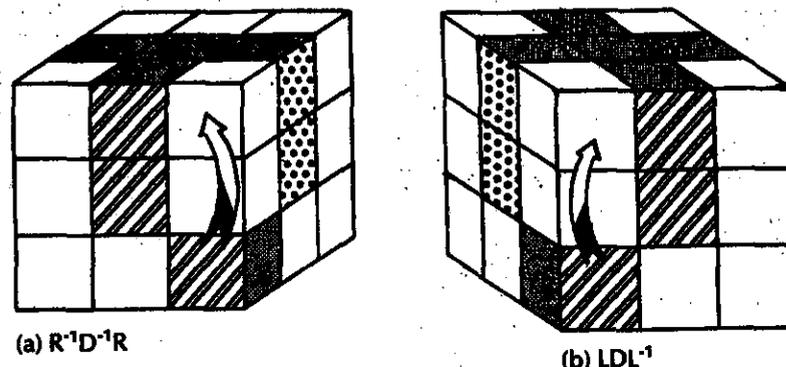


Fig. 3.5. Solving the top corners

face until the spot where the edge belongs is on the left. Then apply L^{-1} and rotate the top face back.)

The remaining possibility is that the edge is on the top face, but in the wrong location or with the wrong orientation. In this situation, simply turn one of the side faces (R , L , F , or B) to move it off the top face and then solve the edge by the methods above.

Now suppose that all four top edges are in place and in the correct orientation. To put a top corner in place, first locate the cubie and the location where it belongs. The easiest case occurs when the cubie is on the bottom layer and its bottom sticker is not the one that is supposed to agree with the top color. If this happens, turn the bottom face until one of the stickers on the desired corner matches that of an adjoining center, as in Fig. 3.5 (a) or (b). The corner in Fig. 3.5(a) may be solved by $R^{-1}D^{-1}R$. (This is another example of a conjugate. You could also use the com-

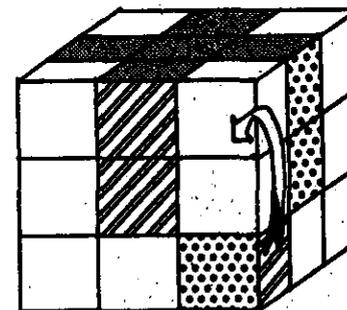


Fig. 3.6. $R^{-1}DFD^2F^{-1}$

mutator $DFD^{-1}F^{-1}$ to obtain the same result.) Fig. 3.5(b) is the mirror image and may be handled by LDL^{-1} .

A more difficult case occurs when the wanted corner is on the bottom layer with the top color down. In this situation, we again rotate the down face until the wanted corner is directly below the spot where it belongs, as in Fig. 3.6. This time $R^{-1}DRFD^2F^{-1}$ will solve it.

If the wanted corner is on the top face, hold the cube so that the corner is on the near upper right. $R^{-1}D^{-1}R$ will put it on the bottom layer. You can now proceed as before to solve it. After all four corners are solved, the cube will look like the cube in Fig. 3.1.

Chapter 4 The Middle Layer

When you have solved the top face, the cube should look like Fig. 3.1 and you are ready to solve the middle layer. In order to solve the middle layer quickly, it is necessary to disrupt one of the top corners. You can restore it after solving the middle layer edges.

To solve the middle layer edges, turn the cube over so that the solved face is now down. Suppose that one of the edges that belongs on the middle layer is on top. If this is the case, rotate the layers until the cube is in either position (a) or (b) of Fig. 4.1. The edge may now be put in place by the conjugate $RU^{-1}R^{-1}$ in case (a), and by $L^{-1}UL$ in case (b).

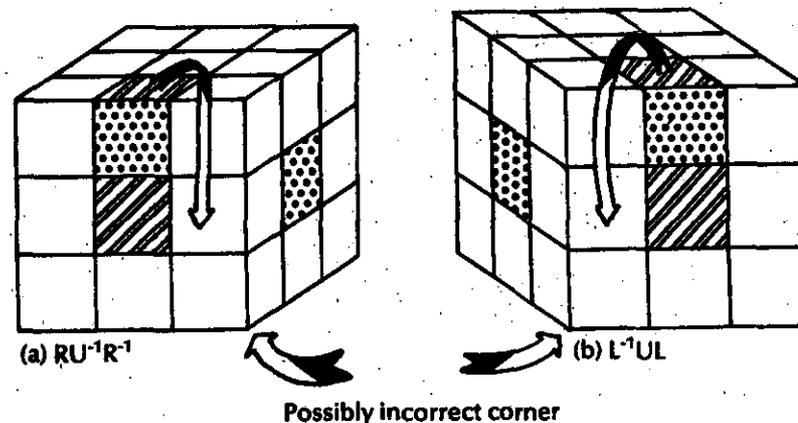


Fig. 4.1. Solving middle layer if bottom is solved except for one corner

This set-up can be easily remembered by following these rules:

1. Turn the mostly solved face down.
2. Turn the possibly incorrect corner closest to you.
3. Turn the middle layer so that the location to be corrected is nearest to you.

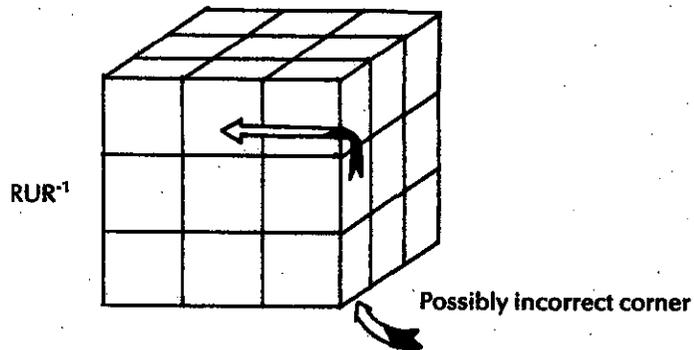


Fig. 4.2. Moving a middle layer edge to the top

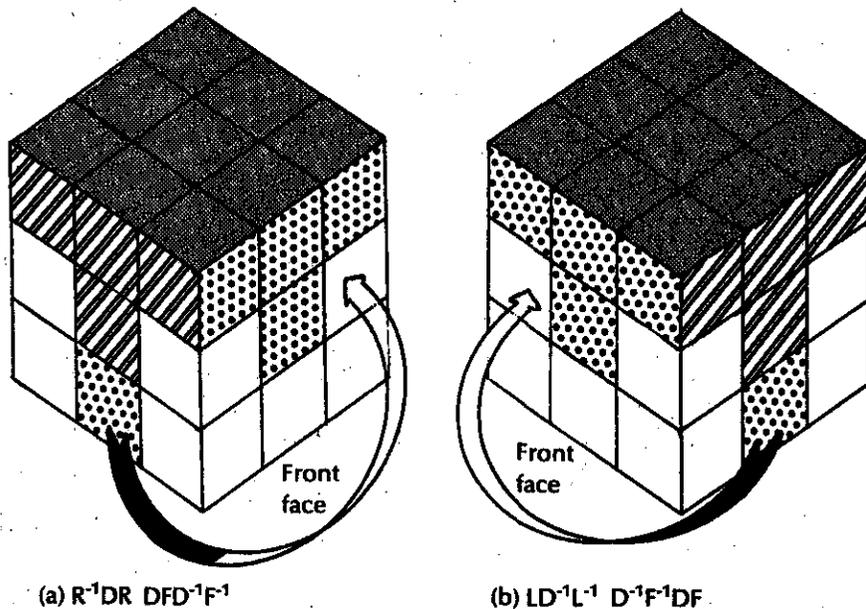


Fig. 4.3. Solving the middle layer

4. Turn the top until the desired edge is in the one near position where the adjacent colors do not match up.

If the desired edge is already on the middle layer but in the wrong place or incorrectly oriented, then the above procedure may be used to move the edge to the top face. That is, turn the middle layer until the desired edge is over the possibly incorrect corner and apply RUR^{-1} as in Fig. 4.2. Use the procedure again to solve the edge.

When three of the four middle layer edges are solved, turn the cube over again so that the mostly completed face is on top. Unless you are uncommonly lucky, one of the top corners will be incorrect. Rotate the layers until the incorrect top corner is above the incorrect middle layer edge. Now use the methods given in chapter 3 to restore the final top corner.

At this point the top two layers should be completely solved except for one middle layer edge. In the interests of speed, it is most efficient to forget about that last edge and proceed to Chapter 5 to solve the bottom corners. If you wish to complete the top two layers, however, you could use the processes in Fig. 4.3. The idea here is to move a top corner down to the bottom layer, and to restore it in a slightly different way. This moves an edge from the bottom layer to the middle layer.

Chapter 5 Bottom Corners

If you have successfully followed the instructions in the preceding chapters, you now have a cube in which the top two layers are completely solved except for possibly one edge on the middle layer. In this chapter, I will explain how to solve the bottom corners. The strategy is to first put them in place while ignoring orientations, and then to orient the corners.

You can view the bottom better if you turn the cube over again so that the scrambled face is now on top. It is always possible to twist the top face so that at least two corners are in place although possibly not oriented. If these two are on the left side, then the other two may be switched by $L^{-1}URU^{-1}LUR^{-1}U^2$, which is essentially the same as the corner 3-cycle in Fig. 2.7.

If two diagonally opposite corners need to be switched, then apply

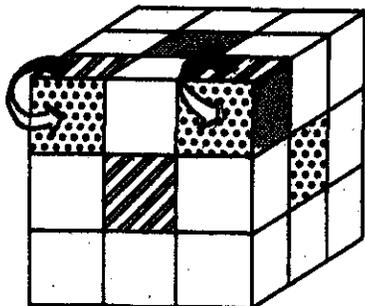
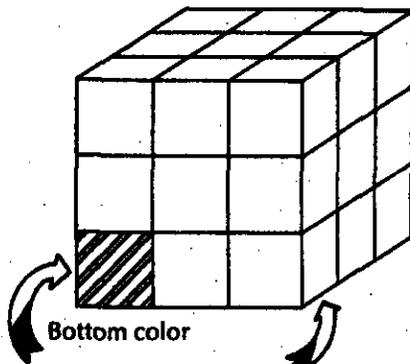
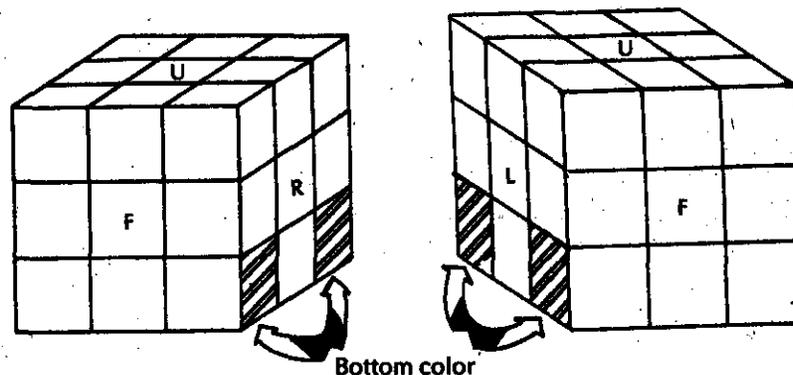


Fig. 5.1. $LDL^{-1}D^{-1}LDL^{-1}$
 $U LD^{-1}L^{-1}DLD^{-1}L^{-1} U^{-1}$



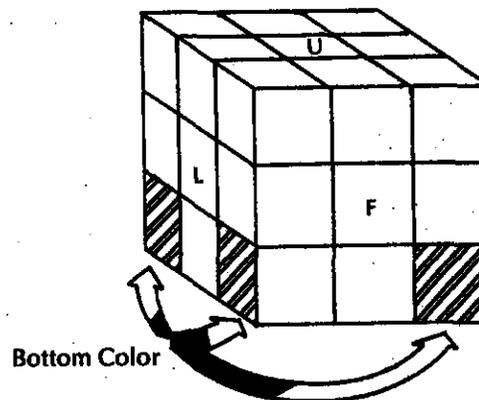
Correctly oriented corner
Fig. 5.2. $LDL^{-1}DLD^2L^{-1}$



Bottom color
Fig. 5.3. $LDL^{-1}DLD^{-1}L^{-1}DLD^2L$

$BLUL^{-1}U^{-1}B^{-1}$. It will then be possible to twist the top so that all of the top corners will be in place, although perhaps not oriented.

Now that the corners are in position, you can use the commutator principle to orient them. Suppose, for example, that two corners need to be oriented as in Fig. 5.1. $LDL^{-1}D^{-1}LDL^{-1}$ will orient the front left upper corner at the expense of disrupting the second and third layers. But if you turn the top and apply the inverse process $LD^{-1}L^{-1}DLD^{-1}L^{-1}$, then the second and third layers will be restored. Turning the top face will put the top corners back in place. The net result is that two of the top corners



Bottom Color
Fig. 5.4. $R^{-1}D^2R^2DR^2DR^2D^2R^{-1}$

have been oriented (by rotations of 120 degrees in opposite directions) and the rest of the cube is left unchanged.

If only two corners need to be oriented, this procedure is the fastest way. If three or four corners need orienting, this method could be used repeatedly but there are faster ways. To use these faster ways, turn the cube over again so that it is the *top* two layers which have been solved (except for possibly one middle layer edge). To orient three corners, hold the cube as in Fig. 5.1 or its mirror image. The situation in Fig. 5.2 is solved by $LDL^{-1}DLD^2L^{-1}$ and the mirror image is solved by $R^{-1}D^{-1}RD^{-1}R^{-1}D^2R$.

If all four bottom corners are incorrectly oriented, then hold the cube so that all four corner stickers which have the bottom color are on the right and left sides, as in Fig. 5.3. This can be solved by $LDL^{-1}DLD^{-1}L^{-1}DLD^2L^{-1}$.

In the remaining case it is possible to hold the cube so that two of the corner stickers having the bottom color are on the left, as in Fig. 5.4. Then $R^{-1}D^2R^2DR^2D^2R^{-1}$ will orient the corners.

Chapter 6 Edges

There are a number of processes which permute three edges without altering the rest of the cubies. These are called "edge 3-cycles." I will give several examples in this chapter, and then explain how these edge 3-cycles may be used to complete the solution to the cube.

The simplest and easiest edge 3-cycle to understand is the commutator of a 180-degree face twist with a slice twist. More specifically, the process $R^2UD^{-1}F^2U^{-1}D$ permutes three middle layer edges as shown in Fig. 6.1.

As explained in Chapter 1, the conjugation principle may be used to obtain other edge 3-cycles. For example, $F^2ULR^{-1}F^2L^{-1}RUF^2$ permutes three edges on the top face as shown in Fig. 6.2 (a). The mirror image is shown in Fig. 6.2 (b). These two 3-cycles do not flip any edges. That is, if the top face is one solid color it will remain that way after these 3-cycles.

If you want a top edge 3-cycle which flips edges, you can use the process shown in Fig. 6.3. This process is more easily understood if we turn

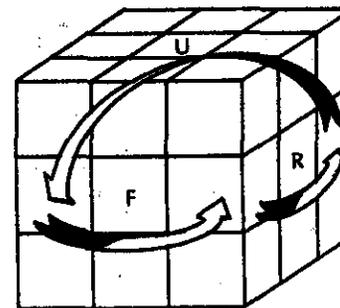
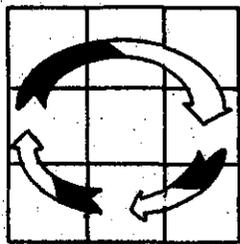


Fig. 6.1. $R^2UD^{-1}F^2U^{-1}D$

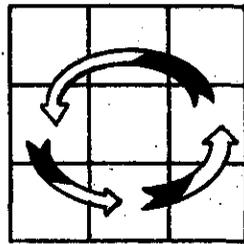
the cube over. A middle slice twist commutator, namely $RL^{-1}FR^{-1}LD^{-1}$, moves the front upper edge to the front down edge. The inverse to the mirror image puts it back in place. The result is the 3-cycle shown in Fig. 6.3 (a). This process could also be viewed as a conjugate to the process shown in Fig. 6.1.

A very fast edge 3-cycle is $LDLDD^{-1}L^{-1}D^{-1}L^{-1}D^{-1}$. It is shown in Fig. 6.4 (a), along with its mirror image, (b). The idea here is to move the bottom corners to the left side, twist the left side, and put the corners back on the bottom. Alternatively, you can think of moving the upper left cubies off of the left face.

A useful conjugate to process 6.4 (a) is given in Fig. 6.5 (a), along with its mirror image, (b).

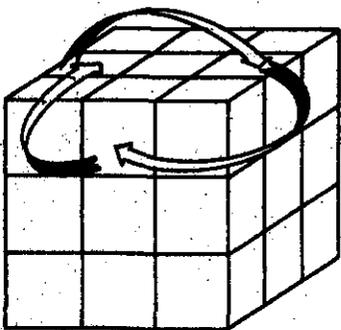


(a) $F^2ULR^{-1}F^2L^{-1}RUF^2$

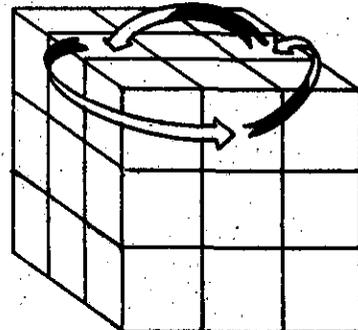


(b) $F^2U^{-1}R^{-1}L^{-1}F^2RL^{-1}U^{-1}F^2$

Fig. 6.2. Top face is shown

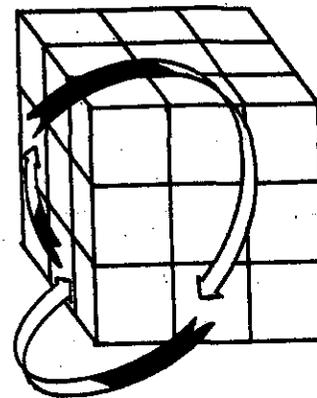


(a) $RL^{-1}BR^{-1}LU^2RL^{-1}BR^{-1}L$

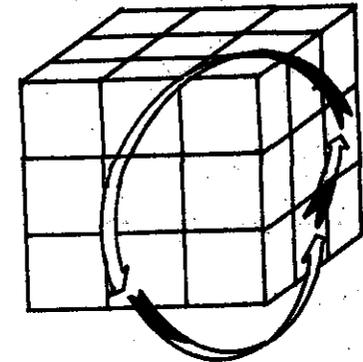


(b) $L^{-1}RB^{-1}LR^{-1}U^2L^{-1}RB^{-1}LR^{-1}$

Fig. 6.3

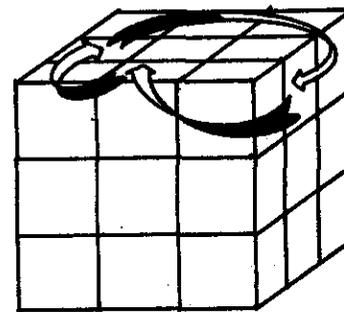


(a) $LDLDD^{-1}L^{-1}D^{-1}L^{-1}D^{-1}$

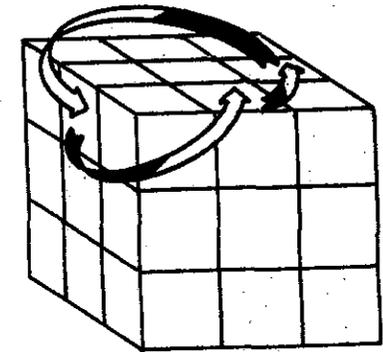


(b) $R^{-1}D^{-1}R^{-1}D^{-1}R^{-1}DRDRD$

Fig. 6.4

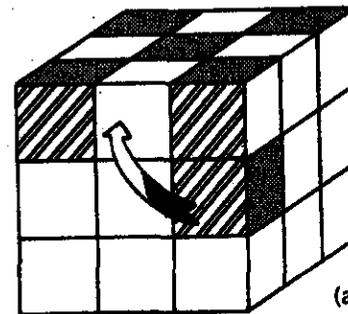


(a) $R^{-1}UFUFUF^{-1}U^{-1}F^{-1}U^{-1}F^{-1}R$

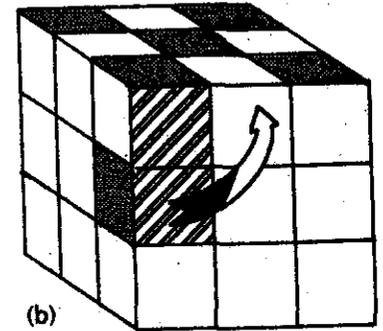


(b) $L^{-1}U^{-1}F^{-1}U^{-1}F^{-1}U^{-1}FUFUF L^{-1}$

Fig. 6.5. A conjugate to process of Fig. 6.4

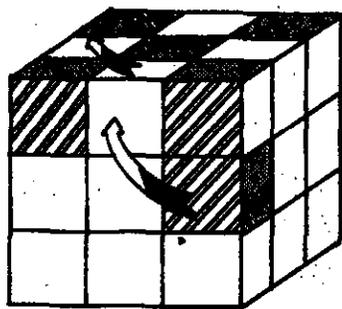


(a)

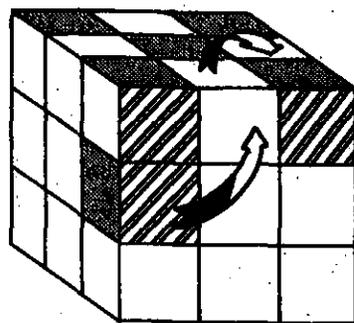


(b)

Fig. 6.6



(a) $UFUFU^1U^1F^1U^1F^1$

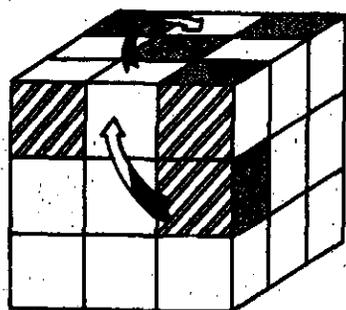


(b) $U^1F^1U^1F^1U^1FUFUF$

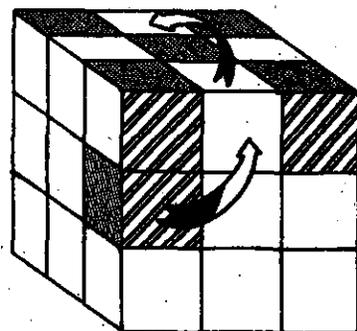
Fig. 6.7

With the techniques described in Chapter 3, you can solve one face; with those of Chapter 4, three of the middle layer edges; and Chapter 5 explained how to finish solving the corners. If you followed all of these steps, you should now have a cube which is solved except for—at most—five edges. You can now finish the cube by repeatedly applying the conjugation principle to the 3-cycles just described.

To illustrate this, suppose that you have five wrong edges: four on top and one on the middle layer. It will usually be the case that the wrong middle layer edge belongs on top. To correct this, rotate the top face until a top corner matches the color of a sticker on the wrong middle layer edge as in Fig. 6.6. (a) or (b). Now look to see where the near upper edge belongs. It may be put there, although with possibly the wrong orienta-



(a) $R^1F^1BLFB^1U^2F^1BLFB^1R^1$



(b) $L^1FB^1R^1F^1BU^2FB^1R^1F^1B^1L$

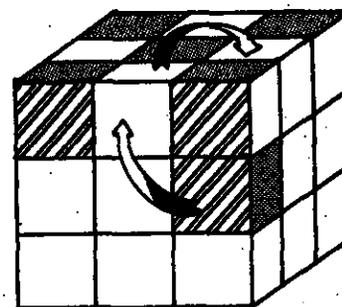
Fig. 6.8

tion, by a process from Fig. 6.7, 6.8, or 6.9. These are conjugates of processes shown in Figs. 6.4, 6.3, and 6.4, respectively.

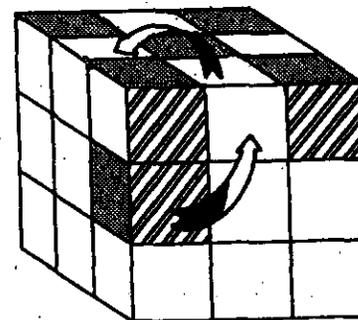
It sometimes happens that an edge is in place but flipped. You can use the commutator principle to flip two edges, as shown in Fig. 2.6, or as follows: The process $FUD^1L^2U^2D^2R$ flips the near edge on the top face without changing anything else on the top face. You can now turn the top face and apply the inverse process. The result will flip two edges on the top face without altering the rest of the cube as in Fig. 6.10.

The conjugation principle can now be used to flip any two edges. Simply rotate a face to put the two edges on the same slice or face and apply one of the above methods.

Therefore, the edge 3-cycles given in this chapter may be used to put the edges in the correct locations, although with possibly the wrong orientations. You can then flip the incorrectly oriented edges, two at a time. Your cube will then be solved.



(a) $U^1FUFUFU^1F^1U^1F^1U^1U^1$



(b) $U^1F^1U^1F^1U^1F^1UFUFU^1U^1$

Fig. 6.9

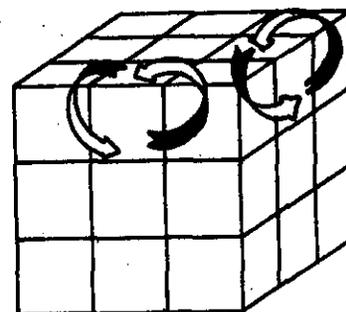


Fig. 6.10. $FUD^1L^2U^2D^2R$
 $R^1D^2U^2L^2DU^1F^1U^1$

Chapter 7 Picking Up Speed

It takes a lot of practice to get your time under one minute, but—in addition to practice—the following tips will increase your speed.

The procedures in this book are designed with speed in mind, so virtually all of them involve only two or three faces. For a process involving only two faces, I have found it best to use one hand for each face. This is particularly true for the frequently used commutator $LDL^{-1}D^{-1}$ where I use my left hand to turn L and L^{-1} and my right hand for D and D^{-1} . With practice, you can do these at a rate of three turns per second.

Most people have a favorite color which they like to solve first. Some care should be taken here; both it and its opposite color should be among the more visible colors. It helps to memorize the relative position of the side colors. If this is too difficult, just memorize which colors are opposite. This will enable you to solve the top face without paying too much attention to the other centers. In fact, since you are essentially doing each

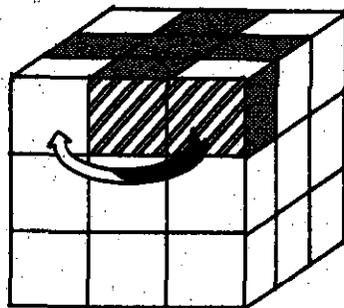
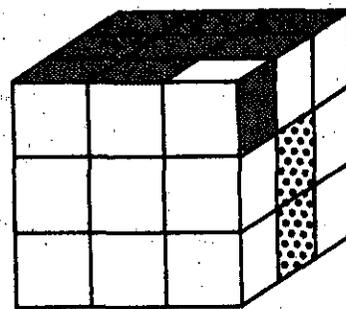
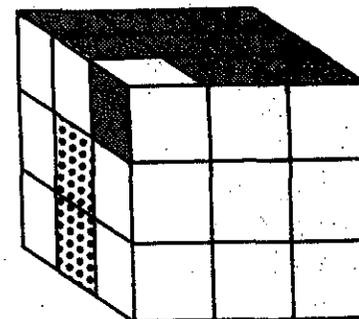


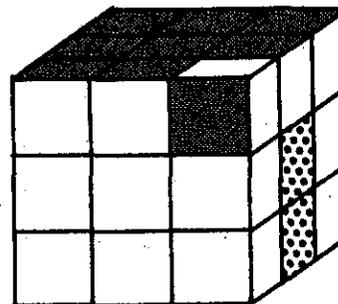
Fig. 7.1. $LR^{-1}D^{-1}RL^{-1}$



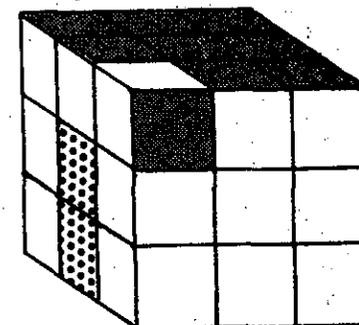
(a) $R^{-1}D^{-1}RDR^{-1}D^{-1}RD$



(b) $LDL^{-1}D^{-1}LDL^{-1}$



(c) $R^{-1}DRD^{-1}R^{-1}DR$



(d) $LD^{-1}L^{-1}DLD^{-1}L^{-1}$

Fig. 7.2

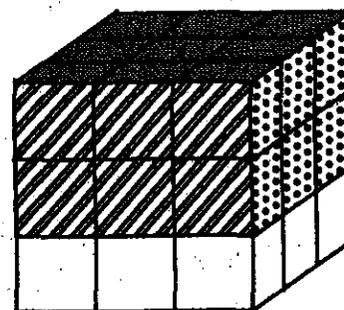
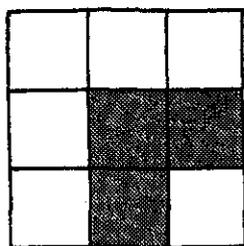
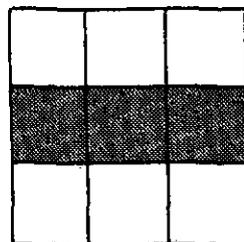


Fig. 7.3. Top two layers completed



(a) $BULU^{-1}L^{-1}B^{-1}$



(b) $BLUL^{-1}U^{-1}B^{-1}$

Fig. 7.4. Top face shown only

layer separately, you need not bother to align the colors of the layers until you have solved each layer.

Whenever you have a choice of cubies to put in place, you should always choose whatever is easiest. For example, the corners to do first are the ones on the bottom with its top sticker on the side; this is the case illustrated in Fig. 3.3. Another easily recognized and quickly solved situation is shown in Fig. 7.1.

In general, flipping edges is a very slow process and should be avoided if your goal is speed. Suppose that you have completed all of the top two layers except for a top corner and the edge beneath it. When the last top corner is put in place, care should be taken that the last middle layer edge is not left flipped. If the process given in Fig. 3.5 does that, then you could use $DFD^{-1}F^{-1}$ for Fig. 3.5(a) and $D^{-1}F^{-1}DF$ for Fig. 3.5(b).

The process given in Fig. 3.5 flips the middle layer edge. If this is not desired, another process such as $R^{-1}DRD^2R^{-1}D^{-1}R$ can be used.

If the last top corner is in place but incorrectly oriented, and the last

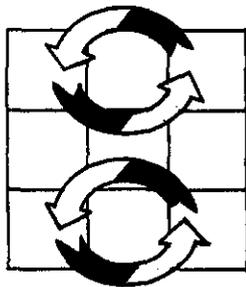


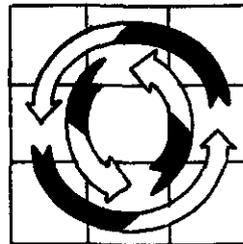
Fig. 7.5 $LDR^{-1}D^{-1}L^{-1}R U^2 R^{-1}LDRD^{-1}L^{-1} U^2$

middle layer edge is on the bottom, then it is possible to complete the top two layers simultaneously. Rotate the layers until the incorrectly oriented top corner is over the incorrect cubie on the middle layer, and the edge which belongs on the middle layer matches the color of a middle layer center. Now hold the cube as in Fig. 7.2 and apply the process given there. (In Fig. 7.2, (b) is the mirror image of (a), (c) is the inverse of (a), and (d) is the mirror image of (c).) This process will complete the top two layers, as shown in Fig. 7.3.

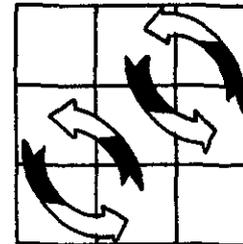
If you are able to complete the top two layers as in Fig. 7.3, turn the cube over and look at what is now the top. Exactly zero, two, or four edges will show the same color as the center. The usual case is two, which may be increased to four by the processes indicated in Fig. 7.4. If you now have four top edges agreeing with the center, then solve the corners.

All but one of the processes given in Chapter 5 preserve the edge orientations. The exceptional case occurs when two diagonally opposite corners need to be interchanged. To interchange two diagonally opposite corners, first switch two pairs of corners as in Fig. 7.5. It will then be possible to put all of the top corners in place (although possibly not oriented) by turning the top face. (The process given in Fig. 7.5 is an application of the commutator principle, since $LDR^{-1}D^{-1}L^{-1}R$ switches the front two corners while leaving the rest of the top face unchanged. Alternatively, you could turn the cube over and apply $F LDL^{-1}D^{-1}LDL^{-1}D^{-1}LDL^{-1}D^{-1}F^{-1}$.)

After orienting the edges and solving the corners, the cube can usually be solved by the process given in Fig. 6.2. The remaining cases are shown in Fig. 7.6. These can be treated by two such 3-cycles, but they may be handled more quickly by the conjugates of processes 2.4 and 8.3 given in Fig. 7.6. $RLU^2L^{-1}R^{-1}F^{-1}B^{-1}U^2BF$ may also be used for Fig. 7.6(a).



(a) $LR^{-1}UF^2U^2F^2U^2FU RL^{-1}$



(b) $R^2L^2U F^2B^2L^2F^2B^2R^2 U^{-1}L^2R^2$

Fig. 7.6

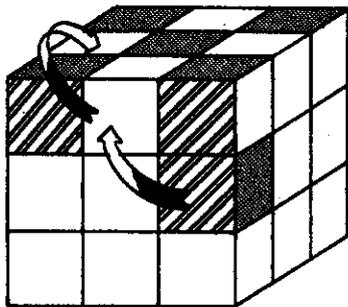
Suppose that you solved the bottom corners before finishing the middle layer. You will then be in a situation where there are at most five wrong edges. Under favorable circumstances, a 3-cycle will solve two edges and another 3-cycle will finish the cube. In any case, it never takes more than three 3-cycles to solve the five edges. Flipping edges can be avoided.

To understand how to use a 3-cycle to solve two edges at once, consider Fig. 6.7(a). The process given there puts the wrong middle layer edge correctly in place at the upper front, but the edge cubie at the upper front might be put in place with the wrong orientation. If this happens, you could use another 3-cycle instead. An appropriate conjugate of the processes given in Figs. 6.2, 6.3, or 6.4 does the job. Three such conjugates are given in Fig. 7.7. You will easily find conjugates to handle the other possibilities.

When applying a particularly complicated use of the conjugation principle, you may get stuck trying to "unconjugate." Of course, this is simply the inverse of what you did earlier—if only you could remember exactly what you *did* earlier. A useful device is to remember just the last twist. Then just concentrate on restoring a face that had been solved before conjugating.

With practice, all of the processes in this book will become automatic. Your hands will be able to perform them without your thinking about them. When this happens, try to look ahead to see what the next process will be. For instance, while putting a middle layer edge in place, look for the other middle layer edges. While correcting the corners, see what needs to be done to the edges. This will only save a few seconds, but if you expect to solve the cube in under one minute, those seconds count.

Fig. 7.8. $LU^2F^3RL^{-1}U^2R^{-1}LF^{-1}U^2L^{-1}$,
or $R^2L^{-1}BR^{-1}LU^2RL^{-1}BR^2L$,
or $L^2F^{-1}LFLFLF^{-1}L^{-1}F^{-1}L$



Chapter 8 Patterns

Besides restoring the cube to its original position, you can create patterns on its surfaces. You can discover many pleasing, even pretty, patterns for yourself by experimenting with a solved cube. Those I consider the best are given in this chapter. You will find your own favorites.

Not all patterns you might think possible are possible. (This is explained in Chapter 10.) But if you know how to solve the cube, theoretically, you should be able to construct directly any possible pattern without first actually solving the cube. In practice, however, it is usually much easier to solve the cube before proceeding to do patterns.

One way to find patterns is to dismantle the cube and reassemble it in pattern form. This, I think most of us would agree, is cheating. Another big drawback to this method is that you may get an impossible pattern from it. As a test, try to restore the cube to its original position. You will be able to restore it if and only if the pattern is possible. (Another method

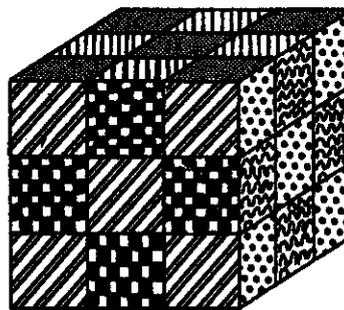


Fig. 8.1. $R^2L^2U^2D^2F^2B^2$

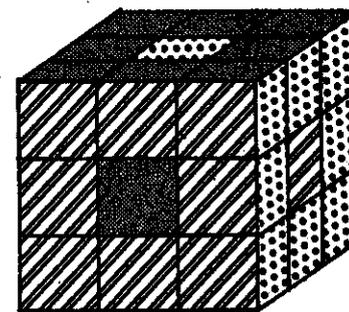


Fig. 8.2. $RL^{-1}FB^{-1}UD^{-1}RL^{-1}$

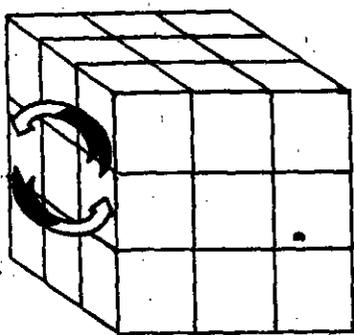


Fig. 8.3. $U^2D^2R^2U^2D^2L^2$

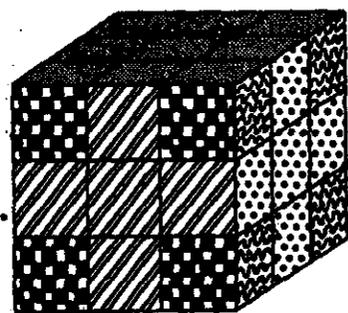
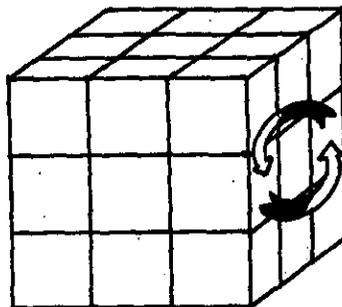


Fig. 8.4. $R^2L^2F^2B^2U^2F^2B^2R^2L^2D^2$

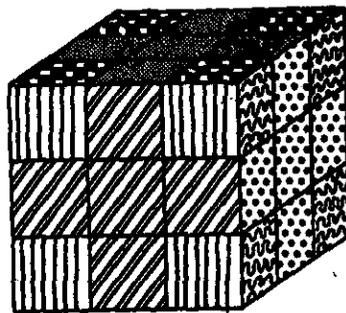


Fig. 8.5. $L R^2L^2F^2B^2U^2F^2B^2R^2L^2D^2 L^{-1}$

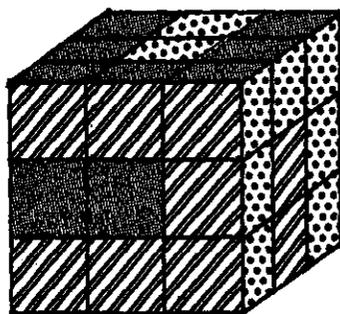


Fig. 8.6. Apply $F^{-1}B^{-1}R^{-1}D^{-1}R^{-1}D^{-1}R^{-1}DRDRD BF$ to the pattern shown in Fig. 8.2.

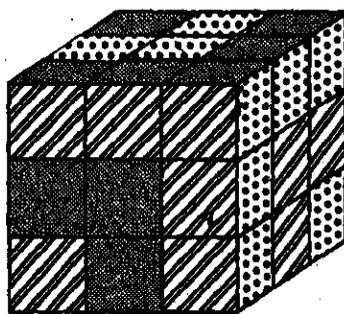


Fig. 8.7. Apply $LR DFD F D F^{-1}D^{-1}F^{-1}D^{-1}F^{-1}R^{-1}L^{-1}$ to the pattern shown in Fig. 8.6.

for determining which patterns are possible will be given in Chapter 10.)

Many nice patterns may be found by starting with a solved cube and making only 180 degree turns or slice twists. For example, turn each of the three slices (in any order) 180 degrees and you will obtain an *X* on each face as in Fig. 8.1. Another example is the commutator of two 90 degree slice twists; this has an *O* on each face and is shown in Fig. 8.2.

The process shown in Fig. 8.3 is very useful. A slight variation of it yields a + on four sides, as in Fig. 8.4. The conjugate by *L* gives Fig. 8.5, which has a + on each of the six faces.

Depending on how you hold the cube, the process given in Fig. 8.5 may be viewed as a process that moves the corners while preserving the edges and centers, or as a process that moves the edges and centers while preserving the corners.

Fig. 8.6 has a *U* on each face and may be obtained from Fig. 8.2 by the conjugate of the process of Fig. 6.4 (b) by $F^{-1}B^{-1}$. You can then obtain a worm-like pattern by applying the edge 3-cycle given in Fig. 8.7. A similar pattern is given in Fig. 8.8.

A more difficult pattern is the "cube within a cube" shown in Fig. 8.9. The other three faces look similar to the three shown in Fig. 8.9. To obtain this pattern, start with the pattern shown in Fig. 8.2. The front upper right and back lower left corners may be re-oriented by a conjugate of a commutator. That is, we know from Chapter 5 how to orient two corners if they are both on top, so put the back down left corner on top. A conjugate of the process given in Fig. 5.1 which does this is given in Fig. 8.10. To finish the pattern, you need two edge 3-cycles, one of which is shown in Fig. 8.11. As usual, you can get this as a conjugate of

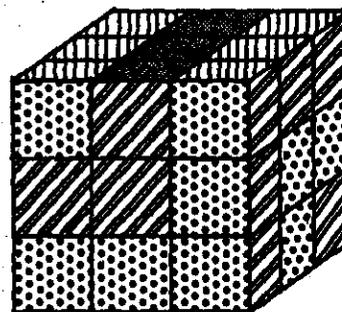


Fig. 8.8. $DR^2L^2D^{-1}R^2U^2D^2L^2U^2D^2$, then $F^{-1}RU^{-1}FR^{-1}U^2D^2LB^{-1}UL^{-1}B D^2U^2$

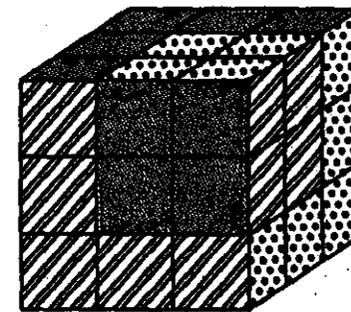


Fig. 8.9

one of the basic 3-cycles given in Chapter 6. A mirror image process will take care of the rest of the cube.

There are several patterns such as Fig. 8.12 which have stripes on the faces, but the best is shown in Fig. 8.13. Each face has three stripes. Three of the faces have three of the colors in each possible order and the other three faces have the remaining three colors in each possible order.

To get the pattern in Fig. 8.13, do the commutator $LDL^{-1}D^{-1}$ three times to interchange two pairs of corners. Interchange the remaining two pairs of corners with another triple commutator. Now four of the edges will appear flipped relative to the adjacent corners. Flip these edges. Stripes can now be obtained by a simple conjugate of the process given in Fig. 8.3. The cube may be restored by repeating each of these steps in any order.

The procedures for constructing patterns in this chapter were chosen because they are easy to understand and because they illustrate general techniques which may be used to construct other patterns. However,

Fig. 8.10. Result of applying $L^2 LDL^{-1}D^{-1}LDL^{-1}U LD^{-1}L^{-1}DLD^{-1}L^{-1}U^{-1}L^2$ to cube shown in Fig. 8.2.

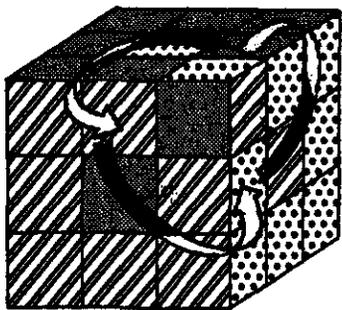
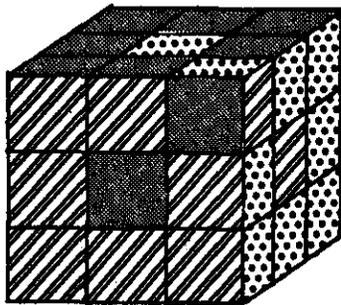


Fig. 8.11. $UR RL^{-1}BR^{-1}LU^2RL^{-1}BR^{-1}L R^{-1}U^{-1}$

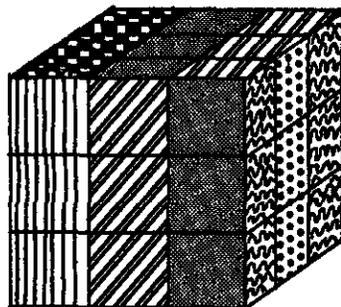


Fig. 8.12. $R^2D^2L^2R^2D^2L^{-1}R$

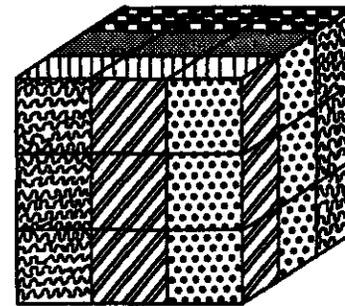
some of the patterns may be constructed more efficiently. For example, Figs. 8.7, 8.8, 8.9, and 8.13 may be obtained from the solved cube by the processes given below:

Fig. 8.7 - $RUR^2D^{-1}RL^{-1}FB^{-1}D^{-1}F^{-1}R^{-1}F^2RU^2FR^2F^{-1}R^{-1}U^{-1}F^{-1}U^2FR$;

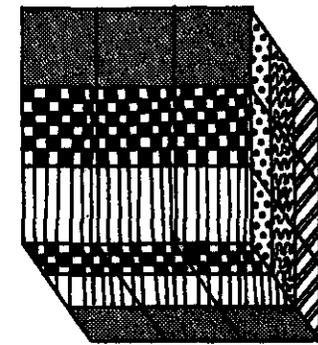
Fig. 8.8 - $BRL^{-1}D^{-1}R^2DR^{-1}LB^{-1}R^2UB^2U^{-1}DR^2D^{-1}$;

Fig. 8.9 - $F^{-1}U^{-1}BU^2B^{-1}URU^2R^{-1}FBDF^{-1}D^2FD^{-1}L^{-1}D^2LB^{-1}$.

Fig. 8.13 - $R^2BUR^2U^{-1}R F^2R^{-1}UR^2U^{-1}B^{-1}R^2 F^2$,
then $U^2FRU^2R^{-1}U B^2 U^{-1}RU^2R^{-1}F^{-1}U^2 B^2$,
then $BL^2R^2F^2B^2R$



Front view



rear view

Fig. 8.13. This pattern is obtained via the following steps:

1. $LDL^{-1}D^{-1}$ three times;
2. $RUR^{-1}U^{-1}$ three times;
3. $RF^{-1}UR^{-1}F UD^{-1} L^{-1}FU^{-1}LF^{-1} DU^{-1}$;
4. $U^{-1}BL^{-1}UB^{-1} RL^{-1} DB^{-1}LD^{-1}B LR^{-1}$
5. $F U^2D^2F^2U^2D^2B^2 F^{-1}$

Chapter 9 Permutations

If Rubik's Cube is dismantled and reassembled in a different way than intended, then it may not be possible to restore it to its original position. In other words, not all arrangements of the cubes can be obtained by turning the faces. For example, it is not possible for all but one of the cubies to be in its correct position and orientation.

To see which positions *are* possible, we must first study permutations. A permutation is any rearrangement of some set of objects. For example, a permutation of [1, 2, 3, 4, 5] can be:

1	2	3	4	5
↓	↓	↓	↓	↓
3	2	5	1	4

In this example, 1 is "sent" to 3, and so forth. Two permutations may be combined to get a third permutation. I call this multiplication, even though numbers are not being multiplied in the usual sense. For example, the permutation may be multiplied by

1	2	3	4	5
↓	↓	↓	↓	↓
5	2	1	3	4

as shown in Fig. 9.1. There, the first permutation sends 3 to 5, for example, and the second sends 5 to 4 so the "product" sends 3 to 4, and so forth.

Any process on the cube may be a permutation of the cubies. The product of two processes is just the process obtained by doing one pro-

cess after another. (Of course, the order in which the processes are done does make a difference. This multiplication is not "commutative." The commutator is an expression of this fact.)

As you shall soon see, any permutation may be regarded as "positive" or "negative." This definition will have the property that when two permutations are multiplied, they satisfy the multiplication table of Fig. 9.2. For example, $\text{pos} \times \text{neg} = \text{neg}$, that is, the product of a positive and a negative permutation is always a negative permutation.

Fig. 9.1. Multiplying two permutations to get the permutation on the right

1	2	3	4	5
↓	↓	↓	↓	↓
3	2	5	1	4
↓	↓	↓	↓	↓
1	2	4	5	3
1	2	3	4	5
↓	↓	↓	↓	↓
1	2	4	5	3

The "identity" permutation

1	2	3	4	5
↓	↓	↓	↓	↓
1	2	3	4	5

is an example of a positive permutation. A "transposition" is a permutation such as

1	2	3	4	5
↓	↓	↓	↓	↓
1	4	3	2	5

which interchanges two items and leaves everything else fixed. Any transposition is defined to be negative.

To see whether some more complicated permutation is positive or negative, all you have to do is "factor" it into transpositions. For example, the more complicated permutation

1	2	3	4	5
↓	↓	↓	↓	↓
3	2	5	1	4

Fig. 9.2. The multiplication table for signs of permutations

×	+	-
+	+	-
-	-	+

may be factored as a product of three transpositions:

1	2	3	4	5
↓	↓	↓	↓	↓
4	2	3	1	5
↓	↓	↓	↓	↓
3	2	4	1	5
↓	↓	↓	↓	↓
3	2	5	1	4

It is therefore negative, since each transposition is negative and

$$\text{neg} \times \text{neg} \times \text{neg} = \text{neg}$$

The above factorization is somewhat arbitrary, of course, and could have been done in many other ways. For instance, omitting the arrows, another factorization is given by:

1	2	3	4	5
2	1	3	4	5
2	1	3	5	4
2	3	1	5	4
3	2	1	5	4
3	2	5	1	4

This time five transpositions were used rather than three. However this does not contradict the original permutation being negative since

$$\text{neg} \times \text{neg} \times \text{neg} \times \text{neg} \times \text{neg} = \text{neg}$$

It is a remarkable fact that if a permutation can be factored into an

odd number of transpositions, then any other factorization of that permutation will also have an odd number of transpositions. And if a permutation can be factored into an even number of transpositions, then any other factorization will have an even number of transpositions.

It follows from this fact that the procedure above defines a consistent method for determining the "sign" of a permutation. That is, a permutation is positive if it can be factored into an even number of transpositions and negative if it can be factored into an odd number of transpositions.

To understand why this remarkable fact should be true, consider a factorization of the identity permutation into transpositions involving only "nearest neighbors," such as

1	2	3	4	5
2	1	3	4	5
2	3	1	4	5
2	3	1	5	4
3	2	1	5	4
3	1	2	5	4
1	3	2	5	4
1	2	3	5	4
1	2	3	4	5

This factorization required an even number of steps, namely eight. This agrees with the fact that the identity is positive.

In any such factorization, the only way "1" could get to the right of "2" is by transposing "1" and "2". And that is the only way "1" could get back to the left of "2". Since "1" begins and ends to the left of "2", the factorization must involve this transposition an even number of times. Likewise, each other transposition must occur an even number of times. Hence the total number of transpositions is even.

Now suppose, for the sake of obtaining a contradiction, that some permutation had a factorization into an even number of nearest neighbor transpositions and also an odd number of nearest neighbor transpositions. By reversing the order of one of these, we could string the factorizations together to get a factorization of the identity. Since an even number plus an odd number is odd, this would be a factorization of the identity into an odd number of nearest neighbor transpositions. But I just proved this to be impossible. Therefore any two factorizations of a

permutation into (nearest neighbor) transpositions must have the same parity (that is, both be even or both be odd). This proves the remarkable fact cited above, at least for the case of nearest neighbor transpositions. (Problem 20 in chapter 11 shows that this last restriction is superfluous.)

In the following chapter, we shall also need some facts about “modular” arithmetic. We will look at the set $[0, +1, -1]$ with the addition table in Fig. 9.3. This is easy to remember if you just pretend that $3 = 0$. For example:

$$1 + 1 = 2 = 3 - 1 = 0 - 1 = -1$$

This may seem somewhat unnatural, but many systems obey such laws. For example, if $+1$ means “clockwise rotation by 120 degrees” and -1 means “counterclockwise rotation by 120 degrees,” then

- $1 + 1 =$ “clockwise rotation by 120 degrees” followed by another “clockwise rotation by 120 degrees”
- $=$ “clockwise rotation by 240 degrees”
- $=$ “counterclockwise rotation by 120 degrees”
- $= -1.$

+	0	+1	-1
0	0	+1	-1
+1	+1	-1	0
-1	-1	0	+1

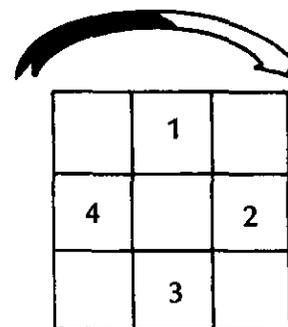
Fig. 9.3. An addition table

Chapter 10 Counting Cube Positions

Why are some Rubik’s Cube positions possible and some impossible? In this chapter, I apply the somewhat theoretical discussion of permutations in Chapter 9 to answer this question. The number of possible positions is shown to be approximately 43 quintillion.

A sequence of turns on the cube can be considered a permutation in several different ways: it permutes the cubies, it permutes the edges, it permutes the corners, it permutes the stickers, etc. If you turn one face 90 degrees, then the edges are permuted as a “4-cycle” (see Fig. 10.1). This is negative, as can be seen by the factorization

1	2	3	4
2	1	3	4
2	3	1	4
2	3	4	1



1	2	3	4
↓	↓	↓	↓
2	3	4	1

Fig. 10.1

If, however, we consider the 90-degree twist as a permutation on the edge stickers, then it is a product of two 4-cycles (see Fig. 10.2). This must be positive since it is a product of two negative permutations.

Clearly, any permutation of the cube is a product of 90-degree face twists. Since any product of positive permutations is positive, any possible position must be an even permutation of edge stickers. In particular, if all of the edge stickers are in place, then the number of incorrectly oriented edges must be even. For example, Fig. 10.3 shows a transposition of two edge stickers. This is negative and hence impossible.

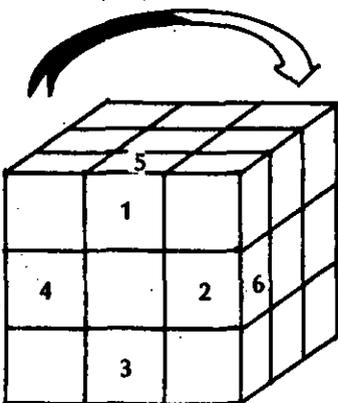
As mentioned, a 90-degree face twist is a 4-cycle on the edges, and therefore negative. Similarly, it is also a 4-cycle on the corners. Thus a 90-degree face twist satisfies the following equation:

$$\text{sign of permutation on edges} = \text{sign of permutation on corners}$$

The multiplication law says that the product of the signs is the sign of the product, so this equation continues to hold for arbitrary sequence of 90 degree face twists. Thus the equation holds for any possible position of the cube.

In particular, if all of the corners are in place, the sign of the permutation on the edges must be positive. For example, Fig. 10.4 is impossible even though the sign of the permutation on the edge stickers is positive.

To study corner orientations, select two colors which occur on opposite faces when the cube is restored to its original position. Call these "good" colors and the other four colors "bad." Then each corner has exactly one good sticker.



1	2	3	4	5	6	7	8
↓	↓	↓	↓	↓	↓	↓	↓
2	3	4	1	6	7	8	5

Fig. 10.2

As we have seen, not all of these are obtainable from the starting position by twisting faces (that is, without taking the cube apart). There were three reasons for this. The first was that the sign of the permutation on the edge stickers is positive. This cut down the number of possibilities by a factor of 2. The other two reasons give factors of 2 and 3. Thus the count above is too many by a factor of $2 \times 2 \times 3 = 12$ and the number of positions obtainable from the initial position is

$$= 43,252,003,274,489,856,000 \approx 4.3 \times 10^{19}$$

This is a large number. It is approximately equal to 100 times the age of the universe in seconds, or the distance in inches that light would travel in 100 years. It is also approximately the number of electrons in a speck of dust.

It is not known what the most efficient method for solving the cube is, but it is possible to show that there are some positions which require at least 19 90-degree face twists to solve.

To prove this, first note that since the cube has 6 faces and each face may be turned clockwise or counterclockwise, there are 12 possible 90-degree face twists. If you start with the solved cube, there are 12 positions which may be obtained by exactly one 90-degree face twist. There are $12 \times 12 = 12^2$ processes involving exactly two 90-degree face twists. These do not all give different positions, since for example RR^{-1} has the same effect as $R^{-1}R$. Therefore, the number of positions obtainable from two 90-degree face twists is less than 12^2 .

Likewise, the number of positions which could be obtained from the solved cube by making exactly 18 90-degree face twists is less than 12^{18} . The number of positions which could be obtained by 18 or fewer 90-degree face twists is less than

$$1 + 12^1 + 12^2 + \dots + 12^{18} \approx 3 \times 10^{19}$$

Therefore, since this is less than the total number of possible cube positions, there must be some positions which require at least 19 90-degree face twists to solve.

Now take a scrambled cube and lay it flat on the table. Define a corner to have "quark number" zero if its good sticker faces either up or down. Let it have quark number +1 if a counterclockwise twist of 120 degrees would change its quark number to zero. Otherwise let it have quark

number -1 . Quark numbers may be added or subtracted according to the rules given at the end of Chapter 9. The quark number of the (scrambled) cube is the sum of the quark numbers of the eight corners.

It is not hard to see that a 90-degree face twist increases the quark numbers of two corners by 1 and decreases the quark numbers of two other corners by 1. Hence the quark number of the cube is unchanged. Since the restored cube has quark number zero, and any possible position is obtained by a sequence of 90-degree face twists, it follows that any possible position must have zero quark number. In particular, if all corners are in place, it is not possible to have exactly one corner oriented incorrectly. All quarks appear in threes, or in quark-antiquark pairs.

We have therefore shown that any possible position of the cube must satisfy the following three rules:

1. The sign of the permutation on the edge stickers is positive.
2. The sign of the permutation on the edge cubies equals the sign of the permutation on the corner cubies.
3. The quark number is zero.

Furthermore, problem 21 of chapter 11 shows that any position satisfying these rules must be possible. Thus you can determine whether or not a position is possible by checking to see if it satisfies the three rules just given.

If you dismantle the cube, how many ways are there to reassemble it? First consider the centers. Any of the six centers can be put face down. This leaves four centers which could be put in front. The remaining centers are determined. Thus there are $6 \times 4 = 24$ ways to arrange the centers.

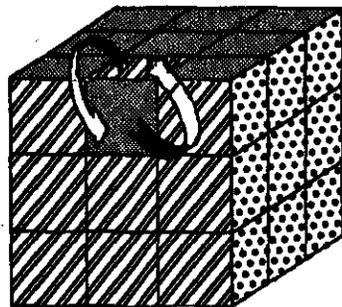


Fig. 10.3 The unseen part is solved.

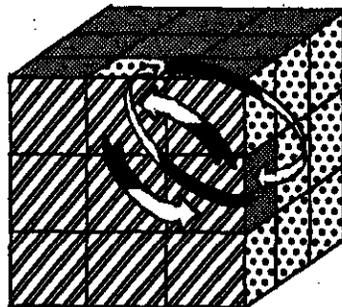


Fig. 10.4

There are 8 places to put the first corner. This leaves 7 places for the next corner, 6 for the following one, etc. Hence there are $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ ways to put the corners in, ignoring orientations. (This quantity is called "8 factorial" and is abbreviated "8!".) Each corner has 3 orientations, so there are $8! \times 3^8$ possibilities altogether.

There are 12 edges and 2 orientations for each edge, so similar reasoning gives $12! \times 2^{12}$ ways to assemble the edges. Thus there are

$$24 \times 8! \times 3^8 \times 12! \times 2^{12}$$

ways to assemble the cube.

For 24 of these positions, each face is a solid color. It is not really fair to count these as different, since any one may be obtained from any other by turning the whole cube (and not twisting any faces). So if we keep the centers fixed, there are

$$8! \times 3^8 \times 12! \times 2^{12}$$

ways to assemble the cube.

Chapter 11 Games and Exercises

1. Race. Try to get your time consistently under one minute.
2. Have a friend twist a solved cube five times. Try to restore it using only five twists.
3. Two or more people can try to restore a scrambled cube as follows: Each person makes six twists in private (out of sight) and hands it to the next person. No communication is allowed. The object is to solve the cube in as few turns as possible. (If this game is too difficult, try playing it with ten twists per person.)
4. The $2 \times 2 \times 2$ version of the usual $3 \times 3 \times 3$ Rubik's Cube can be simulated by paying attention only to the corners. How many twists does it take you to solve the corners while ignoring the edges and centers? As a variation of game no. 2, have a friend solve the corners while leaving the edges and centers scrambled and then make four random twists. The object is to restore the corners using only four twists.
5. Any number of people can play the following game: Consider some pattern such as those given in chapter 8 and see who can obtain the pattern from the solved cube in the fewest number of turns. For example, how many twists do you need to obtain the striped pattern of Fig. 8.13?
6. The process $F^2R^1LD^2R^2L^2FR^2L^2D^2RL^1F^2D^1$ flips each of the four edges on the down face. Use the commutator principle to explain why this process works.
7. Consider the edge 3-cycle shown in Fig. 6.1. There are other sequences of twists which accomplish the same result, such as $U^1R^2UD^1F^2D$. Find one that involves only 180-degree rotations.

8. If you are given a cube which has been scrambled with 180-degree rotations only, can you restore it using only 180-degree rotations? (This is not too hard if you solved problem no. 7.)
9. If all of the cubies are solved except for those on two faces, is it possible to solve the cube by turning only those two faces?
10. If someone makes a random sequence of F and U turns, can you find a method which will restore the cube using only F and U turns?
11. The processes RL , FB , UD , and their inverses are called antislice moves. Find a sequence of antislice moves which flips all of the edges on the U and D faces.
12. Suppose you are given a cube which has been scrambled with antislice moves. Find a method which will restore it with antislice moves.
13. Is there a process which has the same result as U but which only involves R , L , F , B , and D . How about one that only involves four other faces?
14. If you take the cube in its original position and apply some process to it repeatedly, you will eventually restore the cube. Why is this so? How many times are needed for the commutator LDL^1D^1 ? For FR ? For FR^1 ? For $RF^2B^1UB^1$? For $B^2R^2F^2L^2D^2$? For $F^2U^2R^2U^2$? For $R^1D^2RL^1F^2RF^2LR^1D$? (Hint: It is possible to figure out the answer without repeating the process.)
15. If orientations of the centers are considered, then the centers are not really fixed. Suppose you mark your cube (while in the starting position) as in Fig. 11.1. If you then scramble and restore it, you will find that the arrows may not match up. Can you find a

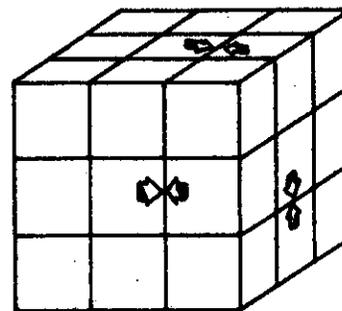


Fig. 11.1

method which will match the arrows? In particular, can you find a process which will rotate two centers 90 degrees and leave everything else fixed?

16. Why is it impossible to rotate one center by 90 degrees and leave all else fixed? (Hint: Use signs of permutations as in Chapter 10.)
17. Suppose a process changes exactly three edges and leaves the rest of the cube unaffected. If we repeat this process three times, will we then necessarily get the original position? How about a process that changes only one corner? Two corners? Three corners? Four corners?
18. Throughout this book I have taken the point of view that the centers are fixed. However, for the sake of this problem, let us consider processes which turn the entire cube. Then there are processes such as the one in Fig. 8.2 which permute the centers while preserving the other cubies. How many permutations of the centers are possible?
19. Show that the cube has 88,580,102,706,155,225,088,000 possible positions if center orientations are considered. (See problem nos. 15 and 16.)
20. Show that any transposition can be factored into an odd number of nearest neighbor transpositions.
21. Suppose that a cube is dismantled and reassembled in a random manner. Show that the cube can be solved if the three rules in Chapter 10 are satisfied.
22. If the cube has been solved except that some of the corners on the top face are incorrectly oriented, then the following method will solve the cube: Apply $LDL^{-1}D^{-1}$ repeatedly until the front upper

left corner is correctly oriented. Then turn the top face and repeat the procedure. When all of the top corners have been oriented, the rest of the cube will be solved as well. Why does this method work?

23. One of the patterns shown in Fig. 11.2 is possible. Which is it? (You can find the answer by applying $RFB^{-1}D^{-1}F^2DF^{-1}BR^{-1}F^2UR^2U^{-1}DR^2FBU^2F^{-1}B^{-1}R^2D^{-1}$ to the solved cube.) For the impossible pattern, how many of the three rules in Chapter 10 are violated?
24. Show that the cube has 3,981,312 possible positions which have the following property: the color of each sticker matches that of the center of either the face it is on or the opposite face. Prove that exactly 663,552 of these can be solved using only 180-degree twists.
25. Show that there must be some positions of the cube which require at least 18 face twists to solve. (In this problem, a face twist may be by 90 or by 180 degrees.)

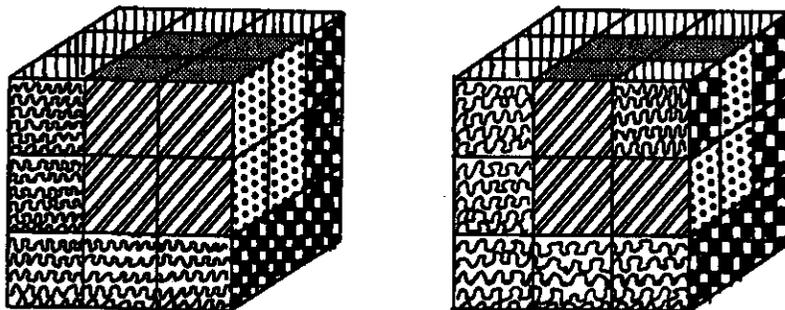


Fig. 11.2. The rear views are similar.

Glossary

- antislice move—One of the processes RL , FB , UD , R^1L^1 , F^1B^1 , or U^1D^1 .
- B —90-degree clockwise twist of the back (not bottom) face. The direction is clockwise as you face the back of the cube.
- center—The cubie in center of a face.
- clockwise—The direction that the hands of a clock would move if the clock were situated on the face of the cube.
- color—The color of a face is the color of its center.
- commutator—If X and Y are processes with inverses X^1 and Y^1 , their commutator is XYX^1Y^1 . See Chapter 2.
- conjugate—The conjugate of Y by X is XYX . See Chapter 2.
- counterclockwise—The direction opposite to clockwise.
- corner—A cubie having three colored stickers.
- cube—The entire Rubik's Cube, not to be confused with the cubies.
- cubies—The small cubes of which Rubik's Cube appears to be composed. There are 8 corners, 12 edges, and 6 centers, for a total of 26 cubies.
- cycle—A permutation that permutes objects in a cyclic manner. An example of a 3-cycle is the permutation of $[1, 2, 3]$ that sends 1 to 2, 2 to 3, and 3 to 1. A corner 3-cycle is given in Fig. 2.7 and an edge 3-cycle in Fig. 6.1. Some 4-cycles are given in Chapter 9.
- D —90-degree clockwise twist of the down face. The twist is clockwise as viewed from below the cube.
- edge—A cubie having two colored stickers.
- F —90-degree clockwise twist of the front face.
- face—A side of the cube. The cube has six faces.
- flip—To reverse orientation (of an edge). See Fig. 1.2.
- inverse—A process that reverses the effect of some other process. See Chapter 2.

- L —90-degree clockwise twist of the left face.
- layer—One of the three horizontal levels. The top and bottom layers each have nine cubies and the middle layer has eight. See Fig. 1.4.
- mirror image process—The process obtained by observing someone in a mirror. The right and left hands get interchanged. See Chapter 2.
- negative permutation—A permutation that can be factored into an odd number of transpositions.
- orient—To put a cubie in its correct orientation. A cubie is correctly oriented if its stickers match the colors of the nearby centers.
- orientation—The way a cubie's stickers are situated relative to the nearby center cubies.
- permutation—A rearrangement. See Chapter 9.
- permute—To rearrange. Each process permutes the cubies.
- positive permutation—A permutation that can be factored into an even number of transpositions.
- possible position of the cube—A position obtainable from the solved cube by a sequence of face twists. Dismantling the cube is not allowed.
- process—A sequence of operations on the cube such as in Fig. 2.1.
- quark—An incorrectly oriented corner cubie is either a quark or an antiquark. See Chapter 10.
- R —90-degree clockwise rotation of the right face.
- slice—Those cubies between opposite faces, as shown in Fig. 1.4. There are three slices, each having four edges and four centers.
- slice twist—A rotation of a slice. Except for a turn of the entire cube, these are equivalent to RL^1 , LR^1 , FB^1 , BF^1 , UD^1 , and DU^1 .
- solve a cubie—To put the cubie in place with the correct orientation.
- solved cube—A cube with each face having one solid color.
- stickers—The colored material glued to the cubies. Each corner has three stickers, each edge has two stickers, and each center has one.
- transposition—A permutation that interchanges two objects and leaves everything else fixed.
- U —90-degree clockwise rotation of the upper face.