ΣΧΕΔΙΑΣΜΟΣ ΚΑΙ ΠΡΟΓΡΑΜ-ΜΑΤΙΣΜΟΣ ΠΑΡΑΓΩΓΗΣ

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PRODUCTION PLANNING AND SCHEDULING
Inventory Control Under Known Time-Varying Demand

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Dynamic Lot Sizing (DLS)

- **Assumptions/notation**
  - Time is broken into discrete time periods $t = 1, 2, \ldots, T$; T is the planning horizon
  - Demand in period t: λ_t (parts), $t = 1, 2, \ldots, T$
  - Shortages are not permitted
  - Infinite production/replenishment rate (instantaneous replenishment)
  - Zero lead time
  - Variable unit production/order cost in period t: c_t (€ per part), $t = 1, 2, \ldots, T$
  - Fixed setup production/order cost in period t: K_t (€ per production run/order initiated in period t), $t = 1, 2, \ldots, T$
  - Interest rate: i (€ per € invested per period)
  - Initial inventory at time zero: I_0

- **Computation**
  - Inventory holding cost rate in period t: $h_t = ic_t$ (€ per part) $t = 1, 2, \ldots, T$

- **Decision**
  - Inventory left over at the end of period t: I_t (parts), $t = 1, 2, \ldots, T$
  - Lot size (reorder quantity) in period t: Q_t (parts), $t = 1, 2, \ldots, T$

DLS: No capacity

- **Problem formulation (no capacity limitation)**

$$\text{Minimize} \quad G = \sum_{t=1}^{T} h_t I_t + K_t Y_t$$

$$\text{subject to:} \left.\begin{aligned} & I_t = I_{t-1} + Q_t - \lambda_t \\ & Q_t \le M \cdot Y_t \\ & I_t, Q_t \ge 0 \\ & Y_t \in \{0,1\} \end{aligned}\right\} \quad t = 1,\ldots,T$$

M : very BIG number

- **Next:**
  - Alternative practical & heuristic lot sizing schemes
  - Optimal solution

DLS: No capacity

- **Prototype example***

  $I_0 = 0$

t	1	2	3	4	5	6	7	8	9	10	Total
λ_t	20	50	10	50	50	10	20	40	20	30	300
K_t	100	100	100	100	100	100	100	100	100	100	
h_t	1	1	1	1	1	1	1	1	1	1	

***Source**: Hopp, W. H., M. L. Spearman. 2000. Factory Physics: Foundations of Manufacturing Management, 2E. McGraw-Hill, Boston, MA (chapter 2).

DLS: No capacity

- **Lot-for-Lot** $Q_t = \lambda_t \quad \Rightarrow \quad Y_t = 1, \quad I_t = 0, \quad t = 1, \ldots, T$

$$G = \sum_{t=1}^{T} K_t$$

t	1	2	3	4	5	6	7	8	9	10	Total
λ_t	20	50	10	50	50	10	20	40	20	30	300
Q_t	**20**	**50**	**10**	**50**	**50**	**10**	**20**	**40**	**20**	**30**	300
I_t											0
Y_t	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	10
K_tY_t	100	100	100	100	100	100	100	100	100	100	1000
h_tI_t											0
G	100	100	100	100	100	100	100	100	100	100	**1000**

DLS: No capacity

- **Fixed Order Quantity (EOQ) lot sizing**

$$\text{EOQ} = \sqrt{\frac{2\bar{K}\bar{\lambda}}{\bar{h}}} = \sqrt{\frac{2(100)(30)}{(1)}} = 75.46 \approx 75$$

t	1	2	3	4	5	6	7	8	9	10	Total
λ_t	20	50	10	50	50	10	20	40	20	30	300
Q_t	**75**		**75**		**75**			**75**			300
I_t	**55**	**5**	**70**	**20**	**45**	**35**	**15**	**50**	**30**		325
Y_t	**1**		**1**		**1**			**1**			4
K_tY_t	100		100		100			100			400
h_tI_t	55	5	70	20	45	35	15	50	30		325
G	155	5	170	20	145	35	15	150	30		**725**

DLS: No capacity

- **Fixed Order Period (EOQ period) lot sizing**

$$\text{EOQ} = \sqrt{\frac{2\bar{K}\bar{\lambda}}{\bar{h}}} = \sqrt{\frac{2(100)(30)}{(1)}} = 75.46 \approx 75 \Rightarrow T_{\text{EOQ}} = \frac{\text{EOQ}}{\bar{\lambda}} = \frac{75}{30} = 2.5 \approx 2$$

$$Q_t = \sum_{n=t}^{t+T_{\text{EOQ}}-1} \lambda_n, \quad t = 1, T_{\text{EOQ}}+1, 2T_{\text{EOQ}}+1, \ldots; \quad 0, \text{otherwise}$$

t	1	2	3	4	5	6	7	8	9	10	Total
λ_t	20	50	10	50	50	10	20	40	20	30	300
Q_t	**70**		**60**		**60**		**60**		**50**		300
I_t	**50**		**50**		**10**		**40**		**30**		180
Y_t	**1**		**1**		**1**		**1**		**1**		5
K_tY_t	100		100		100		100		100		500
h_tI_t	50		50		10		40		30		180
G	150		150		110		140		130		**680**

DLS: No capacity

- **Part Period Balancing**

Idea: Myopic balancing of inventory cost and setup cost

$G_t^h(n) \equiv$ Total inventory holding cost if order placed in period t spans the next n periods, i.e., $Q_t = \lambda_t + \lambda_{t+1} + \ldots + \lambda_{t+n-1}$

e.g., $G_1^h(3) = (\lambda_1)(0) + (\lambda_2)(h_1) + (\lambda_3)(h_1 + h_2) = (20)(0) + (50)(1) + (10)(2) = 70$

$n^*(t) \equiv$ n that most closely matches $G_t^h(n)$ and K_t

t	1	2	3	4	5	6	7	8	9	10	Total
λ_t	20	50	10	50	50	10	20	40	20	30	300
$n = 1$	0			0			0			0	
$n = 2$	50	$G_1^h(n)$		50			40				
$n = 3$	70			70			80				
$n = 4$	220			130			170				
Q_t	80			110 130			80			30	300

DLS: No capacity

- **Part Period Balancing (cont'd)**

t	1	2	3	4	5	6	7	8	9	10	Total
λ_t	20	50	10	50	50	10	20	40	20	30	300
Q_t	**80**			**110**			**80**			**30**	300
I_t	**60**	**10**		**60**	**10**		**60**	**20**			220
Y_t	**1**			**1**			**1**			**1**	4
K_tY_t	100			100			100			100	400
h_tI_t	60	10		60	10		60	20			220
G	160	10		160	10		160	20		100	**620**

DLS: No capacity

- **Silver-Meal Heuristic**

Idea: Myopic minimization of average cost per period

$\bar{G}_t(n) \equiv$ Average inventory holding cost plus setup cost per period if order placed in period t spans the next n periods, i.e., $Q_t = \lambda_t + \lambda_{t+1} + \ldots + \lambda_{t+n-1}$

e.g., $\bar{G}_1(3) = \dfrac{K_1 + (\lambda_2)(h_1) + (\lambda_3)(h_1 + h_2)}{3} = \dfrac{100 + (50)(1) + (10)(2)}{3} = 56.66$

$n^*(t) \equiv$ smallest n such that $\bar{G}_t(n) \leq \bar{G}_t(n+1)$

t	1	2	3	4	5	6	7	8	9	10	Total
λ_t	20	50	10	50	50	10	20	40	20	30	300
$n = 1$	100			100			100			100	
$n = 2$	75			75			70				
$n = 3$	56.66			56.66			60				
$n = 4$	80			57.5			67.5				
Q_t	80			110			80			30	300

$\bar{G}_1(n)$

DLS: No capacity

- **Silver-Meal Heuristic (cont'd)**

t	1	2	3	4	5	6	7	8	9	10	Total
λ_t	20	50	10	50	50	10	20	40	20	30	300
Q_t	**80**			**110**			**80**			**30**	300
I_t	**60**	**10**		**60**	**10**		**60**	**20**			220
Y_t	**1**			**1**			**1**			**1**	4
K_tY_t	100			100			100			100	400
h_tI_t	60	10		60	10		60	20			220
G	160	10		160	10		160	20		100	**620**

Coincidentally, it yields the same total cost as part period balancing

DLS: No capacity

- **Least Unit Cost**

  Idea: Myopic minimization of average cost per part

$\bar{G}_t(n) \equiv$ Average inventory holding cost plus setup cost per part if order placed in period t spans the next n periods, i.e., $Q_t = \lambda_t + \lambda_{t+1} + \ldots + \lambda_{t+n-1}$

e.g., $\bar{G}_1(3) = \dfrac{K_1 + (\lambda_2)(h_1) + (\lambda_3)(h_1 + h_2)}{80} = \dfrac{100 + (50)(1) + (10)(2)}{80} = 2.125$

$n^*(t) \equiv$ smallest n such that $\bar{G}_t(n) \leq \bar{G}_t(n+1)$

t	1	2	3	4	5	6	7	8	9	10	Total
λ_t	20	50	10	50	50	10	20	40	20	30	300
$n = 1$	5			2		10			5		
$n = 2$	2.142			1.5		4			2.6		
$n = 3$	2.125			1.545		2.857					
$n = 4$	2.462					2.888					
Q_t	80			100		70			50		300

$\bar{G}_1(n)$

DLS: No capacity

- **Least Unit Cost (cont'd)**

t	1	2	3	4	5	6	7	8	9	10	Total
λ_t	20	50	10	50	50	10	20	40	20	30	300
Q_t	**80**			**100**		**70**			**50**		300
I_t	**60**	**10**		**50**		**60**	**40**		**30**		220
Y_t	**1**			**1**		**1**			**1**		4
K_tY_t	100			100		100			100		400
h_tI_t	60	10		50		60	40		30		250
G	160	10		150		160	40		130		**650**

DLS: No capacity

- **Optimal Solution**

$$\text{Minimize} \quad G = \sum_{t=1}^{T} h_t I_t + K_t Y_t$$

$$\text{subject to:} \left.\begin{array}{l} I_t = I_{t-1} + Q_t - \lambda_t \\ Q_t \le M\ Y_t \\ I_t, Q_t \ge 0 \\ Y_t \in \{0,1\} \end{array}\right\} \quad t = 1, \ldots, T$$

M : very BIG number

Wagner-Whitin property

$Q_t = 0$ or $Q_t = \lambda_t$ or $Q_t = \lambda_t + \lambda_{t+1}$ or …or $Q_t = \lambda_t + \lambda_{t+1} + \ldots + \lambda_{t+T}$

- **Reference**

Wagner, H.M., T.M. Whitin. 1958. Dynamic version of the economic lot sizing model. Management Science 5 (1) 89-96

DLS: No capacity

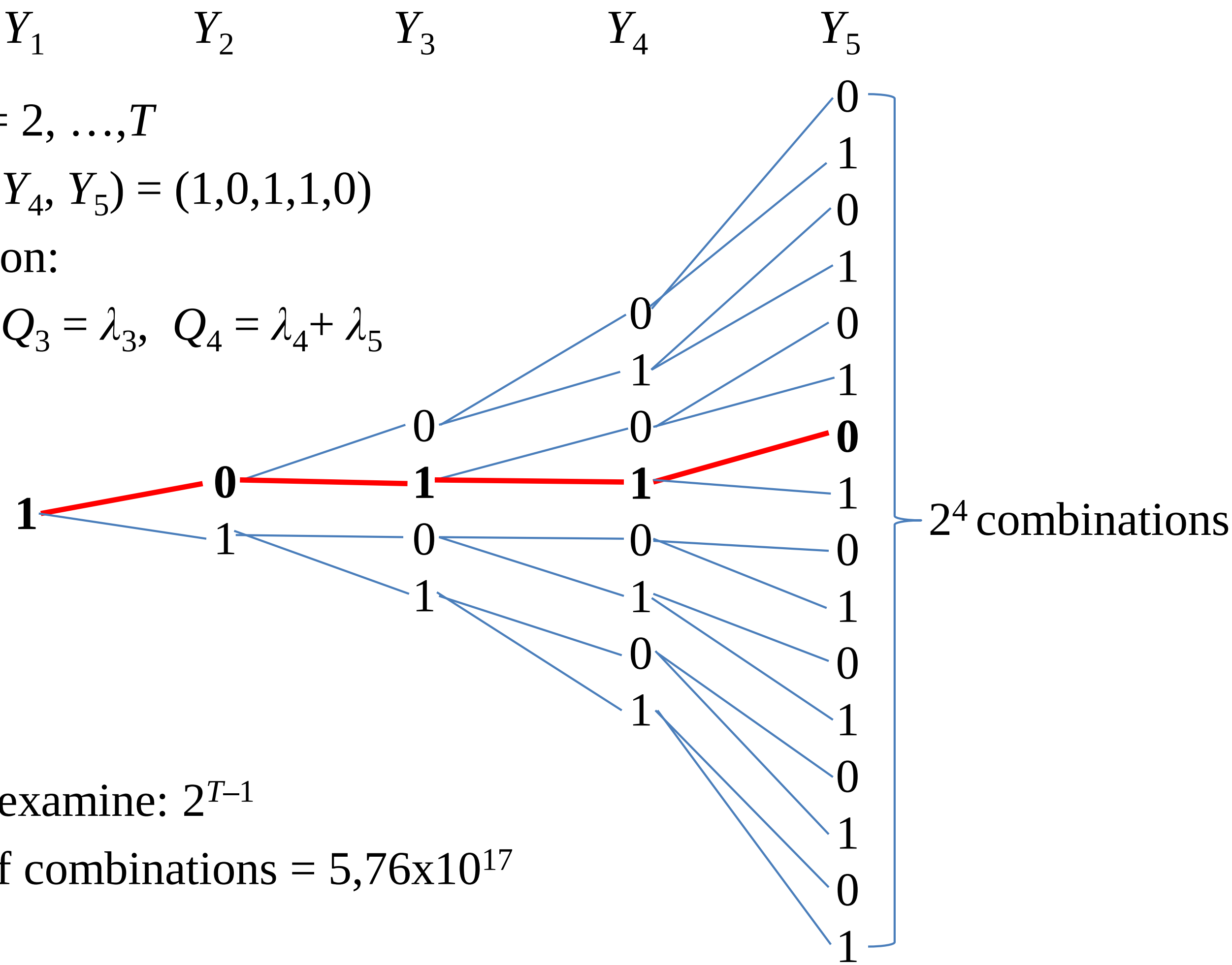
- **Problem is a pure combinatorial problem**

Decision: $Y_t = 0/1,\ t = 2, \ldots, T$

Example: $(Y_1, Y_2, Y_3, Y_4, Y_5) = (1,0,1,1,0)$

Corresponds to solution:

$Q_1 = \lambda_1 + \lambda_2,\ \ Q_2 = 0,\ \ Q_3 = \lambda_3,\ \ Q_4 = \lambda_4 + \lambda_5$



# of combinations to examine: 2^{T-1}

Example: $T = 60$: # of combinations = $5{,}76 \times 10^{17}$

DLS: No capacity

- **Wagner-Whiting procedure**

Forward algorithm: Consider successively t-period problems, $t=1,...,T$

For each problem compute:

$Z_t^* \equiv$ minimum cost in the t-period problem

$j_t^* \equiv$ last period of production in the t-period problem

To compute Z_t^* and j_t^*, define:

$G_t(n) \equiv$ Total inventory holding cost plus setup cost if order placed in period t spans the next n periods, i.e., $Q_t = \lambda_t + \lambda_{t+1} + \ldots + \lambda_{t+n-1}$

e.g., $G_2(3) = K_2 + \lambda_3 h_2 + \lambda_4 (h_2 + h_3)$

Computation of Z_t^* and j_t^*:

$Z_0^* = 0, \quad j_0^* = 1$

$$Z_t^* = \min_{i=j_{t-1}^*,...,t} \{Z_{i-1}^* + G_i(t-i+1)\}, \quad j_t^* = \arg\min_{i=j_{t-1}^*,...,t} \{Z_{i-1}^* + G_i(t-i+1)\}, \quad t=1,\ldots,T$$

DLS: No capacity

- **Wagner-Whitin procedure (cont'd)**

	t	1	2	3	4	5	6	7	8	9	10
	λ_t	20	50	10	50	50	10	20	40	20	30
Last period with production	1	100	150	170	320						
	2		200	210	310						
	3			250	300						
	4				270	320	340	400	560		
	5					370	380	420	540		
	6						420	440	520		
	7							440	480	520	610
	8								500	520	580
	9									580	610
	10										620
	Z_t^*	100	150	170	270	320	340	400	480	520	580
	j_t^*	1	1	1	4	4	4	4	7	7/8	8

DLS: No capacity

- **Wagner-Whiting procedure (cont'd)**

Computations :

$$Z_0^* = 0, \quad j_0^* = 1$$

$$Z_1^* = \min\left\{Z_0^* + G_1(1) = G_1(1) = K_1 = 100, \text{ produce in period } 1\right\}, \quad j_1^* = 1$$

$$Z_2^* = \min\begin{Bmatrix} Z_0^* + G_1(2) = K_1 + h_1\lambda_2 = 100 + 1\cdot 50 = 150 & \text{produce in period } 1 \\ Z_1^* + G_2(1) = Z_1^* + K_2 = 100 + 100 = 200 & \text{produce in period } 2 \end{Bmatrix}, \quad j_2^* = 1$$

$$Z_3^* = \min\begin{Bmatrix} Z_0^* + G_1(3) = K_1 + h_1\lambda_2 + (h_1 + h_2)\lambda_3 = 100 + 1\cdot 50 + 2\cdot 10 = 170 & \text{produce in period } 1 \\ Z_1^* + G_2(2) = Z_1^* + K_2 + h_2\lambda_3 = 100 + 100 + 1\cdot 10 = 210 & \text{produce in period } 2 \\ Z_2^* + G_3(1) = Z_2^* + K_3 = 150 + 100 = 250 & \text{produce in period } 3 \end{Bmatrix}, \quad j_3^* = 1$$

$$Z_4^* = \min\begin{Bmatrix} Z_0^* + G_1(4) = G_1(4) = 320 & \text{produce in period } 1 \\ Z_1^* + G_2(3) = 100 + G_2(3) = 310 & \text{produce in period } 2 \\ Z_2^* + G_3(2) = 150 + G_3(2) = 300 & \text{produce in period } 3 \\ Z_3^* + G_4(1) = 170 + 100 = 270 & \text{produce in period } 4 \end{Bmatrix}, \quad j_4^* = 4$$

DLS: No capacity

- **Wagner-Whiting procedure (cont'd)**

Computations :

$$Z_5^* = \min\left\{\begin{matrix} Z_3^* + G_4(2) = 170 + 150 = 320 \text{ produce in period } 4 \\ Z_4^* + G_5(1) = 270 + 100 = 370 \text{ produce in period } 5 \end{matrix}\right\}, \quad j_5^* = 4$$

$$Z_6^* = \cdots$$

$$\vdots$$

$$Z_{10}^* = \cdots$$

Construct solution going backwards :

$$t = 10: \quad j_{10}^* = 8 \quad \Rightarrow \quad Q_8 = \lambda_8 + \lambda_9 + \lambda_{10} = 90, \quad Q_9 = Q_{10} = 0 \quad \Rightarrow \quad \text{GOTO } t = 7$$

$$t = 7: \quad j_7^* = 4 \quad \Rightarrow \quad Q_4 = \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 = 130, \quad Q_5 = Q_6 = Q_7 = 0 \quad \Rightarrow \quad \text{GOTO } t = 3$$

$$t = 3: \quad j_3^* = 1 \quad \Rightarrow \quad Q_1 = \lambda_1 + \lambda_2 + \lambda_3 = 80, \quad Q_2 = Q_3 = 0$$

DLS: No capacity

- **Wagner-Whiting procedure (cont'd)**

t	1	2	3	4	5	6	7	8	9	10	Total
λ_t	20	50	10	50	50	10	20	40	20	30	300
Q_t	**80**			**130**				**90**			300
I_t	**60**	**10**		**80**	**30**	**20**		**50**	**30**		280
Y_t	**1**			**1**				**1**			3
K_tY_t	100			100				100			300
h_tI_t	60	10		80	30	20		50	30		280
G	170	10		180	30	20		150	30		**580**

DLS: Finite capacity

- **Problem formulation (capacity limitation)**

$$\text{Minimize} \quad G = \sum_{t=1}^{T} h_t I_t + K_t Y_t$$

$$\text{subject to:} \left.\begin{aligned} & I_t = I_{t-1} + Q_t - \lambda_t \\ & Q_t \le C_t \cdot Y_t \\ & I_t, Q_t \ge 0 \\ & Y_t \in \{0,1\} \end{aligned}\right\} \quad t = 1, \ldots, T$$

C_t : maximum production capacity in period t

- **Feasibility condition:**

$$\sum_{i=1}^{t} C_i \ge \sum_{i=1}^{t} \lambda_i, \quad t = 1, \ldots, T$$

- Not obvious how to get a feasible solution

DLS: Finite capacity

- **2-step heuristic procedure for solving the problem**

Step 1: Obtain an initial feasible solution

1. Start with a lot-for-lot initial solution
2. Starting from the first period and working forward, backshift demand from periods in which demand exceeds capacity to prior periods in which there is capacity
3. Repeat for each period in which demand exceeds capacity until schedule is feasible

Step 2: Improve initial feasible solution

1. Start with the feasible lot-for-lot solution developed in step 1
2. Starting from the last period and working backward, determine if it is cheaper to produce the units composing that lot by shifting production to prior periods in which there is excess capacity

DLS: Finite capacity

- **Example***

t	1	2	3	4	5	6	7	8	9	Total
λ_t	100	79	230	105	3	10	99	126	40	792
C_t	120	200	200	400	300	50	120	50	30	
K_t	450	450	450	450	450	450	450	450	450	
h_t	2	2	2	2	2	2	2	2	2	
$\sum_{i=1}^{t} \lambda_i$	100	179	409	514	517	527	626	752	792	
$\sum_{i=1}^{t} C_i$	120	320	520	920	1220	1270	1390	1440	1470	

- Feasibility test satisfied: $\sum_{i=1}^{t} C_i \geq \sum_{i=1}^{t} \lambda_i, \quad t = 1, \ldots, T$

***Source**: Nahmias, S. 2009. Production and Operations Analysis, 6E. McGraw Hill, Boston MA (chapter 7).

DLS: Finite capacity

- **Step 1:** Obtain an initial feasible solution

t	1	2	3	4	5	6	7	8	9	Total
λ_t	100	~~79~~	~~230~~	105	3	10	99	126	40	792
C_t	120	200	200	400	300	50	120	50	30	
		+30	-30							
$\lambda_t(1)$	100	109	200	105	~~3~~	~~10~~	~~99~~	~~126~~	40	
					+15	+40	+21	-76		
$\lambda_t(2)$	100	109	200	105	~~18~~	50	120	50	~~40~~	
					+10				-10	
$\lambda_t(3)$	100	109	200	105	28	50	120	50	30	
Q_t	100	109	200	105	28	50	120	50	30	

DLS: Finite capacity

- **Step 1:** Obtain an initial feasible solution

t	1	2	3	4	5	6	7	8	9	Total
λ_t	100	79	230	105	3	10	99	126	40	792
Q_t	**100**	**109**	**200**	**105**	**28**	**50**	**120**	**50**	**30**	792
I_t		**30**			**25**	**65**	**86**	**10**		216
Y_t	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	
K_tY_t	450	450	450	450	450	450	450	450	450	4050
h_tI_t		60			50	130	172	20		432
G	450	510	450	450	500	580	622	470	450	**4482**

DLS: Finite capacity

- **Step 2:** Improve initial feasible solution

t	1	2	3	4	5	6	7	8	9	Cost improvement
λ_t	100	109	200	105	28	50	120	50	30	
C_t	120	200	200	400	300	50	120	50	30	
Q_t	100	109	200	105	~~28~~	50	120	50	~~30~~	
EC	20	91		295	272					
					+30				-30	450 – 2 30 4 = 210
$Q_t(1)$	100	109	200	105	~~58~~	50	120	~~50~~	0	
EC	20	91		295	242				30	
					+50			-50		450 – 2 50 3 = 150
$Q_t(2)$	100	109	200	105	~~108~~	~~50~~	120	0		
EC	20	91		295	192			50	30	
					+120		-120			450 – 2 120 2 = –30
					+50	-50				450 – 2 50 1 = 350
$Q_t(3)$	100	109	200	105	158	0	120			
EC	20	91		295	142	50		50	30	

DLS: Finite capacity

- **Step 2:** Improve initial feasible solution (cont'd)

t	1	2	3	4	5	6	7	8	9	Cost improvement
λ_t	100	109	200	105	28	50	120	50	30	
C_t	120	200	200	400	300	50	120	50	30	
$Q_t(3)$	100	109	200	105	158	0	120			
EC	20	91		295	142	50		50	30	
				+158	-158					450 – 2 158 1 = 134
$Q_t(4)$	100	109	200	263	0		120			
EC	20	91		137	300	50		50	30	

DLS: Finite capacity

- Evaluate improved solution

t	1	2	3	4	5	6	7	8	9	Total
λ_t	100	79	230	105	3	10	99	126	40	792
Q_t	**100**	**109**	**200**	**263**			**120**			792
I_t		**30**		**158**	**155**	**145**	**166**	**40**		694
Y_t	**1**	**1**	**1**	**1**			**1**			
K_tY_t	450	450	450	450			450			2250
h_tI_t		60		316	310	290	332	80		1388
G	450	510	450	766	310	290	782	80		**3638**