Lecture 2 Canonical Forms or Normal Forms

By a suitable change of the independent variables we shall show that any equation of the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0, \quad (1)$$

where A, B, C, D, E, F and G are functions of the variables x and y, can be reduced to a canonical form or normal form. The transformed equation assumes a simple form so that the subsequent analysis of solving the equation will be become easy.

Consider the transformation of the indpendent variables from (x, y) to (ξ, η) given by

$$\xi = \xi(x, y), \quad \eta = \eta(x, y). \quad (2)$$

Here, the functions ξ and η are continuously differentiable and the Jacobian

$$J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = (\xi_x\eta_y - \xi_y\eta_x) \neq 0 \quad (3)$$

in the domain where (1) holds.

Using chain rule, we notice that

$$\begin{aligned} u_x &= u_\xi\xi_x + u_\eta\eta_x \\ u_y &= u_\xi\xi_y + u_\eta\eta_y \\ u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx} \\ u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy} \\ u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy} \end{aligned}$$

Substituting these expression into (1), we obtain

$$\bar{A}(\xi_x,\xi_y)u_{\xi\xi} + \bar{B}(\xi_x,\xi_y;\eta_x,\eta_y)u_{\xi\eta} + \bar{C}(\eta_x,\eta_y)u_{\eta\eta} = F(\xi,\eta,u(\xi,\eta),u_\xi(\xi,\eta),u_\eta(\xi,\eta)), \quad (4)$$

where

$$\begin{aligned} \bar{A}(\xi_x,\xi_y) &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\ \bar{B}(\xi_x,\xi_y;\eta_x,\eta_y) &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\ \bar{C}(\eta_x,\eta_y) &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2. \end{aligned}$$

An easy calculation shows that

$$\bar{B}^2 - 4\bar{A}\bar{C} = (\xi_x\eta_y - \xi_y\eta_x)^2(B^2 - 4AC). \quad (5)$$

The equation (5) shows that the transformation of the independent variables does not modify the type of PDE.

We shall determine ξ and η so that (4) takes the simplest possible form. We now consider the following cases:

Case I: $B^2 - 4AC > 0$ (Hyperbolic type)
Case II: $B^2 - 4AC = 0$ (Parabolic type)
Case III: $B^2 - 4AC < 0$ (Elliptic type)

Case I: Note that $B^2 - 4AC > 0$ implies the equation $A\alpha^2 + B\alpha + C = 0$ has two real and distinct roots, say λ_1 and λ_2. Now, choose ξ and η such that

$$\frac{\partial \xi}{\partial x} = \lambda_1 \frac{\partial \xi}{\partial y} \quad \text{and} \quad \frac{\partial \eta}{\partial x} = \lambda_2 \frac{\partial \eta}{\partial y}. \tag{6}$$

Then the coefficients of $u_{\xi\xi}$ and $u_{\eta\eta}$ will be zero because

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = (A\lambda_1^2 + B\lambda_1 + C)\xi_y^2 = 0,$$
$$\bar{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = (A\lambda_2^2 + B\lambda_2 + C)\eta_y^2 = 0.$$

Thus, (5) reduces to

$$\bar{B}^2 = (B^2 - AC)(\xi_x\eta_y - \xi_y\eta_x)^2 > 0$$

as $B^2 - 4AC > 0$. Note that (6) is a first-order linear PDE in ξ and η whose characteristics curves are satisfy the first-order ODEs

$$\frac{dy}{dx} + \lambda_i(x, y) = 0, \quad i = 1, 2. \tag{7}$$

Let the family of curves determined by the solution of (7) for $i = 1$ and $i = 2$ be

$$f_1(x, y) = c_1 \quad \text{and} \quad f_2(x, y) = c_2, \tag{8}$$

respectively. These family of curves are called characteristics curves of PDE (5). With this choice, divide (4) throughout by $\bar{B}$ (as $\bar{B} > 0$) and use (7)-(8) to obtain

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = \phi(\xi, \eta, u, u_\xi, u_\eta), \tag{9}$$

which is the canonical form of hyperbolic equation.

Example 1. *Reduce the equation $u_{xx} = x^2 u_{yy}$ to its canonical form.*

Solution. Comparing with (1) we find that $A = 1$, $B = 0$, $C = -x^2$.

The roots of the equations $A\alpha^2 + B\alpha + C = 0$ i.e., $\alpha^2 + x^2 = 0$ are given by $\lambda_i = \pm x$. The differential equations for the family of characteristics curves are

$$\frac{dy}{dx} \pm x = 0.$$

whose solutions are $y + \frac{1}{2}x^2 = c_1$ and $y - \frac{1}{2}x^2 = c_2$. Choose

$$\xi = y + \frac{1}{2}x^2, \quad \eta = y - \frac{1}{2}x^2.$$

An easy computation shows that

$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x, \\
u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx} \\
&= u_{\xi\xi}x^2 - 2u_{\xi\eta}x^2 + u_{\eta\eta}x^2 + u_\xi - u_\eta, \\
u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}, \\
&= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.
\end{aligned}$$

Substituting these expression in the equation $u_{xx} = x^2 u_{yy}$ yields

$$\begin{aligned}
& 4x^2 u_{\xi\eta} = (u_\xi - u_\eta) \\
\text{or} \quad & 4(\xi - \eta)u_{\xi\eta} = \frac{1}{4(\xi - \eta)}(u_\xi - u_\eta) \\
\text{or} \quad & u_{\xi\eta} = \frac{1}{4(\xi - \eta)}(u_\xi - u_\eta)
\end{aligned}$$

which is the required canonical form.

CASE II: $B^2 - 4AC = 0 \Longrightarrow$ the equation $A\alpha^2 + B\alpha + C = 0$ has two equal roots, say $\lambda_1 = \lambda_2 = \lambda$. Let $f_1(x, y) = c_1$ be the solution of $\frac{dy}{dx} + \lambda(x, y) = 0$. Take $\xi = f_1(x, y)$ and η to be the any function of x and y which is independent of ξ.

As before, $\bar{A}(\xi_x, \xi_y) = 0$ and hence from equation (5), we obtain $\bar{B} = 0$. Note that $\bar{C}(\eta_x, \eta_y) \neq 0$, otherwise η would be a function of ξ. Dividing (4) by $\bar{C}$, the canonical form of (2) is

$$u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta). \tag{10}$$

which is the canonical form of parabolic equation.

Example 2. *Reduce the equation $u_{xx} + 2u_{xy} + u_{yy} = 0$ to canonical form.*

Solution. In this case, $A = 1$, $B = 2$, $C = 1$. The equation $\alpha^2 + 2\alpha + 1 = 0$ has equal roots $\lambda = -1$. The solution of $\frac{dy}{dx} - 1 = 0$ is $x - y = c_1$ Take $\xi = x - y$. Choose $\eta = x + y$. proceed as in Example 1 to obtain $u_{\eta\eta} = 0$ which is the canonical form of the given PDE.

CASE III: When $B^2 - 4AC < 0$, the roots of $A\alpha^2 + B\alpha + C = 0$ are complex. Following the procedure as in CASE I, we find that

$$u_{\xi\eta} = \phi_1(\xi, \eta, u, u_\xi, u_\eta). \tag{11}$$

The variables ξ, η are infact complex conjugates. To get a real canonical form use the transformation

$$\alpha = \frac{1}{2}(\xi + \eta), \quad \beta = \frac{1}{2i}(\xi - \eta)$$

to obtain

$$u_{\xi\eta} = \frac{1}{4}(u_{\alpha\alpha} + u_{\beta\beta}), \tag{12}$$

which follows from the following calculation:

$$\begin{aligned} u_\xi &= u_\alpha \alpha_\xi + u_\beta \beta_\xi = \frac{1}{2}u_\alpha + \frac{1}{2i}u_\beta \\ u_{\xi\eta} &= \frac{1}{2}(u_{\alpha\alpha}\alpha_\eta + u_{\alpha\beta}\beta_\eta) + \frac{1}{2i}(u_{\beta\alpha}\alpha_\eta + u_{\beta\beta}\beta_\eta) \\ &= \frac{1}{4}(u_{\alpha\alpha} + u_{\beta\beta}). \end{aligned}$$

The desired canonical form is

$$u_{\alpha\alpha} + u_{\beta\beta} = \psi(\alpha, \beta, u(\alpha, \beta), u_\alpha(\alpha, \beta), u_\beta(\alpha, \beta)). \tag{13}$$

Example 3. *Reduce the equation $u_{xx} + x^2 u_{yy} = 0$ to canonical form.*

Solution. In this case, $A = 1$, $B = 0$, $C = x^2$. The roots are $\lambda_1 = ix$, $\lambda_2 = -ix$. Take $\xi = iy + \frac{1}{2}x^2$, $\eta = -iy + \frac{1}{2}x^2$. Then $\alpha = \frac{1}{2}x^2$, $\beta = y$. The canonical form is

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{2\alpha}u_\alpha.$$

Practice Problems

1. Reduce the following equations to canonical/normal form:

   (A) $2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0$.

   (B) $u_{xx} + yu_{yy} = 0$.

   (C) $u_{xy} + u_x + u_y = 2x$.

2. Show that the equation

   $$u_{xx} - 6u_{xy} + 12u_{yy} + 4u_x - u = \sin(xy)$$

   is of elliptic type and obtain its canonical form.

3. Determine the regions where Tricomi's equation $u_{xx} + xu_{yy} = 0$ is of elliptic, parabolic, and hyperbolic types. Obtain its characteristics and its canonical form in the hyperbolic region.