A Note on Parametrization

The key to parametrizartion is to realize that the goal of this method is to describe the location of all points on a geometric object, a curve, a surface, or a region. This description must be one-to-one and onto: every point must be described once and only once.

1 Parametrization of Curves in R^2

Let us begin with parametrizing the curve C whose equation is given by

$$x^2 + y^2 = 4 \tag{1}$$

i.e., a circle of radius 2 centered at the origin. We start by associating a **position vector r** to each point (x, y) on C through the relation

$$\mathbf{r} = \langle x, y \rangle. \tag{2}$$

The coordinates x and y in (2)are not arbitrary – they are related through equation (1). This means that we are free to assign a value to only one of the coordinates of a typical point on C; the other coordinate must be determined from the equation of the circle. For this reason we say C has one **degree of freedom**.

Choosing x as the parameter for C, we see from (1) that

$$y = \pm\sqrt{4 - x^2},$$

where the positive square root describes those points on C that lie above the x-axis and the negative square root the points below the x-axis. The complete parametrization of C is

$$\mathbf{r}_1(x) = \langle x, \sqrt{4 - x^2} \rangle \quad \text{and} \quad \mathbf{r}_2(x) = \langle x, -\sqrt{4 - x^2} \rangle, \tag{3}$$

where $-2 \leq x \leq 2$ for $\mathbf{r}_1$ and $-2 < x < 2$ for $\mathbf{r}_2$. Note that the points $(-2, 0)$ and $(2, 0)$ are arbitrarily assigned to $\mathbf{r}_1$. We can now use the parametrization of C to determine tangent vectors to C, plot C on a graphics software, or to perform a line integral around C.

Although the paramerization in (3) is adequate for the purpose of describing C, it is not the most convenient description of this curve. A more efficient way to view C is to use polar coordinates to describe its points: $x = 2\cos\theta, y = 2\sin\theta$, with $\theta \in [0, 2\pi)$. So C can also be parametrized as

$$\mathbf{r}_3(\theta) = \langle 2\cos\theta, 2\sin\theta \rangle, \quad \theta \in [0, 2\pi). \tag{4}$$

Note that $\mathbf{r}_3$ in (4) does the job of both $\mathbf{r}_1$ and $\mathbf{r}_2$ in (3).

The parametrizations $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ are just a few ways out of the infinitely many ways that one could describe C. Here are three other parametrizations of the same curve:

$$\mathbf{r}_4(t) = \langle 2\sin t, 2\cos t \rangle, \quad t \in [0, 2\pi), \tag{5}$$

where C is traversed in the clockwise direction,

$$\mathbf{r}_5(u) = \langle -2\sin u, 2\cos u\rangle, \quad u \in [0, 2\pi), \tag{6}$$

where C is traversed in the counterclockwise direction (how is $\mathbf{r}_5$ different from $\mathbf{r}_3$?) and

$$\mathbf{r}_6(w) = \langle 2\sin 2w, 2\cos 2w\rangle, \quad w \in [0, \pi). \tag{7}$$

To understand the difference between $\mathbf{r}_4$ and $\mathbf{r}_6$, compute the speed of a particle traveling around C according to these parametrizations.

Let us now consider parametrizations of other familiar curves. Any two dimensional curve whose equation is given by $y = f(x)$ can be parametrized as

$$\mathbf{r}(x) = \langle x, f(x)\rangle, \quad x \in (a, b), \tag{8}$$

so, for instance, the straight line $y = mx + b$ can be viewed as

$$\mathbf{r}(x) = \langle x, mx + b\rangle. \tag{9}$$

The circle of radius a centered at (b, c) is parametrized as

$$\mathbf{r}(\theta) = \langle b + a\cos\theta, c + a\sin\theta\rangle, \quad \theta \in (0, 2\pi]. \tag{10}$$

The ellipse whose equation is given by $a^2x^2 + b^2y^2 = c^2$ is parametrized as (to see where the following expressions come from, divide $a^2x^2 + b^2y^2 = c^2$ by c^2 and set the term containing x^2 equal to $\cos^2 t$ and the one containing y^2 to $\sin^2 t$)

$$\mathbf{r}(t) = \langle \frac{c}{a}\cos t, \frac{c}{b}\sin t\rangle \quad t \in (0, 2\pi). \tag{11}$$

2 Parametrization of Curves in R^3

Similar to curves in R^2, curves in R^3 still have only one degree of freedom, that is, a single parameter is sufficient to describe the coordinates of a typical point on curves in R^3. As an example, consider the straight line C that connects the two points $P = (1, 2, 1)$ and $Q = (-1, 1, 3)$. Let $\mathbf{P} = \langle 1, 2, 1\rangle$ and $\mathbf{Q} = \langle -1, 1, 3\rangle$. Define $\mathbf{v} = \mathbf{Q} - \mathbf{P} = \langle -2, -1, 2\rangle$. Note that $\mathbf{v}$ is parallel to the line C. So every point S on C can be accessed by the vector

$$\mathbf{S} = \mathbf{P} + t\mathbf{v}$$

for some $t \in R$. So

$$\mathbf{r}(t) = \langle 1, 2, 1\rangle + t\langle -2, -1, 2\rangle, \quad t \in R \tag{12}$$

is a parametrization of C. In terms of coordinates, (12) is equivalent to

$$\begin{cases} x(t) &= 1 - 2t \\ y(t) &= 2 - t \\ z(t) &= 1 + 2t \end{cases} \tag{13}$$

Every straight line C, whether in R^2 or R^3, can be parametrized as

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \quad t \in R \tag{14}$$

where $\mathbf{r}_0$ is the position vector corresponding to a known point on C (such as $\langle 1, 2, 1 \rangle$ in our previous example), and $\mathbf{v}$ is a vector parallel to C. For instance, to find the parametrization of the line of intersection between the two planes $2x - 3y + z = 2$ and $x + y + z = 0$, first we find a point on this line by setting $z = 0$ in the equations of the planes and then solve for x and y to see that $(\frac{2}{5}, -\frac{2}{5}, 0)$ lies on C. Next, we note that the vectors $\mathbf{n}_1 = \langle 2, -3, 1 \rangle$ and $\mathbf{n}_2 = \langle 1, 1, 1 \rangle$ are normal to the planes. Therefore,

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle -4, -1, 5 \rangle$$

is parallel to C. Therefore

$$\mathbf{r}(t) = \langle \frac{2}{5}, -\frac{2}{5}, 0 \rangle + t\langle -4, -1, 5 \rangle \tag{15}$$

is a parametrization of C.

More complicated curves are parametrized similarly. Typical points on a curve C are accessed by a position vector $\mathbf{r}$ of the form

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle.$$

For example, the parametrization $\langle \sin t, \cos t, t \rangle$ describes a helix in R^3. Or the intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 1$ is given by

$$\mathbf{r}(t) = \langle \cos t, \sin t, 1 - \cos t - \sin t \rangle, \quad t \in (0, 2\pi]. \tag{16}$$

3 Parametrization of Surfaces

Surfaces in R^3 are characterized by two degrees of freedom; one is allowed to vary two parameters independently to cover all points on a surface. The simplest examples are surfaces that are graphs of functions f that depend on two variables, $z = f(x, y)$. Such surfaces are often parametrized as

$$\mathbf{r}(x, y) = \langle x, y, f(x, y) \rangle, \quad a < x < b, \quad c < y < d. \tag{17}$$

For example, the surface $z = x^2 + y^2$ over the unit square is parametrized as

$$\mathbf{r}(x, y) = \langle x, y, x^2 + y^2 \rangle, \quad 0 < x < 1, \quad 0 < y < 1. \tag{18}$$

The cylinder $x^2 + y^2 = 1$ is parametrized as

$$\mathbf{r}(\theta, z) = \langle \cos\theta, \sin\theta, z \rangle, \quad \theta \in (0, 2\pi], \quad z \in R, \tag{19}$$

while the cylinder $x^2 + z^2 = 4$ is parametrized as

$$\mathbf{r}(\theta, y) = \langle 2\cos\theta, y, 2\sin\theta \rangle, \quad \theta \in (0, 2\pi], \quad y \in R, \tag{20}$$

The surface of the disk of radius a in the plane $z = b$ centered at the origin is given by

$$\mathbf{r}(u,v) = \langle u\cos v, u\sin v, b\rangle, \quad u \in [0,1], \quad v \in (0,2\pi]. \tag{21}$$

Certain surfaces are best parametrized in spherical coordinates where

$$\begin{cases} x &= \rho\cos\theta\sin\phi, \\ y &= \rho\sin\theta\sin\phi, \\ z &= \rho\cos\phi. \end{cases} \tag{22}$$

For example, the cone $z^2 = x^2 + y^2$ can be parametrized as

$$\mathbf{r}(\rho,\theta) = \frac{\sqrt{2}}{2}\langle\rho\cos\theta, \rho\sin\theta, \rho\rangle, \quad \rho \in R, \quad \theta \in (0,2\pi]. \tag{23}$$

Similarly, the northern hemisphere of radius 3 centered at the origin may be parametrized as

$$\mathbf{r}(\theta,\phi) = 3\langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle, \quad \theta \in (0,2\pi], \quad \phi \in [0,\frac{\pi}{2}]. \tag{24}$$

An alternative way of parametrizing this surface is as follows:

$$\mathbf{r}(x,y) = 3\langle x, y, \sqrt{9 - x^2 - y^2}\rangle, \quad x^2 + y^2 \leq 9. \tag{25}$$

The **boundary** of this surface (the circle of radius 3 in the xy-plane and centered at the origin) is best parametrized using (24) by setting $\phi = \frac{\pi}{2}$ in that relation to get

$$\mathbf{r}(\theta) = 3\langle\cos\theta, \sin\theta, 0\rangle, \quad \theta \in (0,2\pi]. \tag{26}$$

Once a parametrization $\mathbf{r}(u,v)$ of a surface S is known, the vector

$$\mathbf{r}_u \times \mathbf{r}_v$$

defines a **normal** vector to S.

4 Parametrization of Regions in R^3

Regions in R^3 have three degrees of freedom. They are parametrized by $\mathbf{r}(u,v,w)$ where u , v and w take on values in respective intervals. For example, the region bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = -2$ and $z = 1$ is paramterized as

$$\mathbf{r}(r,\theta,z) = \langle r\cos\theta, r\sin\theta, z\rangle, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi, \quad -2 \leq z \leq 1. \tag{27}$$

The boundary of this region consists of three surfaces S_1, S_2 and S_3 given by

$$\begin{cases} S_1: & \mathbf{r}_1(u,v) &= \langle u\cos v, u\sin v, -2\rangle, \quad 0 \leq u \leq 1, \quad 0 \leq v < 2\pi, \\ S_2: & \mathbf{r}_2(u,v) &= \langle u\cos v, u\sin v, 1\rangle, \quad 0 \leq u \leq 1, \quad 0 \leq v < 2\pi, \\ S_3: & \mathbf{r}_3(u,v) &= \langle\cos v, \sin v, u\rangle, \quad -2 \leq u \leq 1, \quad 0 \leq v < 2.\pi. \end{cases} \tag{28}$$

Similarly, the region inside the northern hemisphere of radius 2 is parametrized as follows:

$$\mathbf{r}(\rho,\theta,\phi) = \langle \rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi\rangle, \quad 0 \le \rho \le 2, \quad 0 \le \theta < 2\pi, \quad 0 \le \phi \le \frac{\pi}{2} \tag{29}$$

The boundary of this region consists of two surfaces S_1 and S_2 given by

$$\begin{cases} S_1: & \mathbf{r}_1(\theta,\phi) = 2\langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle, \quad 0 \le \theta < 2\pi, \quad 0 \le \phi < \frac{\pi}{2}, \\ S_2: & \mathbf{r}_2(r,\theta) = \langle r\cos\theta, r\sin\theta, 0\rangle, \quad 0 \le r \le 2, \quad 0 \le \theta < 2\pi. \end{cases} \tag{30}$$