

## Python for Education: Permutations

**Andrzej Kapanowski**

*Marian Smoluchowski Institute of Physics*

*Jagiellonian University, Cracow, Poland*

[andrzej.kapanowski@uj.edu.pl](mailto:andrzej.kapanowski@uj.edu.pl)

### Abstract

Python implementation of permutations is presented. Two classes are introduced: *Perm* for permutations and *Group* for permutation groups. The class *Perm* is based on Python dictionaries and utilize cycle notation. The methods of calculation for the perm order, parity, ranking and unranking are given. A random permutation generation is also shown. The class *Group* is very simple and it is also based on dictionaries. It is mainly the presentation of the permutation groups interface with methods for the group order, subgroups (normalizer, centralizer, center, stabilizer), orbits, and several tests. The corresponding Python code is contained in the modules *perms* and *groups*.

**Keywords:** *permutations, permutation groups, computational group theory.*

### 1. Introduction

Python is a programming language that is used by many companies, universities, and single programmers (Lutz, 2007). Some of its key features are: very clear, readable syntax; high level dynamic data types; exception-based error handling; extensive standard libraries and third party modules; availability for all major operating systems. Python is sometimes called *executable pseudocode*, because it can be used as a prototyping or RAD (rapid application development) language. On the other hand, it was shown that Python can be used as the first language in the computer science curriculum (Zelle, 2004; Downey et al., 2008).

Python can be also used to implement classic algorithms and design new problem-solving algorithms (Hetland, 2010). Although Python is not as fast as C or Java, in many cases it may be fast enough to do the job. It is important how our program *scales* with the input size, what algorithms are used. A solid understanding of algorithm design is a crucial problem and Python stimulates experiments and tests. Python tools as *doctest* and *unittest* can reduce the effort involved in code testing (Arbuckle, 2010; Turnquist, 2011).

In this paper we are interested in computational group theory (CGT) and permutation (perm) groups algorithms (Seress, 1997). Perm groups are the oldest type of representations of groups and perm groups algorithms are among the best developed parts of CGT. The methods developed by Charles C. Sims are at the heart of most of the algorithms (Seress, 2003). Many algorithms have been implemented in GAP, a system for computational

discrete algebra (GAP, 2013). GAP code is written in a high-level, Pascal-like language, and is freely available.

There is also a Python library for symbolic mathematics called SymPy (2013). SymPy has the Combinatorics Module with perms and perm groups. SymPy code contains many advanced Python features and may be difficult to read for a novice programmer. The Combinatorics Module imports some objects from other modules so it is not self-contained. We would like to present a similar code but self-contained, readable, and avoiding the most advanced Python features. A Python code is a considerable part of the paper because it plays a role of a pseudocode and it also presents Python best practices. The code described in the paper is available as the perms-dict project on Google Code (2013).

The paper is organized as follows. In Section 2 basic notions of groups are defined. In Sections 3 and 4 the implementation of perms is presented (*perms* module) and some usage examples are given. Perms are based on Python dictionaries and utilize cycle notation. In Sections 5 and 6 the implementation of perm groups is presented (*groups* module). This implementation is very simple and suitable only for groups of sufficiently small order. It is included to familiarize the reader with the perm groups interface. It is important that almost the same interface can be used for the advanced implementation of perm groups that will be published elsewhere. Conclusions are contained in Section 7.

## 2. Basic notions of groups

A group  $G$  is a set together with an operation  $*$  that combines any two elements from  $G$  to form another element from  $G$  (Wikipedia, 2013b). The operation  $*$  must satisfy four requirements:

- *Closure*. For all  $a, b$  in  $G$ ,  $a * b$  is in  $G$ .
- *Associativity*. For all  $a, b, c$  in  $G$ ,  $(a * b) * c = a * (b * c)$ .
- *Identity element*. There exists  $e$  in  $G$ , such that for all  $a$  in  $G$ ,  $e * a = a * e = a$ .
- *Inverse element*. For each  $a$  in  $G$ , there exists  $\sim a$  in  $G$ , such that  $a * (\sim a) = (\sim a) * a = e$ .

A group  $G$  is called *abelian* if  $a * b = b * a$  for all  $a, b$  in  $G$ . A group  $G$  is *finite* if the set  $G$  has a finite number of elements (the group order  $|G|$  is finite). In this paper, all groups are finite. A subset  $H$  of  $G$  is a *subgroup* of  $G$  if  $H$  is a group together with the operation  $*$  from  $G$ .  $H$  is a *normal subgroup* in  $G$  ( $H \leq G$ ) if  $a * b * (\sim a)$  is in  $H$ , for all  $a$  in  $G$ , for all  $b$  in  $H$ .

If  $S$  is a subset of  $G$ , then we denote by  $\langle S \rangle$  the subgroup generated by  $S$ . The *commutator* of  $a, b$  in  $G$  is  $[a, b] = a * b * (\sim a) * (\sim b)$ . For subgroups  $H, K$  of  $G$ , the commutator of  $H$  and  $K$  is defined as  $[H, K] = \langle [a, b] \mid a \text{ in } H, b \text{ in } K \rangle$ . The commutator  $[G, G]$  is called the *derived subgroup* of  $G$  and it is always a normal subgroup of  $G$ . A group  $G$  is *perfect* if  $[G, G] = G$ . On the other hand, if  $[G, G]$  is trivial, then  $G$  is abelian.

### 3. Interface for permutations

A permutation (perm) is a one-to-one mapping of a set onto itself. If  $p$  and  $q$  are perms such that  $p[i] == j$  and  $q[j] == k$ , the product  $(q * p)[i] == k$ . Note that many authors have the opposite convention (Knuth, 1991). The set of all permutations of any given set  $X$  of  $n$  elements forms the symmetric group  $\text{Sym}(X)$  or  $S_n$ . The order of  $S_n$  is  $n!$ . Any subgroup of a symmetric group  $S_n$  is called a perm group of degree  $n$ .

Perms are often shown as an array with two rows.

```
[ 0  1  2  ...  n-1 ]
[p[0] p[1] p[2] ... p[n-1]]
```

Sometimes, only the second line is used to present a perm (array form). Note that we have  $X = \{0, 1, 2, \dots, n-1\}$ .

The third method of notation is cycle notation. A *cycle* ( $k$ -cycle)  $c$  with the length  $k$  can be written as a Python tuple  $(c[0], c[1], \dots, c[k-1])$  but, in fact, a Python list can be also used in programming perms. It corresponds to the permutation  $q$ , where

$q[c[i]] == c[i + 1]$  for  $0 \leq i < k-1$ ,  $q[c[k - 1]] == c[0]$ .

If  $j$  is not in the cycle then  $q[j] == j$ . A 2-cycle is called a *transposition*. Any permutation can be expressed as a product of disjoint cycles (1-cycles are often omitted). Any cycle can be expressed as a product of transpositions  $(c[0], c[k - 1])(c[0], c[k-2]) \dots (c[0], c[1])$ .

Method name	Short description
<code>Perm()</code>	returns the identity perm
<code>Perm&gt;(*c)(*d)</code>	returns a perm from cycles
<code>Perm(data=[0, 2, 1, 3])</code>	returns a perm from a list
<code>p.is_identity()</code>	returns True for the identity perm
<code>~p</code>	returns the inverse of $p$
<code>p * q</code>	returns the product $p*q$ as a perm
<code>p==q</code>	returns True for the same perms
<code>p[k]</code>	returns the item on the position $k$ in $p$
<code>pow(p, m), p ** m</code>	returns the $m$ -th power of $p$
<code>p.support()</code>	returns a list of integers moved by $p$
<code>p.max()</code>	returns $\max(p.support())$
<code>p.min()</code>	returns $\min(p.support())$
<code>p.list(size)</code>	returns $p$ in array form
<code>p.label(size)</code>	returns the perm string label
<code>p.cycles()</code>	returns a list of perm cycles
<code>p.order()</code>	returns the perm order

p.parity()	returns the parity of p (0 or 1)
p.is_even()	returns True if p is even
p.is_odd()	returns True if p is odd
p.sign()	returns the perm sign (+1 or -1)
p.commutates_with(q)	returns True if $p*q==q*p$
p.commutator(q)	returns the commutator [p,q]
Perm.random(size)	returns a random perm
p.inversion_vector(size)	returns the inversion vector of p
p.rank_lex(size)	returns the lexicographic rank of p
Perm.unrank_lex(size, rank)	returns a perm (lexicographic unranking)
p.rank_mr(size)	returns the Myrvold and Ruskey rank of p
Perm.unrank_mr(size, rank)	returns a perm (Myrvold and Ruskey unranking)

Table 3.1. Interface for perms; p and q are perms, c and d are cycles given as Python tuples or lists.

Let us show some properties of perms that are listed in Table 3.1. Perms are almost always entered and displayed in disjoint cycle notation. The perm size n is undefined because keys not defined explicitly are equal to their values ( $p[i] == i$ ).

```
>>> from perms import Perm
>>> p, q, r = Perm() (0, 1), Perm() (1, 2), Perm() (2, 3)
>>> p.is_odd()
True
>>> p * q
Perm() (0, 1, 2)
>>> q * p
Perm() (0, 2, 1)
>>> p.commutates_with(q)
False
>>> p.commutator(q)
Perm() (0, 2, 1)
>>> (p * p).is_identity()
True
>>> (p * r).cycles()
[[0, 1], [2, 3]]
>>> q.list(size=4)
[0, 2, 1, 3]
>>> p[0], p[4], p[8]
(1, 4, 8)
>>> pow(p * q * r, 1234567890)
Perm() (0, 2) (1, 3)
>>> Perm.random(size=5)
Perm() (0, 2) (1, 4, 3)
>>> q.rank_lex(size=4)
2
>>> q.rank_lex(size=5)
6
>>> Perm.unrank_lex(size=4, rank=20)
Perm() (0, 3, 2)
>>> Perm.unrank_lex(size=5, rank=20)
Perm() (1, 4, 3)
>>> q.rank_mr(size=5)
```

```

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>>> Perm.unrank_mr(5, 20)
Perm()(0, 2, 1, 3, 4)
>>>

```

## 4. Class for permutations

Now we would like to present Python implementation of perms. The code was tested under Python 2.6.

### 4.1. Basic methods

A perm is internally a dictionary where missing keys ( $p[k] == k$ ) are created when they are required. Initially, only the keys with  $p[k] \neq k$  have to be created. The code of the Perm class is fairly self-explanatory. It is inspired by the *Cycle* class from SymPy (SymPy, 2013) but has enhanced functionality. Note that the binary exponentiation algorithm is used for finding powers of perms. All integer powers are allowed. String labels will be used in perm groups.

```

class Perm(dict):
    """The class defining a perm."""

    def __init__(self, data=None):
        """Loads up a Perm instance."""
        if data:
            for key, value in enumerate(data):
                self[key] = value

    def __missing__(self, key):
        """Enters the key into the dict and returns the key."""
        self[key] = key
        return key

    def __call__(self, *args):
        """Returns the product of the perm and the cycle."""
        changed = {}
        n = len(args)
        for i in range(n):
            changed[args[i]] = self[args[(i + 1) % n]]
        self.update(changed)
        return self

    def is_identity(self):
        """Test if the perm is the identity perm."""
        return all(self[key] == key for key in self)

    def __invert__(self):
        """Finds the inverse of the perm."""
        perm = Perm()
        for key in self:
            perm[self[key]] = key
        return perm

    def __mul__(self, other):
        """Returns the product of the perms."""
        perm = Perm()
        # Let us collect all keys.

```

```

    # First keys from other, because self can grow up.
    for key in other:
        perm[key] = self[other[key]]
    for key in self:
        perm[key] = self[other[key]]
    return perm

def __eq__(self, other):
    """Test if the perms are equal."""
    return (self * ~other).is_identity()

def __ne__(self, other):
    """Test if the perms are not equal."""
    return not self == other

def __getitem__(self, key):
    """Finds the item on the given position."""
    return dict.__getitem__(self, key)

def __pow__(self, n):
    """Finds powers of the perm."""
    if n < 0:
        return pow(~self, -n)
    elif n == 0:
        return Perm()
    elif n == 1:
        return self
    elif n == 2:
        return self * self
    else:
        # binary exponentiation
        perm = self
        res = Perm() # identity
        while True:
            if n % 2 == 1:
                res = res * perm
                n = n - 1
            if n == 0:
                break
            if n % 2 == 0:
                perm = perm * perm
                n = n / 2
        return res

def support(self):
    """Returns the elements moved by the perm."""
    return [key for key in self if self[key] != key]

def max(self):
    """Return the highest element moved by the perm."""
    if self.is_identity():
        return 0
    else:
        return max(key for key in self if self[key] != key)

def min(self):
    """Return the lowest element moved by the perm."""
    if self.is_identity():
        return 0
    else:
        return min(key for key in self if self[key] != key)

def list(self, size=None):
    """Returns the perm in array form."""
    if size is None:
        size = self.max() + 1

```

```

    elif size < self.max() + 1:
        raise ValueError("size is too small")
    return [self[key] for key in range(size)]

def label(self, size=None):
    """Returns the string label for the perm."""
    if size is None:
        size = self.max() + 1
    if size > 62:
        raise ValueError("size is too large for labels")
    letters = "0123456789ABCDEFGHIJKLMNOPQRSTUVWXYZ"
    letters = letters + "abcdefghijklmnopqrstuvwxyz_"
    chars = []
    for key in range(size):
        chars.append(letters[self[key]])
    return "".join(chars)

```

Most basic operations require  $O(n)$  time for perms from  $S_n$ . The binary exponentiation takes  $O(n \log(m))$  time for the power  $m$ .

## 4.2. Cycles

The method `cycles()` returns a list of cycles without 1-cycles. It is used to get the string representation of a perm and to compute the order of a perm via the functions `lcm()` and `gcd()` (SymPy, 2013). When a perm is raised to the power of its order it equals the identity perm, `pow(p, p.order()) == Perm()`. Note that the code of the method `order()` is exceptionally compact and transparent.

```

def gcd(a, b):
    """Computes the greatest common divisor."""
    while b:
        a, b = b, a % b
    return a

def lcm(a, b):
    """Computes the least common multiple."""
    return a * b / gcd(a, b)

class Perm(dict):
    # ... other methods ...

    def cycles(self):
        """Returns a list of cycles for the perm."""
        size = self.max() + 1
        unchecked = [True] * size
        cyclic_form = []
        for i in range(size):
            if unchecked[i]:
                cycle = [i]
                unchecked[i] = False
                j = i
                while unchecked[self[j]]:
                    j = self[j]
                    cycle.append(j)
                    unchecked[j] = False
                if len(cycle) > 1:
                    cyclic_form.append(cycle)
        return cyclic_form

    def __repr__(self):

```

```

        """Computes the string representation of the perm."""
        words = ["Perm()"]
        for cycle in self.cycles():
            words.append(str(tuple(cycle)))
        return "".join(words)

    def order(self):
        """Returns the order of the perm."""
        numbers = [len(cycle) for cycle in self.cycles()]
        return reduce(lcm, numbers, 1)

```

### 4.3. Parity

Every permutation can be expressed as a product of transpositions. There are many possible expressions for a given perm but the parity of the transposition number is preserved. All permutations are then classified as even or odd, according to the transposition number. The set of all even permutations from the symmetric group  $\text{Sym}(X)$  forms the alternating group  $\text{Alt}(X)$  or  $A_n$ . The order of  $A_n$  is  $n!/2$ .

```

class Perm(dict):
    # ... other methods ...

    def parity(self):
        """Returns the parity of the perm (0 or 1)."""
        size = self.max() + 1
        unchecked = [True] * size
        # c counts the number of cycles in the perm including 1-cycles
        c = 0
        for j in range(size):
            if unchecked[j]:
                c = c + 1
                unchecked[j] = False
                i = j
                while self[i] != j:
                    i = self[i]
                    unchecked[i] = False
        return (size - c) % 2

    def is_even(self):
        """Test if the perm is even."""
        return self.parity() == 0

    def is_odd(self):
        """Test if the perm is odd."""
        return self.parity() == 1

    def sign(self):
        """Returns the sign of the perm (+1 or -1)."""
        return (1 if self.parity() == 0 else -1)

```

### 4.4. Commutators and random perms

Here we define the commutator of two perms  $p, q$  as  $p*q*(\sim p)*(\sim q)$ . A random perm generator uses the Python *random* module.

```

class Perm(dict):
    # ... other methods ...

```



```

def commutes_with(self, other):
    """Test if the perms commute."""
    return self * other == other * self

def commutator(self, other):
    """Finds the commutator of the perms."""
    return self * other * ~self * ~other

@classmethod
def random(cls, size):
    """Returns a random perm of the given size."""
    import random
    alist = range(size)
    random.shuffle(alist)
    return cls(data=alist)

```

#### 4.5. Ranking and unranking permutations

A ranking function for perms on  $n$  elements assigns a unique integer in the range from 0 to  $n!-1$  to each of the  $n!$  Perms. The corresponding unranking function is the inverse (Myrvold and Ruskey, 2001). The algorithm for ranking perms in lexicographic order uses the inversion vector and it takes  $O(n^2)$  time. The inversion vector consists of elements whose value indicates the number of elements in the perm that are lesser than it and lie on its right hand side (SymPy, 2013). The inversion vector is the same as the Lehmer encoding of a perm.

In 2001, Myrvold and Ruskey presented simple ranking and unranking algorithms for perms that can be computed using  $O(n)$  arithmetic operations (Myrvold and Ruskey, 2001). It is inspired by the standard algorithm for generating a random perm. Myrvold and Ruskey algorithms are shown in functions `rank_mr()` and `unrank_mr()`.

```

def swap(L, i, j):
    """Exchange of two elements on the list."""
    L[i], L[j] = L[j], L[i]

class Perm(dict):
    # ... other methods ...

    def inversion_vector(self, size):
        """Returns the inversion vector of the perm."""
        lehmer = [0] * size
        for i in range(size):
            counter = 0
            for j in range(i + 1, size):
                if self[i] > self[j]:
                    counter = counter + 1
            lehmer[i] = counter
        return lehmer

    def rank_lex(self, size):
        """Returns the lexicographic rank of the perm."""
        lehmer = self.inversion_vector(size)
        lehmer.reverse()
        k = size - 1
        res = lehmer[k]
        while k > 0: # a modified Horner algorithm
            k = k - 1
            res = res * (k + 1) + lehmer[k]

```

```

        return res

    @classmethod
    def unrank_lex(cls, size, rank):
        """Lexicographic perm unranking."""
        alist = [0] * size
        i = 1
        while i < size:
            i = i + 1
            alist[i - 1] = rank % i
            rank = rank / i
        if rank > 0:
            raise ValueError("size is too small")
        alist.reverse() # this is the inversion vector
        E = range(size)
        blist = []
        for item in alist:
            blist.append(E.pop(item))
        return cls(data=blist)

    def rank_mr(self, size):
        """Myrvold and Ruskey rank of the perm."""
        alist = self.list(size)
        blist = (~self).list(size) # inverse
        return Perm._mr_helper(size, alist, blist)

    @classmethod
    def _mr_helper(cls, size, alist, blist):
        """A helper function for Myrvold and Ruskey ranking."""
        # both alist and blist are modified
        if size == 1:
            return 0
        s = alist[size - 1]
        swap(alist, size - 1, blist[size - 1])
        swap(blist, s, size - 1)
        return s + size * cls._mr_helper(size - 1, alist, blist)

    @classmethod
    def unrank_mr(cls, size, rank):
        """Myrvold and Ruskey perm unranking."""
        tmp = range(size)
        while size > 0:
            swap(tmp, size-1, rank % size)
            rank = rank / size
            size = size - 1
        return cls(data=tmp)

```

## 5. Interface for permutation groups

A perm group is a finite group  $G$  whose elements are perms of a given finite set  $X$  (usually numbers from 0 to  $n-1$ ) and whose group operation is the composition of perms (Joyner and Kohel, 2007). The number of elements of  $X$  is called the degree of  $G$ .

Method name	Short description
Group()	returns a trivial group
G.order()	returns the group order
G.is_trivial()	returns True if G is trivial

<code>p in G</code>	returns True if p belongs to G
<code>G.insert(p)</code>	generates new perms in G from p
<code>G.iterperms()</code>	generates perms from G on demand
<code>G.iterlabels()</code>	generates perm labels on demand
<code>G.is_abelian()</code>	returns True if G is abelian
<code>G.subgroup_search(prop)</code>	returns a subgroup
<code>H.is_subgroup(G)</code>	returns True if H is a subgroup of G
<code>H.is_normal(G)</code>	returns True if H is a normal subgroup of G
<code>G.normalizer(H)</code>	returns the normalizer of H in G
<code>G.centralizer(H)</code>	returns the centralizer of H in G
<code>G.center()</code>	returns the center of G
<code>G.commutator(H, K)</code>	returns the commutator of groups
<code>G.derived_subgroup()</code>	returns the derived subgroup of G
<code>G.orbits(points)</code>	returns a list of orbits
<code>G.is_transitive(points)</code>	returns True if G is transitive
<code>G.stabilizer(point)</code>	returns a stabilizer subgroup
<code>G.action(points)</code>	returns an induced group

Table 5.1. Interface for perm groups; G, H, and K are groups, p and q are perms.

Let us show some computations with perm groups using methods listed in Table 5.1. We will find the relation  $1 \leq V_4 \leq A_4 \leq S_4$ , where the unity denotes the trivial group and  $V_4$  is a Klein four-group.

```
>>> from perms import Perm
>>> from groups import Group
>>> s4 = Group()
>>> s4.insert(Perm()(0, 1))
>>> s4.insert(Perm()(0, 1, 2, 3))
>>> s4.order()          # the symmetric group S_4
24
>>> a4 = s4.derived_subgroup()
>>> a4.order()          # the alternating group A_4
12
>>> all(perm.is_even() for perm in a4.iterperms())
True
>>> a4.is_normal(s4)
True
>>> v4 = a4.derived_subgroup()
>>> v4.order()          # the Klein four-group V_4
4
>>> v4.is_abelian()
True
>>> v4.is_normal(a4)
True
>>>
```

## 6. Class for permutation groups

The class *Group* is based on Python dictionaries. All elements of a group are kept, keys are string labels of perms, values are instances of the *Perm* class. It is clear that it is possible to handle only small groups because of the limited computer memory.

### 6.1. Basic methods

In this subsection basic group methods are presented.

```
class Group(dict):
    """The class defining a perm group."""

    def __init__(self):
        """Loads up a Group instance."""
        perm = Perm()
        self[perm.label()] = perm

    order = dict.__len__

    def __contains__(self, perm):
        """ Test if the perm belongs to the group."""
        return dict.__contains__(self, perm.label())

    def iterperms(self):
        """The generator for perms from the group."""
        return self.itervalues()

    def iterlabels(self):
        """The generator for perm labels from the group."""
        return self.iterkeys()

    def is_trivial(self):
        """Test if the group is trivial."""
        return self.order() == 1

    def insert(self, perm):
        """The perm inserted into the group generates new
        perms in order to satisfy the group properties."""
        label1 = perm.label()
        if perm in self:
            return
        old_order = self.order()
        self[label1] = perm
        perms_added = {}
        perms_added[label1] = perm
        perms_generated = {}
        new_order = self.order()
        while new_order > old_order:
            old_order = new_order
            for label1 in perms_added:
                for label2 in self.iterlabels():
                    perm3 = perms_added[label1] * self[label2]
                    label3 = perm3.label()
                    if perm3 not in self:
                        perms_generated[label3] = perm3
            self.update(perms_generated)
            perms_added = perms_generated
```

```

        perms_generated = {}
        new_order = self.order()

    def is_abelian(self):
        """Test if the group is abelian."""
        for perm1 in self.iterperms():
            for perm2 in self.iterperms():
                if not perm1.commutes_with(perm2):
                    return False
        return True

```

## 6.2. Subgroups

A group  $H$  is a *subgroup* of a group  $G$  if all elements of  $H$  belong to  $G$ . The *centralizer* of a subset  $S$  of  $G$  is a set  $C_G(S) = \{g \in G \mid s * g = g * s, s \in S\}$  (Wikipedia, 2013a). It is clear that  $C_G(S) = C_G(\langle S \rangle)$  and that is why the argument of the method `centralizer()` is a group.

The *normalizer* of  $S$  in  $G$  is a set  $N_G(S) = \{g \in G \mid g * s * (\sim g) \in S, s \in S\}$  (Wikipedia, 2013a). We have  $N_G(S) = N_G(\langle S \rangle)$  and that is why the argument of the method `normalizer()` is a group. The centralizer and normalizer of  $S$  are both subgroups of  $G$ . The centralizer  $C_G(S)$  is always a normal subgroup of the normalizer  $N_G(S)$ .

The *center* of  $G$  is a set  $Z(G) = C_G(G)$ . The center of  $G$  is always a normal subgroup of  $G$ . In the case of the abelian group, we get  $Z(G) = G$ . On the other hand, sometimes the center can be trivial.

```

class Group(dict):
    # ... other methods ...

    def is_subgroup(self, other):
        """H.is_subgroup(G) - test if H is a subgroup of G."""
        if other.order() % self.order() != 0:
            return False
        return all(perm in other for perm in self.iterperms())

    def is_normal(self, other):
        """H.is_normal(G) - test if H is a normal subgroup in G."""
        for perm1 in self.iterperms():
            for perm2 in other.iterperms():
                if perm2 * perm1 * ~perm2 not in self:
                    return False
        return True

    def subgroup_search(self, prop):
        """Returns a subgroup of all elements satisfying the property."""
        newgroup = Group()
        for perm in self.iterperms():
            if prop(perm):
                newgroup.insert(perm)
        return newgroup

    def normalizer(self, other):
        """G.normalizer(H) - returns the normalizer of H."""
        return self.subgroup_search(lambda perm: all(
            perm * perm2 * ~perm in other for perm2 in other.iterperms()))

    def centralizer(self, other):
        """G.centralizer(H) - returns the centralizer of H."""

```

```

        return self.subgroup_search(lambda perm: all(
            perm * perm2 == perm2 * perm for perm2 in other.iterperms()))

    def center(self):
        """Returns the center of the group."""
        return self.centralizer(self)

    def commutator(self, group1, group2):
        """Returns the commutator of the groups."""
        newgroup = Group()
        for perm1 in group1.iterperms():
            for perm2 in group2.iterperms():
                newgroup.insert(perm1.commutator(perm2))
        return newgroup

    def derived_subgroup(self):
        """Returns the derived subgroup of the group."""
        return self.commutator(self, self)

```

The `subgroup_search()` method uses a property `prop()` that has to be callable on group elements and it has to return *True* or *False*.

```

# Get A_4 from S_4.
>>> a4 = s4.subgroup_search(lambda perm: perm.is_even())

```

### 6.3. Group action

If  $G$  is a group and  $X$  is a set, then a *group action* of  $G$  on  $X$  is a function  $F: G \times X \rightarrow X$  that satisfies the following two axioms (Wikipedia, 2013c):

- *Identity.*  $F(e, x) = x$  for all  $x$  in  $X$ , where  $e$  denotes the identity element of  $G$ .
- *Associativity.*  $F(g * h, x) = F(g, F(h, x))$  for all  $g, h$  in  $G$  and all  $x$  in  $X$ .

The *orbit* of a point  $x$  in  $X$  is the set  $F(G, x) = \{F(g, x) \mid g \text{ in } G\}$ . There is an equivalence relation defined by saying  $x \sim y$  if and only if there exists  $g$  in  $G$  with  $F(g, x) = y$ . Two elements  $x$  and  $y$  are equivalent if and only if their orbits are the same,  $F(G, x) = F(G, y)$ . The group action is transitive if it has one orbit,  $F(G, x) = X$ .

For every  $x$  in  $X$ , we define the *stabilizer subgroup* of  $x$  as a set

$$\text{Stab}_G(x) = \{g \text{ in } G \mid F(g, x) = x\}.$$

For finite  $G$  and  $X$ , the orbit-stabilizer theorem states that  $|F(G, x)| = |G| / |\text{Stab}_G(x)|$ .

In the case of a perm group  $G$  (perms from  $S_n$ ) we have the standard action  $F(p, k) = p[k]$  for  $p$  in  $G$ ,  $0 \leq k \leq n-1$ . In our Python implementation, an orbit is a list of points, where the points ordering is inessential.

Let us analyze the symmetries of a square (the  $D_4$  group) shown in Figure 6.1. The symmetry group will be constructed from three flips.

0	1	2
3	4	5
6	7	8

Figure 6.1. A symmetry group for a square is  $D_4$ . The elements of the group can be written as perms of integers from 0 to 8.

```
>>> g8 = Group()
>>> g8.insert(Perm()(0, 2)(3, 5)(6, 8))      # horizontal flip
>>> g8.insert(Perm()(0, 6)(1, 7)(2, 8))      # vertical flip
>>> g8.insert(Perm()(1, 3)(2, 6)(5, 7))      # diagonal flip
>>> g8.order()
8
>>> g8.orbits(range(9))
[[0, 6, 8, 2], [1, 7, 3, 5], [4]]
>>> z2 = g8.center()
>>> [perm for perm in z2.iterperms()]
[Perm(), Perm()(0, 8)(1, 7)(2, 6)(3, 5)]
>>> g4 = g8.action([0, 6, 8, 2])
>>> g4.order()                                # the same abstract group D_4
8
>>>
```

The first orbit contains the points at the corners, the second points at the edges, and the third contains the center. The group cannot move a point at a corner onto a point at an edge or at the center. The center of the group consists of a half-turn and the identity.

```
class Group(dict):
    # ... other methods ...

    def orbits(self, points):
        """Returns a list of orbits."""
        used = {}
        orblist = []
        for pt1 in points:
            if pt1 in used:
                continue
            orb = [pt1]      # we start a new orbit
            used[pt1] = True
            for perm in self.iterperms():
                pt2 = perm[pt1]
                if pt2 not in used:
                    orb.append(pt2)
                    used[pt2] = True
            orblist.append(orb)
        return orblist

    def is_transitive(self, points):
        """Test if the group is transitive (has a single orbit)."""
        return len(self.orbits(points)) == 1

    def stabilizer(self, pt):
        """Returns a stabilizer subgroup."""
        return self.subgroup_search(lambda perm: perm[pt] == pt)

    def action(self, points):
        """Return a new group induced by the action."""
        if not self.is_transitive(points):
            raise TypeError("the group is not transitive")
```

```

adict = {}
for i, pt in enumerate(points):
    adict[pt] = i
newgroup = Group()
for perm in self.iterperms():
    newdata = [adict[perm[pt]] for pt in points]
    newgroup.insert(Perm(data=newdata))
return newgroup

```

## 7. Conclusions

In this paper, we presented Python implementation of perms, the *Perm* class (the *perms* module), based on Python dictionaries. It is inspired by the *Cycle* class from SymPy (2013) but has enhanced functionality. The methods of calculation for the perm order, parity, random perms, ranking and unranking perms are given. It is interesting that classic algorithms, such as the Euclidean algorithm and the binary exponentiation, have found the natural application.

The interface for perm groups is also shown by means of the *Group* class (the *groups* module) but the implementation is too simple (and slow) to handle large groups. The Python code (*executable pseudocode*) can serve as an introduction to the group theory and Python programming. We note that almost the same interface can be used for the advanced implementation of perm groups based on Schreier-Sims method (Seress, 2013).

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