Chapter 4

Sequences III

4.1 Roots

We can use the results we've established in the last workbook to find some interesting limits for sequences involving roots. We will need more technical expertise and low cunning than have been required hitherto. First a simple inequality.

Bernoulli's Inequality
When $x > -1$ and n is a natural number,

$$(1+x)^n \geq 1+nx.$$

Strictly Speaking
Bernoulli's Inequality is actually *strict* unless $x = 0$, $n = 0$ or $n = 1$.

Exercise 1 Sketch a graph of both sides of Bernoulli's inequality in the cases $n = 2$ and $n = 3$.

For *non-negative* values of x Bernoulli's inequality can be easily proved using the Binomial Theorem, which expands the left-hand side:

$$\begin{aligned}(1+x)^n &= 1+nx+\frac{n(n-1)}{2}x^2+\frac{n(n-1)(n-2)}{6}x^3 \\ &\quad +\cdots+nx^{n-1}+x^n \\ &\geq 1+nx.\end{aligned}$$

Binomial Theorem
For all real values x and y

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

What difficulties do we have with this line of argument if $x < 0$?

Exercise 2 Finish off the following proof of Bernoulli's Inequality for $x > -1$ using *mathematical induction*. Note down where you use the fact that $x > -1$.

Proof. We want to show that $(1+x)^n \geq 1+nx$ where $x > -1$ and n is a natural number. This is true for $n = 1$ since $(1+x)^1 = 1+x$.

Now assume $(1+x)^k \geq 1+kx$. Then

$$(1+x)^{k+1} = (1+x)^k(1+x) \overset{1+x\geq 0}{\geq} (1+kx)(1+x) = 1+(k+1)x+kx^2 \overset{kx^2\geq 0}{\geq} 1+(k+1)x.$$

We are done.

The Bernoulli Boys
Bernoulli's Inequality is named after Jacques Bernoulli, a Swiss mathematician who used it in a paper on infinite series in 1689 (though it can be found earlier in a 1670 paper by an Englishman called Isaac Barrow).

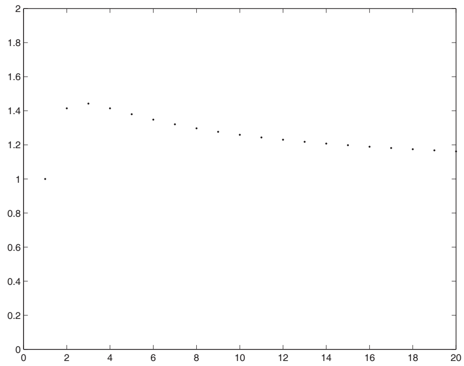

Figure 4.1: The sequence $(n^{1/n})$.

Exercise 3 Use a calculator to explore the sequences $(2^{1/n})$, $(10^{1/n})$ and $(1000^{1/n})$. Repeated use of the square root button will give a subsequence in each case.

Roots
$x^{1/n} = \sqrt[n]{x}$ is the positive n^{th}root of x.

Proposition
If $x > 0$ then $(x^{1/n}) \to 1$.

Example $100000000000^{1/n} \to 1$ and also $0.000000000001^{1/n} \to 1$.

To prove this result you might follow the following fairly cunning steps (although other proofs are very welcome):

Exercise 4

1. First assume that $x \geq 1$ and deduce that $x^{1/n} \geq 1$.
2. Let $a_n = x^{1/n} - 1$ and use Bernoulli's inequality to show that $x \geq 1 + na_n$.
3. Use the Sandwich Rule to prove that (a_n) is a null sequence.
4. Deduce that $(x^{1/n}) \to 1$.
5. Show that $(x^{1/n}) \to 1$ when $0 < x < 1$ by considering $(1/x^{1/n})$.

Auto Roots
$n^{1/n} = \sqrt[n]{n}$ is the n^{th}root of n. Natural numbers approach unity by rooting themselves.

Exercise 5 Use a calulator to explore the limit of $(2^n + 3^n)^{1/n}$. Now find the limit of the sequence $(x^n + y^n)^{1/n}$ when $0 \leq x \leq y$. (Try to find a sandwich for your proof.)

What happens if we first write $n = (n^{1/n})^n = (1+(n^{1/n}-1))^n$ and apply Bernoulli's Inequality? Do we get anywhere?

Proposition
$(n^{1/n}) \to 1$.

See figure 4.1 for a graph of this sequence.

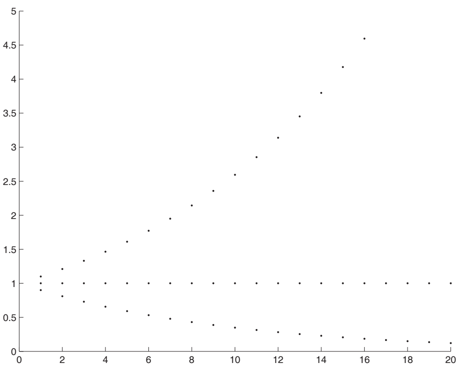

Figure 4.2: The sequence (x^n) with three slightly different values of x.

Proof. The proof is similar to that of the previous lemma but we have to be cunning and first show that $(n^{1/2n}) \to 1$. Since $n \geq 1$ we have $n^{1/2n} \geq 1$. Therefore,

$$\begin{aligned}\sqrt{n} &= (n^{1/2n})^n = (1 + (n^{1/2n} - 1))^n \\ &\geq 1 + n(n^{1/2n} - 1) > n(n^{1/2n} - 1)\end{aligned}$$

using Bernoulli's inequality. Rearranging, we see that $1 \leq n^{1/2n} < \frac{1}{\sqrt{n}} + 1$. So $(n^{1/2n}) \to 1$ by the Sandwich Theorem. Hence $(n^{1/n}) = (n^{1/2n})^2 \to 1$ by the Product Rule. ■

4.2 Powers

Exercise 6 Explore, with a calculator if necessary, and then write down a conjectured limit for the power sequence (x^n). (Warning: you should get four different possible answers depending on the value of x.)

To prove your conjectures you can use Bernoulli's inequality again. Note that if $x > 1$ then $x^n = (1 + (x - 1))^n \geq 1 + n(x - 1)$. To prove your conjecture for $0 < x < 1$ look at the sequence $1/x^n$ and then use Exercise 11 of Chapter 2. Then treat all other values of x such as $x = 0$, $x = 1$, $-1 < x < 0$, and $x \leq -1$.

Many sequences are not exactly powers but grow or shrink at least as fast as a sequence of powers so that we can compare them with (or sandwich them by) a geometric sequence. A useful idea to formalise this is to consider the ratio of two successive terms: a_{n+1}/a_n. If this is close to a value x then the sequence (a_n) might behave like the sequence (x^n). We explore this idea.

Ratio Lemma, "light" version
Let (a_n) be a sequence of positive numbers. Suppose $0 < l < 1$ and $\frac{a_{n+1}}{a_n} \leq l$ for all n. Then $a_n \to 0$.

Proof. We have that

$$a_n \leq la_{n-1} \leq l^2 a_{n-2} \leq ... \leq l^{n-1}a_1.$$

Then $0 \leq a_n \leq l^{n-1}a_1$. Since $l^{n-1}a_1 \to 0$ as $n \to \infty$, and using the Sandwich Theorem (Chapter 3), we obtain that $a_n \to 0$. ■

Exercise 7 Consider the sequence $a_n = 1 + \frac{1}{n}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{1+\frac{1}{n+1}}{1+\frac{1}{n}} = \frac{\frac{n+2}{n+1}}{\frac{n+1}{n}} = \frac{n(n+2)}{(n+1)^2} = \frac{n^2+n}{n^2+2n+1} < 1,$$

for all n. Then $a_n \to 0$ by the Ratio Lemma. Is that really so? If not, what is wrong?

A much more powerful of the above lemma is as follows:

Ratio Lemma, full version
Let (a_n) be a sequence of nonzero numbers. Suppose $0 < l < 1$ and $|\frac{a_{n+1}}{a_n}| \leq l$ eventually. Then $a_n \to 0$.

The precise definition of "$|\frac{a_{n+1}}{a_n}| \leq l$ eventually" is that there exists N such that $|\frac{a_{n+1}}{a_n}| \leq l$ for all $n > N$.
Proof. This proof follows the same lines as those for the "light" version of the lemma, but it looks more difficult. We know that $|a_{n+1}| \leq l|a_n|$ for all $n > N$. Then if $n > N$,

$$|a_n| \leq l|a_{n-1}| \leq l^2|a_{n-2}| \leq ... \leq l^{n-N-1}|a_{N+1}| = l^n \frac{|a_{N+1}|}{l^{N+1}}.$$

By choosing C large enough, we have $|a_n| \leq l^n C$ for all $n \leq N$. Suppose also that $C \geq \frac{|a_{N+1}|}{l^{N+1}}$. Then we have for all n,

$$0 \leq |a_n| \leq l^n C.$$

Since $l^n C \to 0$ as $n \to \infty$, we have $a_n \to 0$ by the Sandwich Theorem (Chapter 3). ■

Examples

1. Show that $\frac{n^2}{2^n} \to 0$
2. Show that $\frac{n!}{1000^n} \to \infty$

Proof

1. The ratio of successive terms is $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2/2^{n+1}}{n^2/2^n} = \frac{1}{2}\left(1+\frac{1}{n}\right)^2 \to \frac{1}{2}$. So, taking $\epsilon = \frac{1}{4}$ in the definition of convergence, we have $\frac{1}{4} \leq \frac{a_{n+1}}{a_n} \leq \frac{3}{4}$ for large n. The Ratio Lemma then implies that $\frac{n^2}{2^n} \to 0$.

Powerful Powers

All increasing power sequences grow faster than any polynomial sequence.

Powerless Powers

All power sequences are powerless against the factorial sequence $(n!)$.

2. Let $a_n = \frac{1000^n}{n!}$. Then $\frac{a_{n+1}}{a_n} = \frac{1000^{n+1}/(n+1)!}{1000^n/n!} = \frac{1000}{n+1} \leq \frac{1}{2}$ for all $n \geq 1999$. The Ratio Lemma says that $\frac{1000^n}{n!} \to 0$ so that $\frac{n!}{1000^n} \to \infty$.

In both of the examples above we showed that $\left(\frac{a_{n+1}}{a_n}\right) \to a$ for $0 \leq a < 1$, and then used this to show that $\frac{a_{n+1}}{a_n} \leq l$ eventually. The following corollary to version 2 of the Ratio Lemma allows us to cut out some of this work.

Corollary
Let $a_0, a_1, a_2, \ldots$ be a sequence of positive numbers. If $\left(\frac{a_{n+1}}{a_n}\right) \to a$ with $0 \leq a < 1$ then $(a_n) \to 0$.

Proof. If $\frac{a_{n+1}}{a_n} \to a$, then for all $\varepsilon > 0$ there exists N such that $\left|\frac{a_{n+1}}{a_n} - a\right| < \varepsilon$ for all $n > N$. We choose $\varepsilon = \frac{1-a}{2}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| < a + \varepsilon = \frac{a}{2} + \frac{1}{2} < 1$$

for all $n > N$. Then $a_n \to 0$ by the Ratio Lemma (full version). ■

Exercise 8 State whether the following sequences tend to zero or infinity. Prove your answers:

1. $\frac{n^{1000}}{2^n}$
2. $\frac{1.0001^n}{n}$
3. $\frac{n!}{n^{1000}}$
4. $\frac{(n!)^2}{(2n)!}$

Exercise 9 Try using the Ratio Lemma to prove that the sequence $\frac{1}{n} \to 0$. Why does the lemma tell you nothing?

The sequences (n^k) for $k = 1, 2, 3, \ldots$ and (x^n) for $x > 1$ and $(n!)$ all tend to infinity. Which is quickest? The above examples suggest some general rules which we prove below.

Exercise 10 Prove that $\left(\frac{x^n}{n!}\right) \to 0$ for all values of x.

Notice that this result implies that $(n!)^{1/n} \to \infty$, since for any value of x, eventually we have that $\left(\frac{x^n}{n!}\right) < 1$ giving that $x < (n!)^{1/n}$.

Exercise 11 Prove that $\left(\frac{n!}{n^n}\right) \to 0$ as $n \to \infty$.

Proof. We use the Ratio Lemma. Let $a_n = \frac{n!}{n^n}$. We have

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{(n+1)!}{n!} \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n} = \frac{1}{(1+\frac{1}{n})^n} \leq \frac{1}{2}.$$

The last inequality follows from $(1+\frac{1}{n})^n \geq 2$ (Bernoulli's inequality). Then $a_n \to 0$ by the Ratio Lemma. Alternatively, we could have remarked that

$$a_n = \frac{n!}{n^n} = \frac{n}{n} \cdot \frac{(n-1)}{n} \cdot \frac{(n-2)}{n} \cdots \frac{2}{n} \cdot \frac{1}{n} \leq \frac{1}{n}.$$

Then $a_n \to 0$ by the Sandwich Theorem. ■

Exercise 12 Find the limit of the sequence $\left(\frac{x^n}{n^k}\right)$ as $n \to \infty$ for all values of $x > 0$ and $k = 1, 2, \ldots$

Exercise 13 Find the limits of the following sequences. Give reasons.

1. $\left(\frac{n^4 11^n + n^9 9^n}{7^{2n}+1}\right)$
2. $\left(\left(4^{10} + 2^n\right)^{1/n}\right)$
3. $\left(\frac{3n^3 + n\cos^2 n}{n^2 + \sin^2 n}\right)$
4. $\left(\left(3n^2 + n\right)^{1/n}\right)$

4.3 Application - Factorials

Factorials $n!$ occur throughout mathematics and especially where counting arguments are used. The last section showed that the factorial sequence $(n!)$ is more powerful than any power sequence (x^n), but earlier you showed that $\frac{n!}{n^n} \to 0$. One gains much information about the speed of divergence of $n!$ with Stirling's formula: $n! \approx n^n e^{-n}$. Here is an amazingly precise version of Stirling's formula:

Stirling & de Moivre

The Stirling formula is due to Abraham de Moivre (1667–1754), a French Protestant who lived in England because of persecutions from the Catholic king Louis XIV. De Moivre could improve the formula thanks to comments by the Scottish James Stirling (1692–1770), who was from a Jacobite family and therefore had difficulties with the English (Jacobites were supporters of the king James II of Britain, who was deposed because of his Catholic faith and went into exile in France).

Theorem
For all $n \geq 1$,

$$\sqrt{2\pi n}\; n^n e^{-n}\left(1 + \tfrac{1}{12n}\right) \leq\; n! \;\leq \sqrt{2\pi n}\; n^n e^{-n}\left(1 + \tfrac{1}{12n} + \tfrac{1}{288n^2}\right).$$

In this section we will get a partial proof of the above theorem by using two clever but very useful tricks. The first trick is to change the product $n! = 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n$ into a sum by taking logarithms: $\log n! = \log 2 + \log 3 + \cdots + \log(n-1) + \log n$.

The second trick, which we shall use repeatedly in future sections, is to approximate the sum by an integral, see figure 4.3. Here is a graph of the function $\log(x)$ with a series of blocks, each of width one, lying underneath the graph. The sum of the areas of all the blocks is $\log 2 + \log 3 + \cdots + \log(n-1)$. But the area of the blocks is less than the area under the curve between $x = 1$ and $x = n$. So we have:

$$\begin{aligned}
\log n! &= \log 2 + \log 3 + \cdots + \log(n-1) + \log n \\
&\leq \int_1^n \log x dx + \log n \\
&= [x \log x - x]_1^n + \log n \\
&= (n+1)\log n - n + 1.
\end{aligned}$$

Taking exponentials of both sides we get the wonderful upper bound

$$n! \leq n^{n+1} e^{-n+1}.$$

This should be compared to claim of the theorem, namely that $n! \approx n^{n+\frac{1}{2}} e^{-n}$.

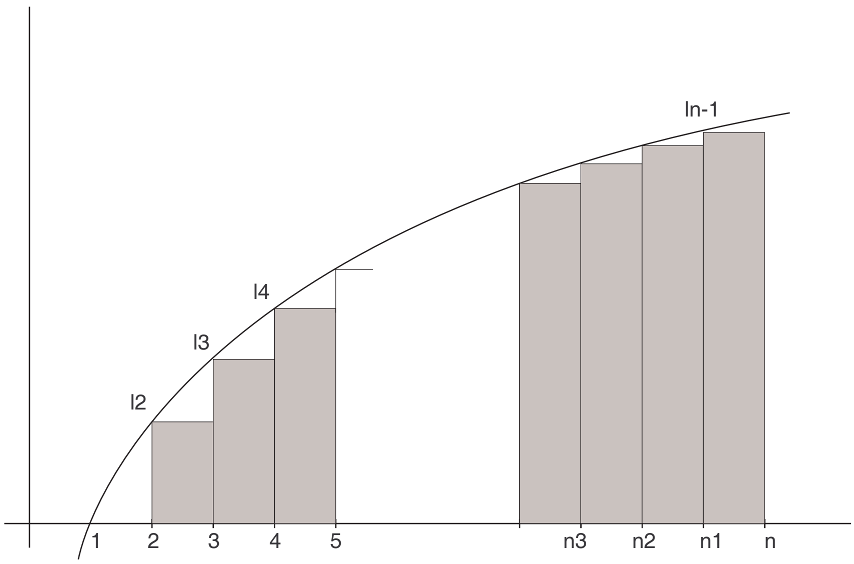

Figure 4.3: Approximating $\int_1^n \log x \mathrm{d}x$ from below.

Exercise 14 Use figure 4.4 to obtain a lower bound on $n!$. In this case the area of the blocks is greater than the area under the graph.

Exercise 15 Use your upper and lower bounds on $n!$ to find the following limits: (i) $\left(\frac{n!2^n}{n^n}\right)$ (ii) $\left(\frac{n!4^n}{n^n}\right)$

4.4 * Application - Sequences and Beyond *

In Chapter 2 we defined a sequence as an infinite list of numbers. However, the concept of an infinite list of other objects is also useful in mathematics. With that in mind, we define a *sequence of objects* to be an infinite list of those objects.

For example, let P_n be the regular n-sided polygon of area 1. Then (P_n) is a sequence of shapes.

The main question we asked about sequences was whether they converged or not. To examine convergence in general, we need to be able to say when two objects are close to each other. This is not always an easy thing to do. However, given a sequence of objects, we may be able to derive sequences of numbers and examine those sequences to learn something about the original sequence. For example, given a sequence of tables, we could look at the number of legs of each table. This gives us a new sequence of numbers which is related to the original sequence of tables. However, different related sequences can behave in wildly different ways.

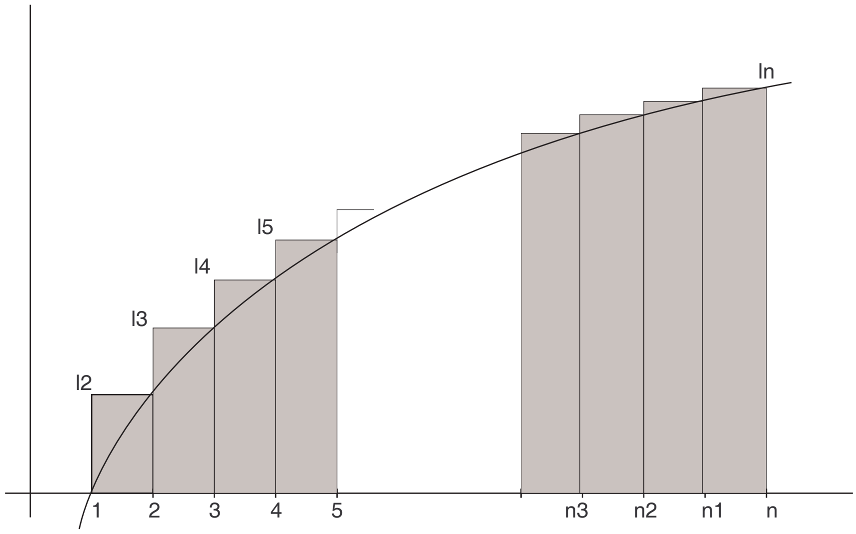

Figure 4.4: Approximating $\int_1^n \log x dx$ from above.

We shall demonstrate this by considering the set of solid shapes in $\mathbb{R}^2$. Given a sequence of shapes, (S_n), two obvious related sequences are the sequence of perimeters, $(p(S_n))$, and the sequence of areas, $(a(S_n))$.
Example Let P_n be the regular n-sided polygon centred at the origin which fits exactly inside the unit circle. This sequence starts with P_3, which is the equilateral triangle.

Let $a_n = a(P_n)$ be the sequence of areas of the polygons. In Chapter 1 we saw that $a_n = \frac{n}{2} \sin\left(\frac{2\pi}{n}\right)$ and in Chapter 2 we saw that this converges to π as $n \to \infty$ which is the area of the unit circle.

Exercise 16 Let $p_n = p(P_n)$ be the perimeter of P_n. Show that $p_n = 2n \sin\left(\frac{\pi}{n}\right) = 2n \sin\left(\frac{2\pi}{2n}\right) = 2a_{2n}$.

Exercise 17 Show that $\lim p_n = 2\pi$.

We see that in both cases we get what we would expect, namely that as the shape looks more and more like the circle, so also the area and perimeter tend to those of the circle.

Exercise 18 Consider the sequence of shapes in figure 4.5. Each is produced from the former by replacing each large step by two half-sized ones. Draw the "limiting" shape. Given that the original shape is a square of area 1, what is the perimeter and area of the n^{th}shape? Compare the limits of these sequences with the perimeter and area of the limiting shape.

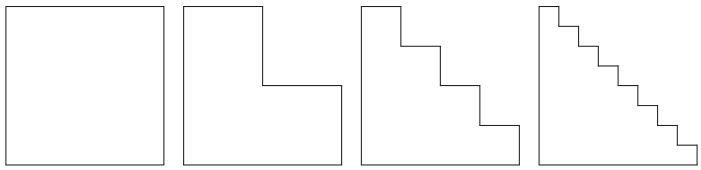

Figure 4.5:

A famous example of this type of behaviour is the Koch curve, see figure 4.6. The initial figure is an equilateral triangle of area A_1 and perimeter p_1. To each side of the triangle is attached another equilateral triangle at the trisection points of the triangle. This process is then applied to each side of the resulting figure and so on.

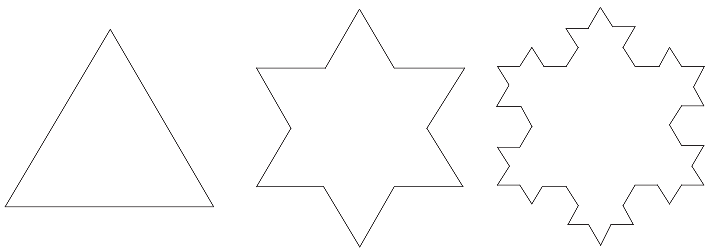

Figure 4.6:

Let p_n be the perimeter of the shape at the n^{th}stage and A_n the area.

Exercise 19

1. What is the number of sides of the shape at the n^{th}stage (for $n = 1$ the answer is 3).
2. Show that $p_{n+1} = \frac{4}{3} p_n$.
3. Prove that $(p_n) \to \infty$.
4. Show that, in making the $(n+1)^{\text{th}}$ shape, each little triangle being added has area $\frac{1}{9^n} A_1$.
5. Show that $A_{n+1} = A_n + \frac{3}{9} \left(\frac{4}{9}\right)^{n-1} A_1$.
6. Prove that $(A_n) \to \frac{8}{5} A_1$. (Recall the sum of a geometric series.)

Check Your Progress
By the end of this chapter you should be able to:

- Understand, memorise, prove, and use a selection of standard limits involving roots, powers and factorials.