3.0 Wave Loading

3.1 Fundamentals

We will first consider a cylindrical structure subject to a uniform current u (i.e. the flow is steady and there is no wave motion).

(i) Inviscid Solution

If the flow is assumed to be inviscid ($\mu = 0$) the equations derived at the beginning of the lecture course apply:

mass continuity $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$ Equation (5a)

irrotationality $$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$ Equation (5b)

If a solution is found in terms of a velocity potential ϕ, such that

$$u = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v = \frac{\partial \phi}{\partial y}$$ Equation (5c)

equation (5b) is satisfied immediately and the continuity expression gives:

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0}$$ Equation (5d)

Laplace's Equation

The boundary conditions require the velocity normal to the cylinder is to be zero on the surface of the cylinder. Hence,

$$u_r = 0 \qquad \text{on} \qquad r = \frac{D}{2}$$

where D is the diameter of the cylinder and u_r the radial velocity component.

Clearly, the solution is best obtained in polar co-ordinates (easy to satisfy the boundary condition). The resulting solution is of the form:

$$\boxed{\phi = u\left(r + \frac{D^2}{4r}\right)\cos\theta \text{ for } r \geq \frac{D}{2}} \qquad \text{Equation (5e)}$$

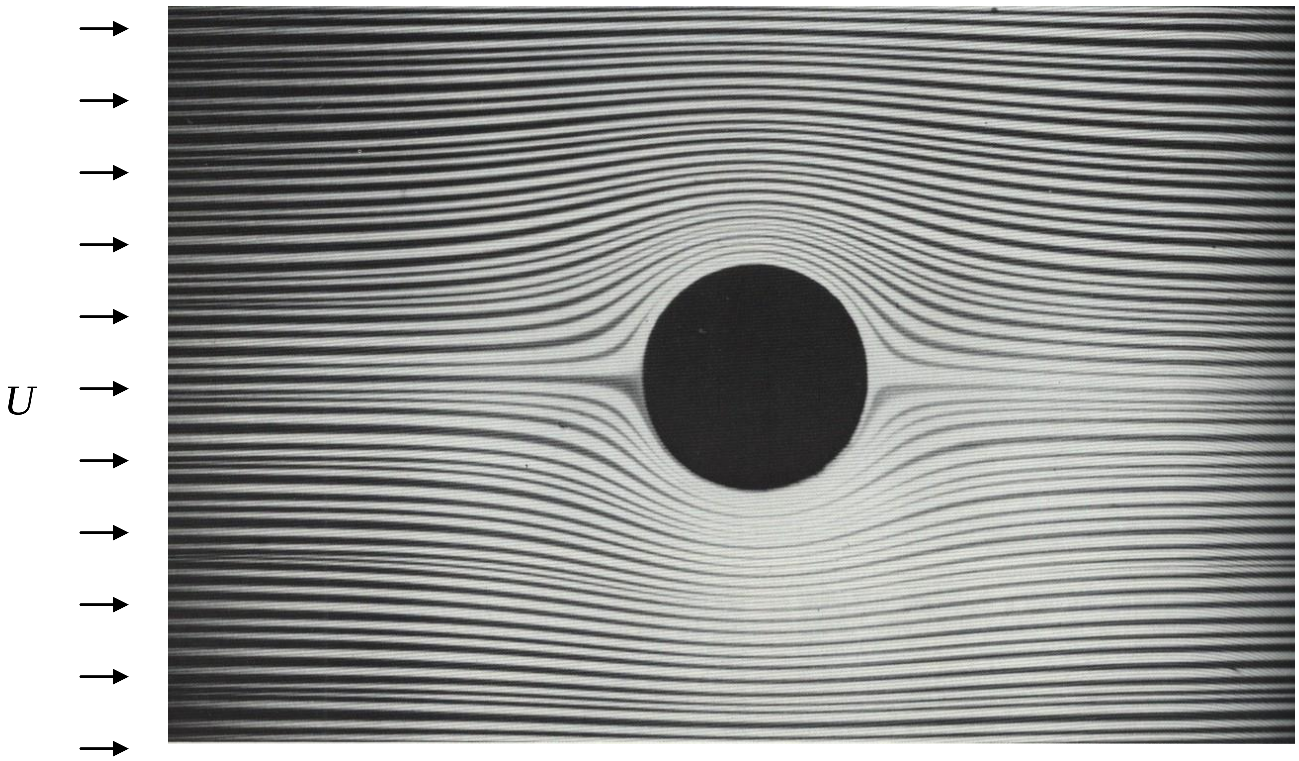
Potential flow around a cylinder

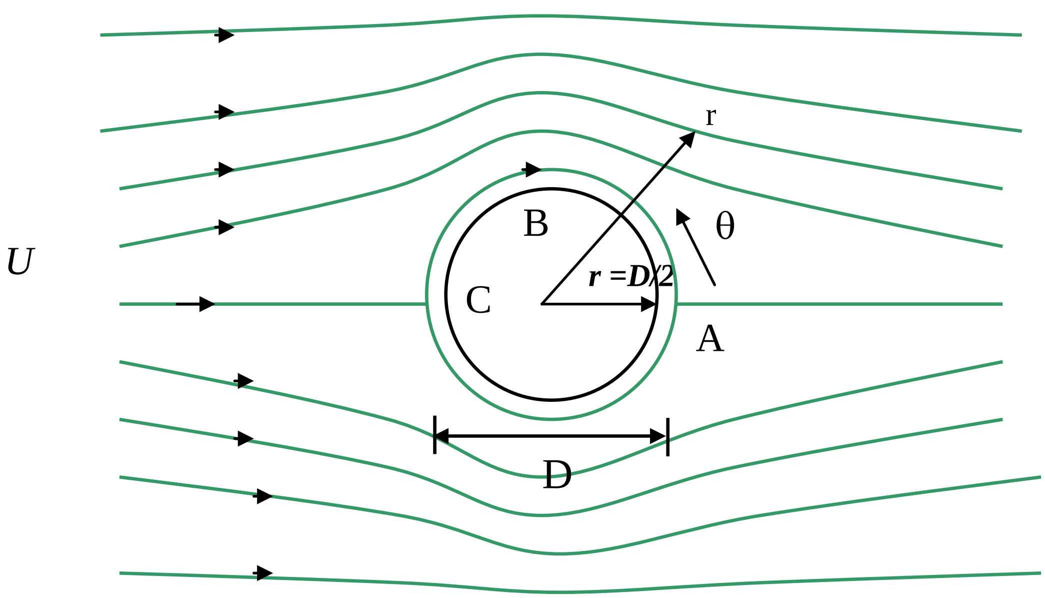
where the two velocity components are given by:

$$\boxed{u_r = \frac{\partial \phi}{\partial r} \text{ and } u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}} \qquad \text{Equation (5f)}$$

Velocity components in Polar Co-ordinates

When plotted out, the so-called potential-flow solution looks like:





Note: Potential-flow solution is inviscid and irrotational.

The velocity on the surface of the cylinder is:

$$u_\theta = -2u\sin\theta$$

Using Bernoulli's equation in its steady form:

$$p + \frac{1}{2}\rho u^2 + \rho g y = \text{constant} \qquad \left\{ \text{i.e. set } \frac{d\phi}{dt} = 0 \right\}$$

gives the pressure distribution as:

$$\boxed{p = \frac{1}{2}\rho u^2\left(1 - 4\sin^2\theta\right)} \qquad \text{Equation (5g)}$$

Inviscid Pressure Distribution

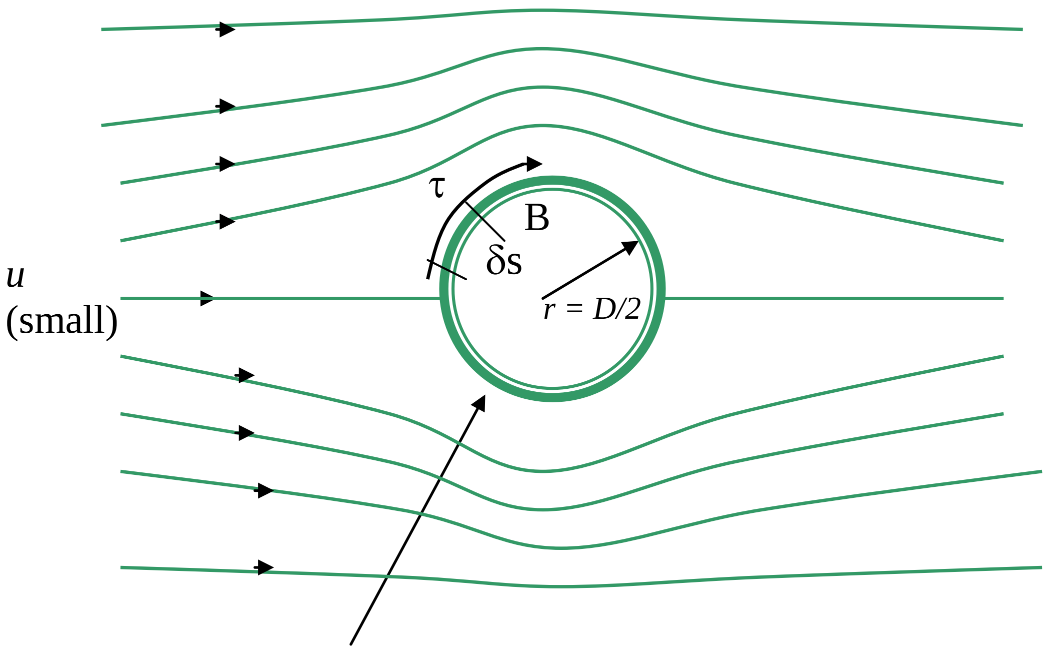
Since the predicted pressure distribution is symmetric, it implies that the total drag force must be zero (i.e. integrate pressure round the surface of a cylinder gives zero net effect).

This is clearly incorrect!!

(ii) ***Incorporating the effects of viscosity***

When we considered wave motion in the near-bed region (p19) we noted that because the bed was impermeable the velocity perpendicular to the bed must be zero ($v = 0$ at $y = -d$). In addition, we decided that because of viscosity (friction) the horizontal velocity component must also reduce to zero at the bed. This change takes place within the so-called boundary layer where friction is important. This produces a ***no-slip*** condition in which fluid in contact with a stationary body must also be stationary.

This same argument applies to the movement of a real fluid around a cylinder. If we consider the case of a very small u,



boundary layer – region within which viscosity produces strong shear

'No slip' boundary condition gives $u_r = u_\theta = 0$ at $r = \dfrac{D}{2}$

Within the boundary layer the shear stress $\propto$ velocity gradient:

$$\tau \propto \frac{\partial u_\theta}{\partial r} \text{ or } \tau = \mu \frac{\partial u_\theta}{\partial r} \qquad \text{Equation (5h)}$$

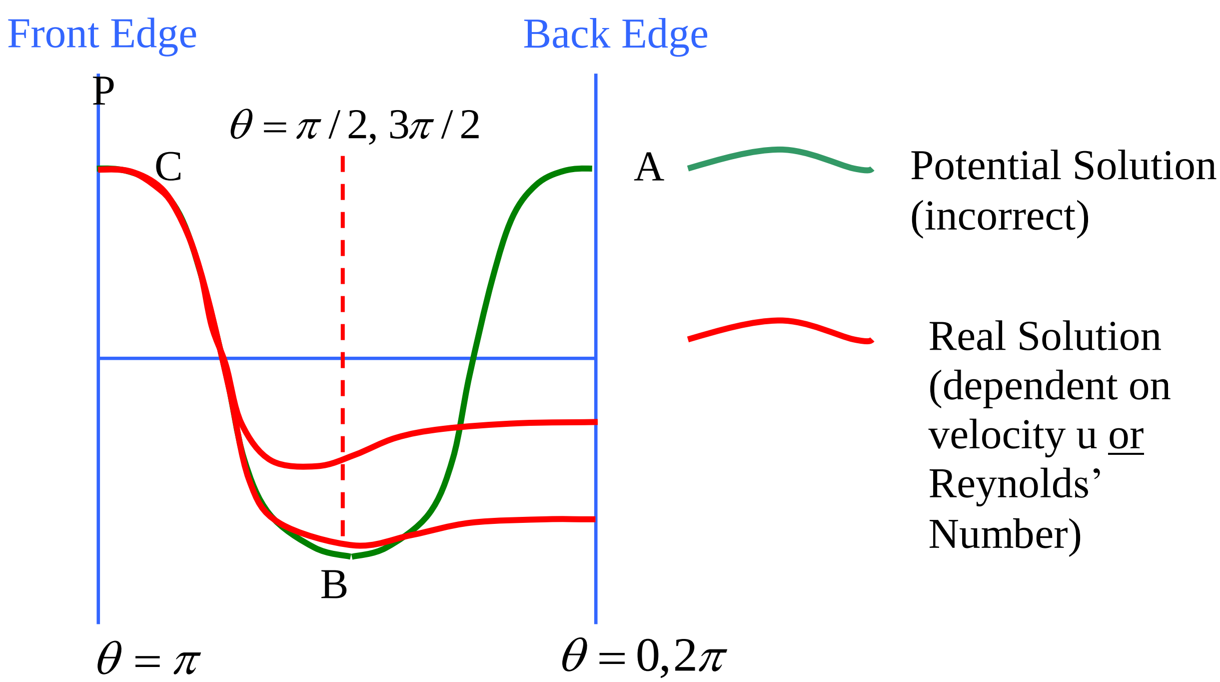
Definition of dynamic viscosity

Note, $\mu = 10^{-3} kgm^{-1}s^{-1}$ for water. Alternatively, one might work in terms of kinematic viscosity, $\nu = \mu/\rho = 10^{-6}\ m^2 s^{-1}$ for water.

Incorporating the effects of viscosity gives a total drag force of:

Drag force = $\int_S \tau_{r=D/2} ds$, where the integral is taken round the surface of the cylinder.

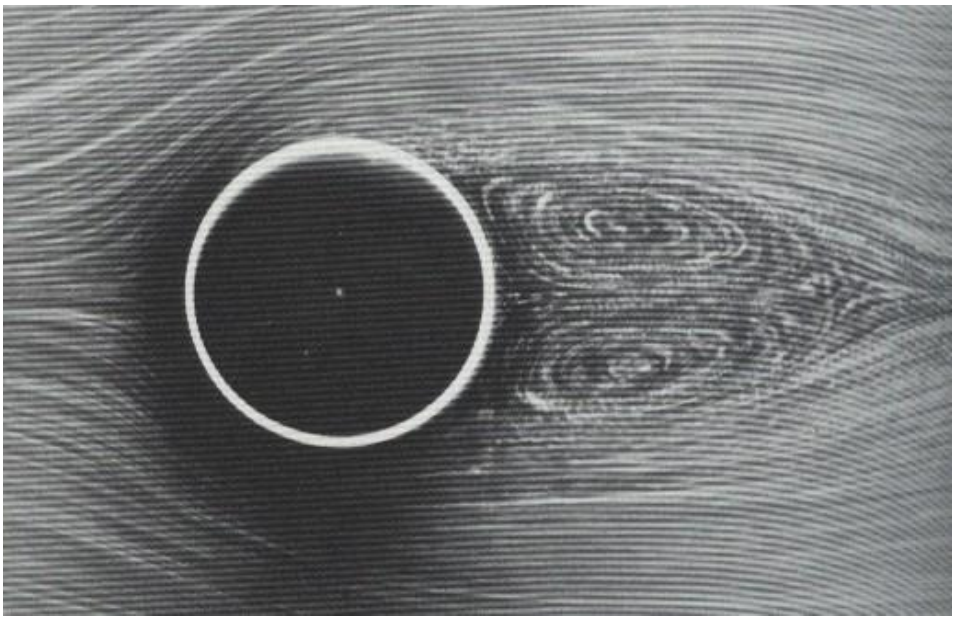
Typical pressure gradient



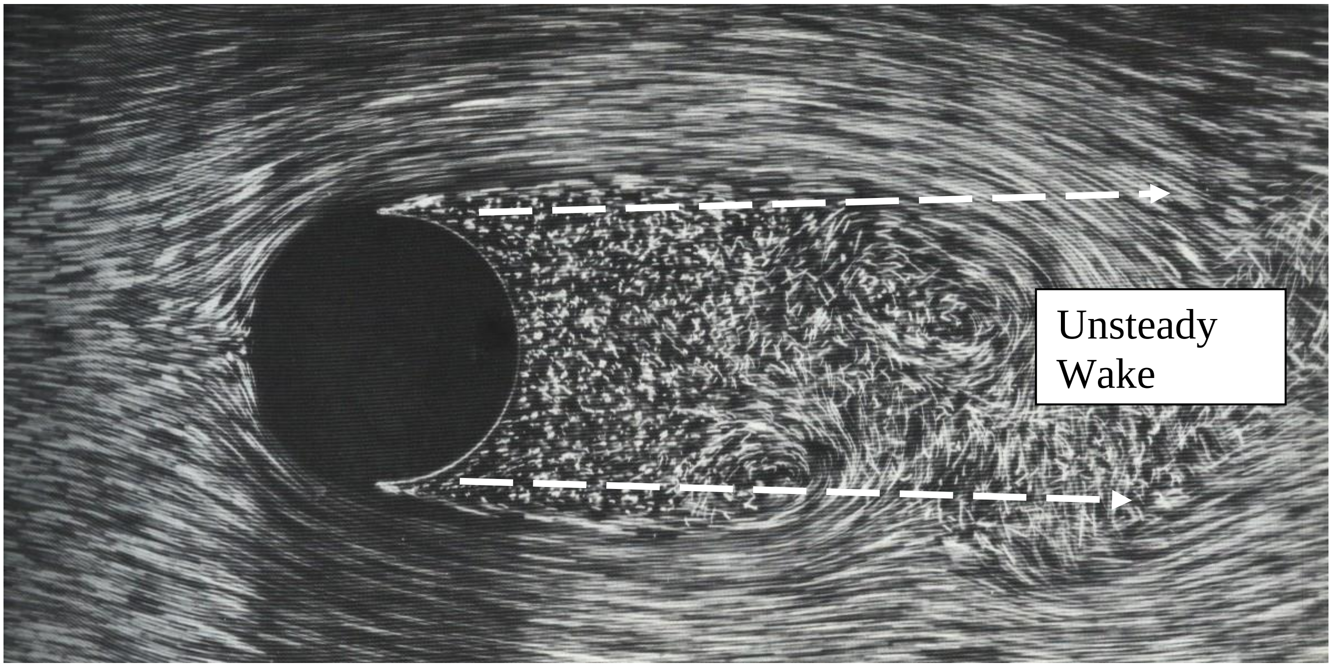
$\theta = \theta, 2\pi$

In real flows the fluid is unable to cope with the adverse pressure gradient between $\theta = 3\pi/2$ and 2π. This causes the boundary layer to separate from the cylinder, resulting in the development of a low pressure region at the back of the cylinder.

Flow separation



As the velocity increases further, the point of separation moves further around the cylinder and an unsteady wake develops:



In this case the large pressure differences between the front and the back of the cylinder produces a significant ***drag force***. It is usual to define this force in terms of a drag coefficient C_d such that:

$$C_d = \frac{F_d}{\left(1/2\rho u^2\right)A} \qquad \text{Equation (5i)}$$

Definition of Drag Coefficent

where F_d is the drag force, *u* the incident velocity and *A* the projected area. In the case of a vertical circular cylinder *A=DL*, where *L* is the length of the cylinder and *D* the diameter of the cylinder.

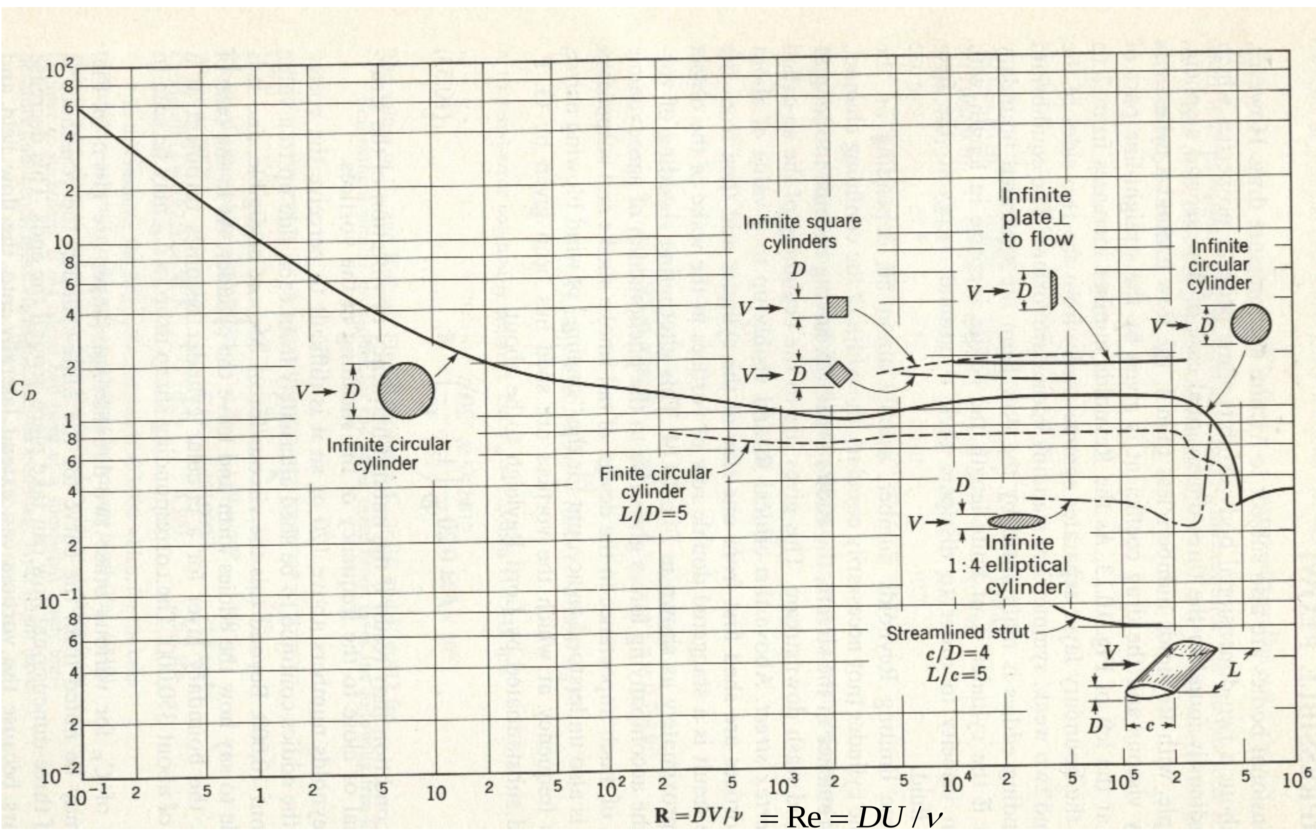
This is analogous to the Lift Coefficient C_L, commonly adopted in aeronautics and also applicable in some Civil Engineering flows.

$$C_L = \frac{F_L}{(1/2\rho u^2)A}$$ Equation (5j)

Definition of Lift Coefficient

Where F_L is the ***Lift Force***.

Experimental measurements indicate that the drag coefficient, C_d, varies with the Reynolds Number $\mathrm{Re} = \rho u \frac{D}{\mu}$, according to:



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Laboratory observations confirm that the nature of the flow around a cylinder is strongly dependent upon the Reynolds Number, Re, of the flow (remember slides).

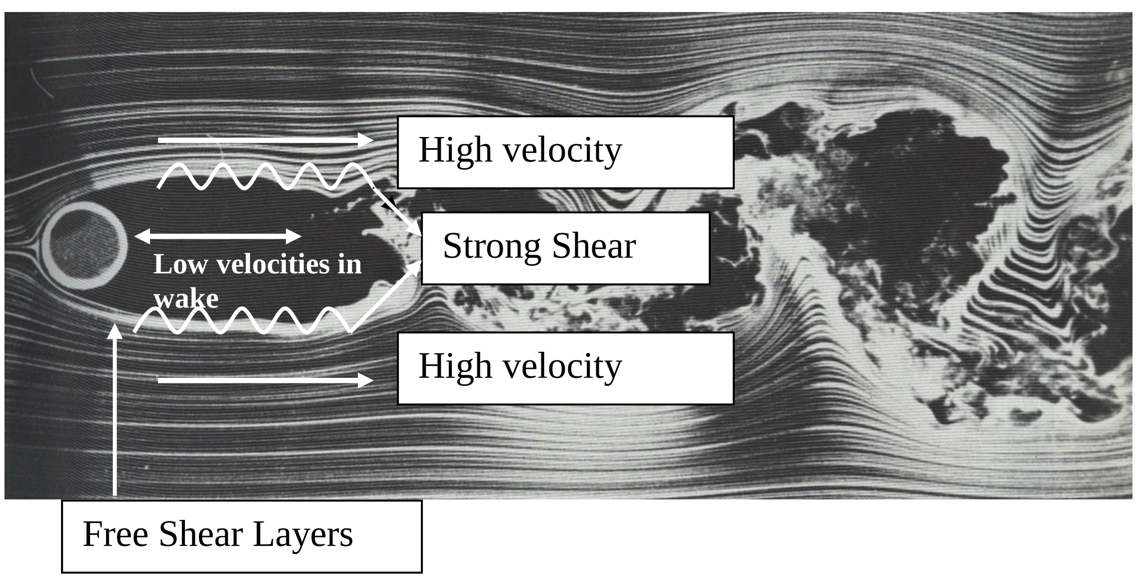
$$\rho \frac{uD}{\mu} \qquad\qquad \nu = \frac{\mu}{\rho}$$

dynamic viscocity

Kinematic viscocity

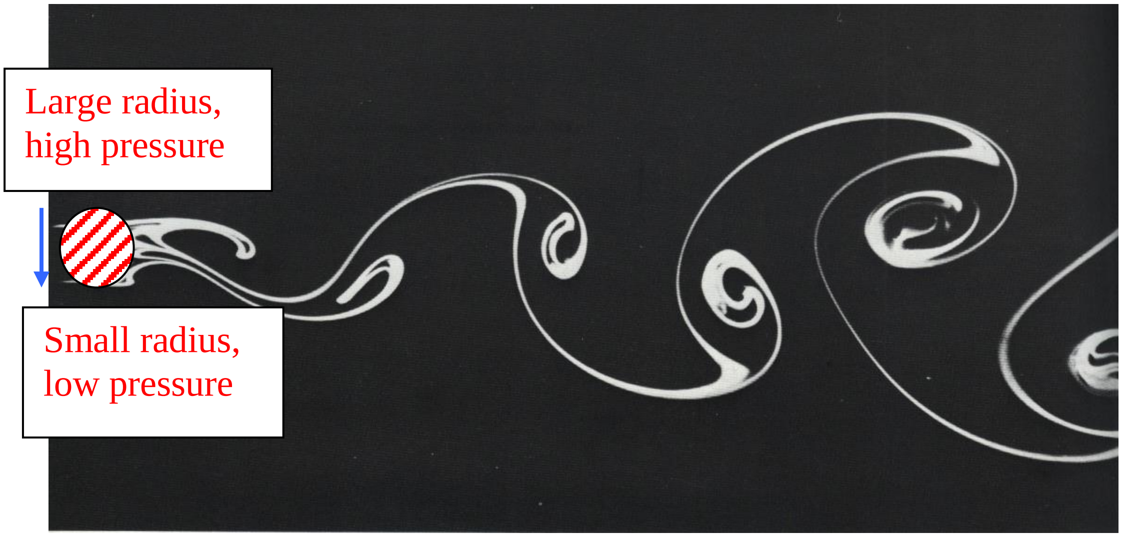
Vortex Shedding

Consider the case of flow around a cylinder at a relatively large Reynolds Number ($\mathrm{Re} \geq 10^3$). In this case flow separation occurs and this can produce an unsteady "lift" force.



The vorticity within the strongly sheared layers will merge into vortices. These vortices tend to shed from alternate sides of the cylinder leading to the formation of a so-called ***Kármán vortex street.***

Kármán vortex street



The pressure within a flow depends upon the radius of curvature of the streamlines. It can be shown that a large streamline curvature (small radius) produced smaller pressures. Hence, the above cylinder will tend to move downwards. When the vortex is shed from the opposite side of the cylinder, the pressures will be reverse, and the cylinder will move upwards.

Vortex-Induced Vibrations VIV

5.2. Fluid loading in Unsteady Flow

We shall now consider the fluid loading produced by a time-dependent flow. The most important example of this type of loading is that due to waves where the velocity components are given by:

$$u = a\omega \frac{\cosh k(z+d)}{\sinh(kd)} \sin(\omega t - kx)$$

$$w = a\omega \frac{\sinh k(z+d)}{\sinh(kd)} \cos(\omega t - kx)$$

If we return to our initial example of potential flow round a cylinder, the streamline pattern was given by:

$$\phi = u\left(r + \frac{D^2}{4r}\right)\cos\theta$$

and the velocity components: $u_r = \frac{\partial \phi}{\partial r}$ and $u_\theta = \frac{-1}{r}\frac{\partial \phi}{\partial \theta}$

In the previous example, we applied the steady form of the Bernoulli equation, found that the pressure distribution was symmetric and thus that there was no force applied.

In the present case we must use the unsteady Bernoulli equation, since the velocity is a function of time, *u(t).*

$$\rho \frac{\partial \phi}{\partial t} + P + \rho g y + \frac{1}{2} \rho u^2 = \text{constant.}$$

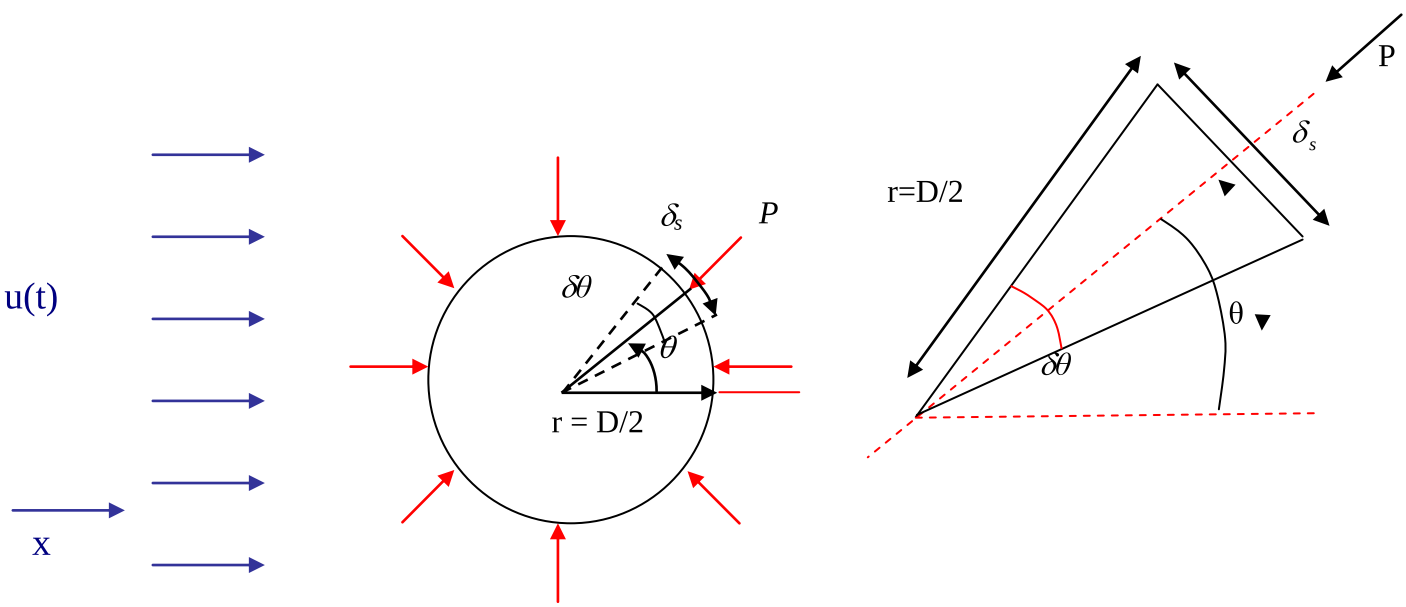
Hence,

$$P = \underbrace{\frac{-1}{2} \rho u^2 \left(1 - 4\sin^2 \theta\right)}_{\text{Previous symmetric term}} - \underbrace{\rho \frac{\partial \phi}{\partial t}}_{\text{Additional unsteady term}}$$

Previous symmetric term

Additional unsteady term

We now require an expression for *F*, the total force in the *x*-direction. Since the pressure distribution is given by:



Integrating the pressure round the cylinder gives:

$$F_x = \int_{\theta=0}^{\theta=2\pi} -P\delta s\cos\theta \quad = \quad \int_{\theta=0}^{\theta=2\pi} \rho\frac{\partial\phi}{\partial t}\cos\theta\frac{D}{2}d\theta$$

Notes
- cosθ is used because x axis always forms an angle θ with δs.
- We have neglected the symmetric term since its integral round the surface of the cylinder will be zero.

However, on the surface of the cylinder:

$$\phi = u(r+\frac{D^2}{4r})\cos\theta \quad = \quad Du\cos\theta \qquad \text{on} \quad r = D/2$$

Hence,

$$\left(\frac{\partial\phi}{\partial t}\right)_{r=D/2} \quad = \quad D\cos\theta\frac{\partial u}{\partial t}$$

Substituting into the above expression gives:

$$F_x = \int_{\theta=0}^{\theta=2\pi} \rho\frac{\partial u}{\partial t}\frac{D^2}{2}\cos^2\theta\, d\theta$$

Hence:

$$\boxed{F_x = 2\rho\frac{\pi D^2}{4}\frac{\partial u}{\partial t}} \qquad \text{Equation (5k)}$$

Definition of inertia force

(Note F_x is a force/unit length with units N/m).

This is the so-called ***inertia force*** and is associated with the unsteady pressure distribution $\left(\dfrac{\partial \phi}{\partial t}\right)$.

This solution corresponds to the very simple case in which separation does not occur. However, at practical values of the Reynolds number, separation does occur and must be taken into account within the description of the inertia force.

Experimental measurements show that the inertia force is given by:

$$F_M = C_M \frac{\rho \pi D^2}{4} \frac{\partial u}{\partial t}$$

where C_M is the ***inertia coefficient*** which varies depending upon the degree of flow separation and wake formation.

Combining both the drag force (F_D) and the inertia force (F_M) we obtain Morrison's equation:

$$F = C_D \frac{1}{2} \rho u |u| D \quad + \quad C_M \rho \frac{\pi D^2}{4} \frac{\partial u}{\partial t}$$

Equation (5l)

Morison's Equation

Note:

- F units Nm^{-1}
- 1^{st} term Drag force per unit length
- 2^{nd} term Inertia force per unit length

Note – ***the modulus sign is used in the drag term to ensure that the force is always in the direction of the flow.***

Keulegan – Carpenter Number (KC)

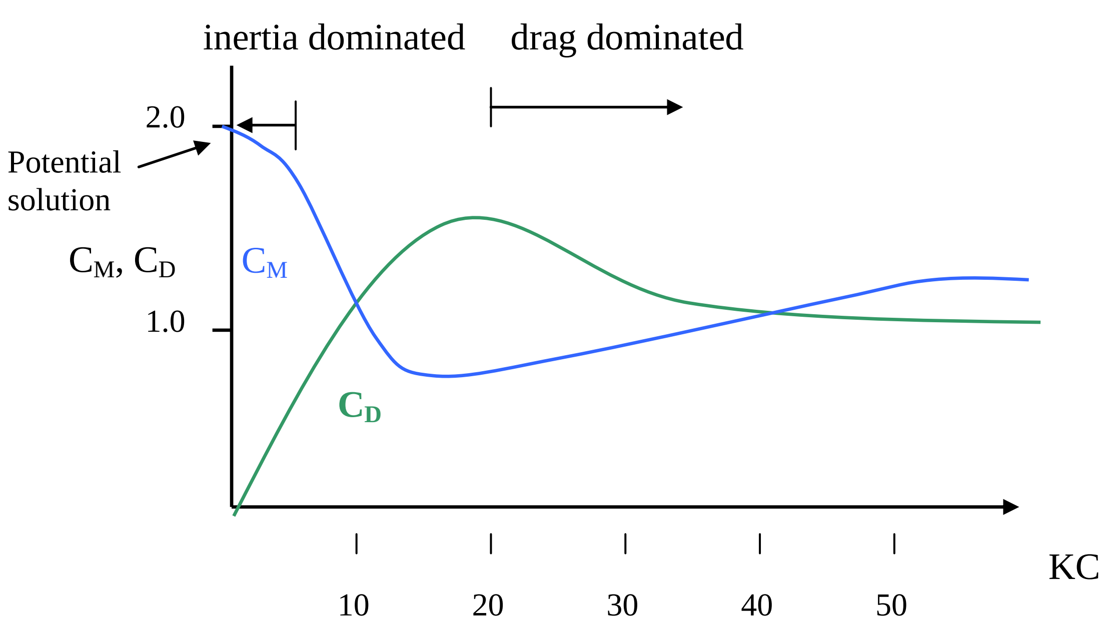
Experimental measurements have shown that the amplitude of the fluid motion relative to the diameter of the cylinder has an important effect on the nature of the induced loading. This ratio is defined as the Keulegan - Carpenter number:

$$\boxed{KC = \frac{UT}{D}}$$ Equation (5m)

Keulegan – Carpenter Number

where: *U* = velocity amplitude
T = period of oscillatory motion
D = diameter of cylinder

Experimental measures indicate that:



IMPORTANT POINTS

(i) ***Small values of KC***: (< 5)

- Velocity amplitude is small relative to diameter of cylinder
- Flow does not separate fully
- In the limiting case where *KC* is very small, the flow becomes almost ***"potential flow"***.
- In this limiting case, C_D → zero and C_M → 2.0 (Idealised case)

(ii) ***Large values of KC:*** (> 20)

- Velocity amplitude is large in comparison to the cylinder diameter
- Both inertial and drag forces are important
- However, since the drag force is proportional to U^2, this will produce the dominant contribution to Morison's equation.

5.3 Applications of Morison's Equation

$$F = C_D \frac{1}{2}\rho u|u|D + C_M \rho \frac{\pi D^2}{4}\frac{\partial u}{\partial t}$$

where:

F - **Force per unit length (Nm^{-1})**

C_D - Drag coefficient — dimensionless

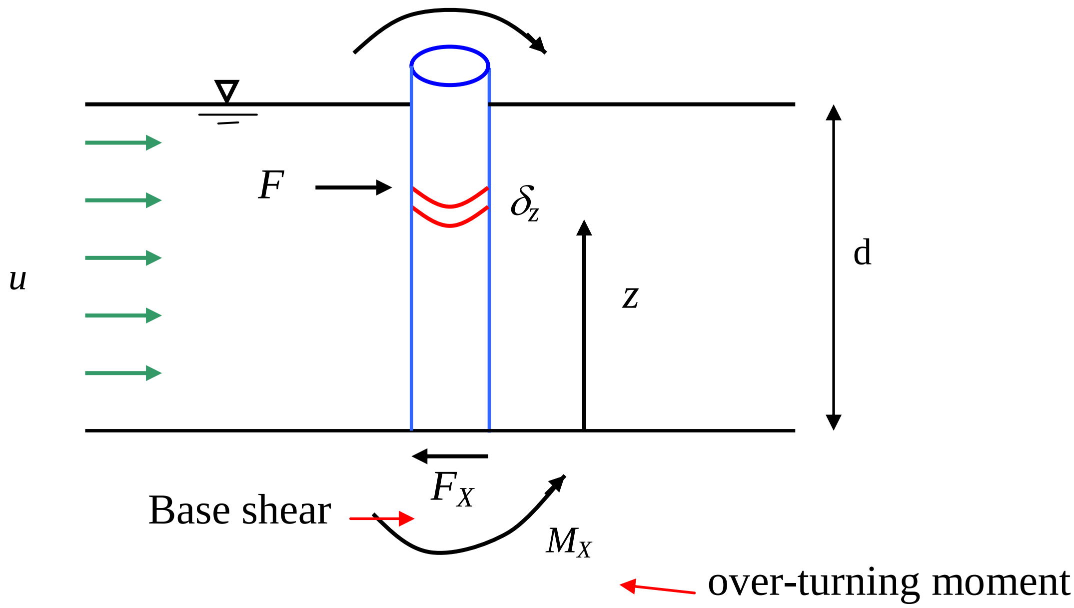
C_M - Inertia coefficient — dimensionless

ρ - Density of the fluid (kgm^{-3})

D - Diameter of cylinder (m)

Example 1

Vertical Cylinder in a Steady Current



In a steady current $\frac{du}{dt} = 0$ $\therefore$ No inertia forces (irrespective of D)

(a) If the current is uniform with depth, $u(z) = \text{constant}$.

$$F_X = \int_{z=0}^{z=d} F dz = \int_{z=0}^{z=d} C_D \frac{1}{2} \rho u |u| D dz$$

$$F_X = C_D \frac{1}{2} \rho u |u| D d$$

$$M_X = \int_{z=0}^{z=d} F z dz = C_D \frac{1}{2} \rho u|u| D \int_{z=0}^{z=d} z dz$$

$$M_X = C_D \frac{1}{2} \rho u|u| D \frac{d^2}{2} = F_X \frac{d}{2}$$

effective moment arm

(b) If the current is non-uniform with depth $u(z) \neq$ constant.

$$F_X = \int_{z=0}^{z=d} F dz = C_D \frac{1}{2} \rho D \int_{z=0}^{z=d} u|u| dz$$

$$F_X = C_D \frac{1}{2} \rho D \int_{z=0}^{z=d} u^2 dz$$

Be careful about the sign!

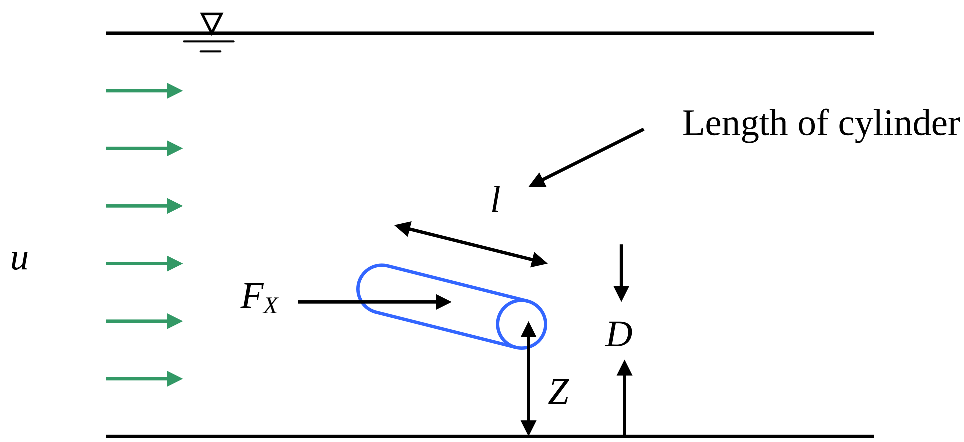
$$M_X = \int_{z=0}^{z=d} F z dz = C_D \frac{1}{2} \rho D \int_{z=0}^{z=d} u^2 z dz$$

If the current varies with depth, you must integrate over the length of the cylinder to calculate the total horizontal force.

Example 2

Horizontal Cylinder is Steady Current

Again, only drag forces are important, $\left(\frac{\partial u}{\partial t} = 0\right)$.



For a cylinder of length l,

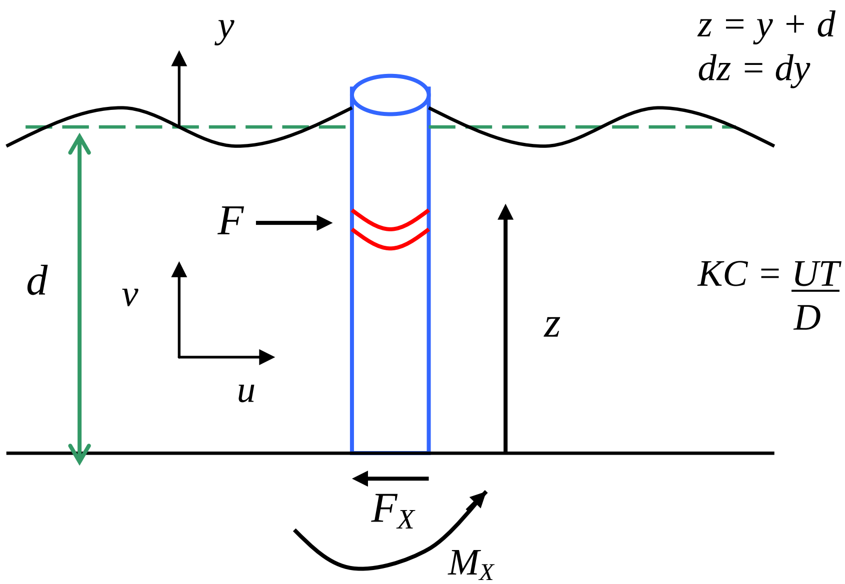
$$F_X = C_D \frac{1}{2} \rho u |u| D l$$

where the velocity *(u)* is determined at the centerline elevation (Z) and is taken as the valve perpendicular to the cylinder axis.

Note: If the horizontal cylinder forms part of a space frame (or jacket-structure) the force F_X will contribute to the total base shear and over-turning moment. In the latter case an appropriate moment arm must be defined.

Example 3

Vertical Cylinder in Waves



If possible, use *KC* number to determine the dominant forces:

Small KC (< 5) - $F = C_M \rho \frac{\pi D^2}{4} \frac{\partial u}{\partial t}$ inertia dominated

(N/m)

Large KC (>20) - $F = C_D \frac{1}{2} \rho u |u| D$ drag dominated

(N/m)

Not always possible since both forces may be important (5 < KC < 20). In this case:

$$F_X = C_D \frac{1}{2} \rho D \int_{z=0}^{z=d} u^2 dz \quad + \quad C_M \rho \frac{\pi D^2}{4} \int_{z=0}^{z=d} \frac{\partial u}{\partial t} dz$$

where $u^2 = \frac{a^2 \omega^2 \cosh^2(kz) \sin^2(\omega t - kx)}{\sinh^2(kd)}$

$$\frac{du}{dt} = \frac{a\omega^2 \cosh(kz)\cos(\omega t - kx)}{\sinh(kd)}$$

Note:

$$\int \cosh(kz)dz = \frac{\sinh(kz)}{k}$$

$$\int \cosh^2(kz)dz = \frac{1}{2}\int(\cosh(2kz)+1)dz = \frac{\sinh(2kz)}{4k} + \frac{z}{2}$$

Alternatively, express hyperbolic functions in exponential form:

$$\sinh(kz) = \left(\frac{e^{kz} - e^{-kz}}{2}\right) \quad ; \quad \cosh(kz) = \left(\frac{e^{kz} + e^{-kz}}{2}\right)$$

This approach is particularly useful in deep water

$$u = a\omega e^{kz}\sin(\omega t - kx) \quad ; \quad w = a\omega e^{kz}\cos(\omega t - kx)$$

In this form the equations are very easy to integrate.
Having integrated Morison's equation:

$$F_X = F_D \sin(\omega t - kx)\left|\sin(\omega t - kx)\right| + F_I \cos(\omega t - kx) \quad \text{----- Eq. (5o)}$$

where F_D - amplitude of the drag force
F_I - amplitude of the inertia force

To find maximum total force take

$$\frac{\partial F_X}{\partial \theta} = 0 \qquad \text{where} \qquad \theta = (\omega t - kx) \quad \text{- phase angle}$$

Solve for θ, and substitute in (5o) to find $(F_X)_{max}$

Note: So far we have taken the upper limit of the integral as:

$$F_X = C_D \frac{1}{2}\rho D \int_{z=-d}^{z=0} u^2 dz + ...$$

However, it might be argued that we should integrate to the instantaneous water surface:

$$F_X = C_D \frac{1}{2} \rho D \int_{z=-d}^{z=\eta} u^2 dz + ...$$

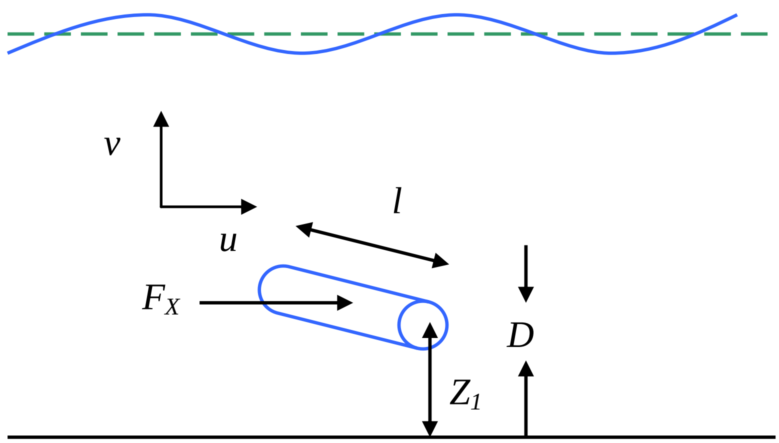
In fact, this will introduce higher-order terms, $O(a^3)$, several components of which have already been ignored. Hence, integration to the water surface is:

- Not strictly correct, but often applied
- There is concern regarding the ability of linear wave theory to predict velocities close to the water surface
- Best approach is to use a ***nonlinear*** wave theory and integrate to η.

Example 4

Horizontal Cylinder in Waves

Again, use the *KC* number to determine relative importance of drag and inertia.



Assuming that the waves are perpendicular to the axis of the cylinder (if not, take the component of the velocity that is).
Horizontal force:

$$F_x = C_D \frac{1}{2}\rho D u_{z=Z_1}\left|u_{z=Z_1}\right| l + C_M \rho \frac{\pi D^2}{4}\left.\frac{\partial u}{\partial t}\right|_{z=Z_1} l$$

Assuming the cylinder is perpendicular to the wave!

Vertical force:

$$F_z = C_D \frac{1}{2}\rho D v_{z=Z_1}\left|v_{z=Z_1}\right| l + C_M \rho \frac{\pi D^2}{4}\left.\frac{\partial v}{\partial t}\right|_{z=Z_1} l$$

Note: In calculating F_z

- Be careful with the choice of C_D and C_M (under some circumstances a circulation may be established).
- Be careful not to confuse vertical forces and lift forces (in the same direction!).