Lecture 20:
Rubik's Cube: The Fundamental Theorem of Cubology

Contents

In this lecture, we present the Fundamental Theorem of Cubology. This is the theorem which gives us a complete understanding of what permutations of the cubies are possible, a solvability criteria, and much more.

20.1 Rubik's Cube - A Model

We now describe a mathematical model of Rubik's cube which is superior to our previous models in a few ways. The difficulty in modeling Rubik's cube comes from the fact that each cubie has a home location *and* orientation. Sometimes we would like to focus on how the cubies have been *permuted* (without focusing on the orientation of the stickers), and other times we would like to focus on how the cubies are *oriented* in the cubicle they occupy. Our model will consist of a 4-tuple $(\rho, \sigma, \boldsymbol{v}, \boldsymbol{w})$, where ρ (and σ) describes how the corner cubies (and edge cubies) are permuted, and $\boldsymbol{v}$ (and $\boldsymbol{w}$) describe how the corner cubies (and edges) are oriented.

We begin by fixing an orientation of the cube in space, that is, we choose an up face and a front face. This can be done in any way whatsoever (in fact, there are 24 different ways to do this), but once an orientation is chosen this will remain fixed for the rest of the discussion. In these notes the orientation we will choose is: blue face up, red face in front. We also assume the classic colouring scheme: blue opposite green, red opposite orange, and yellow opposite white. See Figure 1.

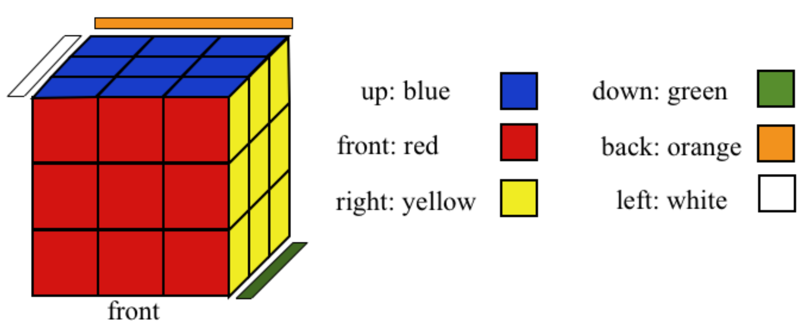


Figure 1: The standard orientation for the cube: blue face up, red face front.

We recall some notation:

- V denotes the set of corner cubies. $|V| = 8$.
- E denotes the set of edge cubies. $|E| = 12$.
- RC_3 denotes the Rubik's cube group.
- $S_V = S_8$ is the symmetric group on the corner cubies.
  (Arbitrarily number the corner cubies - see Figure 2a.)
- $S_E = S_{12}$ is the symmetric group on the edge cubies.
  (Arbitrarily number the edge cubies - see Figure 2b.)

As with our other puzzles, we imagine that both the cubies (pieces), and the cubicles (locations), are numbered. When a cubie is in its home location the cubie number will match the cubicle number. Imagine a fictitious layer of skin around the outside of the cube which stays in play under cube moves, the cubicle numbers are printed on this layer of skin. Any configuration of the cube will give two permutations (ignoring orientation of the cubies): $\rho \in S_8$ which corresponds to how the corner cubies are permuted, and $\sigma \in S_{12}$ which corresponds to how the edge cubies are permuted. See Figure 2.

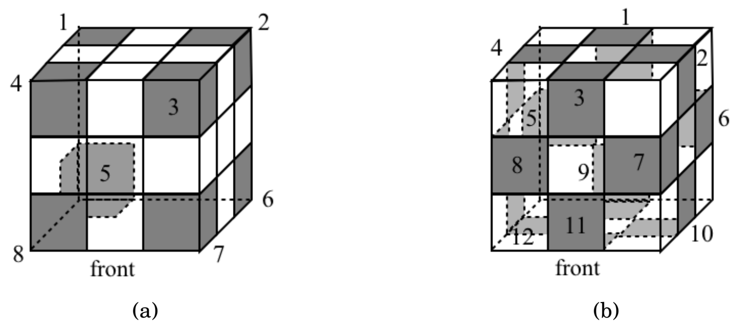


Figure 2: Labeling of the corner and edge cubies.

In order to describe the *orientation* of the corner and edge cubies, we mark one facet of each cubicle with a "+" sign. Again, imagine this marking is on the fictitious layer of skin surrounding the cube. Figure 3 shows how the facets will be marked. See Figure 4 for a 2-dimensional layout of the same marking pattern. The key thing to observe is that every cubicle has exactly one facet marked. We call this marked facet the *primary facet* of the cubicle.

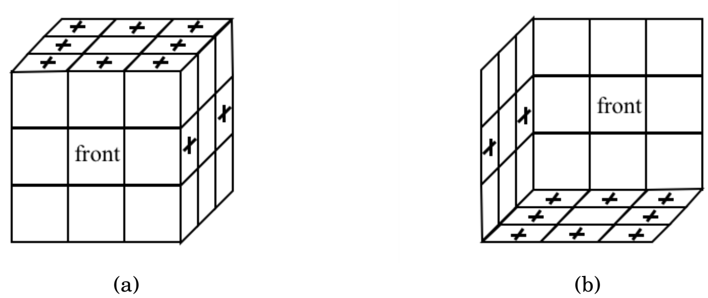


Figure 3: Marking the primary facet of a cubicle.

Next we mark the stickers on each cubie based on their relative position to the primary facet. For this marking, think of the cube in the solved state. For an edge cubie, mark the sticker with a 0 if it is in the primary facet

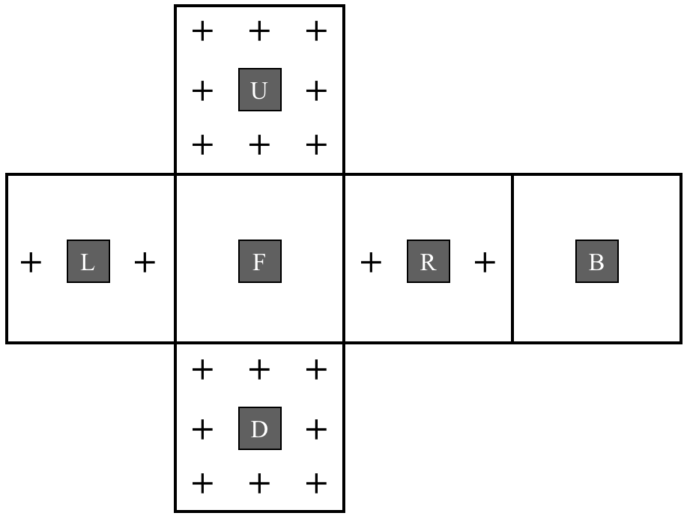


Figure 4: Marking the primary facets of a cubicle.

(i.e. beneath the "+" mark on the skin layer), and mark the other sticker on the same cubie with a 1. For a corner cubie, mark the sticker in the primary facet with a 0, and mark the other two stickers with 1 and 2 as you move in the clockwise direction around the cubie. See Figures 5 and 6.

(a) (b)

Figure 5: Marking the facets of a cubicle for orientation.

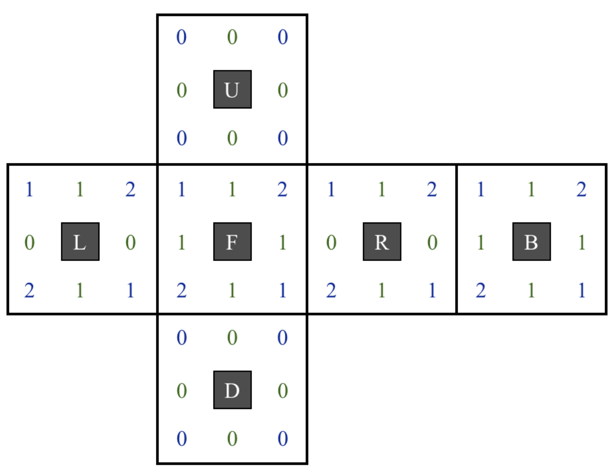


Figure 6: Marking the facets of a cubicle for orientation.

For an arbitrary configuration of the cube, the orientation of the edge pieces can be charcterised by a 12-tuple $\boldsymbol{w} = (w_1, w_2, \ldots, w_{12}) \in C_2^{12} = \{0,1\}^{12}$,[1] where w_i is the number on the sticker of the i^{th} edge cubie that is in the primary facet of the cubicle it occupies. Similarly, the orientation of the corner pieces can be charcterised by an 8-tuple $\boldsymbol{v} = (v_1, v_2, \ldots, v_8) \in C_3^8 = \{0,1,2\}^8$, where v_i is the number on the sticker of the i^{th} corner cubie that is in the primary facet of the cubicle it occupies.

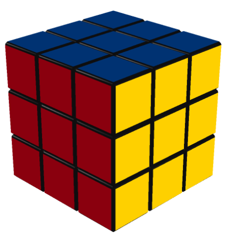
We now have a way to describe the position of all the pieces in any configuration of the cube.

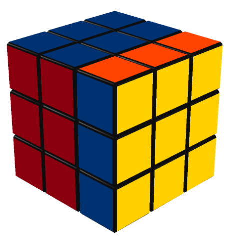
Definition 20.1 (Position vector of a configuration of cube pieces.) *If X is any configuration of Rubik's cube the* ***position vector*** *is a 4-tuple $(\rho, \sigma, \boldsymbol{v}, \boldsymbol{w})$ where $\rho \in S_8$, $\sigma \in S_{12}$ encode the permutations of the cubies, and $\boldsymbol{v} \in C_3^8$ and $\boldsymbol{w} \in\in C_2^{12}$ encode the orientations of the cubies.*

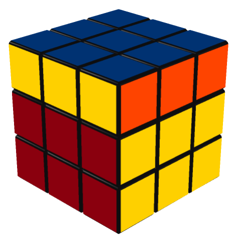
$$\begin{aligned}
\rho \in S_8 : &\quad \rho(i) = j \quad \textit{if corner cubie } i \textit{ moves to cubicle } j. \\
\sigma \in S_{12} : &\quad \sigma(i) = j \quad \textit{if edge cubie } i \textit{ moves to cubicle } j. \\
\boldsymbol{v} = (v_1, v_2, \ldots, v_8) \in C_3^8 = \{0,1,2\}^8 : &\quad v_i \textit{ is the number on the } i^{th} \textit{ corner cubie beneath} \\
&\quad \textit{the "+" mark of the cubicle it occupies.} \\
\boldsymbol{w} = (w_1, w_2, \ldots, w_{12}) \in C_2^{12} = \{0,1\}^{12} : &\quad w_i \textit{ is the number on the } i^{th} \textit{ edge cubie beneath} \\
&\quad \textit{the "+" marking of the cubicle it occupies.}
\end{aligned}$$

For simplicity we will use $\mathbf{0}$ to denote the 8-tuple and 12-tuple $(0, 0, \ldots, 0)$.

Let's look at a few examples where we take a configuration and write it as a 4-tuple.


(a) The solved state cube


(b) Move R^{-1}


(c) Move U

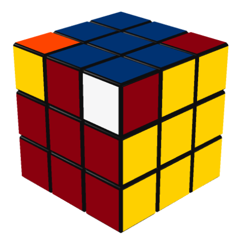

(d) Corner 3-cycle C1'

Figure 7: Configurations for Example 20.1.

Example 20.1 *(a) The solved state cube shown in Figure 7a corresponds to the 4-tuple $(\varepsilon, \varepsilon, \mathbf{0}, \mathbf{0})$, since cubies have not been permuted, nor twisted.*

(b) Consider the cube corresponding to the move R^{-1} as shown in Figure 7b. The corner cubies have been 4-cycled: $\rho = (2,3,7,6)$, and the edge cubies have been 4-cycled: $\sigma = (2,7,10,6)$. To determine the orientation vectors v *and* w*, we look at where each was moved to, one by one. Let's start with the corner cubies. Only 4-corner cubies were moved, namely 2, 3, 7 and 6, therefore we only need to figure out what v_2, v_3, v_7 and v_6 are. All others are 0. Cubie 2 (the blue-orange-yellow cubie) has its orange side in the primary facet now, since the orange side is labeled 1 (see brf facet in Figures 5 and 6) this means $v_2 = 1$. Similarly, cubie 3 (the UFR cubie) is now in cubicle frd and the sticker in facet fru (marked with number 2) is now primary facet dfr. Therefore, $v_3 = 2$. The reader should verify the rest of the components in the orientation vectors:*

$$\boldsymbol{v} = (0,1,2,0,0,2,1,0), \qquad \boldsymbol{w} = (0,1,0,0,0,1,1,0,0,1,0,0).$$

Together with permutations $\rho = (2,3,7,6)$, and $\sigma = (2,7,10,6)$, we have found the 4-tuple position vector.

[1] For a set A, A^n denotes the cartesian product of A with itself n times: $A \times A \cdots \times A$.

(c) *Consider the cube corresponding to the move U as shown in Figure 7c. Since all "+" markings are on the* up-*face each cubie still has the sticker labeled* 0 *in the primary facet. Therefore,*

$$\boldsymbol{v} = \mathbf{0}, \qquad \boldsymbol{w} = \mathbf{0}.$$

The permutations of the cubies are:

$$\rho = (1, 2, 3, 4), \qquad \sigma = (1, 2, 3, 4).$$

(d) *Finally, consider the cube corresponding to the move U as shown in Figure 7d. Edge cubies remained fixed so $\sigma = \varepsilon$ and $\boldsymbol{w} = \mathbf{0}$. The corner cubies are permuted as a 3-cycle $\rho = (2, 4, 3)$ and the orientation vector is:*

$$\boldsymbol{v} = (0, 1, 2, 0, 0, 0, 0, 0).$$

Not every 4-tuple $(\rho, \sigma, \boldsymbol{v}, \boldsymbol{w})$ corresponds to a legal configuration of Rubik's cube (i.e. one that is achievable using basic cube moves). For example, the 4-tuple $(\varepsilon, \varepsilon, \mathbf{0}, (1, 0, 0, \ldots, 0))$ represents a single edge flip (where the edge cubie in the ub position was flipped). This is not possible to do through legal cube moves as we have already seen. Therefore, the set

$$S_8 \times S_{12} \times C_3^8 \times C_2^{12} = \{(\rho, \sigma, \boldsymbol{v}, \boldsymbol{w}) \mid \rho \in S_8, \sigma \in S_{12}, \boldsymbol{v} \in C_3^8, \boldsymbol{w} \in C_3^{12}\} \tag{1}$$

is much larger than the set of legal cube configurations RC_3. In fact, this set is precisely the set of all ways there is to reassemble the cube (assuming you don't take apart the mechanism holding the centres in place, but only disassemble and reassemble edge and corner pieces). We denote set (1) by RC_3^* and call it the **illegal cube group** (as opposed to RC_3 which is the (legal) cube group). Previously we used the notation $\mathcal{A}$ to denote this set, but from now on we will use RC_3^* as a reminder of how it is related to RC_3.

Since $RC_3 \subset RC_3^*$ we'd like to characterize exactly which 4-tuples correspond to legal configurations of the cube. This characterization is known as the First Fundamental Theorem of Cubology.

20.2 The First Fundamental Theorem of Cubology

Theorem 20.1 (First Fundamental Theorem of Cubology) *A position vector $(\rho, \sigma, \boldsymbol{v}, \boldsymbol{w}) \in S_8 \times S_{12} \times C_3^8 \times C_2^{12}$ corresponds to a legal configuration of Rubik's cube if and only if the following three conditions are satisfied.*

(a) *sign*$(\rho) =$ *sign*(σ)

(b) $v_1 + v_2 + \cdots + v_8 = 0 \pmod 3$

(c) $w_1 + w_2 + \cdots + w_{12} = 0 \pmod 2$

In words, this theorem says that a configuration is legal if and only if the permutation of the edge cubies has the same parity as the permutation of the corner cubies, the number of clockwise corner twists is equal to the number of counterclockwise corner twists modulo 3, and edge flips occur in pairs.

Verify for yourself that these conditions are satisfied in each case of Example 20.1. Moreover, in the case of a single edge flip in the ub cubicle, the position vector is $(\varepsilon, \varepsilon, \mathbf{0}, (1, 0, 0, \ldots, 0))$ which doesn't satisfy condition (c) of the theorem, hence it isn't a legal configuration.

Proof: (1) First we show that the three conditions are necessary, i.e. that they hold for every legal configuration. To do this we just need to show these conditions hold for the solved state configuration, and they are preserved under the six basic cube moves R, L, U, D, F, B.

The solved state configuration corresponds to the position vector $(\varepsilon, \varepsilon, \mathbf{0}, \mathbf{0})$ and the three conditions in the theorem are satisfied.

For each of the six moves R, L, U, D, F, B the corresponding position vectors are:

$$
\begin{aligned}
R &\mapsto ((2,6,7,3),(2,6,10,7),(0,1,2,0,0,2,1,0),(0,1,0,0,0,1,1,0,0,1,0,0))\\
L &\mapsto ((1,4,8,5),(4,8,12,5),(2,0,0,1,1,0,0,2),(0,0,0,1,1,0,0,1,0,0,0,1))\\
U &\mapsto ((1,2,3,4),(1,2,3,4),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0,0,0,0,0))\\
D &\mapsto ((5,8,7,6),(9,12,11,10),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0,0,0,0,0))\\
F &\mapsto ((3,7,8,4),(3,7,11,8),(0,0,1,2,0,0,2,1),(0,0,0,0,0,0,0,0,0,0,0,0))\\
B &\mapsto ((1,5,6,2),(1,5,9,6),(1,2,0,0,2,1,0,0),(0,0,0,0,0,0,0,0,0,0,0,0))
\end{aligned}
$$

and the conditions (a)-(c) of the theorem are satisfied. Each pemutation is a 4-cycle which is odd and has sign -1. The sum of the components of each corner orientation vector is either 0 or 6 which is divisible by 3. The sum of the components of each edge orientation vector is either 0 or 4 which is divisible by 2.

If X is a legal configuration with position vector $(\rho, \sigma, \boldsymbol{v}, \boldsymbol{w})$ satisfying (a)-(c) then after applying one of the six basic moves to X (a)-(c) remain satisfied: (a) is satisfied since every one of these moves simultaneously causes a 4-cycle of corners cubies and a 4-cycle of edge cubies, which are both odd. (b) remains satisfied, because components with U and D don't change at all, while with R, L, F, B simultaneously two components are increased by 1 (modulo 3), and two components are reduced by 1 (modulo 3). (c) remains satisfied, because components with U, D, F, B don't change at all, while with R, L simultaneously two components are increased by 1 (modulo 2), and two components are reduced by 1 (modulo 2).

Since every legal configuration is obtainable from the solved state cube through legal cube moves then properties (a)-(c) are satisfied by any legal configuration.

(2) In order to prove these three conditions are sufficient, we have to show that any position vector $(\rho, \sigma, \boldsymbol{v}, \boldsymbol{w})$ satisfying these three properties can be solved using legal cube moves. Our strategy for solving the cube, as laid out in Lecture 19, is enough to prove this part. Let's see why.

Let X be a configuration corresponding to $(\rho, \sigma, \boldsymbol{v}, \boldsymbol{w})$.

(i) Without loss of generality we can assume $\text{sign}(\rho) = \text{sign}(\sigma) = 1$. If not, just apply any single basic quarter-turn of a face, the resulting position vector will now satisfy this parity condition. This means the permutation of corner cubies is even, and therefore can be restored to their home locations using 3-cycles. Also, edge cubies can be restored to their home locations using 3-cycles. Since we can perform any 3-cycle of corner or edge cubies, then we can restore all cubies to their home locations. Call this new configuration X'.

Since the basic cube moves preserve conditions (a)-(c) then the position vector $(\rho', \sigma', \boldsymbol{v}', \boldsymbol{w}')$ for X' satisfies these conditions, and in this case $\rho' = \varepsilon$, $\sigma' = \varepsilon$. All that remains now is to show we can twist the cubies into their proper orientations.

(ii) Condition (c) says that an even number of edge pieces need to be flipped. Since we have moves to flip any pair of edges then we can solve all the edge cubies. Condition (b) says that the number of clockwise corner twists is equal to the number of counterclockwise corner twists modulo 3. So first twist any cw, ccw pairs into their home orientations. The result will be that all remaining corners twists will occur in triples: 3 cw or 3 ccw twists. These can be solved using our corner twisting moves.

Therefore, X is a solvable configuration. This completes the proof of the theorem. $\square$

The First Fundamental Theorem of Cubology is the solvability criteria for Rubik's cube. This is the analogue to the solvability criteria that we developed for all the other puzzles. Moreover, this theorem allows us to compute the size of the group RC_3 simply by counting the number of 4-tuples that satisfy the three conditions.

Corollary 20.1 (The Size of the Cube Group) *The number of positions of the illegal and legal Rubik's Cube groups are:*

$$|RC_3| = |\mathcal{C}| = \frac{|RC_3^*|}{12} = 2^{27} \cdot 3^{14} \cdot 5^3 \cdot 7^2 \cdot 11 = 43,252,003,274,489,856,000 \approx 4.3 \cdot 10^{19}.$$
$$|RC_3^*| = |\mathcal{A}| = 8! \cdot 12! \cdot 3^8 \cdot 2^{12}.$$

Proof: Since $RC_3^* = S_8 \times S_{12} \times C_3^8 \times C_2^{12}$ then $|RC_3^*| = |S_8| \cdot |S_{12}| \cdot |C_3|^8 \cdot |C_2|^{12} = 8! \cdot 12! \cdot 3^8 \cdot 2^{12}$.

For legal positions this number is reduced by

- half by condition (a) in Theorem 20.1, since there are as many even permutations as there are odd ones,
- a third by condition (b), since the orientation of 7 corner cubies can be arbitrarily chosen and this would determine the orientation of the 8^{th},
- half by by condition (c), since the orientation of 11 edge cubies can be arbitrarily chosen and this would determine the orientation of the 12^{th}.

Therefore $|RC_3| = \frac{|RC_3^*|}{12}$. □

How big is this number $|RC_3|$?

If we put $4.3 \cdot 10^{19}$ cubes of 5.6 cm width – each in a different configuration – side by side in a straight line, the length of the line would be $\approx 2.4 \cdot 10^{15}$ kilometres, which is about 255 light years. By way of comparison the star α_1 Centauri is about 4.39 lights years away. Or packed tightly on the surface of the earth the cubes would blanket the earth to a height of 15 metres (see Figure 8). Allowing a second for each turn, it would take 1364 billion years to go though all possible configurations (assuming you don't revisit the same configuration twice). By comparison the universe is around 13 billion years old.

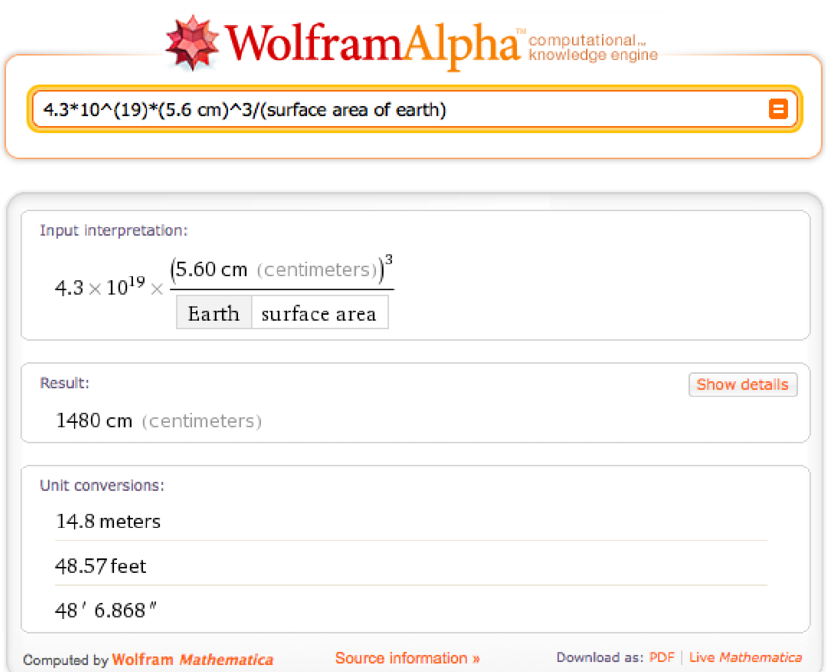


Figure 8: Covering the earth in Rubik's cubes would create a blanket 15m thick. Calculation on wolframalpha.com.

Of course it is not the size of the cube group that make Rubik's cube challenging. After all, if you were given a shuffled deck of 52 playing cards and asked to put them back in order this would be a simple task. Yet there are $52! \approx 8.07 \cdot 10^{67}$ ways the cards could be shuffled, and only one is in the proper order. What makes Rubik's cube challenging is the way the pieces are linked together, and the restrictions this imposes on legal moves.

20.3 The Second Fundamental Theorem of Cubology

We have two different models for a configuration of Rubik's cube: (i) the permutation of the 48 faces, as an element in S_{48}, which also corresponds to the move sequence that was applied to the solved state cube to reach the configuration, and (ii) the 4-tuple position vector. The First Fundamental Theorem of Cubology was about the position vector, which we can now restate in terms of move sequences. This is the Second Fundamental Theorem of Cubology.

Theorem 20.2 (The Second Fundamental Theorem of Cubology) *A move sequence is possible, if and only if the following three conditions are satisfied:*

(a) The total number of cycles of even length (corner and edge cycles) is even.

(b) The number of corners that are twisted clockwise is equal to the number that are twisted counterclockwise modulo 3.

(c) The number of flipped edges is even.

As a consequence we can characterize some impossible move sequences. Notice we have already seen each of these moves to be impossible (some used SAGE to investigate these impossibilities). However, now we have given a formal mathematical proof that these are impossible.

Corollary 20.2 *Each of the following configurations cannot be obtained from the solved state cube through legal cube moves.*

(a) Exactly two edge cubies are swapped.

(b) Exactly two corner cubies are swapped.

(c) Exactly one edge cubie is flipped.

(d) Exactly one corner cubie is twisted.

20.4 When are two assembled cubes equivalent?

Consider the equivalence relation $\sim_{RC_3}$ on the illegal cube group RC_3^* defined by:

$$\begin{aligned} X \sim_{RC_3} Y \quad &\iff \quad X^{-1}Y \in RC_3 \\ &\iff \quad X \text{ can be taken to } Y \text{ through a sequence of legal cube moves (i.e. twists of the 6 faces).} \end{aligned}$$

All this means is that we consider two assembled cubes equivalent if one can be twisted into the other using legal cube moves.

This partitions RC_3^* into equivalence classes: the left cosets of RC_3 in RC_3^*. The class RC_3 is precisely the set of solvable configurations. We'd like to be able to determine (i) all the other left cosets of RC_3, (ii) a set of representatives for $RC_3^*/\sim_{RC_3}$, and (iii) a quick way to determine to which coset a given cube belongs.

By Corollay 20.1 the number of left cosets is $[RC_3^* : RC_3] = \frac{|RC_3^*|}{|RC_3|} = 12$. The First Fundamental Theorem provides a complete characterization of the left cosets. The conditions for a position vector $(\rho, \sigma, \boldsymbol{v}, \boldsymbol{w})$ to be in RC_3 are sign(ρ) = sign(σ) and $v_1 + v_2 + \cdots + v_8 = 0 \pmod 3$ and $w_1 + w_2 + \cdots + w_{12} = 0 \pmod 2$. The other cosets are given by the 12 different ways these conditions can be modified.

For example, modifying the conditions so that

$$\text{sign}(\rho) = \text{sign}(\sigma), \qquad v_1 + v_2 + \cdots + v_8 = 0 \pmod 3, \qquad w_1 + w_2 + \cdots + w_{12} = 1 \pmod 2$$

defines the coset $[X_3] = X_3 RC_3$ represented by a single edge flip X_3 (shown in Figure 9c).

$$\text{sign}(\rho) = \text{sign}(\sigma), \qquad v_1 + v_2 + \cdots + v_8 = 1 \pmod 3, \qquad w_1 + w_2 + \cdots + w_{12} = 0 \pmod 2$$

defines the coset $[X_4] = X_4 RC_3$ represented by a single corner twist in the counterclockwise direction X_4 (shown in Figure 9e).

$$\text{sign}(\rho) \neq \text{sign}(\sigma), \qquad v_1 + v_2 + \cdots + v_8 = 0 \pmod 3, \qquad w_1 + w_2 + \cdots + w_{12} = 0 \pmod 2$$

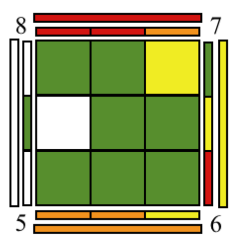
defines the coset $[X_1] = X_1 RC_3$ represented by a swap of two edge cubies X_1, or equivalently a swap of two corner cubies (shown in Figure 9b).

$$\text{sign}(\rho) \neq \text{sign}(\sigma), \qquad v_1 + v_2 + \cdots + v_8 = 2 \pmod 3, \qquad w_1 + w_2 + \cdots + w_{12} = 0 \pmod 2$$

defines the coset $[X_9] = X_9 RC_3$ represented by a swap of two edge cubies, and a clockwise twist of a corner cubie X_9 (shown in Figure 9j). And so on.

Figure 9 shows a set of twelve representative for the left cosets of RC_3 in RC_3^*. This means that a randomly assembled cube can be reduces to exactly one of these 12 possibilities.

Example 20.2 *The following diagram shows a (possibly illegal) configuration of the last layer of Rubik's cube. We assume all other non-visible pieces of the cube are in their home orientations. We'd like to determine which of the configurations in Figure 9 it is equivalent to. To do this it suffices to determine the position vector.*



The corners permutation is $\rho = (6, 7)$ and the edge permutation is $\sigma = \varepsilon$. The corner orientation vector is

$$\boldsymbol{v} = (0, 0, 0, 0, 0, 1, 0, 0)$$

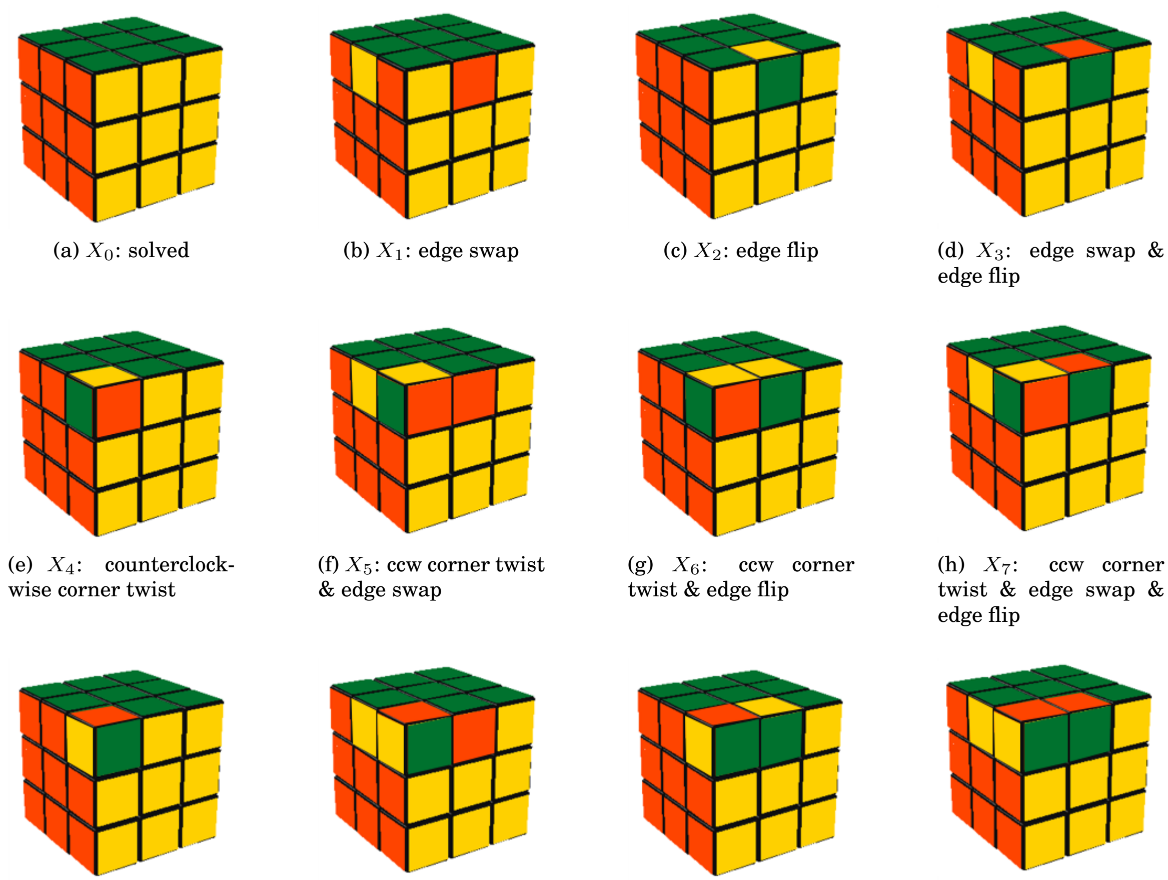
since corner cubie 6 is now in position 7 and twisted counterclockwise, and the edge orientation vector is

$$\boldsymbol{w} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1),$$

since edge cubie 12 is flipped. Therefore

$$\textit{sign}(\rho) \neq \textit{sign}(\sigma), \qquad v_1 + v_2 + \cdots + v_8 = 1 \pmod 3 \qquad w_1 + w_2 + \cdots + w_8 = 1 \pmod 2$$

so it is equivalent to the a configuration where: two edge are swapped, one corner is twisted counterclockwise, and one edge is flipped. This is the configuration in Figure 9h.

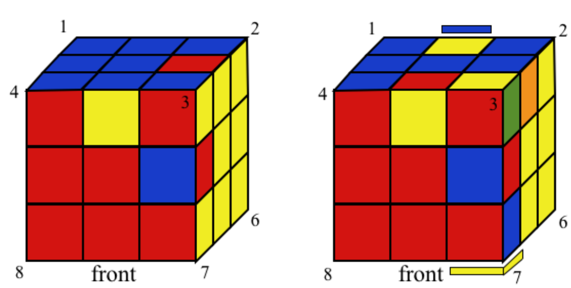


(a) X_0: solved
(b) X_1: edge swap
(c) X_2: edge flip
(d) X_3: edge swap & edge flip
(e) X_4: counterclockwise corner twist
(f) X_5: ccw corner twist & edge swap
(g) X_6: ccw corner twist & edge flip
(h) X_7: ccw corner twist & edge swap & edge flip
(i) X_8: clockwise corner twist
(j) X_9: cw corner twist & edge swap
(k) X_{10}: cw corner twist & edge flip
(l) X_{11}: cw corner twist & edge swap & edge flip

Figure 9: Representatives for the 12 different equivalence classes in $\mathcal{RC}_{\ni}^{*}$

20.5 Exercises

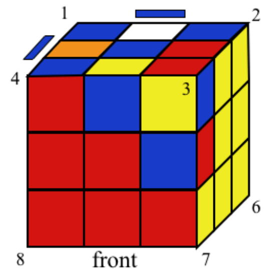
1. For each of the following configurations (i) determine the position vector $(\rho, \sigma, \boldsymbol{v}, \boldsymbol{w}) \in S_8 \times S_{12} \times C_3^8 \times C_2^{12}$, and (ii) determine whether it is a legal (i.e. solvable) configuration. Assume all non-visible cubes are in their home orientations.
(The corner cubies are labeled, see Figure 2 for a labeling of the edge cubies.)



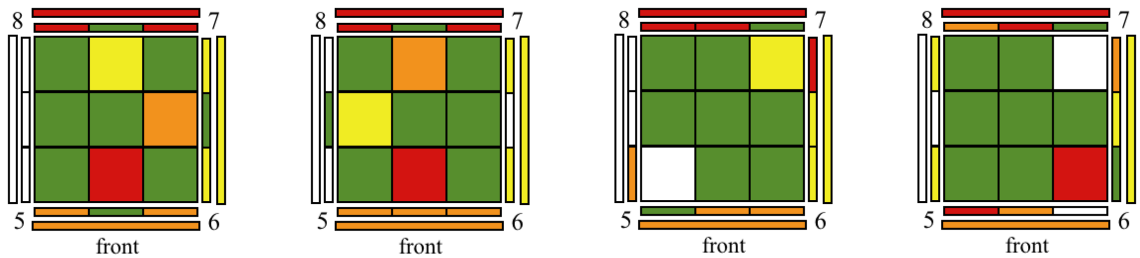
(a) (b)

2. Verify the following configuration is not solvable, by showing the position vector doesn't satisfy the three conditions of Theorem 20.1. Determine the quickest way to disassemble/reassemble it so that it becomes

solvable. That is, decide if you have to swap two pieces, or flip a single edge, or twist a corner, or a combination of these, etc.



3. **Impossible Configurations.** In each part below, a configuration of the last layer is shown. All non-visible cubies are in their home orientations. Show that each configuration is impossible by showing its position vector doesn't satisfy the three conditions of Theorem 20.1.



(a) (b) (c) (d)

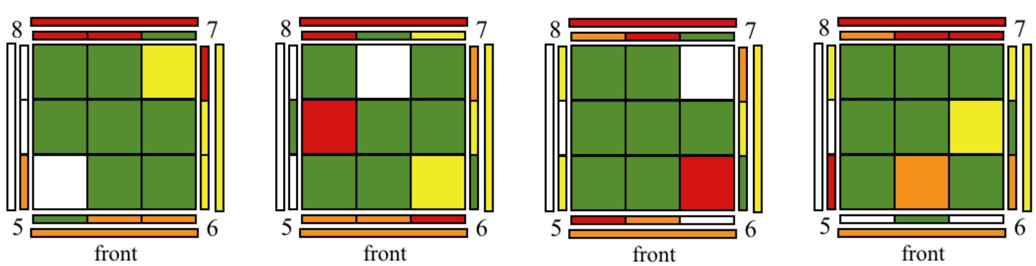
4. For each of the following move sequences determine the position vector $(\rho, \sigma, \boldsymbol{v}, \boldsymbol{w}) \in S_8 \times S_{12} \times C_3^8 \times C_2^{12}$.

   (a) RU
   (b) R^2U^2
   (c) $(R^2U^2)^3$
   (d) $[LD^2L^{-1}, U]$ (a corner 3-cycle)

5. For each of the following position vectors $(\rho, \sigma, \boldsymbol{v}, \boldsymbol{w}) \in S_8 \times S_{12} \times C_3^8 \times C_2^{12}$ draw the corresponding configuration.
(Assume the standard orientation as shown in Figure 1.)
(The puzzles templates file on the webpage includes some Rubik's cube templates.)

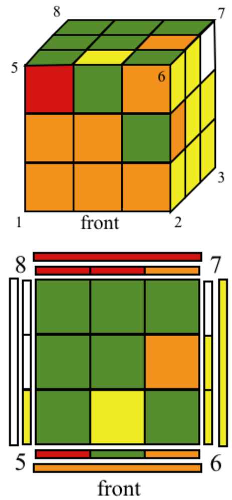
   (a) $(\rho, \sigma, \boldsymbol{v}, \boldsymbol{w}) = ((2,4)(1,3), \varepsilon, (1,1,2,2,0,0,0,0), (0,1,1,0,0,0,0,0,0,0,0,0))$
   (b) $(\rho, \sigma, \boldsymbol{v}, \boldsymbol{w}) = (\varepsilon, (2,3)(6,7), (0,0,0,0,0,0,0,0), (0,0,0,0,0,0,0,0,0,0,0,0))$
   (c) $(\rho, \sigma, \boldsymbol{v}, \boldsymbol{w}) = ((2,4)(3,7), (2,7,3), (0,1,2,1,0,0,2,0), (0,0,1,1,0,0,0,0,0,0,0,0))$

6. In each part below, a configuration Y_i of the last layer for a cube in RC_3^* is shown. The only pieces out of place are the indicated edge pieces, all other non-visible cubies are in there home orientations. Determine the representative X_j (from Figure 9) for the coset to which configuration Y_i belongs. That is, determine $0 \leq j \leq 11$ for which $Y_i \in [X_j] = X_j RC_3$.

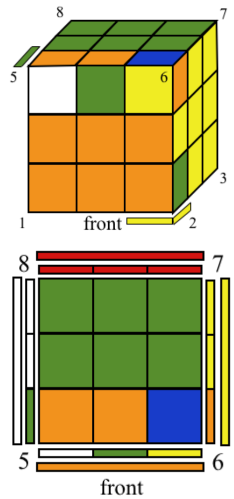


(a) Y_1 (b) Y_2 (c) Y_3 (d) Y_4

7. Are the following two (possibly illegal) configurations equivalent under cube moves? Each cube is drawn from two perspectives. All other non-visible cubes are assumed to be in their home orientations.



(a) X



(b) Y