Math 3150 Lecture Notes
Partial Differential Equations for Engineers

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Chapter 1

Introduction

1.1 Review and Notation

Recall that a *partial derivative* is the derivative of a function (with more than one independent variable) with respect to one of its variables. *i.e.* Suppose that

$$u = f(x, y), \tag{1.1}$$

then the partial derivative of u or $f(x, y)$ with respect to x is

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}, \tag{1.2}$$

and with respect to y is

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}. \tag{1.3}$$

I will occasionally use subscripts as shorthand notation for the partial derivative. That is,

$$u_x = \frac{\partial u}{\partial x} \tag{1.4a}$$

$$u_y = \frac{\partial u}{\partial y} \tag{1.4b}$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} \tag{1.4c}$$

$$u_{xy} = \frac{\partial^2 u}{\partial x \partial y} \tag{1.4d}$$

and so on.

1.2 Definition: Partial Differential Equation

Recall that an *ordinary differential equation* is an equation relating the derivatives of a function (and perhaps the function itself) when the function has only one independent variable. A *partial differential equation* is an equation relating the partial derivatives of a function with perhaps the function itself. For instance,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1.5a}$$

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{1.5b}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{1.5c}$$

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.5d}$$

$$\frac{\partial u}{\partial t} + C(x)\frac{\partial^2 u}{\partial x^2} + Au = Be^{-t} \tag{1.5e}$$

are all partial differential equations. A *solution* to a partial differential equation is of course a function u of multiple independent variables that satisfies the PDE. From here on a partial differential equation will be referred to as a PDE.

1.3 The Order of a PDE

The *order* of a PDE is simply the highest-ordered derivative that appears in the equation. For example,

$$u_t + u_x = Ae^{x-t} \tag{1.6a}$$

$$u_{xx} + u_{yy} = sin(x^2 + y^2) \tag{1.6b}$$

$$u_t + uu_x + u_{xxx} = u^4 \tag{1.6c}$$

$$\tag{1.6d}$$

are 1$^{\text{st}}$, 2$^{\text{nd}}$ and 3$^{\text{rd}}$ order PDEs, respectively.

1.4 Homogeneous and Nonhomogeneous PDEs

A *homogeneous* PDE is one where all of the summands in the equation involve the dependent variable somehow. For instance,

$$u_t + uu_x = -u \tag{1.7a}$$

$$u_t + c(x)u_{xx} = -u^2 \tag{1.7b}$$

$$u_{tt} - c^2 u_{xx} = 0 \tag{1.7c}$$

are homogeneous PDEs. A *nonhomogeneous* PDE has terms (summands) that do not involve the dependent variable, but are simply functions of the independent variables. For instance,

$$u_t + uu_x = Ae^{-t} \tag{1.8a}$$

$$u_{tt} - c^2 u_{xx} + \sin(x) = 0 \tag{1.8b}$$

are nonhomogeneous PDEs. The terms that are only function of the independent variables, are sometimes called *source terms.* For instance, the right hand side of (1.8a) is a source term, as well as the $sin(x)$ term in (1.8b).

1.5 Linear and Nonlinear PDEs

PDEs are either *linear* or *nonlinear.* Technically a linear PDE is one where a linear combination of solutions is also a solution. That is, a PDE is linear if and only if, given any two solutions to the PDE, $u_1(t,x,y)$ and $u_2(t,x,y)$, the linear combination $c_1u_1(t,x,y)+c_2u_2(t,x,y)$, c_1, c_2 arbitrary constants, is also a solution. Note that only a homogeneous PDE is linear by this definition. However, a nonhomogeneous PDE will be referred to as *linear* if the homogeneous part (the PDE neglecting any source terms) is linear. It is easy to determine if a PDE is linear by simply looking at all of the terms in the equation. The PDE is linear if and only if all of the terms are linear functions of the dependent variable and its derivatives. That is,

$$\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = -u \tag{1.9a}$$

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.9b}$$

$$\frac{\partial u}{\partial t} + D\frac{\partial^2 u}{\partial x^2} = x^2 \tag{1.9c}$$

are linear PDEs. The last PDE in (1.9) is linear only by this looser definition where the source term is neglected. (Don't worry—no ambiguous test questions will be given about this. Typically we will only be concerned about the homogeneous and nonhomogeneous distinction for linear PDEs.) The following PDEs are nonlinear,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{1.10a}$$

$$u_t + (\frac{1}{2}u^2)_x = 0 \tag{1.10b}$$

$$u_t + u_x = u^2 \tag{1.10c}$$

since there are terms where the dependent variable and its derivatives are multiplied. (Note that (1.10a) and (1.10b) are the same PDE. Why?).

1.6 Classification of General 2nd-Order Linear PDEs

In this course we will frequently be concerned with 2nd order, linear PDEs (homogeneous and nonhomogeneous) with two independent variables. The general PDE in such a case, where we will assume the independent variables are x and y, can be written as

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \tag{1.11}$$

where A, B, C, D, E, F and G are, in general, functions of x and y (of course this includes constants as well). An equation of the form (1.11) is called

$$\textit{elliptic} \quad \text{if} \quad AC - B^2 > 0, \tag{1.12a}$$

$$\textit{hyperbolic} \quad \text{if} \quad AC - B^2 < 0, \tag{1.12b}$$

$$\textit{parabolic} \quad \text{if} \quad AC - B^2 = 0. \tag{1.12c}$$

Although this may seem like arbitrary semantics, in fact, solutions to PDEs of each class typically share certain qualitative properties, or occur in applications where the physics share some common properties. Note that in the definitions (1.12), x and y enter symmetrically. Therefore, as should be expected, the general class of a PDE does not depend on our choice of coordinate system. Note also that either x or y in (1.11) could be replaced by t if we wanted to consider a PDE with two independent variables—one in space and one in time. That is, we simply consider

$$Au_{xx} + 2Bu_{xt} + Cu_{tt} + Du_x + Eu_t + Fu = G, \tag{1.13}$$

Example 1. *The linear 2nd order wave equation can be written as*

$$u_{tt} - c^2 u_{xx} = 0. \tag{1.14}$$

If we compare (1.14) with the general form (1.13), we see that $A = -c^2$, $B = 0$, *and* $C = 1$. *This gives* $AC - B^2 = -c^2 < 0$, *and we see that the wave equation is hyperbolic.*

Why is (1.14) the "wave equation"? Consider an arbitrary function of one variable $f(\cdot)$. *The only requirement that we will make on* f *is that it can be twice differentiable. Now consider the function* $u(x,t) = f(x - ct)$. *First note that (refresh you memory of the chain rule if you need to)*

$$a\frac{\partial}{\partial t} f(x - ct) = f'(x - ct)(-c) \tag{1.15a}$$

$$\frac{\partial^2}{\partial t^2} f(x - ct) = \frac{\partial}{\partial t}\big(f'(x - ct)(-c)\big) = c^2 f''(x - ct) \tag{1.15b}$$

$$\frac{\partial}{\partial x} f(x - ct) = f'(x - ct) \tag{1.15c}$$

$$\frac{\partial^2}{\partial x^2} f(x - ct) = \frac{\partial}{\partial x}\big(f'(x - ct)\big) = f''(x - ct). \tag{1.15d}$$

Now using these derivatives and plugging $u(t,x) = f(x-ct)$ *into (1.14) gives*

$$\begin{aligned}&\frac{\partial^2}{\partial t^2} f(x - ct) - c^2 \frac{\partial^2}{\partial x^2} f(x - ct)\\ =&c^2 f''(x - ct) - c^2 f''(x - ct) = 0.\end{aligned} \tag{1.16}$$

Therefore, $u(x,t) = f(x-ct)$ *solves the PDE for any function of one variable* $f(\cdot)$ *that is twice differentiable! It is easy to show that* $u(x,t) = f(x + ct)$ *does as well. What does a function* $f(x - ct)$ *behave like? At time* $t = 0$ *the solution is simply* $f(x)$. *As time progresses, the profile* $f(x)$ *translates to the right at a speed* c. *That is*

$$\begin{aligned}&u(x, t_1) = f(x - ct_1)\\ =&f(x - ct_1 + ct_2 - ct_2)\\ =&f((x + c(t_2 - t_1)) - ct_2)\\ =&u(x + c(t_2 - t_1), t_2).\end{aligned} \tag{1.17}$$

Likewise, it is easy to show that if $u(x,t) = f(x + ct)$, *then* u *is the profile of* $f(x)$ *translating to the left at a speed of* c.

Chapter 2

Fourier Series

In this chapter Fourier series will be introduced. Although this is a bit of a departure from PDEs, later we will see that Fourier series are an indispensable tool for solving many linear PDEs. Before describing Fourier series, it is important to have a firm understanding of several concepts described below.

2.1 Some Important Types of Functions

2.1.1 Continuous and Piecewise Continuous Functions

A *continuous* function can be rigorously defined mathematically, however, you can think of a continuous function as one that you can draw the graph of without lifting your pencil from the paper. It can have sharp kinks and points but it must not "jump" from one value to another suddenly. For a continuous function, given any point $x = c$, $f(c^-) = f(c) = f(c^+)$ must hold. The notation $f(c^-)$ is the value of $\lim_{x\to c} f(x)$ when $x < c$ and $f(c^+)$ is the value of $\lim_{x\to c} f(x)$ when $x > c$.

We will also frequently encounter *piecewise continuous* functions. A piecewise continuous function is not necessarily continuous, however, it is the next best thing—it is defined everywhere, and is only discontinuous at a finite number of points on any finite interval. Furthermore, the limit of the function exists as one of the points of discontinuity is approached from either side. Intuitively, a piecewise continuous function is one that is continuous except at points where the function simply jumps from one value to another.

The following function is piecewise continuous

$$f(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ 10 & \text{if} \quad x = 0 \\ 20x & \text{if} \quad x > 0 \end{cases}. \tag{2.1}$$

Note that (2.2) is defined everywhere (including $x = 0$), and is continuous except at $x = 0$. Note also that the limits $f(0^-)$ and $f(0^+)$ both exist. However, $f(0^-) \neq f(0) \neq f(0^+)$. In practice we may not really care if the function is defined at the points of discontinuity. Consider the function

$$f(x) = \begin{cases} -1 & \text{if} \quad x \leq 0 \\ \frac{1}{x} & \text{if} \quad x > 0 \end{cases}. \tag{2.2}$$

The function is not piecewise continuous because the limit, $\lim_{x \to 0^+}$, does not exist.

2.1.2 Smooth and Piecewise Smooth Functions

A *smooth* function is one whose derivative exists (and is continuous) everywhere. A *piecewise smooth* function is simply a function whose derivative is piecewise continuous. At the points of discontinuity in the derivative, the derivative will be undefined. However, if the function is to be piecewise smooth, the limits of the derivative will exist as a point of discontinuity is approached from either side, since, by definition, the derivative is piecewise continuous. Note that a piecewise continuous function may or may not be piecewise smooth. The function $f(x) = |x|$ is continuous, not smooth, but piecewise smooth. The function $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$, but is not piecewise smooth on $[0, \infty)$ because $\lim_{x \to 0} f'(x)$ does not exist.

2.1.3 Periodic Functions

A periodic function is one that satisfies the following condition:

$$f(x) = f(x + T), \tag{2.3}$$

for all x and for some fixed T which depends on the function. The value of T is known as the *period* of the function. Specifically, T should be the smallest positive value for which (2.3) holds. A periodic function with period T is referred to as a T-*periodic function*. Since (2.3) must hold for all x, for any given x we can add T to it, and so (2.3) must imply

$$f(x + T) = f((x + T) + T) = f(x + 2T) = \cdots = f(x + nT), \tag{2.4}$$

for any positive integer n. We could have also subtracted T from any given x, and so (2.4) really holds for *any* integer n, positive or negative. Now we see why T must be the *smallest* positive value for which (2.3) holds. We wouldn't want to refer to a T-periodic function as $2T$-periodic, even though $f(x) = f(x+nT)$ holds for any integer n, because then we wouldn't know if the function is T-periodic. If we know the smallest T for which (2.3) holds, we automatically know that (2.4) holds.

A periodic function is intuitively familiar—the function "repeats itself" after every interval of length T. It is important to recognize that since (2.3) holds for all x, there is no starting and ending point to a periodic function. You can choose any point you like as a starting point. After an interval of length T, the function is back to where you started.

Note that periodicity has nothing to do with continuity and smoothness. However, if a periodic function is continuous, it must be that $f(x_0^+) = f(x_0^-) = f(x_0^- + T)$ for all x_0. That is, it must be true that the limits, $\lim_{x\to x_0}$ from above ($x > x_0$) and $\lim_{x\to x_0+T}$ from below ($x < x_0 + T$), are equal. Draw a picture of this if you need to.

2.1.4 Even and Odd Functions

Recall that an *even* function is a function $f(x)$ that satisfies the following property:

$$f(x) = f(-x). \tag{2.5}$$

If you graph an even function it is symmetric about the origin. Conversely, an *odd* function $f(x)$ satisfies

$$f(x) = -f(-x). \tag{2.6}$$

If you graph an odd function it is anti-symmetric—that is, it's an inverted mirror image on either side of the origin. Note that if an odd function is continuous, it must be that $f(0) = 0$. It should be intuitively clear that if we integrate an odd function over a symmetric interval about the origin, the integral will vanish:

$$\int_{-L}^{L} f(x)\,dx = 0. \tag{2.7}$$

For an even function, it should be clear that the following relation holds:

$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx. \tag{2.8}$$

It is easy to show that the product of two even functions is again an even function. The product of two odd functions is an even function as well. The product of an odd function and an even function is an odd function.

2.2 Properties of the Trigonometric functions

For Fourier series we will be interested in the trigonometric functions

$$1, \cos(x), \cos(2x), \cos(3x), \ldots, \cos(mx), \ldots, \tag{2.9a}$$

$$\sin(x), \sin(2x), \sin(3x), \ldots, \sin(nx), \ldots, \tag{2.9b}$$

Note that $1, \cos(x)$ and $\sin(x)$ are 2π-periodic. The functions $\cos(nx)$ and $\sin(nx)$ are $2\pi/n$-periodic, but note that because of (2.4), this means that all of the functions in (2.9) are 2π periodic. You should also know that all of the sine functions in (2.9) are odd about the origin, and all of the cosine functions are even about the origin.

2.2.1 Orthogonality

Two functions $f(x)$ and $g(x)$, are said to be *orthogonal* on an interval $[a, b]$, if

$$\int_a^b f(x)g(x)\,dx = 0. \tag{2.10}$$

Note that no non-trivial function (a *trivial* function is one that is identically zero everywhere) is orthogonal to itself, since

$$\int_a^b f(x)f(x)\,dx = \int_a^b |f(x)|^2\,dx > 0. \tag{2.11}$$

The trigonometric functions (2.9) are orthogonal on any 2π interval. (We will mostly be interested in symmetrical intervals around the origin, such as $[-\pi, \pi]$.) Specifically, given any nonnegative integers m and n,

$$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = 0 \quad \text{if} \quad m \neq n \tag{2.12a}$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx\,dx = 0 \quad \text{for all } m, n \tag{2.12b}$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = 0 \quad \text{if} \quad m \neq n. \tag{2.12c}$$

The relations (2.12) are a fundamental aspect of Fourier series. You can easily demonstrate (2.12) using simple trig. substitutions and integration. Because of (2.11), we know that $\sin nx$ and $\cos mx$ are not orthogonal to themselves. In fact

$$\int_{-\pi}^{\pi} \cos^2 mx\,dx = \int_{-\pi}^{\pi} \sin^2 nx\,dx = \pi, \tag{2.13}$$

for all $m, n \neq 0$. If you want to perform these integrations, you should recall the following trig. identities:

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] \tag{2.14a}$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right] \tag{2.14b}$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]. \tag{2.14c}$$

2.3 Fourier Series

A Fourier series is an infinite series of the form:

$$f(x) = \sum_{n=0}^{\infty} \left(a_n \cos nx + b_n \sin nx\right). \tag{2.15}$$

Since $\sin 0x = 0$ and $\cos 0x = 1$, we may also write the series as

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right). \tag{2.16}$$

The first thing to note about a Fourier series is that the sum must be 2π-periodic, since it is the sum of functions, all with the common period 2π. So if the function $f(x)$ equals the series, $f(x)$ must also be 2π periodic. (The fundamental period of a function $f(x)$, if it equals a Fourier Series, must be $2\pi/N$ for some $N = 1, 2, 3, \ldots$. Because of (2.4), this means that the function $f(x)$ must have a period of 2π, even if it has a smaller fundamental period $2\pi/N$.)

As we will see, many useful functions can be represented as a Fourier series.

2.3.1 The Euler Formulas for the Coefficients

Suppose that the 2π periodic function $f(x)$ has a Fourier series representation

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \,. \tag{2.17}$$

Assuming that $f(x)$ is some known function, the coefficients a_0, a_n and b_n in the Fourier series (2.17) are given by the following formulas (known as the Euler formulas):

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx \tag{2.18a}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \tag{2.18b}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \tag{2.18c}$$

Example 2. *The 2π-periodic function*

$$f(x) = |x| \quad \textit{if} \quad -\pi \le x \le \pi, \tag{2.19}$$

can be written as a Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \,. \tag{2.20}$$

The coefficients in (2.20) are determined by the formulas (2.18a). We determine the coefficients by simply integrating:

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |x|\, dx \\ &= \frac{1}{\pi} \int_{0}^{\pi} x\, dx = \frac{1}{\pi} \left[\frac{1}{2} x^2 \right]_0^{\pi} \\ \Rightarrow a_0 &= \frac{\pi}{2}. \end{aligned} \tag{2.21}$$

Now for the a_n:

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx\, dx \\ &= \frac{2}{\pi} \int_{0}^{\pi} x \cos nx\, dx, \end{aligned}$$

then, using integration by parts, we have

$$\begin{aligned}\frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx &= \frac{2}{\pi}\left[x\frac{\sin nx}{n}\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi}\frac{\sin nx}{n}\,dx\\ &= \frac{2}{\pi}\left[\frac{\cos nx}{n^2}\right]_0^{\pi}\\ \Rightarrow a_n &= \frac{2}{\pi}\frac{(-1)^n}{n^2} - \frac{2}{\pi n^2}\end{aligned} \tag{2.22}$$

Now, for the b_n we immediately have:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\sin nx\,dx = 0. \tag{2.23}$$

$$\tag{2.24}$$

How do we know? Remember the properties of odd functions. We can now write the Fourier series of $f(x)$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{2\left((-1)^n - 1\right)}{\pi n^2}\right]\cos nx. \tag{2.25}$$

The partial sums $S_N = \sum_{n=0}^{N} a_n \cos nx$ of the Fourier series are shown in Figure 2.1. Notice that even with 3 terms in the series ($a_2 = 0$), the partial sum S_3 approximates $|x|$ fairly well.

Where do the Euler formulas (2.18a) come from? We can derive them easily by using the orthogonality properties of the trigonometric functions on $[-\pi, \pi]$. Suppose that the function $f(x)$ can be represented as a Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos nx + b_n\sin nx\right). \tag{2.26}$$

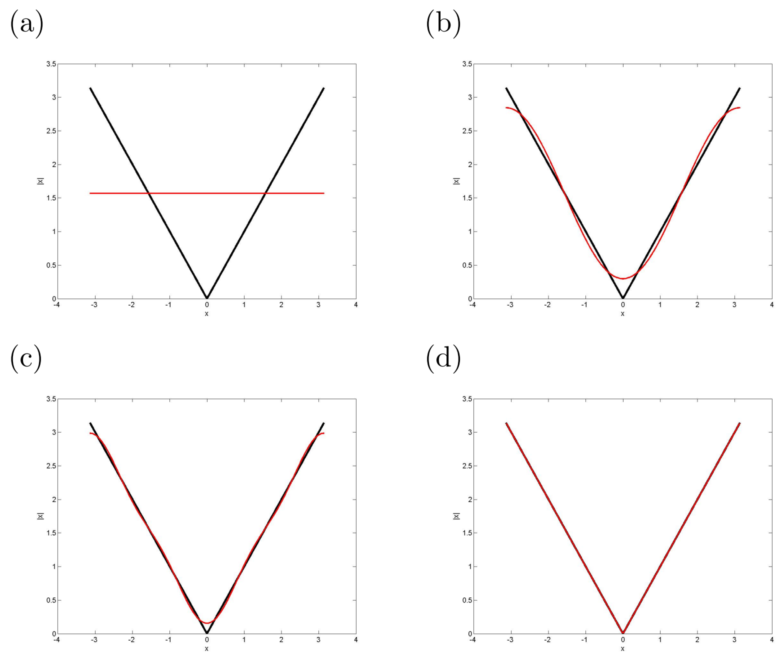

Figure 2.1: Partial sums ($S_N = \sum_{n=0}^{N} a_n \cos nx$) of the Fourier series of $|x|$ are shown in red. (a) 1 term $S_0 = \pi/2$. (b) S_1 (c) S_3 (d) S_{50}.

If we select and arbitrary integer m, we can then take the integral

$$
\begin{aligned}
\int_{-\pi}^{\pi} f(x) \cos mx \, dx &= \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] \cos mx \, dx \\
&= \int_{-\pi}^{\pi} \left[a_0 \cos mx + \sum_{n=1}^{\infty} (a_n \cos nx \cos mx + b_n \sin nx \cos mx) \right] dx \\
&= \int_{-\pi}^{\pi} a_0 \cos mx \, dx + \sum_{n=1}^{\infty} \left(\int_{-\pi}^{\pi} a_n \cos nx \cos mx \, dx + \int_{-\pi}^{\pi} b_n \sin nx \cos mx \, dx \right) \\
&= a_0 \int_{-\pi}^{\pi} \cos mx \, dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \cos mx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \cos mx \, dx \right).
\end{aligned}
\tag{2.27}
$$

Now, because of the orthogonality of the trigonometric functions on $[-\pi, \pi]$, all of the terms in the above sum are zero except for one: $a_m \int_{-\pi}^{\pi} \cos mx \cos mx \, dx$, which is the m^{th} term in the series. Equation (2.27) therefore reduces to simply

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = a_m \int_{-\pi}^{\pi} \cos mx \cos mx \, dx = a_m \pi. \tag{2.28}$$

If we solve for a_m, we have

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \, dx. \tag{2.29}$$

Since m was an arbitrary integer, (2.29) gives the Euler formula for the a_n coefficients. The exact same procedure using 1 and $\sin mx$, gives the formula for a_0 and the b_n coefficients respectively. Note that when we integrate $\int_{-\pi}^{\pi} f(x)\, dx$ to determine a_0, we end up with the equation

$$\int_{-\pi}^{\pi} f(x) \, dx = a_0 \int_{-\pi}^{\pi} 1 \, dx = a_0(2\pi). \tag{2.30}$$

Hence, the reason why the formula for a_0 in (2.18a) has $\frac{1}{2\pi}$ as a constant in front of the integral. In other words, the *norm* of 1: $\int_{-\pi}^{\pi} 1\, dx = 2\pi$ is greater than the *norm* of $\cos mx$: $\int_{-\pi}^{\pi} \cos^2 mx\, dx = \pi$, and the norm of $\sin mx$: $\int_{-\pi}^{\pi} \sin^2 mx\, dx = \pi$.

2.3.2 Fourier Series of Piecewise Continuous Functions

Suppose that we wish to find the Fourier series of a piecewise continuous function $f(x)$, where $f(c^-) \neq f(c^+)$. (In this section we will assume that $f(x)$ is piecewise smooth.) It turns out that regardless of what $f(c)$ is (in fact we don't even need $f(c)$ to be defined), the Fourier series will converge to the average of $f(c^-)$ and $f(c^+)$. That is, if we use the Euler formulas for the coefficients, the following formula holds for all x

$$\frac{f(x^+) + f(x^-)}{2} = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right). \tag{2.31}$$

Note that this is the same as (2.16) since for points of continuity $\frac{1}{2}(f(x^+) + f(x^-)) = f(x)$. This is true even if c is one of the endpoints $x = \pm\pi$. So this means that we can take a function $f(x)$ that is 2π periodic, where $f(-\pi) = f(\pi)$, yet $f(\pi^-) \neq f(-\pi^+)$, compute the Fourier coefficients using the Euler formulas, and the series will converge to $\frac{1}{2}(f(-\pi^+) + f(\pi^-))$ at the endpoints $x = \pm\pi$. However, note that the value of $f(-\pi) = f(\pi)$ (*i.e.* the value right at the single points $x = \pm\pi$) doesn't really matter when computing the Fourier coefficients from the Euler formulas.

2.3.3 Fourier Series of Even and Odd functions

Suppose that we calculate the Fourier series of an odd function $f(x)$. When determining the coefficients from the Euler formulas, note that for a_0 and a_n, the integrals $\int_{-\pi}^{\pi} f(x)\cos nx\,dx$, $n = 0, 1, \ldots$ are all zero, since the integrands are all odd. For b_n, the integrands in $\int_{-\pi}^{\pi} f(x)\sin nx\,dx$ are even functions (odd*odd=even), so the Euler formula could be replaced by

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx. \tag{2.32}$$

For even functions $f(x)$, the opposite is true: the b_n are all zero, and the formula for a_0 and a_n could be replaced by

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx \tag{2.33a}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx. \tag{2.33b}$$

These consequences give rise to what are known as *half-range expansions*, or *sine and cosine series*, instead of the more general Fourier Series. These take the form

The Cosine Series for Functions on $[0, \pi]$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \tag{2.34a}$$

where

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx, \tag{2.34b}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx. \tag{2.34c}$$

The Sine Series for Functions on $[0, \pi]$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \tag{2.35a}$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx. \tag{2.35b}$$

2.3.4 Fourier Series on Arbitrary Intervals

So far we have assumed that our functions are of interest on the interval $[-\pi, \pi]$, (or perhaps $[0, \pi]$ as found in the previous section). You probably wonder, in the real world won't we be interested in functions on arbitrary domains? Of course the answer is yes. However, we can extend our definitions to intervals of arbitrary length and position by a simple change of variables. For instance, suppose we are interested in a function $g(x)$, defined on the interval $x = [3.2, 7.2]$. We can introduce a change of variables, $X = \pi\frac{x-5.2}{2}$, and now you have a function $f(X) = g(x(X))$ defined on $[-\pi, \pi]$.

However, occasionally it will be convenient to extend our concept of a Fourier series directly to the more general case, without making use of a change of variables. We will restrict this to scaling, and note that you can always shift your domain by a change of variables if necessary.

Suppose that we are interested in a function $f(x)$ on the interval $[-L, L]$, where L is some value depending on the problem. Consider the following trigonometric functions

$$\left\{1, \cos(\tfrac{m\pi}{L}x), \sin(\tfrac{n\pi}{L}x)\right\}, \quad \forall\, m, n \in \mathcal{Z}^+. \tag{2.36}$$

($\mathcal{Z}^+$ is the set of all positive integers.) These are really the same as the functions (2.9), except the factor of $(\frac{\pi}{L})$, which scales the functions so that they are $2L$ periodic instead of 2π periodic. You can easily show that (2.36) are orthogonal on the interval $[-L, L]$. We can then express functions on arbitrary intervals as a more general Fourier series, where the coefficients in the series can determined, using the same method as before

Fourier Series on Arbitrary Intervals $[-L, L]$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(\tfrac{n\pi}{L}x) + b_n \sin(\tfrac{n\pi}{L}x)\right). \tag{2.37}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \tag{2.38a}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\tfrac{n\pi}{L}x)\,dx \tag{2.38b}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\tfrac{n\pi}{L}x)\,dx. \tag{2.38c}$$

Of course, the sine and cosine series can be generalized to functions on $[0, L]$, and the formulas are changed accordingly.

The Cosine Series on Arbitrary Intervals $[0, L]$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\tfrac{n\pi}{L}x), \tag{2.39a}$$

where

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \tag{2.39b}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(\tfrac{n\pi}{L}x)\,dx. \tag{2.39c}$$

The Sine Series on Arbitrary Intervals $[0, L]$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(\tfrac{n\pi}{L}x), \tag{2.40a}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(\tfrac{n\pi}{L}x)\,dx. \tag{2.40b}$$

2.3.5 Convergence of Series Computed from Non-Periodic Functions, and Half-Range Series of Non-Even or Non-Odd Functions.

So far we have discussed the fact that we can take a periodic function (of period 2π or later more generally of period $2L$) and write the function as a series—an infinite sum that converges to (equals) the function everywhere, except where the function has a jump discontinuity, in which case the series converges to the average value of the function on either side of the discontinuity. When we compute the formula for the series, that is, determine the coefficients in the series using the Euler formulas, we only needed to use the formula for the function on one of the periodic intervals, say $x = [-L, L]$ (from now on I'll only consider the more general series using L, realizing that this includes the special case where $L = \pi$). So this begs the question: does $f(x)$ have to be periodic to have a Fourier series? The answer is no!

We can use the formula for any function on an interval $x = [-L, L]$ and compute the series as if the function were periodic. The resulting Fourier series itself will be $2L$ periodic, and on the interval $(-L, L)$ the Fourier series will converge to (equal) the function (or the average at a jump discontinuity). The only caveat, is that the Fourier series converges to $\frac{1}{2}(f(-L^+) + f(L^-))$ at $x = \pm L$, as if the function were periodic and $f(-L^+) = f(L^+)$, so that the Fourier series converges to the average value at a jump discontinuity occurring at an endpoint. This means that if we take the Fourier series of a continuous but non-periodic function, where $f(-L) \neq f(L)$, the Fourier series will be only piecewise continuous, with jump discontinuities at the endpoints of each interval $[(2n-1)L, (2n+1)L]$, $n \in \mathcal{Z}$. The Fourier series of a non-periodic function therefore gives us the *periodic extension* of the function. That is, if we take any function $f(x)$ defined on $[-L, L]$, we can define the $2L$ periodic extension $\tilde{f}(x)$ as

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if} \quad -L \leq x \leq L, \\ f(x \pm 2L) & \text{otherwise} \quad . \end{cases} \tag{2.41a}$$

Don't let this recursive mathematical definition get you confused. All it is saying is, "the periodic extension of $f(x)$, *i.e.* $\tilde{f}(x)$, is the same as $f(x)$ on the interval $[-L, L]$, and everywhere else $\tilde{f}(x)$ is $2L$-periodic regardless of what $f(x)$ looks like outside the interval $[-L, L]$." The Fourier series converges to $\tilde{f}(x)$ everywhere (or more generally $\frac{1}{2}(\tilde{f}(x^+) + \tilde{f}(x^-))$ at discontinuities). (If you are having trouble visualizing this mathematical description, draw some graphs as you go.)

Note that for the half range expansions, cosine and sine series, we really only use $f(x)$ on the interval $[0, L]$, since for odd and even functions, we know the value on $[-L, 0]$ from the value on $[0, L]$. This begs the question: can we take any function $f(x)$ (whether it is even, odd, or neither) and calculate its sine or cosine series. The answer is yes! It doesn't matter what the function is outside of $[0, L]$, we can calculate its sine or cosine series as if it was even or odd plus $2L$ periodic. Of course the cosine or sine series will be even or odd respectively outside of the interval $[0, L]$, and both will be $2L$ periodic. However, we may only care about the function on $[0, L]$. The resulting sine series of such a function is known as an *odd periodic extension*, and the resulting cosine series is known as an *even periodic extension*. We will denote these as $\tilde{f}_o(x)$ and $\tilde{f}_e(x)$ respectively. The odd extension of a function defined on $x \geq 0$ can be mathematically defined as

$$f_o(x) = \begin{cases} f(x) & \text{if} \quad x \geq 0 \\ , -f(-x) & \text{otherwise,} \end{cases} \tag{2.42a}$$

and the even periodic extension is defined by

$$f_e(x) = \begin{cases} f(x) & \text{if} \quad x \geq 0 \\ , f(-x) & \text{otherwise.} \end{cases} \tag{2.43a}$$

To define the even or odd periodic extension of $f(x)$ we simply apply the definition of a $2L$ periodic extension of $f_e(x)$ or $f_o(x)$ from $[-L, L]$. That is,

$$\tilde{f}_e(x) = \begin{cases} f_e(x) & \text{if} \quad -L \leq x \leq L, \\ f_e(x \pm 2L) & \text{otherwise,} \end{cases} \tag{2.44a}$$

or

$$\tilde{f}_o(x) = \begin{cases} f_o(x) & \text{if} \quad -L \leq x \leq L, \\ f_o(x \pm 2L) & \text{otherwise} \quad . \end{cases} \tag{2.45a}$$

You should convince yourself that an even periodic extension of a continuous function is still continuous, but an odd periodic extension of a continuous function is only continuous if $f(0) = f(\pi) = 0$.

Chapter 3

The 1D Heat and Wave Equations on Finite Domains—Separation of Variables

In this chapter we will investigate time-dependent PDEs in one spatial dimension with a finite domain. That is, the solutions will be functions of x, on an interval of finite length that change with time. Our solution will therefore be a function, $u(x, t)$, defined on some interval. What u represents will depend on the particular problem we are investigating. Our two most important examples will be the heat equation and the wave equation. We will learn about an important technique called *separation of variables* for these types of problems.

3.1 The Heat Equation

Imagine that we have some heat conducting object that only varies in one spatial dimension. For instance, suppose that we have a metal bar of length L that is uniform in cross section. Assume further that the bar is insulated completely from its surroundings on the sides, yet not necessarily at the ends of the bar. If we use x to denote the coordinate along the length of the bar, it can be shown that the temperature in the bar satisfies the heat equation:

$$u_t - \gamma^2 u_{xx} = 0, \tag{3.1}$$

where γ^2 is a positive constant that depends on the material properties of the bar. Notice that if we add some constant to u, it still satisfies the equation, so we can assume that u represent the difference in temperature from some value that we choose. That is, we will define $u(x,t) = T(x,t) - T_0$, where T_0 is some chosen temperature, perhaps the ambient temperature of the room for instance. (In a moment we will derive (3.1), from some more basic physical assumptions. For now, just take it for granted.)

Now, suppose that we have some way of fixing the temperature at the ends of the bar—for instance, we might have some appliance with a thermostat that can quickly heat or cool the ends of the bar, maintaining a certain temperature at the ends for all time. If we define the left end of the bar to be at $x = 0$, and the right end to be at $x = L$, we can write this condition mathematically:

$$u(0,t) = g_0, \tag{3.2a}$$

$$u(L,t) = g_L, \tag{3.2b}$$

where g_0 is the temperature we choose at the left end of the bar and g_L is the temperature we choose at the right end. The set of conditions (3.2) are known as *boundary conditions.* They tell us what the solution $u(x,t)$ must be at the boundaries of the domain for all time. These conditions are not required to be constant in time—in general g_0 and g_L might be functions of time. If so, their dependence on t will be specified:

$$u(0,t) = g_0(t), \tag{3.3a}$$

$$u(L,t) = g_L(t). \tag{3.3b}$$

Now, suppose that at some starting time, say $t = 0$, we know the temperature profile in the bar. For instance, suppose that we heated the bar with a blowtorch before beginning our experiment, and we determined that the bar has a temperature at $t = 0$,

$$u(x,0) = f(x) \quad \text{for} \quad 0 \le x \le L. \tag{3.4}$$

$$\tag{3.5}$$

The condition (3.4) is an *initial condition.* It tells us the solution on the entire domain at some starting time. Note that $f(x)$ is some function that we are assuming is known. You won't always be given a familiar formula for the initial condition, but you should realize that $f(x)$ represents some known function, not one that you are trying to find. We can summarize this as follows:

$$
\begin{aligned}
&\text{PDE}: && u_t - \gamma^2 u_{xx} = 0, \quad 0 < x < L \qquad 0 < t < \infty && (3.6a)\\
&\text{BCs}: && \begin{cases} u(0,t) = g_0, \\ u(L,t) = g_L, \end{cases} \quad 0 < t < \infty, && (3.6b)\\
&\text{IC}: && u(x,0) = f(x), \quad 0 \le x \le L. && (3.6c)
\end{aligned}
$$

This set of conditions is known as an *initial-boundary value problem*, or sometimes simply a *boundary value problem*. You should remember that it contains three parts. First, the PDE or governing equation (in this case, the heat equation) describes what physical law our system satisfies. The initial condition describes how the system begins, and the boundary conditions describe how the surroundings affect our system. The goal of the problem is to determine $u(x,t)$ on the domain $0 < x < L$ for all time $0 < t < \infty$.

It turns out that the solution to (3.6a) exists and is unique! That means that $u(x,t)$ not only exists, but that there is only one possible answer. Determining existence and uniqueness will often be beyond the scope of this class, but realize that if existence and uniqueness are satisfied, finding *a* solution is finding *the* solution. In general, existence and uniqueness depend on the PDE, the boundary conditions, and the initial conditions. You can usually assume that the problems posed in this class will satisfy existence and uniqueness requirements.

3.1.1 Deriving the Heat Equation from Physical Principles

We can derive the heat equation for a metal bar, from assuming some basic physical principles that we will take for granted—these are known from theory and experiment, as mathematicians we will take them as our starting point. The heat equation follows from conservation of energy. (Often, if not always, PDEs in the real world come from conservation principles. *i.e.* conservation of mass, conservation of momentum, conservation of energy, etc.) The energy in a given length of a metal bar is known to be related to the temperature in the bar, $u(x,t)$, the cross-section of the bar A, the density of the metal ρ and a constant c_0, known as the thermal capacity—a constant depending on the material properties of the bar and relating temperature to energy. (Don't worry too much about these constants—that's not the main point here.) If we integrate between two points in the bar, we get the total energy in that interval of the bar

$$\text{Total Energy in } [x_1, x_2] = \int_{x_1}^{x_2} A\rho c_0 u(x,t)\, dx. \tag{3.7}$$

As you know energy (heat) flows from hotter regions to colder ones. (This should be intuitively very familiar). The larger the gradient in the temperature, the faster the heat will flow. This flow of heat, or flow of anything per unit time, per unit area, is known as a *flux*. If we denote the heat flux ϕ, it turns out that the flux of heat in a metal bar obeys what is known as Fourier's law of heat conduction (he did more than come up with a series):

$$\phi(x) = -K_0 \frac{\partial}{\partial x} u(x,t), \tag{3.8}$$

where the constant K_0 is known as the thermal conductivity. It relates how fast heat can flow in a material. In an insulator K_0 would be small, and in a good conductor K_0 would be large. Note that (3.8) says that heat flows in the positive direction if the slope of temperature is negative, and in the negative direction if the slope is positive. That should make sense.

Now, we consider the change in the amount of energy per unit time in the interval $[x_1, x_2]$. The only way for the energy to change in this interval of the bar, is by heat flowing past the endpoints. The change in the total energy from in $[x_1, x_2]$, per unit time, is therefore

$$\begin{aligned}&\text{Rate of Change in the Total Energy in } [x_1, x_2] = \\ &\qquad A\phi(x_1) - A\phi(x_2) = AK_0 \left(u_x(x_2,t) - u_x(x_1,t) \right).\end{aligned} \tag{3.9}$$

We can also express the change in the total energy per unit time as the time derivative of (3.7), or:

$$\begin{aligned}&\text{Rate of Change in the Total Energy in } [x_1, x_2] = \\ &\qquad \frac{d}{dt} \int_{x_1}^{x_2} A\rho c_0 u(x,t)\, dx.\end{aligned} \tag{3.10}$$

We can therefore equate (3.9) and (3.10), and drop the A from both sides, giving:

$$\frac{d}{dt} \int_{x_1}^{x_2} u(x,t)\, dx = \gamma^2 \left(u_x(x_2,t) - u_x(x_1,t) \right), \tag{3.11}$$

where $\gamma^2 = \rho c_0 / K_0$. The constant γ^2 is always positive. Now, we just need to use a little calculus. First, I can move the time derivative inside the integral on the left hand side of (3.11)

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = \int_{x_1}^{x_2} \frac{\partial}{\partial t} u(x,t)\,dx. \tag{3.12}$$

Second, by using the fundamental theorem of calculus, note that the right hand side of (3.11) can be written

$$(u_x(x_2,t) - u_x(x_1,t)) = \int_{x_1}^{x_2} \frac{\partial}{\partial x} u_x(x,t)\,dx. \tag{3.13}$$

Using these facts, (3.11) becomes

$$\int_{x_1}^{x_2} \frac{\partial}{\partial t} u(x,t)\,dx = \gamma^2 \int_{x_1}^{x_2} \frac{\partial}{\partial x} u_x(x,t)\,dx \tag{3.14a}$$

$$\Rightarrow \int_{x_1}^{x_2} \left[u_t(x,t) - \gamma^2 u_{xx}(x,t)\right] = 0 \tag{3.14b}$$

$$\Rightarrow u_t(x,t) - \gamma^2 u_{xx}(x,t) = 0. \tag{3.14c}$$

Regarding the last equation in (3.14): if a definite integral is zero, obviously that does not necessarily imply that an integrand is zero. However, we chose x_1 and x_2 arbitrarily. Therefore, the integral in the second line of (3.14) is zero on any and every interval we could have chosen. That does imply that the integrand is zero.

3.2 Solving the Heat Equation: Separation of Variables

3.2.1 A Metal Bar with the Temperature fixed at the ends

We are going to solve the following IBVP:

$$\text{PDE}: \quad u_t - \gamma^2 u_{xx} = 0, \quad 0 < x < L \qquad 0 < t < \infty \tag{3.15a}$$

$$\text{BCs}: \quad \begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}, \quad 0 < t < \infty \tag{3.15b}$$

$$\text{IC}: \quad u(x,0) = f(x), \quad 0 \le x \le L. \tag{3.15c}$$

We will use a technique called *separation of variables*, which is a standard and fundamental technique for solving many PDEs. The key step in the separation of variables technique, is to first look for solutions to the PDE only (later we'll consider the BCs and eventually the IC), of the special form

$$u(x,t) = X(x)T(t). \tag{3.16}$$

Of course, not all functions of x and t are of the form (3.16), but that doesn't concern us yet—we just want to find a solution. If (3.16) is a solution to the PDE, then we can plug it in:

$$\begin{aligned}
&\frac{\partial}{\partial t}u - \gamma^2 \frac{\partial^2}{\partial x^2}u = 0 \\
\Rightarrow &\frac{\partial}{\partial t}X(x)T(t) - \gamma^2 \frac{\partial^2}{\partial x^2}X(x)T(t) = 0 \\
\Rightarrow &X(x)T'(t) - \gamma^2 X''(x)T(t) = 0 \\
\Rightarrow &\frac{T'(t)}{\gamma^2 T(t)} - \frac{X''(x)}{X(x)} = 0 \\
\Rightarrow &\frac{T'(t)}{\gamma^2 T(t)} = \frac{X''(x)}{X(x)}. \qquad (3.17)
\end{aligned}$$

Now, notice that we have a function of t only equalling a function of x only. Is this possible? If you think about it, that means that they both must be fixed constants! For now, let's call this constant k. It is known as the *separation constant*. Any value of separation constant would work, and the last equation of (3.17) would be satisfied. The following relationship

$$\frac{T'(t)}{\gamma^2 T(t)} = \frac{X''(x)}{X(x)} = k \tag{3.18}$$

after rearranging, yields two equations:

$$T'(t) - k\gamma^2 T(t) = 0 \tag{3.19a}$$

and

$$X''(x) - kX(x) = 0. \tag{3.19b}$$

These are two ODEs for the functions $T(t)$ and $X(x)$. If we solve them both, we'll have a solution $u(x,t) = T(t)X(x)$ satisfying the PDE. First, recall that the solution to (3.19a) is the exponential function

$$T(t) = Ce^{k\gamma^2 t}, \tag{3.20}$$

where C is some arbitrary constant. Now, there are no requirements on k such that the PDE will be satisfied—any value would work. But let's invoke a restriction on k so that our solutions will at least make sense, physically. If this solution is $u(x,t) = T(t)X(x)$, then, if k is positive, then $k\gamma^2$ is positive, and $T(t)$ would go to infinity as $t \to \infty$. This doesn't make physical sense—we know that the temperature in the bar is not going to keep increasing and increasing. So, we'll place our first restriction on k: we will only consider nonpositive values of k. (If this line of reasoning troubles you, later we will see that the boundary conditions will restrict k to nonpositive values.) In order to remember this, let's say $k = -\lambda^2$. We have

$$T(t) = Ce^{-\lambda^2\gamma^2 t}. \tag{3.21}$$

Now, lets solve (3.19b), which is now

$$X''(x) + \lambda^2 X(x) = 0. \tag{3.22}$$

Recall that the solution to (3.70b) is

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x), \tag{3.23}$$

where A and B are arbitrary constants. We now have a solution to the PDE:

$$u(x,t) = T(t)X(x) = Ce^{-\lambda^2\gamma^2 t}\left(A\cos(\lambda x) + B\sin(\lambda x)\right). \tag{3.24}$$

This solution will satisfy the PDE, no matter what the value of λ is. You can check this for yourself by plugging it into the PDE.

Now, we are free to choose any value of λ. Let's try and pick values such that (3.24) satisfies the boundary conditions. These require

$$u(0,t) = T(t)X(0) = 0, \tag{3.25a}$$
$$u(L,t) = T(t)X(L) = 0. \tag{3.25b}$$

Since $T(t)$ does not depend on x, and we obviously don't want trivial solutions $u(x,t) = 0$, (3.25) is a requirement on $X(x)$

$$X(0) = 0, \tag{3.26a}$$
$$X(L) = 0. \tag{3.26b}$$

If we enforce the left boundary condition we have

$$X(0) = A\cos(\lambda 0) + B\sin(\lambda 0) = A = 0. \tag{3.27}$$

So we now have

$$X(x) = B\sin(\lambda x). \tag{3.28}$$

Enforcing the right boundary condition gives

$$X(L) = B\sin(\lambda L) = 0. \tag{3.29}$$

We certainly don't want $B = 0$ or that would give us a trivial solution for $u(x,t) = T(t)X(x)$. We can satisfy (3.29) if we require that $\lambda = \frac{n\pi}{L}$, for any integer n. We have infinitely many possibilities, so let's denote the constant $\lambda_n = \frac{n\pi}{L}$, and denote one of our solutions as

$$X_n(x) = B_n\sin(\lambda_n x) = B_n\sin(\tfrac{n\pi}{L}x). \tag{3.30}$$

We can now write our solution as

$$\begin{aligned} u_n(x,t) = T_n(t)X_n(x) = C_n e^{-\lambda_n^2\gamma^2 t} B_n \sin(\tfrac{n\pi}{L}x) \\ = b_n e^{-(\frac{n\pi}{L})^2\gamma^2 t}\sin(\tfrac{n\pi}{L}x), \end{aligned} \tag{3.31}$$

where we combine $C_nB_n = b_n$ in the final equation. The subscript on the $u_n(x,t)$ just means that any integer can be chosen and the PDE is satisfied by (3.31), and the boundary conditions are satisfied as well!

Now, the only thing not necessarily satisfied are the initial conditions. But we still have the arbitrary constant b_n to work with. Unless we have very special initial conditions of the form (3.31) for some value of n, simply picking one value for n and b_n will not work. However, recall that the heat equation is linear. So a linear combination of solutions is also a solution, as we discussed in the beginning of the course. So we can add up as many solutions of the form (3.31) as we need (we have one for every integer n). In fact, we can form an infinite sum

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n^2\gamma^2 t}\sin(\tfrac{n\pi}{L}x), \tag{3.32}$$

and expect $u(x,t)$ to be a solution to the PDE. The sum $u(x,t)$ will also satisfy the B.C's, since each $u_n(x,t)$ is zero at $x = 0, L$. (Later we will discuss linear B.C's in general, which are B.C's that are satisfied by linear combinations of functions that satisfy them.) If we now evaluate (3.33) at $t = 0$, we have

$$u(x,0) = \sum_{n=1}^{\infty} u_n(x,0) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n^2\gamma^2 0}\sin(\tfrac{n\pi}{L}x) = \sum_{n=1}^{\infty} b_n\sin(\tfrac{n\pi}{L}x). \tag{3.33}$$

If we can satisfy

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(\tfrac{n\pi}{L}x), \tag{3.34}$$

then we'd have a unique solution, and we'd be done! Is this possible? Yes—it's just a Fourier sine series. We only need to determine the coefficients b_n. We use the formulas for a sine series on $[0, L]$:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(\tfrac{n\pi}{L}x), \tag{3.35a}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(\tfrac{n\pi}{L}x)\,dx. \tag{3.35b}$$

So our solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(\frac{n\pi}{L})^2\gamma^2 t}\sin(\tfrac{n\pi}{L}x), \tag{3.36a}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(\tfrac{n\pi}{L}x)\,dx. \tag{3.36b}$$

If we consider the solution (3.36a) (regardless of the form of the initial condition and hence the values of the constants b_n), we see that as time progresses the value of each term

$$e^{-(\frac{n\pi}{L})^2\gamma^2 t}$$

decreases in magnitude. (Recall that $\gamma^2 > 0$.) In fact, as $t \to \infty$, the solution decays to zero, the value of the temperature at the ends of the bar. So the eventual solution is just $u(x,t) \equiv 0$. Note that if the energy is proportional to the integral of the temperature

$$\text{Energy} \propto \int_0^L u(x,t)\,dx,$$

the energy in the bar goes to zero as $t \to \infty$. The initial energy in the bar was proportional to the integral of the initial condition

$$\text{Initial Energy} \propto \int_0^L f(x)\,dx.$$

If this value was initially less than zero (corresponding to a bar that is initially colder on average than our temperature defined to be 0), the energy in the bar increases as $t \to \infty$. If this value was initially greater than zero (corresponding to a heated bar), then the energy in the bar increases as $t \to \infty$. This should all be physically intuitive.

3.2.2 A Bar that is Insulated on the Ends

Now, suppose that we have a metal bar that, rather than having the temperature at the ends of the bar held at a fixed temperature, suppose that the bar is perfectly insulated on the ends. Recall that we found the heat flux ϕ to be proportional to the slope of the temperature

$$\phi = -K_0 \frac{\partial}{\partial x} u(x,t).$$

If a bar is perfectly insulated at the ends, that means that there is no heat flux at the endpoints. Otherwise, energy would have to be entering the bar or leaving the bar depending on the sign of $\frac{\partial}{\partial x}u(0,t)$ and $\frac{\partial}{\partial x}u(L,t)$. So the appropriate boundary condition for a perfectly insulated bar, is that the slope of the temperature must be zero at the ends. We therefore can cast this problem as an IBVP

$$\text{PDE}: \quad u_t - \gamma^2 u_{xx} = 0, \quad 0 < x < L \qquad 0 < t < \infty \tag{3.37a}$$

$$\text{BCs}: \quad \begin{cases} u_x(0,t) = 0 \\ u_x(L,t) = 0 \end{cases}, \quad 0 < t < \infty \tag{3.37b}$$

$$\text{IC}: \quad u(x,0) = f(x), \quad 0 \le x \le L. \tag{3.37c}$$

Now, we proceed just as before, using the technique of separation of variables. Eventually we end up with our solution for $X(x)$ just as in (3.23)

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x). \tag{3.38}$$

Recall from above that we required that $X(x)$ meet the boundary conditions of our IBVP. We will do the same for this problem. First, since $u_x(0,t) = 0$ and $u_x(L,t) = 0$, we require that $X'(0) = X'(L) = 0$. Differentiating (3.38) gives

$$X'(x) = -A\lambda\sin(\lambda x) + B\lambda\cos(\lambda x). \tag{3.39}$$

First we enforce the left boundary condition

$$X'(0) = -A\lambda \sin(\lambda 0) + B\lambda \cos(\lambda 0) = B\lambda = 0. \tag{3.40}$$

Since we don't want to restrict our solutions to the case where $\lambda = 0$, we must require $B = 0$. So our function $X(x)$ is now

$$X(x) = A\cos(\lambda x) \tag{3.41}$$

Now enforcing the right boundary condition using (3.39) gives

$$X'(L) = -A\lambda \sin(\lambda L) = 0. \tag{3.42}$$

This looks very similar to what we found for the bar in the previous section. We can enforce this condition by requiring

$$\lambda L = n\pi, \tag{3.43}$$

where n is any positive integer or zero. Note that $n = 0$ works, but unlike before that does not give the trivial solution $X(x) = 0$. We therefore denote $\lambda_n = \frac{n\pi}{L}$, and denote our solutions $X(x)$ and $T(t)$ with subscripts

$$X_n(x) = A_n \cos(\tfrac{n\pi}{L} x), \tag{3.44a}$$

$$T_n(t) = C_n e^{-(\frac{n\pi}{L})^2 \gamma^2 t}, \tag{3.44b}$$

so that our separable solution to the PDE is of the form

$$u_n(x,t) = T_n(t) X_n(x) = a_n e^{-(\frac{n\pi}{L})^2 \gamma^2 t} \cos(\tfrac{n\pi}{L} x), \tag{3.44c}$$

where we have combined the arbitrary constants A_n and C_n, into one new one $a_n = A_n C_n$. Now, it is important to realize that (3.44c) is a solution to the PDE, and boundary condition for any integer $n = 0, 1, 2, \ldots,$ and any arbitrary constant a_n. Since the PDE is linear, (and the boundary conditions are linear as explained later), any linear combination of solutions still satisfies the PDE and boundary conditions. We'll take this to the extreme, and assume that an infinite linear combination works as well

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-(\frac{n\pi}{L})^2 \gamma^2 t} \cos(\tfrac{n\pi}{L} x). \tag{3.45}$$

Now as before, all we need to do is make sure that (3.45) satisfies the initial condition and we've found our solution. Is this possible? Sure, we just look at the initial solution

$$u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\tfrac{n\pi}{L} x), \tag{3.46}$$

and note that we can define the coefficients a_n by a cosine series for $f(x)$ on $0 \le x \le L$. So the solution is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-(\frac{n\pi}{L})^2\gamma^2 t} \cos(\tfrac{n\pi}{L}x), \tag{3.47a}$$

where

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \tag{3.47b}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(\tfrac{n\pi}{L}x)\,dx. \tag{3.47c}$$

What happens to the solution as $t \to \infty$? Well, as with the previous problem, all of the terms for $n > 0$ go to zero. However, this time the temperature approaches a single constant value a_0 as $t \to \infty$. What does this mean? Notice from the solution that a_0 represents the average value of the initial temperature. So the solution approaches that value. As for the energy, the energy in the bar remains the same—it is equivalent to the starting energy. You may object that, "we only have shown that the energy for $t = 0$ and $t = \infty$ is the same. We don't know that the energy is the same for all t". Technically you'd be right, but in fact, we can show that the energy is the same for all time by integrating the solution from $x = 0$ to $x = L$. I'd do that now, but I don't want to spoil the fun. I know that you'll enjoy doing that on your own for homework. If you can't handle the suspense—do it now.

3.2.3 Steady-State Solutions

In both of the above examples, we found that the solution approached some function that no longer changed in time. A solution to a PDE that does not change with time is known as a *steady-state solution.* That is, the solution is steady, or unchanging with respect to time. Such a solution is really only a function of x, since it no longer depends on t. We can therefore write a steady-state solution as $u(x,t) = \bar{U}(x)$. For the heat equation in the above two examples, we found that the solution approaches the steady-state solution as time progresses, or $u(x,t) \to \bar{U}(x)$ as $t \to \infty$. However, don't be under the impression that all solutions to all PDEs behave this way. As we'll find with the wave equation, some solutions to PDEs go on changing forever.

What can we say mathematically about a steady-state solution $u(x,t) = \bar{U}(x)$ to the heat equation IBVP? Well, the steady-state solution certainly still obeys the PDE, it's just a special case of solution. If we plug the steady-state solution into the PDE, we have

$$\begin{aligned} &\frac{\partial}{\partial t}\bar{U}(x) - \gamma^2 \frac{\partial^2}{\partial x^2}\bar{U}(x) = 0 \\ &\Rightarrow \gamma^2 \frac{\partial^2}{\partial x^2}\bar{U}(x) = 0 \\ &\Rightarrow \bar{U}''(x) = 0. \end{aligned} \tag{3.48}$$

That is, the second derivative of $\bar{U}$ is zero. Recall from basic calculus, that means that $\bar{U}(x)$ has no concavity, or, it's just a straight line. The solution $\bar{U}(x)$ must also satisfy the boundary conditions, which apply for all $t > 0$. In the first example we looked at, the boundary conditions would require that $\bar{U}(0) = \bar{U}(L) = 0$. Since $\bar{U}(x)$ is a straight line, the only such straight line is the constant 0. In the second example of an insulated bar, the slope was zero at the ends. For this problem, the steady-state solution might be a constant other than 0, since any constant (in x) has zero slope everywhere.

3.2.4 A More General Form of Linear Boundary Conditions

So far we've looked at homogeneous boundary conditions, of the form

$$u(0,t) = u(L,t) = 0,$$

and

$$u_x(0,t) = u_x(L,t) = 0.$$

The boundary conditions are *homogeneous* since they equal 0. These are both a special case of a much more general set of linear boundary conditions of the form

$$\alpha_1 u(0,t) + \beta_1 u_x(0,t) = g_1(t), \tag{3.49a}$$

$$\alpha_2 u(L,t) + \beta_2 u_x(L,t) = g_2(t), \tag{3.49b}$$

where it is assumed that not both of α_1 and β_1 are zero, and the same for α_2 and β_2. If $g_1(t) = g_2(t) = 0$, the boundary conditions (3.49) are said to be homogeneous. Otherwise, they are nonhomogeneous. You should verify the following: these linear boundary conditions are satisfied by an arbitrary linear combination of functions that satisfy the boundary conditions, if and only if, they are homogeneous. This is not true in general if $g_1(t)$ or $g_2(t)$

is nonzero. When we used separation of variables, we needed the PDE and BC's to be linear and homogeneous. See if you can determine at what stage this requirement was needed. (Hint: at some point we assumed that an infinite sum of solutions was still a solution.)

3.2.5 Non-Homogeneous Boundary Conditions

If we require homogeneous boundary conditions in order to be able to use separation of variables, then what do we do in the case of non-homogeneous boundary conditions? In this section we'll look at a few simple cases that we can deal with given the tools that we have so far. Later we will see how we can deal with the general problem of non-homogeneous BC's of any kind of the form (3.49). Consider the heat equation with nonzero Dirichlet BCs:

$$\text{PDE :} \quad u_t - \gamma^2 u_{xx} = 0, \quad 0 < x < L, \quad 0 < t < \infty, \tag{3.50a}$$

$$\text{BCs :} \quad \begin{cases} u(0,t) = A, \\ u(L,t) = B, \end{cases} \quad 0 < t < \infty, \tag{3.50b}$$

$$\text{IC :} \quad u(x,0) = f(x), \quad 0 \le x \le L, \tag{3.50c}$$

where A and B are constants. Now, what we are going to try and do is convert this nonhomogeneous problem for $u(x,t)$ to a homogeneous problem for a new variable that is related to $u(x,t)$ in some known way. Here's how we will proceed: we will let $u(x,t)$ be the sum of two functions

$$u(x,t) = \tilde{u}(x,t) + h(x), \tag{3.51}$$

where $h(x)$ is some known function that we are going to choose. Note that in this step we are not assuming anything about the solution $u(x,t)$, even though we will specify what $h(x)$ is. Think of it this way, we are really just defining the new quantity $\tilde{u}(x,t)$ by the equation

$$\tilde{u}(x,t) = u(x,t) - h(x). \tag{3.52}$$

Now, we are going to choose $h(x)$ to be the steady-state solution satisfying the nonhomogeneous boundary conditions. Recall that $\bar{U}(x)$ is the steady-state solution—a straight line satisfying the boundary conditions. Using a general formula for a straight line, we write

$$\bar{U}(x) = mx + b, \tag{3.53}$$

and we'll determine m and b such that the boundary conditions are satisfied. For the left boundary condition, we require

$$\bar{U}(0) = b = A, \tag{3.54}$$

which determines the value of b, and for the right boundary we require

$$\begin{aligned} \bar{U}(L) &= mL + A = B \\ \Rightarrow m &= \frac{B-A}{L}, \end{aligned} \tag{3.55}$$

which just says that the slope is rise over run. So the steady-state solution is

$$\bar{U}(x) = \frac{B-A}{L}x + A. \tag{3.56}$$

You might have been able to immediately write the equivalent formula for $\bar{U}(x)$

$$\bar{U}(x) = \frac{A(L-x)}{L} + \frac{Bx}{L}, \tag{3.57}$$

but to each their own. Now, we write the solution as

$$u(x,t) = \tilde{u}(x,t) + \bar{U}(x), \tag{3.58}$$

where $\bar{U}(x)$ is now known. Incidentally, the function $\tilde{u}(x,t)$ is known as the *transient part* of the solution. If $u(x,t)$ approaches the steady-state solution, then the transient solution must go to zero. Now, we use the IBVP (3.50) to find another IBVP for $\tilde{u}(x,t)$. First, if $u(x,t)$ satisfies the PDE, then we can plug it in

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} - \gamma^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0, \\ \Rightarrow &\frac{\partial}{\partial t}(\tilde{u}(x,t) + \bar{U}(x)) + -\gamma^2 \frac{\partial^2}{\partial x^2}(\tilde{u}(x,t) + \bar{U}(x)) = 0 \\ \Rightarrow &\frac{\partial \tilde{u}(x,t)}{\partial t} + \frac{\partial \bar{U}(x)}{\partial t} - \gamma^2 \frac{\partial^2 \tilde{u}(x,t)}{\partial x^2} - \gamma^2 \frac{\partial^2 \bar{U}(x)}{\partial x^2}, \\ \Rightarrow &\frac{\partial \tilde{u}(x,t)}{\partial t} - \gamma^2 \frac{\partial^2 \tilde{u}(x,t)}{\partial x^2} = 0, \end{aligned} \tag{3.59}$$

so $\tilde{u}(x,t)$ obeys the exact same PDE. (Note: we could have determined this along another line of reasoning—the steady state solution $\bar{U}(x)$ and

$u(x,t)$ both satisfy the PDE, which is linear, and $\tilde{u}(x,t)$ is just a linear combination of $\bar{U}(x)$ and $u(x,t)$.) Now, $u(x,t)$, the solution we are looking for, must satisfy the original boundary conditions. Of course $\bar{U}(x)$ satisfies the boundary conditions—that's how we determined it. So it should be clear that $\tilde{u}(x,t)$ must satisfy homogeneous boundary conditions, since $\tilde{u}(x,t) = u(x,t) - \bar{U}(x)$. That is, at either boundary we have

$$\tilde{u}(0,t) = u(0,t) - \bar{U}(0) = A - A = 0, \tag{3.60a}$$
$$\tilde{u}(L,t) = u(L,t) - \bar{U}(L) = B - B = 0. \tag{3.60b}$$

Now, for the initial conditions we have

$$u(x,0) = f(x), \quad \text{for} \quad 0 \le x \le L, \tag{3.61}$$

so for the initial conditions for $\tilde{u}(x)$ we have

$$\tilde{u}(x,0) = u(x,0) - \bar{U}(x) = f(x) - \bar{U}(x). \tag{3.62}$$

Now, we can write the new IBVP as

$$\text{PDE}: \quad \tilde{u}_t - \gamma^2 \tilde{u}_{xx} = 0, \quad 0 < x < L, \quad 0 < t < \infty, \tag{3.63a}$$
$$\text{BCs}: \quad \begin{cases} \tilde{u}(0,t) = 0, \\ \tilde{u}(L,t) = 0, \end{cases} \quad 0 < t < \infty, \tag{3.63b}$$
$$\text{IC}: \quad \tilde{u}(x,0) = g(x), \quad 0 \le x \le L, \tag{3.63c}$$

where

$$g(x) = f(x) - \bar{U}(x) = f(x) - \frac{B-A}{L}x - A. \tag{3.64}$$

Once we solve this problem, using separation of variables, we can determine $u(x,t)$ immediately since it is just $u(x,t) = \tilde{u}(x,t) + \bar{U}(x)$. In fact, this form of the solution is quite nice since it shows us the steady state and transient parts of the solution.

3.3 The Wave Equation

We will now look at the one-dimensional wave equation on a finite interval. Suppose that our physical system is an elastic string tied at two points, say $x = 0$, and $x = L$. The string is taught, so that when in equilibrium the string is a straight flat line. If the string is displaced from this flat line, we will refer to the vertical displacement at a given point $0 \leq x \leq L$, and at a given time, $t > 0$, as $u(x,t)$. If we make some physical assumptions about the string and it's displacement profile, it can be shown (see the derivation in the book), that the string obeys the PDE:

$$\frac{\partial^2}{\partial t^2}u - c^2\frac{\partial^2}{\partial x^2}u = 0,$$

where c is a parameter that depends upon physical properties of the string. As we will see, c is actually the wave speed of the system (the speed at which wave profiles propagate.) We confine our system with boundary conditions, just as we did for the heat equation. If we enforce that the string is tied securely at the endpoints, we have the boundary conditions

$$(0,t) = 0, \tag{3.65a}$$

$$u(L,t) = 0, \quad \text{for} \quad 0 < t, \tag{3.65b}$$

since clearly the value at the endpoints is the equilibrium value, which we have already chosen to be zero. Just as we did for the heat equation, we will need initial conditions in order to determine a unique solution to our physical problem. (The solution we are looking for is of course the displacement of the string for all time, $u(x,t)$). For wave equation it turns out, though proving this is beyond the scope of this class, that we need two initial conditions rather than just one in order to have a IBVP with a unique solution. You should find this believable since, for the heat equation we had only one derivative of u with respect to time, and for the wave equation we have two—u_{tt} rather than just u_t in the PDE. (Recall that if you know the derivative of a function $f'(t)$, then you can determine $f(t)$ up to an arbitrary constant, and if you know $f(t)$ at some value, say $f(0)$, then you can find $f(t)$ for all t. If you only know $f''(t)$, then you need to know $f(0)$ and $f'(0)$ to determine $f(t)$ for all t. You can think of the PDEs as being analogous to knowing something about the derivatives of the function you are looking for. For the heat equation you had a relationship for u_t, but for the wave equation, you only have a relationship for u_{tt}. So for the former you only needed $u(x,0)$, but for the latter you will also need $u_t(x,0)$.) So we will specify the initial conditions as

$$u(x,0) = f(x), u_t(x,0) = g(x). \tag{3.66}$$

The function $f(x)$ is the initial displacement of the string, or the initial solution just as with the heat equation, and the function $g(x)$ specifies the initial (vertical) velocity of the string at a given point x. So our initial boundary value problem is

$$\begin{aligned} &\text{PDE}: && u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < L \qquad 0 < t < \infty && (3.67a)\\ &\text{BCs}: && \begin{cases} u(0,t) = 0, \\ u(L,t) = 0, \end{cases} \quad 0 < t < \infty && (3.67b)\\ &\text{ICs}: && \begin{cases} u(x,0) = f(x), \\ u_t(x,0) = g(x), \end{cases} \quad 0 \le x \le L. && (3.67c) \end{aligned}$$

The IBVP (3.67a) is guaranteed to have a *unique* solution (given reasonable restrictions on $f(x)$ and $g(x)$). Again, proving that is way beyond the scope of this class, but you can just take my word for it.

We can solve (3.67a) by using separation of variables like we did for the heat equation. Recall that we begin by looking for solutions, to the PDE only, of the form

$$u(x,t) = X(x)T(t).$$

Then we plug that into the PDE since we are assuming that it is a solution and just want to find its form

$$\begin{aligned} &\frac{\partial^2}{\partial t^2}u - c^2\frac{\partial^2}{\partial x^2}u = 0\\ \Rightarrow &\frac{\partial^2}{\partial t^2}X(x)T(t) - c^2\frac{\partial^2}{\partial x^2}X(x)T(t) = 0\\ \Rightarrow &X(x)T''(t) - c^2X''(x)T(t) = 0\\ \Rightarrow &\frac{T''(t)}{c^2T(t)} - \frac{X''(x)}{X(x)} = 0\\ \Rightarrow &\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)}. \end{aligned} \tag{3.68}$$

We then note that a function of only t can equal a function of only x, only if they are both just constants. So we denote this constant k, which gives

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = k. \tag{3.69}$$

After rearranging, this yields two equations:

$$T''(t) - kc^2T(t) = 0 \tag{3.70a}$$

and

$$X''(x) - kX(x) = 0. \tag{3.70b}$$

Now if we were to solve these two ODEs, using any value of k, we would determine a solution to the PDE $u(x,t) = T(t)X(x)$. However, we want our solution to meet the boundary conditions. We note that if $u(x,t) = T(t)X(x)$ is to meet the boundary conditions, and not be the trivial solution $T(t)X(x) = 0$, then $X(x)$ must meet the boundary conditions. In your homework you were asked to show that $X(x)$ can only meet the boundary conditions if $k < 0$. So we will denote $k = -\lambda^2$, to remember that it is negative. If we solve (3.70a) and (3.70b) after replacing k with $-\lambda^2$, we have

$$T(t) = A\sin(c\lambda t) + B\cos(c\lambda t), \tag{3.71a}$$

$$X(x) = C\sin(\lambda x) + D\cos(\lambda x), \tag{3.71b}$$

where A, B, C and D are arbitrary constants. We chose k to be negative in order to meet the boundary conditions. Now we will enforce the boundary conditions on $X(x)$. The left boundary condition requires

$$X(0) = C\sin(0) + D\cos(0) = D = 0. \tag{3.72}$$

So we now have that $X(x) = C\sin(\lambda x)$, since the coefficient for the cosine, D, must be zero in order to meet the left B.C. Now for the right B.C., we have

$$X(L) = C\sin(\lambda L) = 0, \tag{3.73}$$

which implies that either $C = 0$, or $\lambda L = n\pi$ for any integer n. Since setting $C = 0$ gives the trivial solution, $X(x) = 0$, we require that $\lambda = \frac{n\pi}{L}$ in order to meet the boundary condition. One way to think about this is: you could have picked any value of λ and you would have found a solution to the PDE. If you pick a value $\lambda = \frac{n\pi}{L}$, then you still found a solution to the PDE, plus it meets the boundary conditions if you set $D = 0$. Since n can be any integer, you in fact found infinitely many solutions to the PDE that meet the boundary conditions. They are all of the form

$$u_n(x,t) = T_n(t)X_n(x) = \left[A_n\sin(c\lambda_n t) + B_n\cos(c\lambda_n t)\right] C_n\sin(\lambda_n x), \tag{3.74}$$

where $\lambda_n = \frac{n\pi}{L}$, and A_n, B_n and C_n are arbitrary constants that you can choose. Now, because the PDE is linear, and the boundary conditions are linear and homogeneous, we can take an arbitrary linear combination of solutions of the form (3.74) with different values of n, and that is still a solution to the PDE and meets the B.C.'s. If we take this to the limit (literally), we have a solution as well

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \sin(c\tfrac{n\pi}{L}t) + B_n \cos(c\tfrac{n\pi}{L}t) \right] C_n \sin(\tfrac{n\pi}{L}x). \tag{3.75}$$

Now, these terms in the sum have arbitrary (they can be anything) constants A_n, B_n and C_n. But, really there are only two arbitrary constants, since you can distribute C_n into the terms in the brackets. So we might as well write this sum as

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \sin(c\tfrac{n\pi}{L}t) + B_n \cos(c\tfrac{n\pi}{L}t) \right] \sin(\tfrac{n\pi}{L}x). \tag{3.76}$$

To put this another way, if you think you could have chosen three constants, A_n, B_n and C_n, you really just chose two— A_nC_n and B_nC_n. So we just redefined A_n and B_n to be A_nC_n and B_nC_n. (If this seems confusing you are thinking too hard about it.) Okay, so we have a solution with arbitrary constants that meets the B.C.'s. But we have two initial conditions that we need to meet. With the heat equation we only had to meet one I.C., $u(x,0) = f(x)$, but we only had one arbitrary constant for each term in the sum then. Now we have two, so we have more flexibility. If we try and impose the first I.C., we arrive at the condition

$$\begin{aligned} u(x,0) = \sum_{n=1}^{\infty} &\left[A_n \sin(c\tfrac{n\pi}{L}0) + B_n \cos(c\tfrac{n\pi}{L}0) \right] \sin(\tfrac{n\pi}{L}x) \\ &= \sum_{n=1}^{\infty} B_n \sin(\tfrac{n\pi}{L}x) = f(x). \end{aligned} \tag{3.77}$$

This is just like before with the heat equation! The condition can be met since we know that a sine series can equal a function if we define the coefficients right. So if we choose the arbitrary constants to be

$$B_n = \frac{2}{L}\int_0^L f(x)\sin(\tfrac{n\pi}{L}x)dx, \tag{3.78}$$

the condition is met. Now, we still have the other condition to meet. We need $u_t(x,0) = 0$. We differentiate $u(x,t)$ with respect to t

$$\begin{aligned}
\frac{\partial}{\partial t}u(x,t) =& \frac{\partial}{\partial t}\sum_{n=1}^{\infty}\left[A_n \sin(c\tfrac{n\pi}{L}t) + B_n \cos(c\tfrac{n\pi}{L}t)\right]\sin(\tfrac{n\pi}{L}x) \\
=& \sum_{n=1}^{\infty}\frac{\partial}{\partial t}\left[A_n \sin(c\tfrac{n\pi}{L}t) + B_n \cos(c\tfrac{n\pi}{L}t)\right]\sin(\tfrac{n\pi}{L}x) \\
=& \sum_{n=1}^{\infty}\left[A_n c\tfrac{n\pi}{L} \cos(c\tfrac{n\pi}{L}t) - B_n c\tfrac{n\pi}{L} \sin(c\tfrac{n\pi}{L}t)\right]\sin(\tfrac{n\pi}{L}x). \qquad (3.79)
\end{aligned}$$

Now we try and impose the other I.C., which requires that

$$\begin{aligned}
\frac{\partial}{\partial t}u(x,0) = \sum_{n=1}^{\infty}\left[A_n c\tfrac{n\pi}{L} \cos(c\tfrac{n\pi}{L}0) - B_n c\tfrac{n\pi}{L} \sin(c\tfrac{n\pi}{L}0)\right]\sin(\tfrac{n\pi}{L}x) \\
= \sum_{n=1}^{\infty}\left[A_n c\tfrac{n\pi}{L}\right]\sin(\tfrac{n\pi}{L}x) = g(x). \qquad (3.80)
\end{aligned}$$

The $c\frac{n\pi}{L}$ is a given constant, but the A_n is completely arbitrary, so we can meet this condition as well if we define the A_n appropriately. A sine series for $g(x)$

$$g(x) = \sum_{n=1}^{\infty} b_n \sin(\tfrac{n\pi}{L}x), \qquad (3.81)$$

has coefficients

$$b_n = \frac{2}{L}\int_0^L g(x)\sin(\tfrac{n\pi}{L}x)\,dx. \qquad (3.82)$$

Therefore, we can enforce our I.C. if

$$A_n c\tfrac{n\pi}{L} = \frac{2}{L}\int_0^L g(x)\sin(\tfrac{n\pi}{L}x)\,dx. \qquad (3.83)$$

Or, we define A_n by

$$A_n = \frac{2}{cn\pi}\int_0^L g(x)\sin(\tfrac{n\pi}{L}x)\,dx. \qquad (3.84)$$

So now we write a final solution as

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \sin(c\tfrac{n\pi}{L}t) + B_n \cos(c\tfrac{n\pi}{L}t)\right] \sin(\tfrac{n\pi}{L}x), \tag{3.85a}$$

where now the coefficients are no longer arbitrary, but defined by

$$A_n = \frac{2}{cn\pi} \int_0^L g(x) \sin(\tfrac{n\pi}{L}x)\, dx, \tag{3.85b}$$

and,

$$B_n = \frac{2}{L} \int_0^L f(x) \sin(\tfrac{n\pi}{L}x) dx. \tag{3.85c}$$

There is a very physically enlightening interpretation of this mathematical solution. Note that our solution is an infinite sum of all of the separable solutions that we found

$$u_n(x,t) = \left[A_n \sin(c\tfrac{n\pi}{L}t) + B_n \cos(c\tfrac{n\pi}{L}t)\right] \sin(\tfrac{n\pi}{L}x). \tag{3.86}$$

What do each of these separable solutions behave like? Well, before answering that question, note that $u_n(x,t)$ is of the form

$$u_n(x,t) = T_n(t) \sin(\tfrac{n\pi}{L}x), \tag{3.87}$$

where $T_n(t)$ is just a function of t, not x. So at any given time, this is just some constant multiplying $\sin(\frac{n\pi}{L}x)$, which always has the same profile with respect to x. So we might say that $T_n(t)$ is the amplitude of $\sin(\frac{n\pi}{L}x)$. In fact it is a time dependent amplitude, but still just an amplitude. If we note the form of $T_n(t)$, we see that it oscillates in time. So our solution is a sum of solutions, each of this form—a sine function of x that oscillates in time. Each one of these separable solutions is a standing wave—a wave with a fixed profile and time-dependent amplitude. (The sum of multiple standing waves is NOT necessarily a standing wave, so our solution is not necessarily a standing wave.) Each separable solution is in fact a special type of standing wave, known as a *mode*. Note that each mode has a wavelength of $2L/n$. For $n = 1$, the mode is known as the *fundamental mode*. The first four modes are shown in Figure 3.1. You should also contemplate the time dependence of the amplitude of each mode. Do the modes oscillate at the same frequency in time?

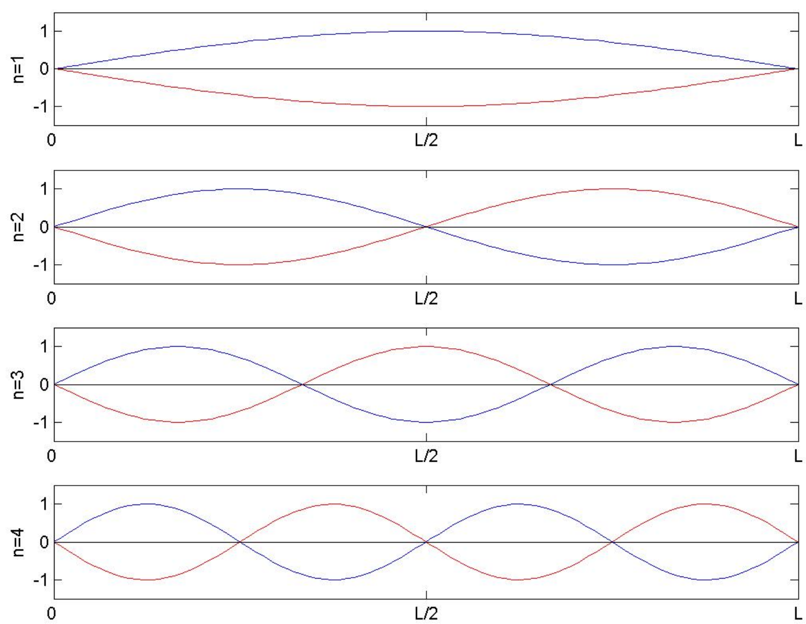

Figure 3.1: The first four modes of the wave equation on $[0, L]$. The top graph is the fundamental mode, $A_1 \sin(\frac{\pi}{L}x)$. The lower graphs are the second ($n = 2$), third ($n = 3$) and fourth ($n = 4$) modes. These are sometimes referred to as overtones of the fundamental mode. That is, for a given length L, the functions $A_n \sin(\frac{n\pi}{L}x)$ for all integers $n > 1$, are overtones of $A_1 \sin(\frac{\pi}{L}x)$.

Example 3. *Let's consider an example. Suppose that we solving problem (3.67a) and we are given initial conditions*

$$f(x) = 2\sin(\tfrac{\pi}{L}x) + \frac{1}{2}\sin(\tfrac{3\pi}{L}x) \tag{3.88a}$$

$$g(x) = 0. \tag{3.88b}$$

So we have started with a nonzero initial profile in the string, but the string is initially motionless. Now, our solution is given by (3.85a), so all we need to do is determine the arbitrary coefficients. Obviously $A_n = 0, \forall n$*, since* $g(x) = 0$*. For the* B_n *we have*

$$B_n = \frac{2}{L}\int_0^L f(x)\sin(\tfrac{n\pi}{L}x)\,dx = \frac{2}{L}\int_0^L \left[2\sin(\tfrac{\pi}{L}) + \frac{1}{2}\sin(\tfrac{3\pi}{L}x)\right]\sin(\tfrac{n\pi}{L}x)\,dx. \tag{3.89}$$

Now, before going on, you should recognize that we are trying to determine the Fourier sine series of a function that is a few of the basis functions in the series. Note that the sine series we are trying to find is

$$2\sin(\tfrac{\pi}{L}) + \frac{1}{2}\sin(\tfrac{3\pi}{L}x) = \sum_{n=1}^{\infty} B_n \sin(\tfrac{n\pi}{L}x). \tag{3.90}$$

You should be able to look at this and determine what the coefficients are. Convince yourself that they are $B_1 = 2$*,* $B_3 = \frac{1}{2}$ *and* $B_n = 0$ *for all other* n*. Remember, anytime you are writing a Fourier series for a function that is a simple finite linear combination of basis functions in the series, you should immediately know the coefficients. But, for the skeptical, let's integrate anyway. We have*

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L 2\sin(\tfrac{\pi}{L})\sin(\tfrac{n\pi}{L}x)\,dx + \frac{2}{L}\int_0^L \frac{1}{2}\sin(\tfrac{3\pi}{L}x)\sin(\tfrac{n\pi}{L}x)\,dx \\ &= \frac{4}{L}\int_0^L \sin(\tfrac{\pi}{L})\sin(\tfrac{n\pi}{L}x)\,dx + \frac{1}{L}\int_0^L \sin(\tfrac{3\pi}{L}x)\sin(\tfrac{n\pi}{L}x)\,dx. \end{aligned} \tag{3.91}$$

Now, again, you should recognize that we can use orthogonality of our basis functions to determine these integrals. We have two integrals, each one is an integral of the form $\int_0^L \sin(\frac{m\pi}{L}x)\sin(\frac{n\pi}{L}x)\,dx$*, which are zero unless* $m = n$*, in which case we have integrals of* $(\sin(\frac{n\pi}{L}x))^2$ *on over* $[0, L]$*. That integral is always* $\frac{L}{2}$*. So we have*

$$B_2 = 2, B_3 = \frac{1}{2}, B_n = 0, \forall n \neq 2, 3. \tag{3.92}$$

So our solution becomes a finite sum with only two terms:

$$u(x,t) = 2\cos(\tfrac{c\pi}{L}t)\sin(\tfrac{\pi}{L}x) + \frac{1}{2}\cos(\tfrac{c3\pi}{L}t)\sin(\tfrac{3\pi}{L}x). \tag{3.93}$$

Chapter 4

The 2D Heat and Wave Equations on Rectangular Domains and 2D Fourier Series

In this chapter we will look at the heat and wave equations in two dimensions, on a fine domain. In one dimension there was only one type of finite domain—an interval. But in 2D a finite domain could be any bounded connected region. For simplicity of solution we will consider the domain to be a rectangle—$0 < x < L$ and $0 < y < H$. For more complex domains, it is not always possible to analytically solve even simple linear PDEs. Often numerical methods on computers are needed if the domain is not a simple geometric shape. However, even in designing the algorithms for these numerical methods, it is important to understand the solution to PDEs on simpler domains such as a rectangle. Therefore, you should realize the material in this class is important for solving real world problems—even if these simple examples do not seem realistic to you.

4.1 The Wave Equation on a Rectangle

Consider a system that is governed by the wave equation in 2D. The solution we are looking for is a time dependent surface $z = u(x, y, t)$. At any given time, it is just a surface (more specifically a function of x and y). However, it moves with time. At any given point (x_0, y_0), $u(x_0, y_0, t)$ is the displacement of the surface from some value defined to be zero, $u_t(x_0, y_0, t)$ is the velocity in the z direction at that point, and $u_{tt}(x_0, y_0, t)$ is the acceleration and so on. If this surface obeys the wave equation on some domain (say a rectangle) then we have

$$u_{tt} - c^2(u_{xx} + u_{yy}) = 0. \tag{4.1}$$

An example might be a rectangular drum. The surface u is the displacement of the drum skin from the equilibrium value defined to be zero. To have a unique solution we need boundary conditions (on all four boundaries) and two initial conditions just like in 1D. So the IBVP takes the form

$$\text{PDE: } u_{tt} - c^2(u_{xx} + u_{yy}) = 0, \quad t > 0, \quad 0 < x < L, \quad 0 < y < H, \tag{4.2a}$$

$$\text{BC's} \begin{cases} u(0, y, t) = 0, \quad u(L, y, t) = 0, \quad t > 0, \quad 0 < y < h, \\ u(x, 0, t) = 0, \quad u(x, H, t) = 0, \quad t > 0, \quad 0 < x < L, \end{cases} \tag{4.2b}$$

$$\text{IC's} \begin{cases} u(x, y, 0) = f(x, y), \quad 0 < x < L, \quad 0 < y < H, \\ u_t(x, y, 0) = g(x, y), \quad 0 < x < L, \quad 0 < y < H. \end{cases} \tag{4.2c}$$

Now, it turns out that we can use separation of variables to solve this IBVP just as we did in 1D. This time, our separable solution will be the product of three functions

$$u(x, y, t) = T(t)X(x)Y(y). \tag{4.3}$$

We plug it into the PDE to determine conditions on $T(t)X(x)Y(y)$ so that it will indeed be a solution. This gives (dropping independent variables)

$$T''XY - c^2TX''Y - c^2TXY'' = 0. \tag{4.4}$$

Dividing by c^2TXY and rearranging gives

$$\frac{T''}{c^2T} = \frac{X''}{X} + \frac{Y''}{Y}. \tag{4.5}$$

So we have a function only of t equalling a function of only x and y. So this must be a constant. We'll denote it $-k^2$, giving

$$\frac{T''}{c^2T} = \frac{X''}{X} + \frac{Y''}{Y} = -k^2. \tag{4.6}$$

You might argue, "how do we know it's negative?" It doesn't need to be to satisfy the PDE. However, later we'd find that it must be in order to satisfy

BCs. So I'll spare you the suspense. Now, if we take the last of the two equations and rearrange it we have

$$\frac{X''}{X} = -k^2 - \frac{Y''}{Y}. \tag{4.7}$$

Again, we have a function only of x equalling a function only of y. So we can set both sides equal to a constant

$$\frac{X''}{X} = -k_x^2, \tag{4.8}$$

$$\frac{Y''}{Y} + k^2 = k_x^2. \tag{4.9}$$

We could follow the same procedure for $Y(y)$ and we'd have

$$\frac{Y''}{Y} = -k_y^2, \tag{4.10}$$

$$\frac{X''}{X} + k^2 = k_y^2. \tag{4.11}$$

So note that we now have an ODE for each of T, X and Y. The negative constants are arbitrary, but they must be related by $k^2 = k_x^2 + k_y^2$. If we solve each of these ODEs, we have

$$T(t) = A\sin(kct) + B\cos(kct), \tag{4.12a}$$

$$X(x) = C\sin(k_x x) + D\cos(k_y y), \tag{4.12b}$$

$$Y(y) = E\sin(k_y y) + F\cos(k_y y). \tag{4.12c}$$

The separable solution is then

$$u(x, y, t) = T(t)X(x)Y(y), \tag{4.13}$$

using the formulas (4.12). Now, we try and meet the BCs. We have

$$u(0, y, t) = T(t)X(0)Y(y) = 0, \qquad 0 < y < h, \tag{4.14a}$$

$$u(L, y, t) = T(t)X(L)Y(y) = 0, \qquad 0 < y < h, \tag{4.14b}$$

$$u(x, 0, t) = T(t)X(x)Y(0) = 0, \qquad 0 < x < L, \tag{4.14c}$$

$$u(x, H, t) = T(t)X(x)Y(H) = 0, \qquad 0 < x < L. \tag{4.14d}$$

Note that in each case, if we avoid the trivial solution, the BCs must be met by only one of the three functions—the one evaluated at the boundary. So

(4.14) will be satisfied by nontrivial solutions if

$$u(0,y,t) = X(0) = 0, \qquad 0 < y < h, \tag{4.15a}$$
$$u(L,y,t) = X(L) = 0, \qquad 0 < y < h, \tag{4.15b}$$
$$u(x,0,t) = Y(0) = 0, \qquad 0 < x < L, \tag{4.15c}$$
$$u(x,H,t) = Y(H) = 0, \qquad 0 < x < L. \tag{4.15d}$$

For $X(x)$ and $Y(y)$ we have a two-point homogeneous boundary conditions. So it follows that

$$X_n(x) = C_n \sin(\tfrac{n\pi}{L}x), \tag{4.16}$$
$$Y_m(y) = E_m \sin(\tfrac{m\pi}{H}y), \tag{4.17}$$

where m and n are arbitrary integers. Note that for each one of these functions, the boundary conditions were used just as they were in one dimension. Note also that $k_y = \frac{m\pi}{H}$ and $k_x = \frac{n\pi}{L}$ for arbitrary integers n and m. The only requirement is that for each separable solution, the value of k, used in the solution for $T(t)$, must be $k = \sqrt{(\frac{n\pi}{L})^2 + (\frac{m\pi}{H})^2}$. We can denote this k with two indices, $k_{mn} = \sqrt{(\frac{n\pi}{L})^2 + (\frac{m\pi}{H})^2}$. So our separable solution, which we will denote with two indices is

$$u_{mn}(x,y,t) = T_{mn}(t)X_n(x)Y_m(y) \tag{4.18}$$
$$= [A_{mn}\sin(k_{mn}ct) + \cos(k_{mn}ct)]\sin(\tfrac{n\pi}{L}x)\sin(\tfrac{m\pi}{H}y), \tag{4.19}$$

where $k_{mn} = \sqrt{(\frac{n\pi}{L})^2 + (\frac{m\pi}{H})^2}$. Now, like always, we can form a linear combination of these solutions and expect it to still satisfy the linear PDE and the homogeneous BCs. But this time we have two indices that can be chosen arbitrarily. That is, for each value of n we have infinitely many values of m to choose from, and vice versa. So we can express this infinite sum as a sum over two indices, m and n. So we have

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} [A_{mn}\sin(k_{mn}ct) + B_{mn}\cos(k_{mn}ct)]\sin(\tfrac{n\pi}{L}x)\sin(\tfrac{m\pi}{H}y). \tag{4.20}$$

Now, the only thing left to do is try and satisfy the initial conditions. Let's just plug in $t = 0$ and set it equal to $f(x,y)$ see what happens

$$u(x,y,0) = f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{mn}\sin(\tfrac{n\pi}{L}x)\sin(\tfrac{m\pi}{H}y). \tag{4.21}$$

Is this possible? Well, it looks a little like a sine series, but it has two functions. First, I'll just make the claim that something called a 2D sine series exists. That is, we can write an arbitrary function $f(x, y)$ as an infinite sum of products of $\sin(\frac{n\pi}{L}x)$ and $\sin(\frac{m\pi}{H}y)$ That is,

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{mn} \sin(\tfrac{n\pi}{L}x) \sin(\tfrac{m\pi}{H}y), \tag{4.22}$$

if we define the coefficients by

$$B_{mn} = \frac{4}{LH} \int_0^H \int_0^L f(x, y) \sin(\tfrac{m\pi}{H}y) \sin(\tfrac{n\pi}{L}x)\, dx\, dy. \tag{4.23}$$

It turns out that this is indeed true, as long as $f(x, y)$ is a suitable function (it has to be continuous with continuous partial derivatives—but we won't worry about those details.)

We can show this using 1D Fourier sine series as well. Suppose that

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{mn} \sin(\tfrac{n\pi}{L}x) \sin(\tfrac{m\pi}{H}y). \tag{4.24}$$

Let's write a few terms in the inner sum to get a feel for this double sum

$$f(x, y) = \sum_{n=1}^{\infty} \left[B_{1n} \sin(\tfrac{\pi}{H}y) + B_{2n} \sin(\tfrac{2\pi}{H}y) + B_{3n} \sin(\tfrac{3\pi}{H}y) + \cdots +\right] \sin(\tfrac{n\pi}{L}x). \tag{4.25}$$

Now, think of this inner sum as a single coefficient in a sine series in x. You can think of the inner sum as a function of y, or you can just imagine evaluating $f(x, y)$ at some point y_0 for the time being. That is

$$f(x, y_0) = \sum_{n=1}^{\infty} c_n(y_0) \sin(\tfrac{n\pi}{L}x), \tag{4.26}$$

where $c_n(y_0) = \sum_{m=1}^{\infty} B_{mn} \sin(\frac{m\pi}{H}y_0)$. This should work just like a 1D sine series, so we need

$$c_n(y_0) = \frac{2}{L} \int_0^L f(x, y_0) \sin(\tfrac{n\pi}{L}x)\, dx. \tag{4.27}$$

That is, for each value of $y = y_0$, $f(x, y_0)$ is just a function of x, and $c_n(y_0)$ is just a constant. However, we know that for any value of y_0 equation (4.26) will be satisfied. So we might as well allow y to vary. Since $c_n(y)$ is actually a series we have

$$\sum_{m=1}^{\infty} B_{mn} \sin(\tfrac{m\pi}{H} y) = \frac{2}{L} f(x,y) \sin(\tfrac{n\pi}{L} x)\, dx. \tag{4.28}$$

Can we isolate B_{mn} for a specific value of m and n? We can use orthogonality of $\sin(\frac{m\pi}{H} y)$ for different values of m. If we multiply both sides of (4.28) by $\sin(\frac{j\pi}{H} y)$, where j is some integer, and then integrate (and multiply by $\frac{2}{H}$) we have

$$\begin{aligned} &\frac{2}{H} \int_0^H \sin(\tfrac{j\pi}{H} y) \left[\sum_{m=1}^{\infty} B_{mn} \sin(\tfrac{m\pi}{H} y) \right] dy \\ &\qquad = \frac{2}{H} \int_0^H \sin(\tfrac{j\pi}{H} y) \left[\frac{2}{L} \int_0^L f(x,y) \sin(\tfrac{n\pi}{L} x)\, dx \right] dy. \end{aligned} \tag{4.29}$$

This can be rearranged to yield

$$\begin{aligned} &\sum_{m=1}^{\infty} B_{mn} \left[\frac{2}{H} \int_0^H \sin(\tfrac{j\pi}{H} y) \sin(\tfrac{m\pi}{H} y)\, dy \right] \\ &\qquad = \frac{4}{LH} \int_0^H \int_0^L f(x,y) \sin(\tfrac{j\pi}{H} y) \sin(\tfrac{n\pi}{L} x)\, dx\, dy. \end{aligned} \tag{4.30}$$

Now, by orthogonality the integrals in the sum on the left are zero, unless $j = m$. So only one term remains in the infinite sum, and we have

$$\begin{aligned} &B_{jn} \frac{2}{H} \int_0^H \sin^2(\tfrac{j\pi}{H} y)\, dy \\ &\qquad = \frac{4}{LH} \int_0^H \int_0^L f(x,y) \sin(\tfrac{j\pi}{H} y) \sin(\tfrac{n\pi}{L} x)\, dx\, dy. \end{aligned} \tag{4.31}$$

Now, the integral on the left is equal to $\frac{H}{2}$. So we are left with

$$B_{jn} = \frac{4}{LH} \int_0^H \int_0^L f(x,y) \sin(\tfrac{j\pi}{H} y) \sin(\tfrac{n\pi}{L} x)\, dx\, dy. \tag{4.32}$$

Since j was an arbitrary integer, and now we have a formula for B_{jn} for any integers j and n, let's rename j by m since that what we had originally. This gives

$$B_{mn} = \frac{4}{LH} \int_0^H \int_0^L f(x,y) \sin(\tfrac{m\pi}{H} y) \sin(\tfrac{n\pi}{L} x)\, dx\, dy. \tag{4.33}$$

Note that this is the same as (4.23), so we've proved it essentially (given a few assumptions about iterated integrals and convergence that we won't worry about).

Now, we still have the A_{mn} to determine, but we also have the other initial condition to meet. If we differentiate (4.20) with respect to time, we have

$$\begin{aligned}\frac{\partial}{\partial t}u(x,y,t) \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[k_{mn}cA_{mn}\cos(k_{mn}ct) - k_{mn}cB_{mn}\sin(k_{mn}ct)\right] \\ &\qquad \sin(\tfrac{n\pi}{L}x)\sin(\tfrac{m\pi}{H}y). \quad (4.34)\end{aligned}$$

Evaluating this at $t = 0$ and setting it equal to $g(x, y)$ gives

$$g(x,y) = \frac{\partial}{\partial t}u(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_{mn}cA_{mn}\sin(\tfrac{n\pi}{L}x)\sin(\tfrac{m\pi}{H}y). \quad (4.35)$$

Now that we believe in 2D Fourier sine series, we recognize this as a 2D sine series with coefficients

$$k_{mn}cA_{mn} = \frac{4}{LH}\int_0^H\int_0^L g(x,y)\sin(\tfrac{m\pi}{H}y)\sin(\tfrac{n\pi}{L}x)\,dx\,dy. \quad (4.36)$$

Rearranging gives

$$A_{mn} = \frac{4}{LHk_{mn}c}\int_0^H\int_0^L g(x,y)\sin(\tfrac{m\pi}{H}y)\sin(\tfrac{n\pi}{L}x)\,dx\,dy. \quad (4.37)$$

So we have our solution

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} [A_{mn}\sin(k_{mn}ct) + B_{mn}\cos(k_{mn}ct)]\sin(\tfrac{n\pi}{L}x)\sin(\tfrac{m\pi}{H}y), \tag{4.38a}$$

where,

$$k_{mn} = \sqrt{(\tfrac{n\pi}{L})^2 + (\tfrac{m\pi}{H})^2}, \tag{4.38b}$$

and,

$$B_{mn} = \frac{4}{LH}\int_0^H \int_0^L f(x,y)\sin(\tfrac{m\pi}{H}y)\sin(\tfrac{n\pi}{L}x)\,dx\,dy, \tag{4.38c}$$

and,

$$A_{mn} = \frac{4}{LHk_{mn}c}\int_0^H \int_0^L g(x,y)\sin(\tfrac{m\pi}{H}y)\sin(\tfrac{n\pi}{L}x)\,dx\,dy \tag{4.38d}$$

Chapter 5

Elliptic Equations on Rectangles

In this chapter we will look at the two classic elliptic PDEs—Laplace's equation and Poisson's equation. These equations arise in many applications, such as fluid dynamics, electrostatics, and just about any other topic in physics. However, we won't always talk about a specific application the way we did for the heat and wave equation. Partly because arriving at these equations often requires some theory from the science of the application that is beyond the scope of this class.

So, for this chapter we will just assume that there is some function $u(x, y)$ that we are interested in. In case you are curious of some examples, in electrostatics this function might be the electric potential (the voltage)—a scalar function in space. The gradient of this function is the electric field. In fluid dynamics this function might be something known as the fluid potential—a scalar function the gradient of which is the fluid velocity vector field. Don't worry about it if you aren't familiar with these things.

5.1 Laplace's Equation

The first elliptic PDE we will look at is Laplace's equation. It has the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{5.1}$$

Note that this PDE has no time variable—it is time independent. The solution $u(x, y)$ is not a function of time—it does not vary in time, it is just a surface that we are looking for. You can think of lots of functions that satisfy this PDE—any function that is linear since both second derivatives would be zero. However, there are lots of functions that are not linear that satisfy (5.1). If $\frac{\partial^2 u}{\partial y^2}$ is equal and opposite to $\frac{\partial^2 u}{\partial x^2}$ then u satisfies (5.1). Like before, conditions will narrow down the possible solutions so that there is only one unique solution. Of course, since there is no time variable, there will be no initial conditions. There are only boundary conditions. The form of the boundary conditions will depend on the application and the domain that the application dictates. In real world applications Laplace's equation arises in all sorts of geometries—in semiconductors, blood vessels, underground oil reservoirs, around airplane wings...you name it. But for now, we'll just look at Laplace's equation on a rectangle. That is, suppose we are interested in a system that is governed by Laplace's equation on a rectangle. That is,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < L, \quad 0 < y < L. \tag{5.2}$$

In order to complete this system and ensure a unique solution we need boundary conditions on $u(x, y)$ along the boundaries of our domain. That is, the function $u(x, y)$ must be specified on the boundaries of the rectangle

$$\begin{cases} u(0, y) = g_1(y), & u(L, y) = g_2(y), \\ u(x, 0) = f_1(x), & u(x, H) = f_2(x) \end{cases} \tag{5.3}$$

Together with the PDE these BCs constitute our boundary value problem, or BVP, which has a unique solution

$$\text{PDE:} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < L, \quad 0 < y < L, \tag{5.4a}$$

$$\text{BC's:} \quad \begin{cases} u(0, y) = g_1(y), & u(L, y) = g_2(y), \\ u(x, 0) = f_1(x), & u(x, H) = f_2(x). \end{cases} \tag{5.4b}$$

This is the equivalent of our IBVP from before.

Let's digress briefly to discuss some common notation. Laplace's equation is commonly written as

$$\nabla^2 u = 0. \tag{5.5}$$

This differential operator ∇^2 is known as the *Laplacian*. Where does this notation come from? Consider the gradient of u

$$\nabla u = \mathbf{i}\frac{\partial u}{\partial x} + \mathbf{j}\frac{\partial u}{\partial y}. \tag{5.6}$$

Now consider the divergence of this vector

$$\nabla \cdot (\nabla u) = (\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}) \cdot (\mathbf{i}\frac{\partial u}{\partial x} + \mathbf{j}\frac{\partial u}{\partial y}) \tag{5.7}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \tag{5.8}$$

The gradient followed by the divergence is kind of like the same operator acting twice, so notationally it makes sense to write ($\nabla \cdot \nabla = \nabla^2$). We'll use this notation since it is commonplace (and it saves me a lot of typing).

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\laplace u=0
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vs.

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\frac{\partial^2 u}{\partial x ^2} + \frac{\partial^2 u}{\partial y ^2} =0
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Sometimes, especially in the mathematics community, you will see the Laplacian written as Δ rather than ∇^2. However, I don't like that notation since it might be confused with delta for a change in something. Plus, the "2" reminds me that second derivatives are involved.

Now, before we get to solving our BVP (5.4) in general, we must consider one with very simple BCs. Consider the BVP

$$\text{PDE:} \quad \nabla^2 u = 0, \quad 0 < x < L, \quad 0 < y < H, \tag{5.9a}$$

$$\text{BC's:} \quad \begin{cases} u(0,y) = 0, & u(L,y) = 0, \\ u(x,0) = 0, & u(x,H) = f_2(x). \end{cases} \tag{5.9b}$$

All of the BCs are zero, except along the edge $y = L$, where the solution equals some specified function of x. How are we going to solve this BVP? You guessed it—separation of variables. We assume solutions of the form

$$u(x,y) = X(x)Y(y), \tag{5.10}$$

and then plug that into the PDE to establish conditions on $X(x)$ and $Y(y)$. We have

$$X''(x)Y(y) + X(x)Y''(y) = 0. \tag{5.11}$$

We then divide by $X(x)Y(y)$ to separate the functions, which lead to

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0, \tag{5.12}$$

which means that each ratio must equal a constant. So we have

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = k. \tag{5.13}$$

Now, note that whatever sign k has, the solutions to the resulting ODEs for $X(x)$ and $Y(y)$ will be different. In one case $X(x)$ will be oscillatory and $Y(y)$ will be exponential, or vice versa. So what do we chose? Consider the BCs. We need $X(x)$ to meet two homogeneous BCs.—that is $X(0) = X(L) = 0$. But $Y(y)$ is only zero at $y = H$. We'll worry about $Y(y)$ later, but for now, we know (from previous homework) that the only way $X(x)$ can be nontrivial and meet the BCs is if $k < 0$. So we replace k with $-\lambda^2$, which gives the two ODEs

$$X''(x) + \lambda^2 X(x) = 0, \tag{5.14a}$$

$$Y''(y) - \lambda^2 Y(y) = 0. \tag{5.14b}$$

By now you should know the two solutions

$$X(x) = A\sin\lambda x + B\cos\lambda x, \tag{5.15a}$$

$$Y(y) = C\sinh\lambda y + D\cosh\lambda y. \tag{5.15b}$$

Now, we need our product solution $u(x, y) = X(x)Y(y)$ to meet the homogeneous boundary conditions. So we have

$$u(0, y) = X(0)Y(y) = 0, \tag{5.16}$$

$$u(L, y) = X(L)Y(y) = 0. \tag{5.17}$$

We don't want to set $Y(y) = 0$, as that would produce the trivial solution. So we insist that $X(0) = X(L) = 0$. This should be familiar and you should recognize that in order to meet the BCs, $D = 0$ and $\lambda = \frac{n\pi}{L}$ for some integer n. So for $X(x)$ we have infinitely many solutions, one for each integer n (each denoted with a subscript)

$$X_n(x) = A_n \sin(\tfrac{n\pi}{L}x). \tag{5.18}$$

This also means that

$$Y_n(y) = C_n \sinh \tfrac{n\pi}{L} y + D_n \cosh \tfrac{n\pi}{L} y \tag{5.19}$$

Now, since the Laplace equation is linear, and the boundary conditions on $X_n(x)$ are homogeneous, we can form a linear combination of the solutions

$u_n(x,t) = X_nY_n$. If we take this idea to the limit, we have a series solution

$$u(x,y) = \sum_{n=1}^{\infty} \left[C_n \sinh \tfrac{n\pi}{L} y + D_n \cosh \tfrac{n\pi}{L} y\right] \sin(\tfrac{n\pi}{L} x), \tag{5.20}$$

that satisfies the PDE and the boundary conditions in x, or

$$u(0,y) = 0, \tag{5.21a}$$
$$u(L,y) = 0. \tag{5.21b}$$

Now, we try and meet the boundary conditions in y. First, lets look at the lower boundary by plugging $y = 0$ into the solution (5.20). We have

$$0 = u(x,0) = \sum_{n=1}^{\infty} \left[C_n \sinh \tfrac{n\pi}{L} 0 + D_n \cosh \tfrac{n\pi}{L} 0\right] \sin(\tfrac{n\pi}{L} x) \quad = \sum_{n=1}^{\infty} [D_n 1] \sin(\tfrac{n\pi}{L} x). \tag{5.22}$$

So we must have $D_n = 0$ for all integers n. Now for the other BC at $y = H$. We have (using what we now know about the D_n)

$$f_2(x) = u(x,H) = \sum_{n=1}^{\infty} C_n \sinh \tfrac{n\pi}{L} H \sin(\tfrac{n\pi}{L} x). \tag{5.23}$$

This should look familiar. The series on the right is a sine series with coefficients $C_n \sinh plnH$. So if we define the C_n, such that $C_n \sinh plnH$ are the coefficients of a sine series of $f(x)$, then the condition (5.23) will be satisfied. So we need

$$C_n \sinh \tfrac{n\pi}{L} H = \frac{2}{L} \int_0^L f_2(x) \sin(\tfrac{n\pi}{L} x)\, dx, \tag{5.24}$$

or

$$C_n = \frac{2}{L \sinh \frac{n\pi}{L} H} \int_0^L f_2(x) \sin(\tfrac{n\pi}{L} x)\, dx. \tag{5.25}$$

This gives us our solution

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sinh(\tfrac{n\pi}{L} y) \sin(\tfrac{n\pi}{L} x), \tag{5.26}$$

where

$$A_n = \frac{2}{L \sinh \frac{n\pi}{L} H} \int_0^L f_2(x) \sin(\tfrac{n\pi}{L} x)\, dx. \tag{5.27}$$

I've switched the C_n to an A_n, for reasons that will be clear below.

Now, again consider Laplace's equation with nonzero boundary conditions on only one of the 4 boundaries, but this time let's say we want to solve

$$\text{PDE:} \quad \nabla^2 u = 0, \quad 0 < x < L, \quad 0 < y < H, \tag{5.28a}$$

$$\text{BC's:} \quad \begin{cases} u(0,y) = 0, & u(L,y) = 0, \\ u(x,0) = f_1(x), & u(x,H) = 0. \end{cases} \tag{5.28b}$$

We would of course proceed in exactly the same manner as above—with separation of variables. Eventually we'd reach the point of (5.20)

$$u(x,y) = \sum_{n=1}^{\infty} \left[C_n \sinh \tfrac{n\pi}{L} y + D_n \cosh \tfrac{n\pi}{L} y \right] \sin(\tfrac{n\pi}{L} x). \tag{5.29}$$

Last time we found that in order to meet the boundary condition $u(x,0) = 0$ we set $D_n = 0$, since the sinh is zero at $y = 0$. This time, it's not quite that easy since we now need $u(x,H) = 0$. In general, we need to form a relationship between C_n and D_n such that $C_n \sinh \frac{n\pi}{L} H + D_n \cosh \frac{n\pi}{L} H = 0$. We could solve this, and then eliminate one of C_n or D_n from (5.29). This would leave one arbitrary degree of freedom—one constant. However, if we did this, and then used the exponential definitions of the sinh and cosh to simplify such an expression, we could actually reduce this to

$$\left[C_n \sinh \tfrac{n\pi}{L} y + D_n \cosh \tfrac{n\pi}{L} y \right] = B_n \sinh(\tfrac{n\pi}{L}(H - y)). \tag{5.30}$$

Feel free to do this to convince yourself. So for our solution we now have

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sinh(\tfrac{n\pi}{L}(H - y)) \sin(\tfrac{n\pi}{L} x). \tag{5.31}$$

Like before, we only have one last boundary condition to meet. This time

$$u(x,0) = f_1(x). \tag{5.32}$$

Again, we simply plug $y = 0$ into (5.31) and set it equal to $f_1(x)$, which gives

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sinh(\tfrac{n\pi}{L} H) \sin(\tfrac{n\pi}{L} x) = f_1(x). \tag{5.33}$$

We then recognize this as a sine series with coefficients $B_n \sinh(\frac{n\pi}{L}H)$. These are still just constants, and we set them equal to

$$B_n \sinh(\tfrac{n\pi}{L}H) = \frac{2}{L}\int_0^L f_1(x)\sin(\tfrac{n\pi}{L}x)\,dx, \tag{5.34}$$

or

$$B_n = \frac{2}{L\sinh(\frac{n\pi}{L}H)}\int_0^L f_1(x)\sin(\tfrac{n\pi}{L}x)\,dx. \tag{5.35}$$

This defines our B_n, and we have a solution

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sinh(\tfrac{n\pi}{L}(H-y))\sin(\tfrac{n\pi}{L}x), \tag{5.36a}$$

where,

$$B_n = \frac{2}{L\sinh(\frac{n\pi}{L}H)}\int_0^L f_1(x)\sin(\tfrac{n\pi}{L}x)\,dx. \tag{5.36b}$$

Now, suppose that we have a nonzero boundary condition at one of the vertical sides $x = 0$. That is, we wish to solve

$$\text{PDE:}\quad \nabla^2 u = 0, \quad 0 < x < L, \quad 0 < y < H, \tag{5.37a}$$

$$\text{BC's:}\quad \begin{cases} u(0,y) = 0, & u(L,y) = g_2(y), \\ u(x,0) = 0, & u(x,H) = 0. \end{cases} \tag{5.37b}$$

Note that this is exactly the same a problem we've already done, except that the roles of x and y are reversed. If we went through the process of separation of variables, we'd eventually arrive at (5.13)

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = k. \tag{5.38}$$

However, this time we have two homogeneous boundary conditions at two different y points, $y = 0$ and $y = H$. So this time, we need to choose the separation constant differently so that we get sines and cosines in y and cosh's and sinh's in x. So we set $k = \lambda^2$ rather than $-\lambda^2$ as before. Solving those ODE's for $X(x)$ and $Y(y)$ gives

$$Y(y) = A\cos(\lambda y) + B\sin(\lambda y), \tag{5.39}$$

$$X(x) = C\sinh(\lambda x) + D\cosh(\lambda x). \tag{5.40}$$

The boundary conditions at $y = 0$ and $y = H$ now imply that $Y(0) = Y(H) = 0$, since otherwise we'd have to set $X(x) = 0$. This time that implies that $A = 0$ and $\lambda = \frac{n\pi}{H}$ for some integer n. So we have, using subscripts to denote the different choices of n

$$Y_n(y) = B_n \sin(\tfrac{n\pi}{H} y), \tag{5.41}$$

$$X_n(x) = C_n \sinh(\tfrac{n\pi}{H} x) + D_n \cosh(\tfrac{n\pi}{H} x). \tag{5.42}$$

Again we we form an infinite sum of these solutions $u_n(x, y) = X_n(x)Y_n(y)$,

$$u(x, y) = \sum_{n=1}^{\infty} \left[C_n \sinh(\tfrac{n\pi}{H} x) + D_n \cosh(\tfrac{n\pi}{H} x)\right] \sin(\tfrac{n\pi}{H} y), \tag{5.43}$$

(where we've absorbed the definition of B_n into C_n and D_n). Now, we first enforce the condition $u(0, y) = 0$, which gives

$$0 = u(0, y) = \sum_{n=1}^{\infty} [D_n 1] \sin(\tfrac{n\pi}{H} y). \tag{5.44}$$

So we have that all the D_n are zero. The last boundary condition gives

$$u(L, y) = g_2(y) = \sum_{n=1}^{\infty} C_n \sinh(\tfrac{n\pi}{H} L) \sin(\tfrac{n\pi}{H} y). \tag{5.45}$$

This is just a sine series on $0 \leq y \leq H$, for $g_2(y)$ with coefficients $C_n \sinh(\frac{n\pi}{H} L)$. So we set them equal to

$$C_n \sinh(\tfrac{n\pi}{H} L) = \frac{2}{H} \int_0^H g_2(y) \sin(\tfrac{n\pi}{H} y)\, dy, \tag{5.46}$$

or

$$C_n = \frac{2}{H \sinh(\frac{n\pi}{H} L)} \int_0^H g_2(y) \sin(\tfrac{n\pi}{H} y)\, dy. \tag{5.47}$$

This gives the solution

$$u(x, y) = \sum_{n=1}^{\infty} C_n \sinh(\tfrac{n\pi}{H} x) \sin(\tfrac{n\pi}{H} y), \tag{5.48}$$

where,

$$C_n = \frac{2}{H \sinh(\frac{n\pi}{H} L)} \int_0^H g_2(y) \sin(\tfrac{n\pi}{H} y)\, dy \tag{5.49}$$

Now, suppose we have

$$\text{PDE:} \quad \nabla^2 u = 0, \quad 0 < x < L, \quad 0 < y < H, \tag{5.50a}$$

$$\text{BC's:} \quad \begin{cases} u(0,y) = g_1(y), & u(L,y) = 0, \\ u(x,0) = 0, & u(x,H) = 0. \end{cases} \tag{5.50b}$$

If you follow similar procedures to those above, you should be able to determine the solution to be

$$u(x,y) = \sum_{n=1}^{\infty} D_n \sinh(\tfrac{n\pi}{H}(L-x)) \sin(\tfrac{n\pi}{H}y), \tag{5.51}$$

where,

$$D_n = \frac{2}{H \sinh(\frac{n\pi}{H}L)} \int_0^H g_1(y) \sin(\tfrac{n\pi}{H}y)\, dy \tag{5.52}$$

Now, lets consider the full Laplace problem—that is, say we have nonhomogeneous Dirichlet BC's on all four boundaries:

$$\text{PDE:} \quad \nabla^2 u = 0, \quad 0 < x < L, \quad 0 < y < H, \tag{5.53a}$$

$$\text{BC's:} \quad \begin{cases} u(0,y) = g_1(y), & u(L,y) = g_2(y), \\ u(x,0) = f_1(x), & u(x,H) = f_2(x). \end{cases} \tag{5.53b}$$

How can we solve this? We simply turn it into four separate problems. That is, we assume that

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y). \tag{5.54}$$

Then we find the four solutions as follows. For $u_1(x,y)$ we solve

$$\text{PDE:} \quad \nabla^2 u_1 = 0, \quad 0 < x < L, \quad 0 < y < H, \tag{5.55a}$$

$$\text{BC's:} \quad \begin{cases} u_1(0,y) = 0, & u_1(L,y) = 0, \\ u_1(x,0) = 0, & u_1(x,H) = f_2(x). \end{cases} \tag{5.55b}$$

For $u_2(x,y)$ we choose one of the other boundaries to be nonzero, and the other three boundaries to be zero and so on. It is easy to verify that the sum of all four of these solutions is the solution to the full problem. It satisfies the PDE since

$$\nabla^2(u) = \nabla^2(u_1 + u_2 + u_3 + u_4) = \nabla^2 u_1 + \nabla^2 u_2 + \nabla^2 u_3 + \nabla^2 u_4 = 0.$$

Further, the sum $u = u_1 + u_2 + u_3 + u_4$ satisfies all four boundary conditions, since at each boundary only one of the four solutions is nonzero, and it equals the specified nonzero boundary condition on that boundary.

5.2 Poisson's Equation

Poisson's Equation is the same as Laplace's equation accept that there is a nonhomogeneous term on the right hand side. That is

$$\nabla^2 u = f(x, y), \tag{5.56}$$

where $f(x, y)$ is some known *forcing function.* Systems that obey Poisson's equation are essentially the same as those that obey Laplace's equation, except that some external force acts on the system. For the example of a drumskin, or a stretched membrane, the function $f(x, y)$ would represent a force acting vertically on the membrane as a function of x and y. Another example is of an electrostatic system—the potential or voltage is governed by Laplace's equation in a vacuum. However, in the presence of a charge density distribution $f(x, y)$, the voltage or potential is governed by Poisson's equation. Typically Poisson's equation comes with boundary conditions just like Laplace's equation. For instance, for a system on a rectangle we might have the BVP

$$\nabla^2 u = f(x, y), \quad 0 < x < L, \quad 0 < y < H, \tag{5.57a}$$

$$\text{BCs: } \begin{cases} u(x, 0) = f_1(x), & 0 < x < L, \\ u(x, H) = f_2(x), & 0 < x < L, \\ u(0, y) = g_1(y), & 0 < y < H, \\ u(L, y) = g_2(y), & 0 < y < H. \end{cases} \tag{5.57b}$$

To solve (5.57), we separate the problem into two separate problems. That is, we assume the solution is the sum of two parts

$$u(x, y) = u_1(x, y) + u_2(x, y). \tag{5.58}$$

For $u_1(x, y)$ we solve Laplace's equation with nonhomogeneous BCs and for $u_2(x, y)$ we solve Poisson's equation with homogeneous BCs. That is, $u_1(x, y)$ is the solution to the BVP

$$\nabla^2 u = 0, \quad 0 < x < L, \quad 0 < y < H, \tag{5.59a}$$

$$\text{BCs } \begin{cases} u(x, 0) = f_1(x), & 0 < x < L, \\ u(x, H) = f_2(x), & 0 < x < L, \\ u(0, y) = g_1(y), & 0 < y < H, \\ u(L, y) = g_2(y), & 0 < y < H, \end{cases} \tag{5.59b}$$

and $u_2(x,y)$ is the solution to the BVP

$$\nabla^2 u = f(x,y), \quad 0<x<L, \quad 0<y<H, \tag{5.60a}$$

$$\text{BCs}\begin{cases} u(x,0)=0, & 0<x<L,\\ u(x,H)=0, & 0<x<L,\\ u(0,y)=0, & 0<y<H,\\ u(L,y)=0, & 0<y<H. \end{cases} \tag{5.60b}$$

(You should convince yourself that the sum $u = u_1 + u_2$ will solve the IBVP (5.57). Note that it will have the correct values on the boundaries since $u_1 + u_2 = u_1$ on the boundaries, and u_1 satisfies the BCs. Also, note that $\nabla^2 u = \nabla^2 u_1 + \nabla^2 u_2 = 0 + \nabla^2 u_2 = f(x,y)$.) We saw how to solve (5.59) in the last section. Now we will see how to solve (5.60). Note that the solution needs to be zero on all four boundaries. Note that the solution to Laplace's equation with such BCs is merely the trivial solution—$u(x,y)=0$. However, the forcing function for Poisson's equation will produce a nontrivial solution. We will just assume that the solution is of the form

$$u_2(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\sin(\tfrac{n\pi}{L}x)\sin(\tfrac{m\pi}{H}y), \tag{5.61}$$

for arbitrary constants A_{nm}. Why? Well, there are a couple of ways to answer this. First, we know from experience with the 2D wave and heat equations that this sum will be zero on all of boundaries $x = 0, L$ and $y = 0, H$. But why will it be a solution to Poisson's equation? Well, let's just try it and see. If we take the Laplacian of our proposed solution, we have

$$\nabla^2 u_2 = \nabla^2 \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\sin(\tfrac{n\pi}{L}x)\sin(\tfrac{m\pi}{H}y) \tag{5.62}$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\nabla^2\sin(\tfrac{n\pi}{L}x)\sin(\tfrac{m\pi}{H}y) \tag{5.63}$$

$$= -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\left[(\tfrac{n\pi}{L})^2\sin(\tfrac{n\pi}{L}x)\sin(\tfrac{m\pi}{H}y) + (\tfrac{m\pi}{H})^2\sin(\tfrac{n\pi}{L}x)\sin(\tfrac{m\pi}{H}y)\right] \tag{5.64}$$

$$= -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\left((\tfrac{n\pi}{L})^2 + (\tfrac{m\pi}{H})^2\right)\sin(\tfrac{n\pi}{L}x)\sin(\tfrac{m\pi}{H}y). \tag{5.65}$$

Our proposed solution then satisfies the PDE if

$$-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\left((\tfrac{n\pi}{L})^2+(\tfrac{m\pi}{H})^2\right)\sin(\tfrac{n\pi}{L}x)\sin(\tfrac{m\pi}{H}y) = f(x,y). \tag{5.66}$$

Is this possible? We can choose A_{nm} any way we like. Since we've seen double Fourier series, we know that we can represent any function $f(x,y)$ on a rectangle by

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\sin(\tfrac{n\pi}{L}x)\sin(\tfrac{m\pi}{H}), \tag{5.67}$$

where

$$B_{nm} = \frac{4}{LH}\int_0^L\int_0^H f(x,y)\sin(\tfrac{n\pi}{L}x)\sin(\tfrac{m\pi}{H}y)\,dy\,dx. \tag{5.68}$$

So looking at (5.66), we see that we have a solution to (5.60) if we define

$$-A_{nm}\left((\tfrac{n\pi}{L})^2+(\tfrac{m\pi}{H})^2\right) = \frac{4}{LH}\int_0^L\int_0^H f(x,y)\sin(\tfrac{n\pi}{L}x)\sin(\tfrac{m\pi}{H}y)\,dy\,dx \tag{5.69}$$

$$\Rightarrow A_{nm} = \frac{-4}{LH\left((\tfrac{n\pi}{L})^2+(\tfrac{m\pi}{H})^2\right)}\int_0^L\int_0^H f(x,y)\sin(\tfrac{n\pi}{L}x)\sin(\tfrac{m\pi}{H}y)\,dy\,dx. \tag{5.70}$$

So now we have our full solution to (5.57)

$$u(x,y) = u_1(x,y)+u_2(x,y), \tag{5.71a}$$

where $u_1(x,y)$ satisfies Laplace's equation with the nonhomogeneous BCs and

$$u_2(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\sin(\tfrac{n\pi}{L}x)\sin(\tfrac{m\pi}{H}y), \tag{5.71b}$$

where

$$A_{nm} = \frac{-4}{LH\left((\tfrac{n\pi}{L})^2+(\tfrac{m\pi}{H})^2\right)}\int_0^L\int_0^H f(x,y)\sin(\tfrac{n\pi}{L}x)\sin(\tfrac{m\pi}{H}y)\,dy\,dx. \tag{5.71c}$$

It turns out that there is another method for solving Poisson's equation called the *method of eigenfunction expansion*. However, for Poisson's equation it is more difficult than what was just shown. The advantage is that the method of eigenfunction expansion is just a single series in either x or y. See you book for details.

We will cover the method of eigenfunction expansion in the next chapter in a more general setting since it is useful for many problems, not just Poisson's equation.

Chapter 6

The Method of Eigenfunction Expansion

In this chapter we will consider a new method called *eigenfunction expansion.* The name may seem foreign until you learn what an eigenfunction is. As we will see, this method is easy if you understand separation of variables. If you recall the steps of separation of variables, the method of eigenfunction expansion is quite easy. Furthermore, as we will see, the method of eigenfunction expansion can be used for all of the problems we've done so far, as well as for nonhomogeneous PDEs. Before considering the method, let's introduce an *eigenvalue problem* in differential equations.

6.1 Eigenvalue Problems in Differential Equations

Eigenvalue problems occur in many areas of mathematics, particularly problems in linear algebra and linear differential equations. Both of these areas of mathematics involve linear operators, and those operators have what are called eigenvalues. Don't worry too much about the theory, but in case you are curious, this might help explain what these are.

Recall form linear algebra, that if you have a square matrix A, it can have scalars called eigenvalues associated with it. These eigenvalues have corresponding vectors called eigenvectors. If we denote a vector v, the eigenvalues and eigenvectors satisfy

$$Av = \lambda v, \tag{6.1}$$

where λ is a scalar called an eigenvalue of A. A given $m \times m$ matrix A can have at most m distinct eigenvectors and eigenvalues. We do not consider

$v = 0$ to be an eigenvector, even though $Av = \lambda v$ whenever $v = 0$. However, it is possible to have an eigenvalue $\lambda = 0$, for $v \neq 0$. If you know A and you want to find the eigenvalues and eigenvectors of A, that is known as an eigenvalue problem in linear algebra.

There are also eigenvalue problems in differential equations. For instance, if you want to find functions $y(x)$ and scalars μ, such that

Eigenvalue Problem in Differential Equations

$$\frac{\partial^2}{\partial x^2} y(x) = \mu y(x), \tag{6.2a}$$

subject to (6.2b)

$$y(0) = y(L) = 0, \tag{6.2c}$$

that is called an eigenvalue problem also. Note that $\frac{\partial^2}{\partial x^2}$ is analogous to the matrix A in the linear algebra case —both are called linear operators. Note that $y(x)$ is analogous to the eigenvectors v in (6.1). In this case $y(x)$ is not a vector, it is a function, so it is called an eigenfunction. The scalar constants μ are called eigenvalues. In the differential equations case (6.2a) we have boundary conditions on $y(x)$. There was nothing analogous in the linear algebra case—at least nothing on the surface. In fact, there is something analogous. In the linear algebra case, the vectors v had to be of the right size. If A is $m \times m$, then v had to have m components. That is $v \in \mathbb{R}^m$. So for a given matrix A, v had to belong to a certain *space*. In the differential equations case, we don't allow $y(x)$ to be any old function. If we did, then we could solve the ODE in (6.2a) for any eigenvalue μ. However, the boundary conditions restrict the eigenvalues by restricting the eigenfunctions to functions $y(x)$ that satisfy $y(0) = y(L) = 0$. These functions belong to a certain space, just like the vectors in the linear algebra case.

The system (6.2a) should look familiar. We've encountered eigenvalue problems whenever we used separation of variables. For any of the PDEs we've discussed, anytime homogeneous boundary conditions arise at two distinct points—sometimes called a *two-point boundary value problem*—we've had to solve an eigenvalue problem. If the two homogeneous boundary conditions were at two distinct points x, we've usually called the function $X(x)$.

The boundary value problem looked like this

$$X''(x) = kX(x), \tag{6.3a}$$

$$X(0) = X(L) = 0. \tag{6.3b}$$

Note that this is an eigenvalue problem exactly the same as (6.2a). You've found for this problem, many times, that $k \leq 0$ in order to meet the BCs. So we've called it $-\lambda^2$. Later we found that $\lambda = \frac{n\pi}{L}$ in order to meet the BCs, and $X(x) = A\sin(\frac{n\pi}{L}x)$. What we were doing is solving an eigenvalue problem—we just didn't call it that. In fact, the eigenvalues of the problem are $k = -\lambda^2 = -(\frac{n\pi}{L})^2$ and the associated eigenfunctions are $X(x) = A\sin(\frac{n\pi}{L}x)$. For solving PDEs, we really don't care about the eigenvalues so much, we just need the eigenfunctions which depend on the eigenvalues. Note that this problem has infinitely many eigenvalues and eigenfunctions—one for every positive integer. For this problem we do not consider $A = \sin(\frac{0\pi}{L}x) = 0$ to be an eigenfunction. For different boundary conditions, $k = 0$ might in fact be an eigenvalue. For instance, consider the problem

$$X''(x) = kX(x), \tag{6.4a}$$

$$X'(0) = X'(L) = 0. \tag{6.4b}$$

This is also an eigenvalue problem with different boundary conditions. For this problem, we found that the eigenvalues were again $k = -\lambda^2 = -(\frac{n\pi}{L})^2$ and the eigenfunctions were $X(x) = \cos(\frac{n\pi}{L}x)$. However this time, $k = 0$ is an eigenvalue. Why? Because the eigenfunction for $n = 0$, $A\cos(\frac{0\pi}{L}x) = A$ is not equal to zero. So we consider it an eigenfunction with eigenvalue $k = 0$, since in this case $X''(x) = 0X(x) = 0$, even though $X(x) = A \neq 0$.

6.2 The Method of Eigenfunction Expansion

The idea behind the method of eigenfunction expansion, is to first consider only the boundary value problem that arises given a PDE and boundary conditions, ignoring any nonhomogeneous terms in the PDE. That is, given any of the PDEs we've seen so far, you consider the eigenvalue problem that would arise if you performed separation of variables on the PDE. Typically, you will already know what eigenvalue problem arises, and the solution to that eigenvalue problem, simply because you have done separation of variables already on the problem. This is best understood by an example.

Suppose we are given the following IBVP for the heat equation

$$u_t - \gamma^2 u_{xx} = 0, \tag{6.5a}$$

$$u(0,t) = u(L,t) = 0, \tag{6.5b}$$

$$u(x,0) = f(x). \tag{6.5c}$$

We know from experience from using separation of variables, that we will arrive at two problems, one for $X(x)$ and one for $T(t)$. The problem for $X(x)$ looks like

$$X''(x) = kX(x), \tag{6.6}$$

$$X(0) = X(L) = 0. \tag{6.7}$$

This is an eigenvalue problem. The problem for $T(t)$ looked like this

$$T'(t) = k\gamma^2 T(t). \tag{6.8}$$

This is not an eigenvalue problem. Why? Because there are no boundary conditions on $T(t)$ that would restrict k to certain values. In this class the only kind of eigenvalue problem we will encounter are two-point homogeneous boundary value problems.

Now the central idea behind eigenfunction expansion is to assume that the solution is an infinite sum of the eigenfunctions of the problem. That is, for the heat equation, the eigenfunctions—the solutions to (6.6) are $X_n(x) = A_n \sin(\frac{n\pi}{L}x)$. You assume that the solution to (6.5) is

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t) \sin(\tfrac{n\pi}{L}x). \tag{6.9}$$

Note that since the eigenfunctions $X(x) = \sin(\frac{n\pi}{L}x)$ are merely functions of x, the t dependence on $u(x,t)$ must be in the coefficients of (6.9). Now, we just need to figure out what the functions $c_n(t)$ are so that (6.9) satisfies the PDE and initial conditions in (6.5). We plug our solution (6.9) into the PDE to see if we can derive a condition on $c_n(t)$. Plugging in gives

$$\begin{aligned}
&\frac{\partial}{\partial t}\sum_{n=1}^{\infty} c_n(t)\sin(\tfrac{n\pi}{L}x) - \gamma^2 \frac{\partial^2}{\partial x^2}\sum_{n=1}^{\infty} c_n(t)\sin(\tfrac{n\pi}{L}x) \\
=&\sum_{n=1}^{\infty} c_n'(t)\sin(\tfrac{n\pi}{L}x) - \gamma^2 \sum_{n=1}^{\infty} c_n(t)\frac{\partial^2}{\partial x^2}\sin(\tfrac{n\pi}{L}x) \\
=&\sum_{n=1}^{\infty} c_n'(t)\sin(\tfrac{n\pi}{L}x) + \gamma^2 (\tfrac{n\pi}{L})^2 \sum_{n=1}^{\infty} c_n(t)\frac{\partial^2}{\partial x^2}\sin(\tfrac{n\pi}{L}x) \\
=&0.
\end{aligned} \tag{6.10}$$

This isn't very revealing yet, but if group the terms in the two sums we have

$$\sum_{n=1}^{\infty} \left[c_n'(t) + (\gamma \tfrac{n\pi}{L})^2 c_n(t)\right] \sin(\tfrac{n\pi}{L}x) = 0. \tag{6.11}$$

Note that this is a sine series that equals zero. All of the junk in the brackets are the coefficients of the series. If the series equals 0 then all of the junk in brackets must equal 0 for all integers n. Setting this equal to zero gives

$$c_n'(t) + (\gamma \tfrac{n\pi}{L})^2 c_n(t) = 0, \tag{6.12}$$

for every positive integer n. Note that this is an ODE for every function $c_n(t)$. If we could solve the ODE, then we'd have a solution given by (6.9). First, recall from ODEs, that in order to uniquely solve an ODE like (6.12), we need to know an initial condition on $c_n(t)$. That is we need to know $c_n(0)$ for all n. How do we get that? Naturally, you should suspect the initial conditions on $u(x,t)$. If we plug $t = 0$ into (6.9), we have

$$(x,0) = \sum_{n=1}^{\infty} = c_n(0) \sin(\tfrac{n\pi}{L}x) = f(x). \tag{6.13}$$

This is a sine series with coefficients $c_n(0)$. So in order to meet the initial conditions on $u(x,t)$, we require that

$$c_n(0) = \frac{2}{L}\int_0^L f(x)\sin(\tfrac{n\pi}{L}x)\,dx. \tag{6.14}$$

This gives us our initial conditions for the ODEs for $c_n(t)$. So we solve

$$c_n'(t) + (\gamma \tfrac{n\pi}{L})^2 c_n(t) = 0, \tag{6.15a}$$

$$\text{with}, c_n(0) = \frac{2}{L}\int_0^L f(x)\sin(\tfrac{n\pi}{L}x)\,dx. \tag{6.15b}$$

The solution is

$$c_n(t) = A_n e^{-(\gamma \frac{n\pi}{L})^2 t}, \tag{6.16}$$

for some arbitrary constant A. By plugging in $t = 0$, we note that $A = c_n(0)$ (plus you should remember that from ODEs). So our solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin(\tfrac{n\pi}{L}x) = \sum_{n=1}^{\infty} A_n e^{-(\gamma\frac{n\pi}{L})^2 t}\sin(\tfrac{n\pi}{L}x), \tag{6.17a}$$

$$\text{where,} \tag{6.17b}$$

$$A_n = c_n(0) = \frac{2}{L}\int_0^L f(x)\sin(\tfrac{n\pi}{L}x)\,dx. \tag{6.17c}$$

Note that this is exactly the solution we found when using separation of variables.

The real beauty of the method of eigenfunction expansion is that we can use it to solve nonhomogeneous PDEs. Suppose that our heat equation has a nonhomogeneous term. That is, we wish to solve

$$u_t - \gamma^2 u_{xx} = p(x, y), \tag{6.18a}$$
$$u(0, t) = u(L, t) = 0, \tag{6.18b}$$
$$u(x, 0) = f(x). \tag{6.18c}$$

We use the method of eigenfunction expansion to solve (6.18) as follows. We consider the eigenvalue problem of the associated homogeneous problem—the IBVP (6.5). So again we propose a solution that is a sum of the eigenfunctions

$$u(x, t) = \sum_{n=1}^{\infty} c_n(t) \sin(\tfrac{n\pi}{L} x). \tag{6.19}$$

Like before, we plug this into the PDE to see what conditions $c_n(t)$ must satisfy. Plugging in gives

$$\begin{aligned}
&\frac{\partial}{\partial t} \sum_{n=1}^{\infty} c_n(t) \sin(\tfrac{n\pi}{L} x) - \gamma^2 \frac{\partial^2}{\partial x^2} \sum_{n=1}^{\infty} c_n(t) \sin(\tfrac{n\pi}{L} x) \\
=& \sum_{n=1}^{\infty} c_n'(t) \sin(\tfrac{n\pi}{L} x) - \gamma^2 \sum_{n=1}^{\infty} c_n(t) \frac{\partial^2}{\partial x^2} \sin(\tfrac{n\pi}{L} x) \\
=& \sum_{n=1}^{\infty} c_n'(t) \sin(\tfrac{n\pi}{L} x) + \gamma^2 (\tfrac{n\pi}{L})^2 \sum_{n=1}^{\infty} c_n(t) \frac{\partial^2}{\partial x^2} \sin(\tfrac{n\pi}{L} x) \\
=& p(x, y).
\end{aligned} \tag{6.20}$$

Now regrouping terms gives

$$\sum_{n=1}^{\infty} \left[c_n'(t) + (\gamma \tfrac{n\pi}{L})^2 c_n(t) \right] \sin(\tfrac{n\pi}{L} x) = p(x, y). \tag{6.21}$$

This doesn't really tell us anything yet. But, if we expand the function $p(x, y)$ in a sine series we have

$$p(x, y) = \sum_{n=1}^{\infty} \rho_n(t) \sin(\tfrac{n\pi}{L} x), \tag{6.22}$$

and

$$\sum_{n=1}^{\infty}\left[c_n'(t)+(\gamma\tfrac{n\pi}{L})^2c_n(t)\right]\sin(\tfrac{n\pi}{L}x)=\sum_{n=1}^{\infty}\rho_n(t)\sin(\tfrac{n\pi}{L}x). \tag{6.23}$$

Now we can regroup terms and we have

$$\sum_{n=1}^{\infty}\left[c_n'(t)+(\gamma\tfrac{n\pi}{L})^2c_n(t)-\rho_n(t)\right]\sin(\tfrac{n\pi}{L}x)=0. \tag{6.24}$$

Now, we have a sine series that equals zero. Like before, this implies that the coefficients are equal to zero. This gives

$$c_n'(t)+(\gamma\tfrac{n\pi}{L})^2c_n(t)=\rho_n(t). \tag{6.25}$$

So the only difference when we have a nonhomogeneous PDE, is that the resulting ODEs for the $c_n(t)$ are nonhomogeneous. To solve (6.25) you should review the use of an integrating factor. For the nonhomogeneous system we would determine the initial conditions on $c_n(t)$ exactly as before. That is, we still have

$$c_n(0)=\frac{2}{L}\int_0^L f(x)\sin(\tfrac{n\pi}{L}x)\,dx. \tag{6.26}$$

Chapter 7

Fourier Transforms

In this chapter we will look at Fourier transforms. These are related in some ways to Fourier series. First we will look at what a Fourier transform is generally, and finally we will see how these can be used to solve PDEs on infinite domains.

7.1 Fourier Transforms

A Fourier transform is a type of integral transform. If we take a function $f(x)$ defined on $-\infty < x < \infty$, we define its Fourier transform $\hat{\mathbf{f}}(k)$ by the integral

$$\hat{\mathbf{f}}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx. \tag{7.1}$$

There are a couple of things to note here. First, the function $f(x)$ must have certain properties in order for its Fourier transform $\hat{\mathbf{f}}(k)$ to exist. This is because the integral (7.1) may not exist because it goes to infinity. We will not concern ourselves too much with this—we will only deal with functions $f(x)$ that do have Fourier transforms. For the curious, it is sufficient for $f(x)$ to belong to the L^1 or L^2 spaces, though not always necessary. That means that the following integrals exist respectively

$$\int_{-\infty}^{\infty} |f(x)| \, dx, \tag{7.2}$$

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx. \tag{7.3}$$

For the not so curious, don't worry about this, just always assume that the Fourier transform of $f(x)$ exists. Make sure that you understand—what we do when we Fourier transform a function $f(x)$ is we remove the dependence on x by integrating it out, and we recover a new function of k. We simply use the notation $\hat{\mathbf{f}}(k)$, to remind us that $\hat{\mathbf{f}}(k)$ is the transform of $f(x)$. It is in general a totally different function. The function $f(x)$ must be defined for all x, and the Fourier transform $\hat{\mathbf{f}}(k)$ is defined for all real values of k. It turns out that if we know the Fourier transform $\hat{\mathbf{f}}(k)$ of a function $f(x)$, we can recover the original function $f(x)$. This is known as the inverse Fourier transform—so named because it inverts the transformation giving the original function back again. Therefore, we consider the Fourier transform pairs

$$\hat{\mathbf{f}}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx \tag{7.4a}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\mathbf{f}}(k) e^{ikx}\, dk. \tag{7.4b}$$

The second integral is the inverse Fourier transform. The factors of $\frac{1}{\sqrt{2\pi}}$ are merely convention. We could multiply either integral by a constant, as long as we multiply the other by the inverse of that constant. There are several conventions commonly used. If you use a table of integral transforms, make sure that you note the convention used in that table, which will typically be given at the top of the table. If it differs from the convention you want to use, simply multiply the transform or inverse transform appropriately.

7.2 Relationship to Fourier Series

Fourier transforms may seem a bit abstract and unrelated to anything we've done so far. In fact, they are related to Fourier series in some ways. Recall the complex form of Fourier series for functions defined on $-L < x < L$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}x}, \tag{7.5a}$$

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i\frac{n\pi}{L}x}\, dx. \tag{7.5b}$$

You may note that the integral which gives c_n is exactly analogous to the Fourier transform in (7.4) which gives $\hat{\mathbf{f}}(k)$. In fact, you can think if the coefficients in a Fourier series as being analogous to a Fourier transform.

For the coefficients, we had one for every integer n. You can think of the coefficients as being functions in a sense of the integers n. When a set is indexed by discrete values, we usually put the dependence on the index as a subscript. That is, we write c_n not $c(n)$, since n is only allowed to be an integer. But you could think of the coefficients in the Fourier series as a "function" of n. To recover the function $f(x)$ from the Fourier coefficients we add up all of the discrete modes $e^{i\frac{n\pi}{L}x}$ multiplied by the corresponding coefficients c_n.

Now, the following is a heuristic exercise intended to help you understand the relationship between Fourier series and Fourier transforms. This is not intended to be a rigorous derivation of Fourier transforms so you do not need to worry too much if you don't follow these steps. But try to get the general idea. Suppose that a function $f(x)$ is defined on $-L < x < L$. We use an infinite, yet discrete, number of different modes $e^{i\frac{n\pi}{L}x}$ in the Fourier series (7.5) to represent $f(x)$. (Recall that $e^{i\frac{n\pi}{L}x}$ is a sine and cosine.) The term $\frac{n\pi}{L}$ is known as the wave number—it is like a frequency in space. The wave number of a sinusoidal function represents the number of radians per unit length. So, the wave number divided by 2π is the number of cycles per unit length. The wave number is often denoted with a k. (For sinusoidal functions in time, the frequency is analogous to k, but is usually denoted with an ω.) Now, note that two adjacent modes, *i.e.* n and $n+1$, differ in wave number by $\frac{\pi}{L}$. So the larger the domain, the larger the value of L and the closer adjacent modes are in wave number. Suppose that L gets very very large. Then, for any real value you choose you will be able to find an integer n such that $\frac{n\pi}{L}$ is very close to that number. So if L approaches infinity, their are modes with wave numbers $\frac{n\pi}{L}$ that are close to every real number. Suppose that we denote these wave numbers $\frac{n\pi}{L} = k_n$. We can then denote the sum in (7.5) as

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{c}(k_n) e^{ik_n x}, \tag{7.6}$$

where $\hat{c}(k_n) = c_n$. Now, note that $\Delta\frac{n\pi}{L} = \frac{\pi}{L}\Delta n$, where Δ means the difference in two adjacent wave numbers. Of course $\Delta n = n + 1 - n = 1$. So $\frac{L}{\pi}\Delta k_n = \frac{L}{\pi}\Delta\frac{n\pi}{L} = 1$. We multiply (7.6) by this factor which equals unity

$$f(x) = \frac{L}{\pi} \sum_{n=-\infty}^{\infty} \hat{c}(k_n) e^{ik_n x} \Delta k_n. \tag{7.7}$$

Now, we define $\hat{\mathbf{f}}(k_n) = \frac{L}{\pi}\hat{c}(k_n)$. Using these new definitions in (7.5) gives

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{\mathbf{f}}(k_n) e^{ik_n x} \Delta k_n, \tag{7.8a}$$

$$\hat{\mathbf{f}}(k_n) = \frac{L}{\pi}\frac{1}{2L}\int_{-L}^{L} f(x)e^{-ik_n x}\,dx = \frac{1}{2\pi}\int_{-L}^{L} f(x)e^{-ik_n x}\,dx. \tag{7.8b}$$

Now, if we take the limit as $L \to \infty$, then the sum should become an integral over all real values of k_n. So we drop the subscript on k, think of it as a real variable, and we have

$$f(x) = \int_{-\infty}^{\infty} \hat{\mathbf{f}}(k)e^{ikx}\,dk, \tag{7.9a}$$

$$\hat{\mathbf{f}}(k) = \frac{L}{\pi}\frac{1}{2L}\int_{-L}^{L} f(x)e^{-ikx}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx. \tag{7.9b}$$

This is actually a Fourier transform pair. The constants are just a different convention from that introduced in (**??**). If we multiply the bottom integral by $\sqrt{2\pi}$ and divide the top integral by $\sqrt{2\pi}$ we have

$$f(x) = \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\mathbf{f}}(k)e^{ikx}\,dk, \tag{7.10a}$$

$$\hat{\mathbf{f}}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx. \tag{7.10b}$$

So this means that you can think of Fourier transforms as Fourier series for functions defined on the entire real axis, not just $-L < x < L$. Accordingly, for Fourier transforms, there aren't coefficients defined for discrete modes, the coefficients become a function of a continuous variable k representing the wave numbers which have every real value. Rather than summing up discrete wave modes, you must integrate all of the modes.

7.3 Properties of Fourier Transforms

It is the properties of Fourier Transforms that makes then useful for solving PDEs. In this section some of those properties are described. First, note that "Fourier transform" has been used as a noun and a verb. *The* Fourier transform of a function $f(x)$ is another function $\mathbf{f}(\hat{k})$. The Fourier transform can also be thought of as the *act* of transforming $f(x)$. This action can be thought of as an operator acting on $f(x)$. This operator we can denote $\mathcal{F}[\cdot] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\cdot] e^{-ikx}\, dx$. That is,

$$\mathcal{F}[f(x)] = \hat{\mathbf{f}}(k). \tag{7.11}$$

Likewise, we can think of the inverse Fourier transform as the original function or as an operator that returns the original function. We can denote the operator as $\mathcal{F}^{-1}[\cdot]$. That is,

$$\mathcal{F}^{-1}[\mathbf{f}(\hat{\mathbf{k}})] = f(x). \tag{7.12}$$

7.3.1 Transforms of Derivatives

The first property that is important is the relationship between Fourier transforms of functions and Fourier transforms of the derivatives of those functions. That is, how is the Fourier transform of $f(x)$ related to the transform of $f'(x)$ and $f''(x)$ etc. Note that if we differentiate

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\mathbf{f}}(k) e^{ikx}\, dk \tag{7.13}$$

we have

$$\frac{d}{dx} f(x) = \frac{d}{dx} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\mathbf{f}}(k) e^{ikx}\, dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\mathbf{f}}(k) \frac{d}{dx} e^{ikx}\, dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\mathbf{f}}(k) ike^{ikx}\, dk. \tag{7.14}$$

If we denote $ik\hat{\mathbf{f}}(k) = \hat{\mathbf{g}}(k)$, we have

$$f'(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\mathbf{g}}(k) e^{ikx}\, dx. \tag{7.15}$$

So, it must be that $\hat{\mathbf{g}}(k) = (ik)\hat{\mathbf{f}}(k)$ is the Fourier transform of $f'(x)$. We could make the same argument about $f''(x)$, that is, that the Fourier transform of $f''(x) = (ik)\hat{\mathbf{g}}(k) = (ik)^2\hat{\mathbf{f}}(k)$. We could continue recursively for $\frac{d^n f(x)}{dx^n}$, giving the result that

$$\mathcal{F}[\frac{d^n f(x)}{dx^n}] = (ik)^n \hat{\mathbf{f}}(k). \tag{7.16}$$

For instance, you should note that $\mathcal{F}[f''(x)] = -k^2\hat{\mathbf{f}}(k)$.

7.3.2 Linearity

The Fourier transform is linear. This just means that

$$\mathcal{F}[af(x)+bg(x)] = a\mathcal{F}[f(x)] + b\mathcal{F}[g(x)] = a\hat{\mathbf{f}}(k) + b\hat{\mathbf{g}}(k). \tag{7.17}$$

This is easy to see by simply considering the integrals that define the transform

$$\mathcal{F}[af(x)+bg(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[af(x)+bg(x)]e^{-ikx}\,dx \tag{7.18}$$

$$= a\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx + b\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{ikx}\,dx \tag{7.19}$$

$$= a\hat{\mathbf{f}}(k) + b\hat{\mathbf{g}}(k). \tag{7.20}$$

7.3.3 Non-transformed Variables

The transform of $f(x)$ is a function $\hat{\mathbf{f}}(k)$. What if we take the transform, with respect to x, of the function $f(x,t)$? The result is a transform that is a function of k and t. The variable t is unaffected. Just think of it this way—at any given time the function $f(x,t)$ is just a function of x with a given transform. At another time the function $f(x,t)$ is different and so the transform is different. So the function and the transform are both functions of time. This is easy to see mathematically:

$$\hat{\mathbf{f}}(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x,t)e^{-ikx}\,dx. \tag{7.21}$$

At any given time we could invert the transform $\hat{\mathbf{f}}(k,t)$ to recover $f(x,t)$. So we still have

$$f(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\mathbf{f}}(k,t)e^{ikx}\,dk. \tag{7.22}$$

Now, what about derivatives with respect to time? If we differentiate (7.22) with respect to time we have

$$\frac{\partial}{\partial t}f(x,t) = \frac{\partial}{\partial t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\mathbf{f}}(k,t)e^{ikx}\,dk \tag{7.23}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\mathbf{f}}_t(k,t)e^{ikx}\,dx. \tag{7.24}$$

So note that the transform of the derivative of a function with respect to a variable other than x is just the corresponding derivative of the transform. That is,

$$\mathcal{F}[f_t(x,t)] = \hat{\mathbf{f}}_t(k,t). \tag{7.25}$$

7.3.4 Products of Transforms and Convolutions

Suppose that we have two functions $f(x)$ and $g(x)$. Suppose that the transform of each is $\hat{\mathbf{f}}(k)$ and $\hat{\mathbf{g}}(k)$ respectively. Now, suppose we consider the function that is the product of the two transforms $\hat{\mathbf{f}}(k)\hat{\mathbf{g}}(k)$. Is is true that the inverse transform of $\mathcal{F}^{-1}[\hat{\mathbf{f}}(k)\hat{\mathbf{g}}(k)]$ is $f(x)g(x)$? The answer is no in general. If we consider the transform of $h(x) = f(x)g(x)$, we have

$$\mathcal{F}[h(x)] = \hat{\mathbf{h}}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)g(x)e^{-ikx}\,dx. \tag{7.26}$$

But the product $\hat{\mathbf{f}}(k)\hat{\mathbf{g}}(k)$ is

$$\begin{aligned} \hat{\mathbf{f}}(k)\hat{\mathbf{g}}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)e^{-ikx}\,dx & (7.27) \\ &\neq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)g(x)e^{-ikx}\,dx = \hat{\mathbf{h}}(k). & (7.28) \end{aligned}$$

So in general

$$\mathcal{F}[f(x)g(x)] \neq \mathcal{F}[f(x)]\mathcal{F}[g(x)], \tag{7.29}$$

and therefore, in general

$$\mathcal{F}^{-1}[\hat{\mathbf{f}}(k)\hat{\mathbf{g}}(k)] \neq \mathcal{F}^{-1}[\hat{\mathbf{f}}(k)]\mathcal{F}^{-1}[\hat{\mathbf{g}}(k)]. \tag{7.30}$$

In other words, if you want to invert a transform that is a product of two functions, unfortunately you can't simply invert each function separately and expect the product of the inverses to be the inverse of the products. However, there is hope. If we denote $\mathcal{F}^{-1}[\hat{\mathbf{f}}(k)\hat{\mathbf{g}}(k)] = p(x)$, can we relate $p(x)$ to $f(x)$ and $g(x)$, even though $p(x) \neq f(x)g(x)$? The answer is yes! We will see that $p(x)$ is something called the convolution of $f(x)$ and $g(x)$. If we look at $\mathcal{F}[f(x)]\mathcal{F}[g(x)]$, we have

$$\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx\right)\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)e^{-ikx}\,dx\right). \tag{7.31}$$

So far this is just the product of two functions of k. Each integral must be performed over all x before multiplying the other one. It is permissable to turn this into an iterated integral, as long as we relabel one of the x variables—since the two x's in (7.31) are not a single variable in an iterated integral. To avoid confusion we rewrite (7.31) as

$$\mathcal{F}[f(x)]\mathcal{F}[g(x)] = \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)e^{-iks}\,ds\right)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(u)e^{-iku}\,du\right)$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)e^{-iks}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(u)e^{-iku}\,du\right)ds. \tag{7.32}$$

Now, the inner integral in (7.32) is over all u at every given value of s in the outer integral. So we can make the change of variables $u = x - s$, where s is fixed and x varies over all real values. So therefore $x - s$ will still vary over all values and $du = dx$. This gives

$$\mathcal{F}[f(x)]\mathcal{F}[g(x)] =\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)e^{-iks}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-s)e^{-ik(x-s)}\,ds\right)dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)g(x-s)\,ds\right)e^{-ikx}\,dx \tag{7.33}$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} p(x)e^{-ikx}\,dx \tag{7.34}$$

$$. \tag{7.35}$$

This is actually the result we are looking for. Note that the inner integral is a function of x since the s is integrated out. Therefore (7.33) is the Fourier transform of the function

$$p(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)g(x-s)\,ds. \tag{7.36}$$

This function $p(x)$ is called the convolution of f and g. It is denoted by $f * g(x)$. It is important to note that $f * g(x)$ is just another function of x. So note we have shown that the product of two Fourier transforms is the same as the Fourier transform of the convolution of the two functions. In other words, we have shown that

$$\mathcal{F}[f(x)]\mathcal{F}[g(x)] = \mathcal{F}[f * g(x)]. \tag{7.37}$$

Since we are assuming that all of the transforms and inverse transforms exist, this also means that the inverse transform of $\mathcal{F}[f(x)]\mathcal{F}[g(x)]$ is the inverse transform of $\mathcal{F}[f * g(x)]$, which is of course $f * g(x)$. Stated another way, the inverse transform of a product of transforms, $\hat{\mathbf{f}}(k)\hat{\mathbf{g}}(k)$, is the convolution of the original functions, not the product of the functions. This is known as the convolution theorem of Fourier transforms

Convolution Theorem of Fourier Transforms

$$\mathcal{F}^{-1}[\hat{\mathbf{f}}(k)\hat{\mathbf{g}}(k)] = f * g(x), \tag{7.38a}$$

or more explicitly,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\mathbf{f}}(k)\hat{\mathbf{g}}(k)e^{ikx}\,dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)g(x-s)\,ds. \tag{7.38b}$$

You might have noticed something perplexing. The product of transforms doesn't matter on the order, that is, products obviously commute

$$\hat{\mathbf{f}}(k)\hat{\mathbf{g}}(k) = \hat{\mathbf{g}}(k)\hat{\mathbf{f}}(k). \tag{7.39}$$

So how do we know if the inverse transform of $\hat{\mathbf{f}}(k)\hat{\mathbf{g}}(k)$ is $f * g(x)$ or $g * f(x)$? The answer is obviously that they are the same. That is,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)g(x-s)\,ds = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(s)f(x-s)\,ds. \tag{7.40}$$

We could easily show this explicitly by a change of variables in either integral. This should be remembered even outside of PDEs—the convolution of two functions commutes in the sense that $f * g(x) = g * f(x)$.

7.3.5 The Transforms of Gaussian Functions

Often Fourier transform pairs can be either calculated directly via integration, or perhaps looked up in a table. That is, given $f(x)$ what is $\hat{\mathbf{f}}(k)$ or vice versa. There are some functions where it is worth memorizing what the pairs are. One example of an important transform pair is that of the Gaussian function $f(x) = e^{-\frac{ax^2}{2}}$. Recall that this function is the normal curve—the bell-shaped curve important in statistics and elsewhere. (It's important in PDEs too.) It turns out that the transform of a Gaussian is again another Gaussian. (This is not usually the case—for a given $f(x)$, the transform $\hat{\mathbf{f}}(k)$ is usually a completely different function.) The pairs are

The Gaussian Function and Transform

$$f(x) = e^{-\frac{ax^2}{2}} \Leftrightarrow \hat{\mathbf{f}}(k) = \frac{1}{\sqrt{a}} e^{-\frac{k^2}{2a}} \tag{7.41}$$

The book has a simple and nice proof of (7.41) on page 406, using ODEs. We are assuming that $a > 0$, otherwise the transform wouldn't exist as the function would blow-up at infinity. The constant $a > 0$, which indicates the spread of $f(x)$ (related to the *standard deviation* in statistics lingo) is inversely related to the spread of the transform. That is, if the function is a very broadly spread then the transform will be peaked around $k = 0$ and vice-versa. (In fact, if we allowed $f(x)$ to be a delta function by letting $a \to \infty$, then the transform would be a constant. We wouldn't be able to invert this transform, but it is theoretically important nevertheless. You might want to read about generalized functions or distributions described in chapter 7.)

7.4 Using Transforms to Solve PDEs

7.4.1 The Heat Equation

We now know enough about Fourier transforms to be able to use them to solve PDEs on infinite domains. Here is the basic idea. Suppose that you are given a PDE on an infinite domain. Let's first look at the heat equation

$$u_t - \gamma^2 u_{xx} = 0, \qquad -\infty < x < \infty, \quad t > 0, \tag{7.42}$$

$$u(x,0) = f(x), \qquad -\infty < x < \infty. \tag{7.43}$$

(Digression: Since we are on an infinite domain, we often don't write boundary conditions explicitly since there aren't any real boundaries. However, when using Fourier transforms, we have to assume that the solution approaches zero at infinity so that it's transform exists. This is a type of boundary condition, but it usually isn't stated explicitly. In the real world, you have to know from context whether this assumption is valid given your system. If it is not, you must use some other solution technique. Most of time in applications, it is the correct assumption.) Now, the function $u_t(x,t)$ is just a function of x and t, so it can be transformed with respect to x. The same goes for $u_{xx}(x,t)$. The sum $u_t(x,t) - \gamma^2 u_{xx}(x,t)$ is just a function too. So we can transform the whole thing—both sides of the PDE. The transform of 0 is of course just 0. So we have

$$\mathcal{F}[u_t - \gamma^2 u_{xx}] = 0. \tag{7.44}$$

Now, using the linearity property we have

$$\mathcal{F}[u_t] - \gamma^2 \mathcal{F}[u_{xx}] = 0. \tag{7.45}$$

Then, using what we know about the properties of derivatives of transforms gives

$$\frac{\partial}{\partial t}\mathcal{F}[u] - \gamma^2 (ik)^2 \mathcal{F}[u] = 0. \tag{7.46}$$

Denoting the transforms as functions of k, and noting that $(ik)^2 = -k^2$, gives

$$\frac{\partial}{\partial t}\hat{\mathbf{u}}(k,t) + (k\gamma)^2 \hat{\mathbf{u}}(k,t) = 0. \tag{7.47}$$

Why did we do this? Well, consider this: if we can find the transform $\hat{\mathbf{u}}(k,t)$, then we essentially know the solution since all we'd have to do is invert the transform. That is, knowing $\hat{\mathbf{u}}(k,t)$ is almost as good as knowing $u(x,t)$ since we are only an integral away from $u(x,t)$. If finding $\hat{\mathbf{u}}(k,t)$ is a lot easier than finding $u(x,t)$, then we might as well solve for the transform and invert it. Note that we have an ODE with respect to t for the function $\hat{\mathbf{u}}(x,t)$ which we can easily solve. What we have in effect done, is convert a problem that is a PDE for $u(x,t)$ into a problem that is an ODE for $\hat{\mathbf{u}}(k,t)$. Solving (7.47) gives

$$\hat{\mathbf{u}}(k,t) = Ae^{-(k\gamma)^2 t}. \tag{7.48}$$

Now, we just need to find the arbitrary coefficient A. However, you should note the following: A is not a function of t, but it could be a function of k! So really, we should write the solution as

$$\hat{\mathbf{u}}(k,t) = A(k)e^{-(k\gamma)^2 t}, \tag{7.49}$$

and note that we need to find the function $A(k)$. Note that plugging in $t = 0$ isolates $A(k)$

$$\hat{\mathbf{u}}(k,0) = A(k). \tag{7.50}$$

So we see that $A(k)$ corresponds to the transform of the solution—evaluated at $t = 0$. Is this just the transform—of the solution evaluated at $t = 0$? Of course it is, as you can see by plugging in $t = 0$ into

$$\hat{\mathbf{u}}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)e^{-ikx}\, dx. \tag{7.51}$$

So $A(k)$ must be the transform of the initial conditions:

$$A(k) = \hat{\mathbf{f}}(k). \tag{7.52}$$

We now know completely the transform of the solution, assuming that we can calculate $\hat{\mathbf{f}}(k)$

$$\hat{\mathbf{u}}(k,t) = \hat{\mathbf{f}}(k)e^{-(k\gamma)^2 t}. \tag{7.53}$$

Now, the solution is only an integral away. That is

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\mathbf{u}}(k,t)e^{ikx}\, dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\mathbf{f}}(k)e^{-(k\gamma)^2 t}e^{ikx}\, dk. \tag{7.54}$$

However, in practice you should note something about this: $\hat{\mathbf{f}}(k)e^{-(k\gamma)^2 t}$ will typically be a more complex function that $\hat{\mathbf{f}}(k)$ and $e^{-(k\gamma)^2 t}$ individually. So, it might be easier to invert them separately rather than as a product. Not to mention, you already know the inverse of $\hat{\mathbf{f}}(k)$—it is just $f(x)$! So it is much less work to use the convolution formula

$$u(x,t) = \mathcal{F}^{-1}[\hat{\mathbf{f}}(k)e^{-(k\gamma)^2 t}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s)g(s-x)\, ds, \tag{7.55}$$

where $f(x)$ is the initial condition, and $g(x)$ is $\mathcal{F}^{-1}[e^{-(k\gamma)^2 t}]$. Note that we never even needed to compute $\hat{\mathbf{f}}(k)$. Also, it should be easy to find $\mathcal{F}^{-1}[e^{-(k\gamma)^2 t}]$, since it is a Gaussian. We use (7.41), letting

$$\begin{aligned} \frac{1}{2a} &= \gamma^2 t, \\ \Rightarrow a &= \frac{1}{2\gamma^2 t}, \end{aligned} \tag{7.56}$$

and noting that $\hat{\mathbf{g}}(k) = \sqrt{a}(\frac{1}{\sqrt{a}}e^{-\frac{k^2}{2a}})$. This gives the result

$$g(x) = \sqrt{a}e^{-\frac{ax^2}{2}} = \sqrt{\frac{1}{2\gamma^2 t}}e^{-\frac{\frac{1}{2\gamma^2 t}x^2}{2}} = \frac{1}{\gamma\sqrt{2t}}e^{-\frac{x^2}{4\gamma^2 t}}. \tag{7.57}$$

Now we have our solution in terms of a single convolution

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) g(x-s)\, ds, \tag{7.58}$$

where $g(x)$ is given by (7.57). The function (7.57) is known as the heat kernel, since it provides the solution to the heat equation by simply convolving it with the initial conditions. This is an example of what is known as Duhamel's principle. For the curious, this is similar to, and in fact can be shown to be equivalent to, the use of Green's functions for nonhomogeneous PDEs. If these names sound familiar and you are curious of what they are, I encourage you to read about them in your book. Depending on $f(x)$, the integral (7.58) can be easy...or difficult. Occasionally it must be done numerically, but solving a PDE numerically by simply performing an integral is still typically preferable to other methods. Integrals can also often be analyzed by what are called asymptotic methods—a way to approximate the integral as close as one likes.

7.4.2 The Wave Equation

Let's look at one more example of the use of Fourier transforms—the wave equation on an infinite domain

$$u_{tt} - c^2 u_{xx} = 0, \qquad -\infty < x < \infty, \quad t > 0, \tag{7.59}$$

$$u(x,0) = f(x), \qquad -\infty < x < \infty, \tag{7.60}$$

$$u_t(x,0) = g(x), \qquad -\infty < x < \infty. \tag{7.61}$$

As on a finite domain, we now need two initial conditions since we have two time derivatives. Taking the same approach as for the heat equation, we transform the entire PDE

$$\mathcal{F}[u_{tt} - c^2 u_{xx}] = 0, \tag{7.62}$$

which gives an ODE for the transform of the solution

$$\frac{\partial^2}{\partial t^2}\hat{\mathbf{u}}(k,t) - c^2 (ik)^2 \hat{\mathbf{u}}(k,t) = 0, \tag{7.63}$$

or

$$\frac{\partial^2}{\partial t^2}\hat{\mathbf{u}}(k,t) + (ck)^2 \hat{\mathbf{u}}(k,t) = 0. \tag{7.64}$$

As you might have expected, we now have a second order ODE rather than a first order ODE as with the heat equation. We solve it with ease

$$\hat{\mathbf{u}}(k,t) = A\sin(ckt) + B\cos(ckt). \tag{7.65}$$

Again, the arbitrary coefficients may be functions of k, since the derivative in the ODE was with respect to t. We therefore have

$$\hat{\mathbf{u}}(k,t) = A(k)\sin(ckt) + B(k)\cos(ckt). \tag{7.66}$$

Now we just need to determine the functions $A(k)$ and $B(k)$. If we consider the inverse transform of $\hat{\mathbf{u}}(k,t)$ (that is, the solution) we have

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\mathbf{u}}(k,t)e^{ikx}\,dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left[A(k)\sin(ckt) + B(k)\cos(ckt)\right]e^{ikx}\,dk. \tag{7.67}$$

Now, plugging in $t = 0$ gives

$$u(x,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\mathbf{u}}(k,0)e^{ikx}\,dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} B(k)e^{ikx}\,dk. \tag{7.68}$$

So we see that $B(k)$ is just the transform of the initial function, or $B(k) = \hat{\mathbf{f}}(k)$. Differentiating (7.70) with respect to t gives

$$u_t(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\mathbf{u}}_t(k,t)e^{ikx}\,dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left[A(k)ck\cos(ckt) - B(k)ck\sin(ckt)\right]e^{ikx}\,dk. \tag{7.69}$$

Evaluating this at $t = 0$ gives

$$u_t(x,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\mathbf{u}}_t(k,0)e^{ikx}\,dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)cke^{ikx}\,dk. \tag{7.70}$$

We see that the function $A(k)ck$ is just the transform of $u_t(x,0) = g(x)$. So we write $A(k)ck = \hat{\mathbf{g}}(k)$, or $A(k) = \frac{1}{ck}\hat{\mathbf{g}}(k)$. Now, we could write the solution as

$$\begin{aligned} u(x,t) =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left[\frac{\hat{\mathbf{g}}(k)}{ck}\sin(ckt) + \hat{\mathbf{f}}(k)\cos(ckt)\right]e^{ikx}\,dk \\ =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left[\hat{\mathbf{g}}(k)\frac{\sin(ckt)}{ck}\right]e^{ikx}\,dk + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left[\hat{\mathbf{f}}(k)\cos(ckt)\right]e^{ikx}\,dk. \end{aligned} \tag{7.71}$$

We have two integrals, but again, we should make use of the convolution formula, since then we won't even need to transform $f(x)$ or $g(x)$. For the first integral, we think of it as the product of $\hat{\mathbf{g}}(k)$ with the function $\frac{\sin(ckt)}{ck}$, since the former we know the inverse transform to be simply $g(x)$. Now, we just need to figure out the inverse transforms of $\frac{\sin(ckt)}{ck}$ and $\cos(ckt)$, and then we can write the solution in terms of two convolution integrals. (By inverse transform, we mean the functions of x, that when transformed give $\frac{\sin(ckt)}{ck}$ and $\cos(ckt)$.) **You are not responsible for the understanding the following material regarding determining the final result—except you should memorize the final result.**

First, you might think that you could simply integrate the functions $\frac{\sin(ckt)}{ck}$ and $\cos(ckt)$ using the definition of the inverse transform. However, it turns out that those integrals don't exist in the classic sense! That is, we can not recover the functions of x from there transforms, by integrating! If you find this confusing, it is. This is a bit beyond the scope of this class, but rest assured, all we need in order to use the convolution formula, is to find functions of x, that when transformed give us $\frac{\sin(ckt)}{ck}$ and $\cos(ckt)$. Furthermore, by "functions" we'll be a bit flexible and consider things like the delta function. First, consider the Fourier transform

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2} \left[\delta(x-a) + \delta(x+a)\right] e^{-ikx}\, dx, \tag{7.72}$$

where $a > 0$ is an arbitrary constant. Integrating gives

$$\frac{1}{\sqrt{2\pi}} \left(\frac{1}{2} \left[e^{ika} + e^{-ika} \right] \right) = \frac{1}{\sqrt{2\pi}} \cos(ka). \tag{7.73}$$

So it must be that the inverse Fourier transform of $\frac{1}{\sqrt{2\pi}} \cos ka$ is the sum of two delta functions $\frac{1}{2}\left[\delta(x-a) + \delta(x+a)\right]$! Or, the inverse transform of $\cos(ka)$ is just $\frac{\sqrt{2\pi}}{2}\left[\delta(x-a) + \delta(x+a)\right]$ So, noting that $a = ct$ in our transform of interest, gives the function we are looking for, whose transform is $\cos(ckt)$. Let's denote the function $v(x)$ and note that

$$v(x) = \frac{\sqrt{2\pi}}{2} \left[\delta(x-ct) + \delta(x+ct)\right]. \tag{7.74}$$

Now, consider the step function that is equal to one on a symmetric interval around the origin and zero elsewhere

$$h(x) = \begin{cases} 1, & \text{if} \quad |x| < a, \\ 0, & \text{if} \quad |x| > a. \end{cases} \tag{7.75}$$

Now, if we consider the Fourier transform of such a function we have

$$\begin{aligned}
\mathcal{F}[h(x)] =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x) e^{-ikx}\, dx \\
=& \int_{-a}^{a} e^{-ikx}\, dx \\
=& \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-ikx}}{-ik} \right] \Bigg|_{-a}^{a} \\
=& \frac{2}{\sqrt{2\pi}} \left(\frac{1}{k} \frac{(e^{ika} - e^{-ika})}{2i} \right) = \qquad \frac{2}{\sqrt{2\pi}} \frac{\sin(ka)}{k} = \hat{\mathbf{h}}(k)
\end{aligned}$$

So, if we recognize that $\frac{\sin(ctk)}{ck}$ is $\frac{\sqrt{2\pi}}{2c}\hat{\mathbf{h}}(k)$, where $a = ct$, then the inverse transform of $\frac{\sin(ctk)}{ck}$ must be $\frac{\sqrt{2\pi}}{2} h(x)$, with $a = ct$. Let's denote it $H(x)$, where

$$H(x) = \begin{cases} \frac{\sqrt{2\pi}}{2c}, & \text{if} \quad |x| < ct, \\ 0, & \text{if} \quad |x| > ct. \end{cases} \tag{7.76}$$

Now, we just use the convolution formula to determine the inverse transform of $\hat{\mathbf{g}}(k)\frac{\sin(ckt)}{ck}$ and $\hat{\mathbf{f}}(k)\cos(ckt)$, noting that the inverse transform of $\hat{\mathbf{g}}(k)$ is of course $g(x)$, inverse transform of $\frac{\sin(ckt)}{ck}$ is $H(x)$, the inverse transform of $\hat{\mathbf{f}}(k)$ is of course $f(x)$ and the inverse transform of $\cos(ckt)$ is $v(x)$. We therefore have

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-s) v(s)\, ds + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s) H(x-s)\, ds. \tag{7.77}$$

Now, if we plug in the formulas for $v(s)$ and $H(s)$ we have

$$u(x,t) = \frac{1}{2} \left[f(x+ct) + f(x-ct) \right] + \frac{1}{2c} \int_{x-s=ct}^{x-s=-ct} g(s)\, ds. \tag{7.78}$$

Note the upper and lower limits on the integral. Those arise because $H(x-s)$ is zero when $x - s > ct$ and $x - s < -ct$. Of course the limits are really on the integration variable, s, but I've written it this way first, so it is clear where they arise from. If we solve the integration limits for s, we have

$$\boxed{u(x,t) = \frac{1}{2} \left[f(x+ct) + f(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds.} \tag{7.79}$$

You may be unimpressed by this formula if you have never seen it before. However, this is known as D'Alembert's formula. It arises is several different derivations of the solution to the wave equation. It actually tells us a lot about the solution physically. Note what it implies: the solution $u(x, t)$ at any given x and t can be entirely determined by the initial displacement at two points, $x_1 = x + ct$ and $x_2 = x - ct$, as well as the integral of the initial velocity between those two points. The interval between these two points at $t = 0$ is known as the *domain of dependence* for the solution u at the point x and time t. We will draw a few pictures in class that will help clarify this. You need only commit to memory (or your notecard) the formula (7.79), and know how to use it.