Math 201 Lecture 28: Sturm-Liouville Theory

Mar. 19. 2012

- Many examples here are taken from the textbook. The first number in () refers to the problem number in the UA Custom edition, the second number in () refers to the problem number in the 8th edition.

0. Review

- **Method of Separation of Variables.**

  Given equation

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2} + P(x,t), \qquad a < x < b; \qquad u(x,0) = f(x), \qquad + \text{boundary conditions} \tag{1}$$

1. Require $X(x)\,T(t)$ to solve the homogeneous equation

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2} \tag{2}$$

   which leads to eigenvalue problem for X:

$$X'' - K\,X = 0 + \text{boundary conditions.} \tag{3}$$

   Solve it to get X_n and K_n. Note that the natural range of n is **not** always $1, 2, 3, \ldots$

2. Expand

$$f(x) = \sum_n f_n\,X_n. \tag{4}$$

   Expand

$$P(x,t) = \sum_n p_n(t)\,X_n. \tag{5}$$

3. Solve

$$T_n' - \beta\,K\,T_n = p_n(t), \qquad T_n(0) = f_n \tag{6}$$

   to obtain T_n.

4. Write down the solution

$$u(x,t) = \sum_n T_n(t)\,X_n(x). \tag{7}$$

- We have seen how this method works when $f(x)$ and $P(x,t)$ are already given in the form

$$f(x) = \sum_n f_n\,X_n; \qquad P(x,t) = \sum_n p_n(t)\,X_n. \tag{8}$$

  However in general this is not the case.

- Question: For arbitrary $f(x)$, is it possible to write it as $f(x) = \sum_n f_n\,X_n$ with X_n's the eigenfunctions obtained in Step 1? If so, how?

- Answer: Yes. See below.

1. Sturm-Liouville Theory

- **Sturm-Liouville theory**, developed almost 200 years ago by Jacques Charles François Sturm (1803 – 1855) and Joseph Liouville (1809 – 1882) studies the following problem: Given an general eigenvalue problem

$$-(p(x)\,X')' + q(x)\,X = \lambda\,w(x)\,X, \qquad a < x < b \tag{9}$$

with boundary conditions

$$\alpha_1 X(a) + \beta_1 X'(a) = 0; \qquad \alpha_2 X(b) + \beta_2 X(b) = 0. \tag{10}$$

What can we say about the eigenvalues/eigenfunctions?

Theorem 1. (Sturm-Liouville, Woolly version) *The following hold true:*

1. *The eigenvalues are countable, and can be ordered by their sizes.*
2. *For each eigenvalue λ_n, the eigenfunction can be written as $C\, X_n$, where C is an arbitrary constant.*
3. *The X_n's are "orthogonal" in the following sense:*

$$\int_a^b X_m(x)\, X_n(x)\, w(x)\, \mathrm{d}x = 0 \text{ whenever } m \neq n. \tag{11}$$

4. *The X_n's are "complete" in the following sense: Any reasonable $f(x)$ (for example, bounded) has exactly one representation as linear combination of X_n's:*

$$f(x) = \sum_n f_n\, X_n. \tag{12}$$

The "=" here means

$$\lim_{N \to \infty} \int \left| f(x) - \sum_{n<N} f_n\, X_n \right| \mathrm{d}x = 0. \tag{13}$$

Remark 2. We intentionally choose not to present the precise version.

Example 3. Consider the eigenvalue problem

$$X'' - K\, X = 0; \qquad X(0) = X(L) = 0; \tag{14}$$

We know that the eigenfunctions are

$$X_n = \sin\left(\frac{n\, \pi\, x}{L}\right), \qquad n = 1, 2, 3, \ldots \tag{15}$$

Then from the above theorem we know that any $f(x)$ can be written as

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\, \pi\, x}{L}\right). \tag{16}$$

We will see later that this expansion has a name: Fourier Sine Series.

Example 4. Consider the eigenvalue problem

$$X'' - K\, X = 0; \qquad X'(0) = X'(L) = 0. \tag{17}$$

We know that the eigenfunctions are

$$X_n = \cos\left(\frac{n\, \pi\, x}{L}\right), \qquad n = 0, 1, 2, 3, \ldots \tag{18}$$

So the above theorem tells us any $f(x)$ can be written as

$$f(x) = \sum_{n=0}^{\infty} f_n \cos\left(\frac{n\, \pi\, x}{L}\right). \tag{19}$$

Such expansion is called: Fourier Cosine Series.

Remark 5. It should be emphasized that, naturally, the f_n's change when we pick a different set of X_n's.

- How to compute f_n's.
  - Problem: Determine f_n's in
$$f(x) = \sum_n f_n X_n. \tag{20}$$
  - Idea: Use "orthogonality":
$$\int_a^b X_m(x)\, X_n(x)\, w(x)\, \mathrm{d}x = 0 \text{ when } m \neq n. \tag{21}$$
  - Let's set a particular n_0 and try to find out f_{n_0}. As we try to use the above orthogonality, naturally we multiply both sides of
$$f(x) = \sum_n f_n X_n. \tag{22}$$
by $X_{n_0}(x)\, w(x)$, and then integrate from a to b. We have
$$\begin{aligned} \int_a^b f(x)\, X_{n_0}(x)\, w(x)\, \mathrm{d}x &= \int_a^b \left[\sum_n f_n X_n\right] X_{n_0}(x) w(x)\, \mathrm{d}x \\ &= \sum_n f_n \int_a^b X_n(x) X_{n_0}(x) w(x)\, \mathrm{d}x. \end{aligned} \tag{23}$$
As
$$\int_a^b X_n(x) X_{n_0}(x) w(x)\, \mathrm{d}x = 0 \text{ for all } n \neq n_0 \tag{24}$$
we see that the right hand side in fact has exactly one nonzero term:
$$\int_a^b X_{n_0}(x)^2\, w(x) \mathrm{d}x. \tag{25}$$
Thus we reach
$$\int_a^b f(x)\, X_{n_0}(x)\, w(x)\, \mathrm{d}x = f_{n_0} \int_a^b X_{n_0}(x)^2\, w(x) \mathrm{d}x. \tag{26}$$
and consequently
$$f_{n_0} = \frac{\int_a^b f(x)\, X_{n_0}(x)\, w(x)\, \mathrm{d}x}{\int_a^b X_{n_0}(x)^2\, w(x) \mathrm{d}x}. \tag{27}$$
- Special cases most relevant to us:
  - In all our problems the equation in the eigenvalue problem is
$$X'' - K X = 0 \Longrightarrow w(x) = 1. \tag{28}$$
So the formula becomes
$$f_n = \frac{\int_0^L f(x)\, X_n(x)\, \mathrm{d}x}{\int_0^L X_n(x)^2\, \mathrm{d}x}. \tag{29}$$
  - Fourier Cosine and Fourier Sine Series.
    Note that, as soon as we know X_n's, the denominator $\int_0^L X_n(x)^2\, \mathrm{d}x$ can be calculated beforehand, without knowledge of $f(x)$.
    - Fourier Cosine Series.
      In this case
$$X_n = \cos\left(\frac{n\,\pi\,x}{L}\right). \qquad n = 0, 1, 2, 3, \ldots \tag{30}$$
We have
$$\int_0^L \left[\cos\left(\frac{n\,\pi\,x}{L}\right)\right]^2 \mathrm{d}x = \int_0^L \frac{\cos\left(\frac{2\,n\,\pi\,x}{L}\right) + 1}{2}\, \mathrm{d}x = \frac{L}{2}. \tag{31}$$

Note that the above calculation is wrong when $n=0$. We have to calculate the $n=0$ case separately:

$$\int_0^L 1^2 \, \mathrm{d}x = L. \tag{32}$$

So the f_n's in the Fourier Cosine expansion

$$f(x) = \sum_{n=0}^{\infty} f_n \cos\left(\frac{n\pi x}{L}\right) \tag{33}$$

are given by

$$f_0 = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x; \qquad f_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \text{ for } n = 1, 2, 3, \ldots \tag{34}$$

A more popular way of writing it is setting $a_0 = 2 f_0$, and $a_n = f_n$ to get a universal formula

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \qquad n = 0, 1, 2, 3, \ldots \tag{35}$$

The Fourier cosine series then reads

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right). \tag{36}$$

– Fourier Sine Series.

In this case

$$X_n = \sin\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, 3, \ldots \tag{37}$$

Similar calculation as in the previous case gives

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi x}{L}\right) \Longrightarrow f_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \qquad n = 1, 2, 3, \ldots \tag{38}$$

○ Notation: Often, to emphasize the relation between Fourier Cosine/Sine series and Fourier series, the following notation is used:

– Fourier Cosine:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \ n = 0, 1, 2, 3, \ldots \tag{39}$$

– Fourier Sine:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \ n = 1, 2, 3, \ldots \tag{40}$$

We will see in the next two lectures the reason of choosing the letters a, b.

2. EXAMPLES

- Fourier Cosine:

Example 6. (10.4.13; 10.4 13) Compute the Fourier cosine series for

$$f(x) = e^x, \qquad 0 < x < 1. \tag{41}$$

Solution. We have $T = 1$. First

$$a_0 = \frac{2}{1}\int_0^1 e^x \,\mathrm{d}x = 2\,(e-1). \tag{42}$$

next

$$\begin{aligned} a_n &= 2\int_0^1 e^x \cos(n\pi x)\,\mathrm{d}x \\ &= 2\int_0^1 \cos(n\pi x)\,\mathrm{d}e^x \\ &= 2\left[\cos(n\pi x)\,e^x|_0^1 + n\pi\int_0^1 e^x \sin(n\pi x)\,\mathrm{d}x\right] \\ &= 2\,[e\,(-1)^n - 1] + 2\,n\,\pi\int_0^1 \sin(n\pi x)\,\mathrm{d}e^x \\ &= 2\,[e\,(-1)^n - 1] + 2\,n\,\pi\left[e^x \sin(n\pi x)|_0^1 - n\pi\int_0^1 e^x\cos(n\pi x)\,\mathrm{d}x\right] \\ &= 2\,[e\,(-1)^n - 1] - 2\,(n\pi)^2\int_0^1 e^x \cos(n\pi x)\,\mathrm{d}x \\ &= 2\,[e\,(-1)^n - 1] - (n\pi)^2\,a_n. \end{aligned} \tag{43}$$

Therefore

$$a_n = \frac{2\,[e\,(-1)^n - 1]}{1 + (n\pi)^2}. \tag{44}$$

So the Fourier cosine series is given by

$$e^x = e - 1 + \sum_{n=1}^{\infty} \frac{2\,[e\,(-1)^n - 1]}{1 + (n\pi)^2} \cos(n\pi x). \tag{45}$$

- Fourier Sine:

Example 7. (10.4.7; 10.4 7) Compute the Fourier sine series for

$$f(x) = x^2, \qquad 0 < x < \pi. \tag{46}$$

Solution. We have $T = \pi$. Compute

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi x^2 \sin(n x)\,\mathrm{d}x \\ &= -\frac{2}{n\pi}\int_0^\pi x^2\,\mathrm{d}\cos(n x) \\ &= -\frac{2}{n\pi}\left[x^2\cos(n x)|_0^\pi - 2\int_0^\pi \cos(n x)\,x\,\mathrm{d}x\right] \\ &= -\frac{2}{n\pi}\left[\pi^2(-1)^n - \frac{2}{n}\int_0^\pi x\,\mathrm{d}\sin(n x)\right] \\ &= -\frac{2}{n\pi}\left[\pi^2(-1)^n - \frac{2}{n}\left(x\sin(n x)|_0^\pi - \int_0^\pi \sin(n x)\,\mathrm{d}x\right)\right] \\ &= -\frac{2}{n\pi}\left[\pi^2(-1)^n - \frac{2}{n}\frac{1}{n}\cos(n x)|_0^\pi\right] \\ &= -\frac{2}{n\pi}\left[\pi^2(-1)^n - \frac{2}{n^2}[(-1)^n - 1]\right] \\ &= \frac{2\pi}{n}(-1)^{n+1} + \frac{4}{n^3\pi}[(-1)^n - 1]. \end{aligned} \tag{47}$$

Therefore the Fourier sine series is

$$x^2 = \sum_{n=1}^{\infty}\left[\frac{2\pi}{n}(-1)^{n+1} + \frac{4}{n^3\pi}[(-1)^n - 1]\right]\sin(n x). \tag{48}$$

- More exotic examples.

Example 8. (Mixed boundary conditions) Expand $f(x) = x$ into $\sum f_n X_n$ with X_n eigenfunctions of

$$X'' - K X = 0, 0 < x < \pi; \qquad X(0) = X'(\pi) = 0. \tag{49}$$

Solution. We have already solved (in the previous lecture) the eigenfunctions:

$$X_n = \sin\left(\frac{2\,n+1}{2}\,x\right), \qquad n = 0, 1, 2, 3, \ldots \tag{50}$$

Now prepare:

$$\int_0^\pi X_n^2 = \int_0^\pi \frac{1-\cos\left((2\,n+1)\,x\right)}{2}\,\mathrm{d}x = \frac{\pi}{2} - \frac{1}{2\,(2\,n=1)}\left[\sin\left(2\,n+1\right)\pi - \cos 0\right] = \frac{\pi}{2} = \frac{\pi}{2}. \tag{51}$$

Thus

$$\begin{aligned} f_n &= \frac{2}{\pi}\int_0^\pi x\sin\left(\frac{2\,n+1}{2}\,x\right)\mathrm{d}x \\ &= \frac{2}{\pi}\int_0^\pi x\,\mathrm{d}\left[-\frac{2}{2\,n+1}\cos\left(\frac{2\,n+1}{2}\,x\right)\right] \\ &= -\frac{4}{\pi\,(2\,n+1)}\left[\left(x\cos\left(\frac{2\,n+1}{2}\,x\right)\right)|_0^\pi - \int_0^\pi \cos\left(\frac{2\,n+1}{2}\,x\right)\mathrm{d}x\right] \\ &= -\frac{4}{\pi(2\,n+1)}\left[\pi\cos\left(\frac{2\,n+1}{2}\,\pi\right) - \frac{2}{2\,n+1}\sin\left(\frac{2\,n+1}{2}\,x\right)|_0^\pi\right] \\ &= -\frac{4}{\pi\,(2\,n+1)}\left[\pi\cdot 0 - \frac{2}{2\,n+1}\left[\sin\left(\frac{2\,n+1}{2}\,\pi\right) - 0\right]\right] \\ &= \frac{8}{\pi\,(2\,n+1)^2}\sin\left(\frac{2\,n+1}{2}\,\pi\right) \\ &= \frac{8}{\pi\,(2\,n+1)^2}\,(-1)^n. \end{aligned} \tag{52}$$

So the expansion is

$$x = \sum_{n=0}^\infty \frac{8}{\pi\,(2\,n+1)^2}\,(-1)^n \sin\left(\frac{2\,n+1}{2}\,x\right). \tag{53}$$

3. Pointwise Convergence?

- Recall that $f(x) = \sum f_n\,X_n$ in the above means

$$\lim_{N\to\infty}\int\left|f(x) - \sum_{n<N} f_n\,X_n\right|\mathrm{d}x = 0. \tag{54}$$

However in some applications we would like to know for a particular x_0, what is the value of

$$\left[\sum_n f_n\,X_n\right](x_0) = \lim_{N\to\infty}\sum_{n<N} f_n\,X_n(x_0) \tag{55}$$

and

$$\lim_{N\to\infty}\int\left|f(x) - \sum_{n<N} f_n\,X_n\right|\mathrm{d}x = 0 \textbf{ does not imply } \lim_{N\to\infty}\sum_{n<N} f_n\,X_n(x_0) = f(x_0). \tag{56}$$

Therefore we need to study "pointwise convergence" property of the expansions.

- For general X_n in the Sturm-Liouville Theorem, the situation seems quite complicated and I am yet to be sure of the existence of a complete theory.
- However, for the X_n's obtained from the eigenvalue problem $X'' - K\,X = 0 +$ boundary conditions, we know exactly what $\lim_{N\to\infty}\sum_{n<N} f_n\,X_n(x_0)$ is. In the next lecture we will reveal this through the study of Fourier series, which is the expansion of $f(x)$ using $\cos\left(\frac{2\,n\,\pi\,x}{L}\right)$ and $\sin\left(\frac{2\,n\,\pi\,x}{L}\right)$.
- Note that $\cos\left(\frac{2\,n\,\pi\,x}{L}\right)$ and $\sin\left(\frac{2\,n\,\pi\,x}{L}\right)$ are actually eigenfunctions to

$$X'' - K\,X = 0 \tag{57}$$

with periodic boundary condition (which is not included in the standard Sturm-Liouville theorem!),

4. Orthonormal Set of Functions

- Any set of **nonzero** functions $\{f_n\}_{n=1}^{\infty}$ satisfying, for some $w \geqslant 0$

$$\int_a^b f_m\, f_n\, w\, \mathrm{d}x = 0 \text{ whenever } n \neq m \tag{58}$$

is said to be an **orthogonal system** with respect to weight w on the interval $[a, b]$.

If furthermore we have

$$\int_a^b f_n^2\, w\, \mathrm{d}x = 1, \qquad n = 1, 2, 3, \ldots \tag{59}$$

then $\{f_n\}$ is called an **orthonormal system**.

The main property of an orthogonal system is that if

$$f(x) \sim c_1\, f_1 + c_2\, f_2 + \cdots \tag{60}$$

then the coefficients can be determined through

$$c_m = \frac{\int f(x)\, f_m(x)\, w(x)\, \mathrm{d}x}{\int f_m^2(x)\, \mathrm{d}x}. \tag{61}$$

If $\{f_n\}$ is furthermore orthonormal, then

$$c_m = \int f(x)\, f_m(x)\, w(x)\, \mathrm{d}x. \tag{62}$$

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Example 9. (10.3.26, 10.3.27) Show that the set of functions

$$\left\{\cos\frac{\pi}{2}\, x, \sin\frac{\pi}{2}\, x, \ldots, \cos\frac{(2\, n-1)\, \pi}{2}\, x, \sin\frac{(2\, n-1)\, \pi}{2}\, x, \ldots\right\} \tag{63}$$

is an orthonormal system on $[-1, 1]$ with respect to the weight function $w(x) \equiv 1$.

Then find the orthogonal expansion for

$$f(x) = \begin{cases} 0 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases} \tag{64}$$

in terms of this orthonormal system.

Solution.

- Verify orthonormality.
  1. Integrating product of different functions gives 0.

     We compute for $n \neq m$, $n, m = 1, 2, \ldots$ (note that $w = 1$)

$$\int_{-1}^{1} \cos\frac{(2\, n-1)\, \pi\, x}{2} \cos\frac{(2\, m-1)\, \pi\, x}{2} \cdot 1\, \mathrm{d}x = \frac{1}{2}\int_{-1}^{1} \cos\left((n+m-1)\, \pi\, x\right) \mathrm{d}x + \frac{1}{2}\int_{-1}^{1} \cos\left((n-m)\, \pi\, x\right) \mathrm{d}x \tag{65}$$

As neither $n+m-1$ nor $n-m$ is zero, we have

$$\int_{-1}^{1} \cos\left((n+m-1)\, \pi\, x\right) \mathrm{d}x = \frac{1}{(n+m-1)\, \pi} \sin\left((n+m-1)\, \pi\, x\right)\Big|_{-1}^{1} = 0, \tag{66}$$

$$\int_{-1}^{1} \cos\left((n-m)\, \pi\, x\right) \mathrm{d}x = \frac{1}{(n-m)\, \pi} \sin\left((n-m)\, \pi\, x\right) \mathrm{d}x = 0. \tag{67}$$

Similarly we compute

$$\int_{-1}^{1} \sin\frac{(2\, n-1)\, \pi\, x}{2} \sin\frac{(2\, m-1)\, \pi\, x}{2} \cdot 1\, \mathrm{d}x = 0 \tag{68}$$

for $n \neq m$.

Finally we can compute

$$\int_{-1}^{1} \cos \frac{(2n-1)\pi x}{2} \sin \frac{(2m-1)\pi x}{2} \cdot 1 \,\mathrm{d}x = 0 \tag{69}$$

Note that this time $n = m$ is OK.

2. Integrating the square of any function in the list gives 1.

   We compute

$$\int_{-1}^{1} \left(\cos \frac{(2n-1)\pi x}{2}\right)^2 \cdot 1 \,\mathrm{d}x = \frac{1}{2} \int_{-1}^{1} [1 + \cos((2n-1)\pi x)] \,\mathrm{d}x = 1. \tag{70}$$

   Similarly we have

$$\int_{-1}^{1} \left(\sin \frac{(2n-1)\pi x}{2}\right)^2 \mathrm{d}x = 1. \tag{71}$$

Thus the set of functions is an orthonormal system.

- Orthogonal expansion for

$$f(x) = \begin{cases} 0 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases} \tag{72}$$

Recall that if

$$f(x) \sim c_1 f_1 + c_2 f_2 + \cdots \tag{73}$$

then the coefficients can be determined through

$$c_m = \frac{\int f(x) f_m(x) w(x) \,\mathrm{d}x}{\int f_m^2(x) \,\mathrm{d}x}. \tag{74}$$

In case of our system we write

$$f(x) = \sum_{n=1}^{\infty} \left\{ a_n \cos \frac{(2n-1)\pi x}{2} + b_n \sin \frac{(2n-1)\pi x}{2} \right\} \tag{75}$$

and compute

$$\begin{aligned} a_n &= \int_{-1}^{1} f(x) \cos \frac{(2n-1)\pi x}{2} \,\mathrm{d}x \\ &= \int_{0}^{1} \cos \frac{(2n-1)\pi x}{2} \,\mathrm{d}x \\ &= \frac{2}{(2n-1)\pi} \sin \frac{(2n-1)\pi x}{2} \Big|_0^1 \\ &= \frac{2}{(2n-1)\pi} \sin(n\pi - \pi/2) \\ &= \frac{2(-1)^{n+1}}{(2n-1)\pi}. \end{aligned} \tag{76}$$

and

$$\begin{aligned} b_n &= \int_{-1}^{1} f(x) \sin \frac{(2n-1)\pi x}{2} \,\mathrm{d}x \\ &= \int_{0}^{1} \sin \frac{(2n-1)\pi x}{2} \,\mathrm{d}x \\ &= -\frac{2}{(2n-1)\pi} \cos \frac{(2n-1)\pi x}{2} \Big|_0^1 \\ &= \frac{2}{(2n-1)\pi}. \end{aligned} \tag{77}$$

Thus finally we have

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2}{(2\,n-1)\,\pi} \left[(-1)^{n+1} \cos \frac{(2\,n-1)\,\pi\,x}{2} + \sin \frac{(2\,n-1)\,\pi\,x}{2} \right]. \tag{78}$$

5. NOTES AND COMMENTS

- Note that the boundary conditions

$$\alpha_1\, y(0) + \beta_1\, y'(0) = 0; \qquad \alpha_2\, y(L) + \beta_2\, y(L) = 0. \tag{79}$$

covers Dirichlet, Neumann, and Mixed boundary conditions, but not periodic boundary conditions.