Lecture 22: Symmetry & Counting I: The Orbit-Stabilizer Theorem

Contents

In this lecture we discuss how to use group theory to *count like a professional*: we look at an application of cosets to determine the size of a permutation group. In particular, we discover a straightforward way to count the number of symmetries of various geometric objects.

22.1 Orbits & Stablizers

In this section we will take a look at how permutation groups act on various structures.

It will be helpful to extend the definition of a permutation from finite sets of numbers $\mathbb{Z}_n$, to arbitrary sets. Let X be a nonempty set. A **permutation** α of X is a bijection $\alpha : X \to X$. The set of all permutations of X is called the **symmetric group of** X and is denoted by S_X:

$$S_X = \{\alpha \mid \alpha : X \to X \text{ is a bijection}\}.$$

If $X = \mathbb{Z}_n = \{1, 2, \ldots, n\}$ then we simply denoted $S_{\mathbb{Z}_n}$ by S_n.

Definition 22.1 (Stabilizer of a Point) *Let G be a subgroup of S_X. For each $i \in X$, let*

$$\mathit{stab}_G(i) = \{\alpha \in G \mid \alpha(i) = i\}.$$

We call $\mathit{stab}_G(i)$ the ***stabilizer of*** *i* ***in*** *G.*

We can check that $\mathrm{stab}_G(i)$ is a subgroup of G. Since ε fixes every element in X it is definitely in $\mathrm{stab}_G(i)$. Let $\alpha, \beta \in G$, then $\alpha(i) = i$ and $\beta(i) = i$. It then follows that $\alpha^{-1}(i) = i$ and $(\alpha\beta)(i) = \beta(\alpha(i)) = \beta(i) = i$, hence $\alpha^{-1}, \alpha\beta \in \mathrm{stab}_G(i)$. Therefore $\mathrm{stab}_G(i) < G$.

Definition 22.2 (Orbit of a Point) *Let G be a subgroup of S_X. For each $i \in X$, let*

$$orb_G(i) = \{\alpha(i) \mid \alpha \in G\}.$$

We call $orb_G(i)$ the ***orbit of** i **under** G.*

Example 22.1 *If $G = S_4$, then $stab_{S_4}(3)$ is the set of all permutation in S_4 which fixes 3. There are $4! = 24$ permutations in S_4 but only the ones that don't have 3 in their disjoint cycle form fix 3. Therefore,*

$$\begin{aligned} stab_{S_4}(3) &= \{\varepsilon, (1,2), (1,4), (2,4), (1,2,4), (1,4,2)\} \\ &= S_{\{1,2,4\}}. \end{aligned}$$

Notice we used the notation $S_{\{1,2,4\}}$ to denote the set of all permutations of the set $\{1,2,4\}$.

Example 22.2 *Let*

$$\begin{aligned} G &= \langle (1,2,3)(4,5,6)(7,8) \rangle \\ &= \{\varepsilon, (1,2,3)(4,5,6)(7,8), (1,3,2)(4,6,5), (7,8), (1,2,3)(4,5,6), (1,3,2)(4,6,5)(7,8)\}. \end{aligned}$$

be a group of permutation on $X = \{1,2,3,4,5,6,7,8\}$. Then

$$\begin{array}{ll} orb_G(1) = \{1,2,3\} & stab_G(1) = \{\varepsilon, (7,8)\} \\ orb_G(2) = \{2,3,1\} & stab_G(2) = \{\varepsilon, (7,8)\} \\ orb_G(3) = \{3,1,2\} & stab_G(3) = \{\varepsilon, (7,8)\} \\ orb_G(4) = \{4,5,6\} & stab_G(4) = \{\varepsilon, (7,8)\} \\ orb_G(5) = \{5,6,4\} & stab_G(5) = \{\varepsilon, (7,8)\} \\ orb_G(6) = \{6,4,5\} & stab_G(6) = \{\varepsilon, (7,8)\} \\ orb_G(7) = \{7,8\} & stab_G(7) = \{\varepsilon, (1,2,3)(4,5,6), (1,3,2)(4,6,5)\} \\ orb_G(8) = \{8,7\} & stab_G(8) = \{\varepsilon, (1,2,3)(4,5,6), (1,3,2)(4,6,5)\} \end{array}$$

In each case notice that $stab_G(i)$ is a subgroup of G. Also notice that orbits are either disjoint or equal. Moreover, the distinct orbits:

$$\{1,2,3\}, \{4,5,6\}, \{7,8\}$$

form a partition of X.

Let G be a group of permutations on X, and define a relation on X by:

$$x \sim_G y \iff y = \alpha(x) \text{ for some } \alpha \in G. \tag{1}$$

Then $\sim_G$ is an equivalence relation (see Exercise 1), and the equivalence class of an element $x \in X$ is its orbit:

$$[x] = \text{orb}_G(x).$$

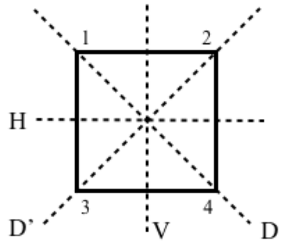
Since equivalence classes partition the set, this indicates that our observation in Example 22.2 were not coincidence. Orbits will always be the same or disjoint, and distinct orbit classes will partition X.

Example 22.3 *Recall that D_4, the dihedral group of the square, is the group of all symmetries of the square (see Figure 1a). The elements are the rotations $R_0, R_{90}, R_{180}, R_{270}$, and the reflections H, V, D, D'. We can view D_4 as a group of permutations on the vertices of the square. Here we identify the vertices of the square with the set $X = \{1,2,3,4\}$. See Figure 1b. Since vertex 1 can be taken to any other vertex by a rotation then the orbit of 1 is all of X: $orb_{D_4}(1) = \{1,2,3,4\}$.*

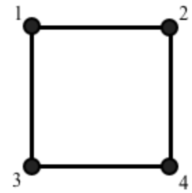
The stabilizer of 1 is:

$$stab_{D_4}(1) = \{R_0, D\}.$$

Similarly, we have $stab_{D_4}(2) = stab_{D_4}(3) = \{R_0, D'\}$.

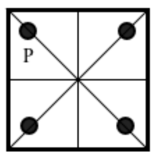


(a) Reflection elements in D_4

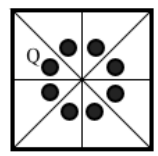


(b) Orbit of vertex 1

Figure 1: The group D_4 acting as a permutation group on the set of vertices.



(a) Orbit of point P under action of D_4



(b) Orbit of point Q under action of D_4

Figure 2: The group D_4 acting as a permutation group on the set of points enclosed by the square.

Example 22.4 *Building on the previous example, we may view D_4 as a group of permutations of the points X enclosed by the square. Figure 2a illustrates the orbit of the point P and Figure 2b illustrates the orbit of the point Q under D_4. Notice stab$_{D_4}(P) = \{R_0, D\}$, and stab$_{D_4}(Q) = \{R_0\}$.*

We can also view D as a group of permutations on the set of 4 line segments h, v, d, d' shown in Figure 3. Then

$$\begin{array}{ll}
\textit{orb}_{D_4}(h) = \{h, v\} & \textit{stab}_{D_4}(h) = \{R_0, R_{180}, H, V\} \\
\textit{orb}_{D_4}(v) = \{h, v\} & \textit{stab}_{D_4}(v) = \{R_0, R_{180}, H, V\} \\
\textit{orb}_{D_4}(d) = \{d, d'\} & \textit{stab}_{D_4}(d) = \{R_0, R_{180}, D, D'\} \\
\textit{orb}_{D_4}(d') = \{d, d'\} & \textit{stab}_{D_4}(d') = \{R_0, R_{180}, D, D'\}
\end{array}$$

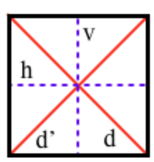


Figure 3: Orbit classes of the group D_4 acting as a permutation group on the set of line segments h, v, d, d'.

Example 22.5 *Let RC_3 be the Rubik's cube group, and let X be the set of all cubies of Rubik's cube. X can be partitioned into edge cubies E, corner cubies V, and centre cubies C. If x denotes the uf edge cubie, then since it is possible to move it to the location of any other edge cubie, then orb$_{RC_3}(x) = E$. Also, since centre cubies don't move under cube moves, the orbit of each centre cubie is just a set of size 1.*

Example 22.6 *Again, let RC_3 be the Rubik's cube group, but now let X be the set of all facets of Rubik's cube. Recall $|X| = 48$. The Rubik's cube group can be viewed as a group of permutations of the set X (we have made use of this fact frequently already). Let x be the facet on the* up *layer of the uf cubie. In our numbering system we denoted this facet by $x = 7$. Since an edge cubie can be moved to the location of any other edge cubie, and with either orientation, then the orbit of x is every edge-facet. Therefore, $|orb_{RC_3}(7)| = 24$. The next theorem will tell us that $|stab_{RC_3}(7)| = \frac{|RC_3|}{24}$.*

Looking back at the examples we can observe an obvious relationship between the sizes of G, $\text{orb}_G(i)$, and $\text{stab}_G(i)$: we always get $|\text{orb}_G(i)| \cdot |\text{stab}_G(i)|$ equal to the size of G. This is true in general and is stated in the next theorem.

Theorem 22.1 (Orbit-Stabilizer Theorem) *Let G be a subgroup of S_X. Then for any i in X,*

$$|G| = |orb_G(i)| \cdot |stab_G(i)|.$$

Proof: Since $\text{stab}_G(x)$ is a subgroup of G, we know from Lagrange's Theorem that

$$|G|/|\text{stab}_G(x)| = \text{ the number of distinct right cosets of } \text{stab}_G(x) \text{ in } G.$$

So we need to show that the number of right cosets equals the number of elements in $\text{orb}_G(x)$. To this end define

$$\psi : \{(\text{stab}_G(x))\alpha \mid \alpha \in G\} \to \text{orb}_G(x)$$

by

$$\psi(\text{stab}_G(x)\ \alpha) = \alpha(x).$$

Our goal is to show that ψ is a bijection.

(a) **ψ is well defined.** We have

$$\begin{aligned} \text{stab}_G(x)\ \alpha = \text{stab}_G(x)\ \beta \quad &\implies \quad \alpha = \gamma\beta \quad \text{for some } \gamma \in \text{stab}_G(x) \\ &\implies \quad \alpha(x) = (\gamma\beta)(x) = \beta(\gamma(x)) \\ &\implies \quad \alpha(x) = \beta(x) \quad \text{since } \gamma \in \text{stab}_G(x). \end{aligned}$$

(b) **ψ is injective.** Let $\alpha, \beta \in G$, we have

$$\begin{aligned} \psi(\text{stab}_G(x)\ \alpha) = \psi(\text{stab}_G(x)\ \beta) \quad &\implies \quad \alpha(x) = \beta(x) \\ &\implies \quad \beta^{-1}(\alpha(x)) = x \\ &\implies \quad (\alpha\beta^{-1})(x) = x \\ &\implies \quad \alpha\beta^{-1} \in \text{stab}_G(x) \\ &\implies \quad \text{stab}_G(x)\ \alpha = \text{stab}_G(x)\ \beta. \end{aligned}$$

(c) **ψ is surjective.** Let $y \in \text{orb}_G(x)$. Then for some $\alpha \in G$ we have $y = \alpha(x)$. Therefore,

$$\psi(\text{stab}_G(x)\ \alpha) = \alpha(x) = y,$$

and so ψ is surjective.

Therefore ψ is a bijection, and so it follows that

$$\begin{aligned} |\text{orb}_G(x)| &= |\{(\text{stab}_G(x))\alpha \mid \alpha \in G\}| \\ &= \text{ the number of right cosets of } \text{stab}_G(x) \text{ in } G \\ &= |G|/|\text{stab}_G(x)|, \end{aligned}$$

which implies

$$|G| = |\text{orb}_G(i)| \cdot |\text{stab}_G(i)|.$$

□

We now consider a few applications of this theorem.

22.2 Permutations Acting on Sets: Application of the Orbit-Stabilizer Theorem

The orbit-stabilizer theorem (Theorem 22.1) is a counting theorem. It enables one to determine the number of elements in a set. We will now see how this theorem will help us determine the number of rotational symmetries of some familiar 3-dimensional objects.

For a object X we let G_X be the group of all rotational symmetries of X. That is, the set of all ways the object can be picked up, rotated, and placed back on a table in front of you, so that it looks as though it wasn't moved. For each of the objects below we will determine $|G_X|$.

22.2.1 Rotation Group of a Tetrahedron

Let G_T be the group of all rotational symmetries of a regular tetrahedron.

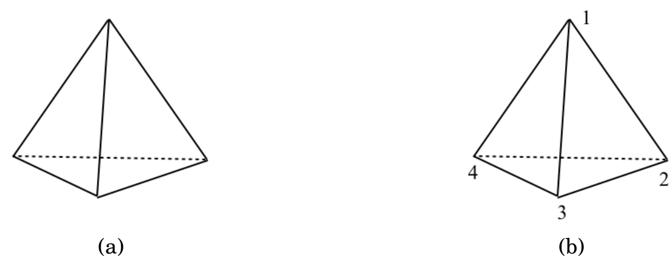


Figure 4: regular tetrahedron.

Let V_T be the set of 4 vertices of the tetrahedron, labeled as in Figure 4b. Then each rotation in G_T induces a permutation on V_T. That is, each element of G_T gives a permutation in $S_{V_T} = S_4$. Vertex 1 can be taken to any other vertex by a rotation, so the orbit of vertex 1 is $\text{orb}_{G_T}(1) = \{1, 2, 3, 4\}$, and therefore $|\text{orb}_{G_T}(1)| = 4$. The stabilizer of 1 consists satisfies $|\text{stab}_{G_T}(1)| = 3$, and the rotations in the stabilizer are: the identity, and two rotations corresponding to the permutations $(2, 3, 4)$ and $(2, 4, 3)$. Therefore, by the orbit-stabilizer theorem:

$$|G_T| = |\text{orb}_{G_T}(1)| \cdot |\text{stab}_{G_T}(1)| = 4 \cdot 3 = 12.$$

The 12 rotations of G_T are shown in Figure 5. Each rotation is described by the permutation it induces on the vertices. It is clear from this description that $G_T \approx A_4$.

22.2.2 Rotation Group of a Cube

Let G_C be the group of all rotational symmetries of a cube.

We can view G_C as a groups of permutations of the 8 corners, that is, as a subgroup of S_8. Observe that

$$\text{orb}_{G_C}(1) = \{1, 2, 3, 4, 5, 6, 7, 8\} \quad \Rightarrow \quad |\text{orb}_{G_C}(1)| = 8$$

and that

$$\text{stab}_{G_C}(1) = \{\varepsilon, (2, 4, 5)(3, 8, 6), (2, 5, 4)(3, 6, 8)\} \quad \Rightarrow \quad |\text{stab}_{G_C}(1)| = 3.$$

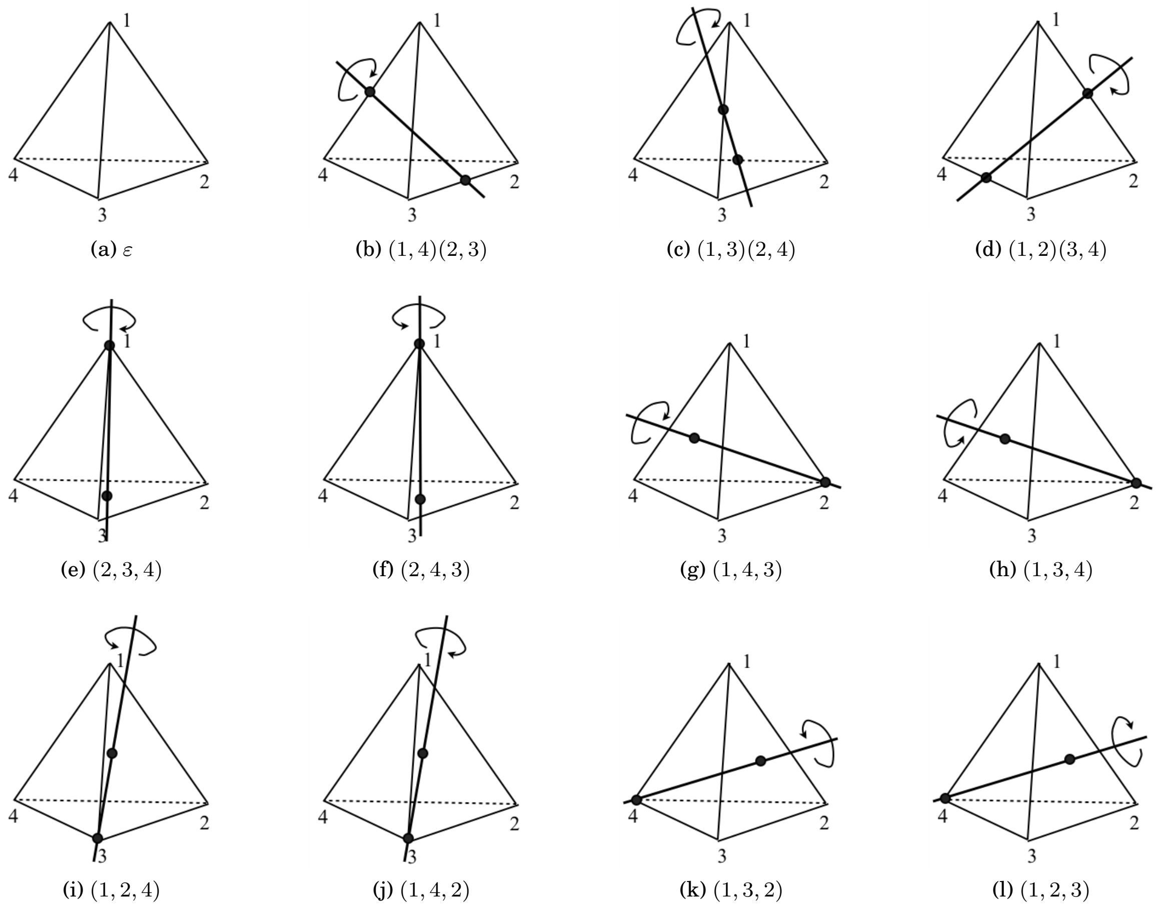


Figure 5: All 12 rotational symmetries of a regular tetrahedron

Figure 6: cube.

The elements of the stabilizer are the rotations about an axis through vertices 1 and 7.

Therefore, by the orbit stabilizer theorem:

$$|G_C| = |\text{orb}_{G_C}(1)| \cdot |\text{stab}_{G_C}(1)| = 8 \cdot 3 = 24.$$

Recall the symmetric group S_4 has 24 elements. Perhaps G_C is S_4 in disguise. To see if it is we should find 4 things in the cube that G_C permutes. There are 4 diagonals as shown in Figure 7, and each rotation of the cube permutes these diagonals. In fact, each rotation of the cube can be described precisely by how these diagonals are permuted. Therefore $G_C \approx S_4$.

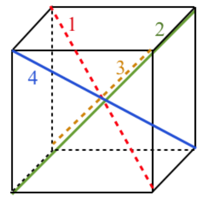


Figure 7: Viewing G_C as a group of permutations on the diagonals 1, 2, 3, 4.

22.2.3 Rotation Group of an Octahedron

Let G_O be the group of all rotational symmetries of a regular octahedron.

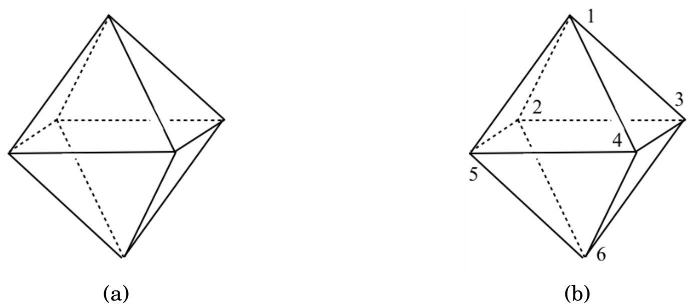


Figure 8: regular octahedron.

We can view G_O as a groups of permutations of the 6 vertices, that is as a subgroup of S_6. Observe that

$$\text{orb}_{G_O}(1) = \{1,2,3,4,5,6\} \quad \Rightarrow \quad |\text{orb}_{G_O}(1)| = 6$$

and that

$$\text{stab}_{G_O}(1) = \{\varepsilon, (2,3,4,5), (2,4)(3,5), (2,5,4,3)\} \quad \Rightarrow \quad |\text{stab}_{G_O}(1)| = 4.$$

The elements of the stabilizer are the rotations about an axis through vertices 1 and 6.

Therefore, by the orbit stabilizer theorem:

$$|G_C| = |\text{orb}_{G_O}(1)| \cdot |\text{stab}_{G_O}(1)| = 6 \cdot 4 = 24.$$

It is no coincidence that this is the same size as the group of symmetries of the cube. Figure 9 shows the octahedron sitting inside the cube (join midpoints of every two squares by a line). This means that $G_C \approx G_O$. The cube and the octahedron are referred to as *dual solids*.

22.2.4 Rotation Group of an Dodecahedron

Let G_D be the group of all rotational symmetries of a regular dodecahedron.

We can view G_D as a groups of permutations of the 20 vertices, that is as a subgroup of S_{20}. Observe that

$$\text{orb}_{G_D}(1) = \{1,2,3,\ldots,20\} \quad \Rightarrow \quad |\text{orb}_{G_D}(1)| = 20$$

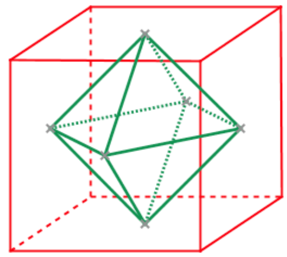
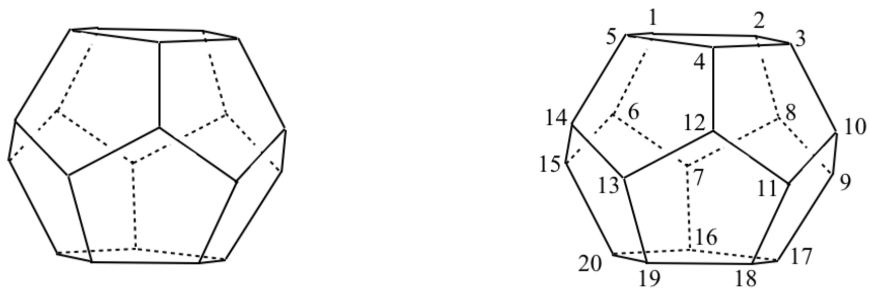


Figure 9: The octahedron is dual to the cube, so $G_O \approx G_C$.



(a) (b)

Figure 10: regular dodecahedron.

and that

$$|\text{stab}_{G_D}(1)| = 3.$$

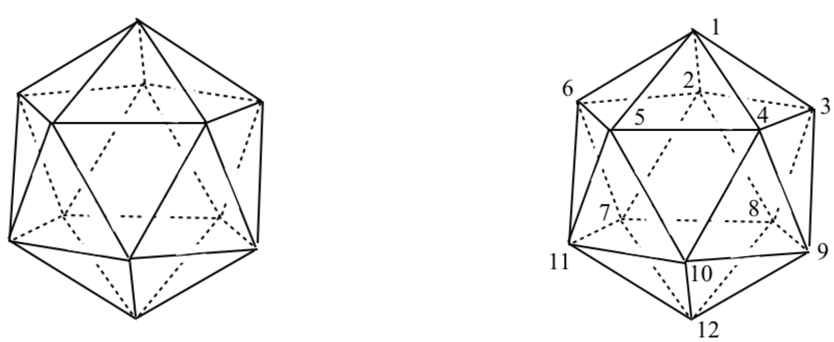
The elements of the stabilizer are the rotations about an axis through vertices 1 and 18.

Therefore, by the orbit stabilizer theorem:

$$|G_C| = |\text{orb}_{G_D}(1)| \cdot |\text{stab}_{G_D}(1)| = 20 \cdot 3 = 60.$$

22.2.5 Rotation Group of an Icosahedron

Let G_I be the group of all rotational symmetries of a regular icosahedron.



(a) (b)

Figure 11: regular icosahedron.

We can view G_I as a groups of permutations of the 12 vertices, that is as a subgroup of S_{20}. Observe that

$$\text{orb}_{G_I}(1) = \{1, 2, 3, \ldots, 12\} \quad \Rightarrow \quad |\text{orb}_{G_I}(1)| = 12$$

and that

$$|\text{stab}_{G_I}(1)| = 5.$$

The elements of the stabilizer are the rotations about an axis through vertices 1 and 12.

Therefore, by the orbit stabilizer theorem:

$$|G_C| = |\text{orb}_{G_I}(1)| \cdot |\text{stab}_{G_I}(1)| = 20 \cdot 3 = 60.$$

It is no coincidence that this is the same size as the group of symmetries of a regular dodecahedron. Figure 12 shows the octahedron sitting inside the cube (join midpoints of every two squares by a line). This means that $G_I \approx G_D$.

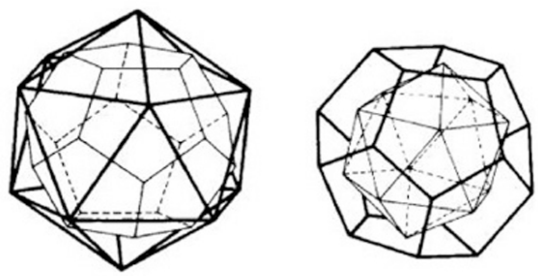


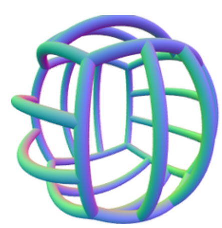
Figure 12: The icosahedron is dual to the dodecahedron, so $G_I \approx G_D$.

22.2.6 Rotation Group of an Soccer Ball, Basket Ball, Volley Ball, and Tennis Ball

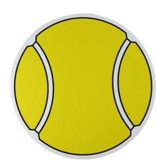
The balls used in soccer, basketball, volleyball, and tennis have district patterns on their surface. We can use the orbit-stabilizer theorem to determine the rotational groups of symmetries of these patterns.

(a) soccer ball

(b) basket ball



(c) volley ball



(d) tennis ball

Figure 13: Familiar sports balls.

For each ball, pick an object on the ball: either a point, or shape. Determine the size of the orbit and stabilizer of the point/shape and verify the results in the Table 1.

It will help if you have a physical ball in your hands. For the soccer ball, there are 12 pentagons (the black faces), and 20 hexagons. See Figure 14 for an unfolded view of the soccer ball.

In case you are interested, the rotational group of the soccer ball is A_5.

In nature, the helix is the structure that occurs most often. The second most commonly found structures are polyhedrons made from pentagons and hexagons, such as the dodecahedron and the truncated icosahedron

ball	**size of group of rotations**
soccer ball	60
basket ball	4
volley ball	12
tennis ball	4

Table 1: The size of the rotational group for various playing balls.

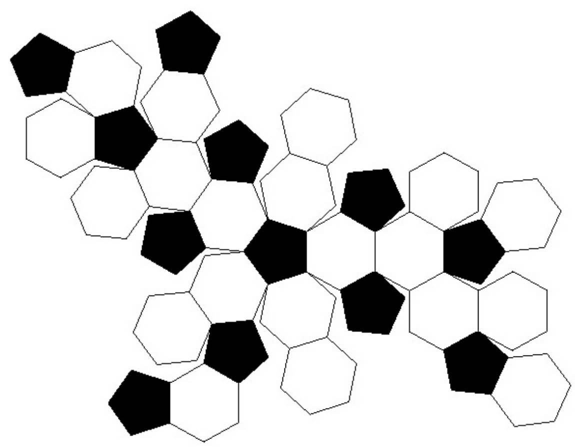
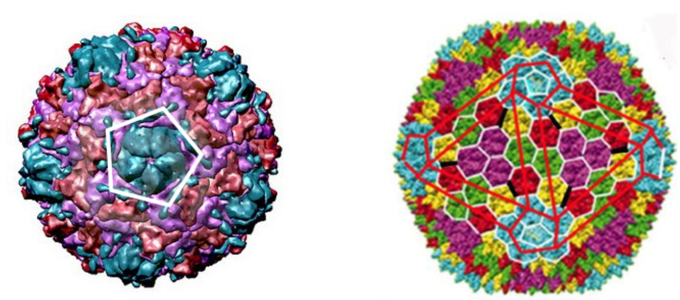


Figure 14: A soccer ball unfolded.

(soccer ball). Although it is impossible to enclose a space with hexagons along, adding 12 pentagons will be sufficient to enclose the space (like the soccer ball). Many viruses have this kind of structure (Figure 15). [1]



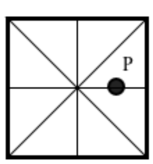
(a) rhinovirus (common cold) (b) Archaeal virus

Figure 15: Viruses.

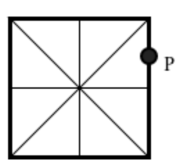
[1]John Galloway, *Nature's Second-Favourite Structure.* New Scientist 114 (March 1988); 36-39

22.3 Exercises

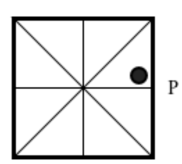
1. Prove the relation defined in (1) is an equivalence relation.
2. Let RC_3 be the Rubik's cube group and let H be the subgroup generated by the product $\alpha = UR$.
$$H = \langle UR \rangle.$$
Let X be the set of all cubies of Rubik's cube.
   (a) If x denotes the ufr corner cubie, determine $\text{orb}_H(x)$.
   (b) If y denotes the uf edge cubie, determine $\text{orb}_H(y)$.
   (c) How many elements do $\text{stab}_H(x)$ and $\text{stab}_H(y)$ have?
3. Instead of considering the set of vertices of the tetrahedron, consider how G_T permutes the 6 edges of the tetrahedron. By picking one edge, say the edge 12, the edge between vertices 1 and 2, verify that $|\text{orb}_{G_T}(12)| \cdot |\text{stab}_{G_T}(12)| = 12$.
4. Consider how G_T permutes the 3 triangular faces of the tetrahedron. That is, consider G_T as a subgroup of S_3. By picking one face, say the face $f_{1,2,3}$ containing vertices 1, 2 and 3, verify that $|\text{orb}_{G_T}(f_{1,2,3})| \cdot |\text{stab}_{G_T}(f_{1,2,3})| = 12$.
5. Instead of considering the set of vertices of the dodecadedron, consider how G_D permutes the 30 edges of the dodecahedron. That is, consider G_D as a subgroup of S_{30}. By picking one edge, say the edge 12, the edge between vertices 1 and 2, verify that $|\text{orb}_{G_D}(12)| \cdot |\text{stab}_{G_D}(12)| = 60$.
6. Consider how G_D permutes the 12 pentagonal faces of the dodecahedron. That is, consider G_D as a subgroup of S_{12}. By picking one face, say the face f containing vertices $1, 2, 3, 4, 5$, verify that $|\text{orb}_{G_D}(f)| \cdot |\text{stab}_{G_D}(f)| = 60$.
7. For each of the following objects, describe each element of the group of rotations as a single rotation. (Similar to what was done for the tetrahedron in Figure 5.)
   (a) cube
   (b) octahedron
8. Let G be the group of rotations of a rectangular box of dimensions $1 \times 2 \times 3$. Describe each element of G as a rotation.
9. Let G be the group of rotations of a rectangular box of dimensions $1 \times 2 \times 2$. Describe each element of G as a rotation.
10. The group D_4 acts as a group of permutations of the points enclosed by the square shown below. (The axis of symmetry are drawn for reference purposes.) For each square, locate the points in the orbit of the indicated point P under the action of D_4. In each case, determine the stabilizer of P.



(a)



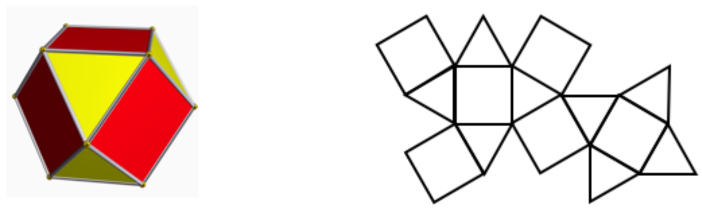
(b)



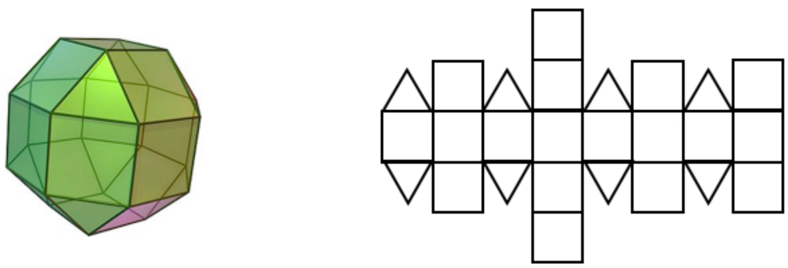
(c)

11. A soccer ball has 20 faces that are regular hexagons and 12 faces that are regular pentagons (see Figures 13a and 14). Use the orbit stabilizer theorem to explain why a soccer ball cannot have $60°$ rotational symmetry about a line through the centres of two opposite hexagonal faces.

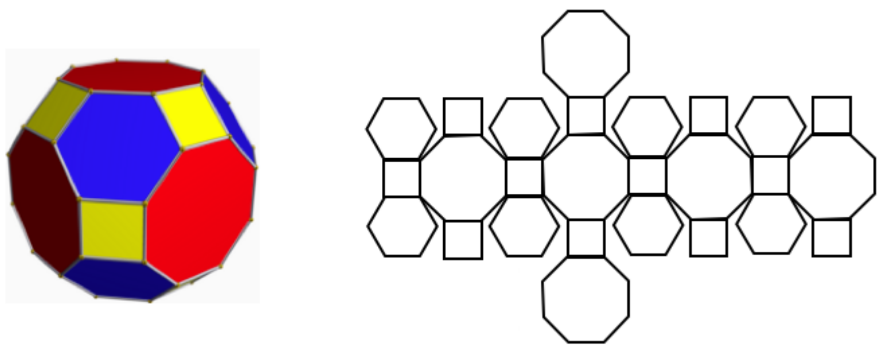
12. For each of the solids below, determine the number of rotational symmetries. (In the figures each solid is also shown as "unfolded".)



(a) cuboctahedron



(b) (small) rhombicuboctahedron



(c) great rhombicuboctahedron or truncated cuboctahedron