3. Bayesian Approach for Parameter Estimation in Structural Dynamics Using Modal Data

3.1 Introduction

The Bayesian framework for parameter estimation is used to address the problem of estimating the uncertainty in the values of the parameters of structural dynamics models based on the measured modal data. The model class used to represent the structural behavior is considered to be linear. Prediction errors, measuring the fit between the measured and the model predicted modal properties, are modeled by Gaussian distributions.

3.2 Bayesian Parameter Estimation Utilizing Modal Data

Let $D = \{\hat{\omega}_r^{(k)}, \hat{\boldsymbol{\phi}}_r^{(k)} \in R^{N_0}, r = 1, \cdots, m, k = 1, \cdots, N_D\}$ be the measured modal data from a structure, consisting of modal frequencies $\hat{\omega}_r^{(k)}$ and modeshape components $\hat{\boldsymbol{\phi}}_r^{(k)}$ at N_0 measured DOFs, where m is the number of observed modes and N_D is the number of modal data sets available. Consider a parameterized class of linear structural models used to model the dynamic behavior of the structure and let $\boldsymbol{\theta} \in R^{N_\theta}$ be the set of free structural model parameters to be identified using the measured modal data. Let also $\{\omega_r(\boldsymbol{\theta}), \boldsymbol{\phi}_r(\boldsymbol{\theta}) \in R^{N_d}, r = 1, \cdots, m\}$, where N_d is the number of model degrees of freedom (DOF), be the predictions of the modal frequencies and modeshapes obtained for a particular value of the parameter set $\boldsymbol{\theta}$ by solving the eigenvalue problem corresponding to the model mass and stiffness matrices $M(\boldsymbol{\theta})$ and $K(\boldsymbol{\theta})$, respectively, that is,

$$[K(\boldsymbol{\theta}) - \omega_r^2(\boldsymbol{\theta}) M(\boldsymbol{\theta})] \, \boldsymbol{\phi}_r(\boldsymbol{\theta}) = \mathbf{0} \tag{3.1}$$

The objective in a modal-based Bayesian structural identification methodology is to estimate the uncertainties in the values of the parameter set $\boldsymbol{\theta}$ so that the modal data

$\{\omega_r(\boldsymbol{\theta}),\ \phi_r(\boldsymbol{\theta}),\ r=1,\cdots,m\}$ predicted by the linear class of models best matches, in some sense, the experimentally obtained modal data in D.

The Bayesian approach uses probability distributions to quantify the plausibility of each possible value of the model parameters $\boldsymbol{\theta}$. Using Bayes' theorem, the updated (posterior) probability distribution $p(\boldsymbol{\theta} \mid D,\boldsymbol{\sigma},\mathsf{M})$ of the model parameters $\boldsymbol{\theta}$ based on the inclusion of the measured data D, the modeling assumptions M and the value of a parameter set $\boldsymbol{\sigma}$, is obtained as follows:

$$p(\boldsymbol{\theta} \mid D,\boldsymbol{\sigma},\mathsf{M}) = c\ p(D \mid \boldsymbol{\theta},\boldsymbol{\sigma},\mathsf{M})\ p(\boldsymbol{\theta} \mid \boldsymbol{\sigma},\ M) \tag{3.2}$$

where $p(D \mid \boldsymbol{\theta},\boldsymbol{\sigma},\mathsf{M})$ is the probability of observing the data from a model corresponding to a particular value of the parameter set $\boldsymbol{\theta}$ conditioned on the modeling assumptions M and the value of $\boldsymbol{\sigma}$, $p(\boldsymbol{\theta} \mid \boldsymbol{\sigma},\ M)$ is the initial (prior) probability distribution of a model, and c is a normalizing constant selected such that the PDF $p(\boldsymbol{\theta} \mid D,\boldsymbol{\sigma},\mathsf{M})$ integrates to one. Herein, the modeling assumptions M refer to the structural modeling assumptions as well as those used to derive the probability distributions $p(D \mid \boldsymbol{\theta},\boldsymbol{\sigma},\mathsf{M})$ and the prior $p(\boldsymbol{\theta} \mid \boldsymbol{\sigma},\ M)$. The parameter set $\boldsymbol{\sigma}$ contains all parameters that need to be defined in order to completely specify the modeling assumptions M. Measured data are accounted for in the updated estimates through the term $p(D \mid \boldsymbol{\theta},\boldsymbol{\sigma},\mathsf{M})$, while any available prior information is reflected in the term $p(\boldsymbol{\theta} \mid \boldsymbol{\sigma},\ M)$. From the experience is usually assumed $p(\boldsymbol{\theta} \mid \boldsymbol{\sigma},\ M) = \pi(\boldsymbol{\theta}) = constant$ (a non-informative prior distribution). Other prior distribution can be assumed as well. In order to simplify the notation, the dependence of the probability distributions on M is dropped in the analysis that follows.

The form of $p(D \mid \boldsymbol{\theta},\boldsymbol{\sigma},\mathsf{M}) \equiv p(D \mid \boldsymbol{\theta},\boldsymbol{\sigma})$ is derived by using a probability model for the prediction error vector $\boldsymbol{e}^{(k)} = [e_1^{(k)},\cdots,e_m^{(k)}]$, $k=1,\cdots,N_D$, defined as the difference between the measured modal quantities involved in D for all $r=1,\cdots,m$ modes and the corresponding modal quantities predicted from a model that corresponds to a particular value of the parameter set $\boldsymbol{\theta}$. Specifically, the prediction error $\boldsymbol{e}_r^{(k)} = [e_{\omega_r}^{(k)}\ \ \boldsymbol{e}_{\phi_r}^{(k)}]$ is given separately for the modal frequencies and the modeshapes by the prediction error equations:

$$e_{\omega_r}^{(k)} = \hat{\omega}_r^{(k)} - \omega_r(\boldsymbol{\theta}), \qquad r=1,\ldots,m \tag{3.3}$$

$$e_{\phi_{jr}}^{(k)} = \hat{\phi}_{jr}^{(k)} - \phi_{jr}(\boldsymbol{\theta}), \qquad j=1,\ldots,N_0\ ,\ r=1,\ldots,m \tag{3.4}$$

where $e_{\omega_r}^{(k)}$ and $e_{\phi_{jr}}^{(k)}$ are respectively the prediction errors for the modal frequency and modeshape components of the r-th mode, $k=1,\cdots,N_D$, m is the number of identified modes and N_0 is the number of sensors.

In order to simplify the notation, equation (3.4) can be rewritten in the vector form

$$\boldsymbol{e}_{\phi_r}^{(k)} = \hat{\boldsymbol{\phi}}_r^{(k)} - \beta_r^{(k)} L_0 \boldsymbol{\phi}_r(\boldsymbol{\theta}) \qquad r = 1,\ldots,m \tag{3.5}$$

where $\beta_r^{(k)} = \hat{\boldsymbol{\phi}}_r^{(k)T} \boldsymbol{\phi}_r / \boldsymbol{\phi}_r^T \boldsymbol{\phi}_r$ is a normalization constant that accounts for the different scaling between the measured and the predicted modeshape for given parameter set $\boldsymbol{\theta}$ and L_0 is a $N_0 \times N_d$ matrix (N_d is the total number of the model degrees of freedom) of ones and zeros that maps the model DOFs to the measured degrees of freedom.

Following the Bayesian methodology, the predictions errors are modeled by zero-mean Gaussian vector variables. Specifically, the prediction error $e_{\omega_r}^{(k)}$ for the r-th modal frequency is assumed to be a zero mean Gaussian variable, $e_{\omega_r}^{(k)} \sim N(0, \sigma_{\omega_r}^2 \hat{\omega}_r^{(k)2})$, with standard deviation $\sigma_{\omega_r} \hat{\omega}_r^{(k)}$. The prediction error parameter σ_{ω_r} represents the fractional difference between the measured and the model predicted frequency of the r-th mode. The prediction error for the r-th truncated modeshape vector $\boldsymbol{e}_{\phi_r}^{(k)} \in R^{N_0}$ is also assumed to be zero mean Gaussian vector, $\boldsymbol{e}_{\phi_r}^{(k)} \sim N(\boldsymbol{0}, C_{\phi_r}^{(k)})$, with covariance matrix $C_{\phi_r}^{(k)} \in R^{N_0 \times N_0}$, where $N(\boldsymbol{\mu}, \Sigma)$ denotes the multidimensional normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix Σ. In the analysis that follows, a diagonal covariance matrix $C_{\phi_r}^{(k)} = \sigma_{\phi_r}^2 \left\|\hat{\boldsymbol{\phi}}_r^{(k)}\right\|_{N_0}^2$ is assumed, where $\left\|\hat{\boldsymbol{\phi}}_r^{(k)}\right\|_{N_0}^2 = \left\|\hat{\boldsymbol{\phi}}_r^{(k)}\right\|^2 / N_0$, $\|\cdot\|$ is the usual Euclidian norm and I is the identity matrix. The prediction error parameter σ_{ϕ_r} represents the difference between the measured and the model predicted component of the r-th modeshape relative to an average value $\left\|\hat{\boldsymbol{\phi}}_r^{(k)}\right\|_{N_0}$ of the modeshape components. The parameters σ_{ω_r} and σ_{ϕ_r}, represent the prediction error estimates of the measured modal frequencies and modeshapes involved in D.

3.3 Formulation for $p(D \mid \boldsymbol{\theta}, \boldsymbol{\sigma})$ Using the Gaussian Probability Distribution for Model Prediction Error

In the analysis that follows, the parameter set $\boldsymbol{\sigma}$ is taken to contain the parameters σ_{ω_r} and σ_{ϕ_r}, $r = 1,\ldots,m$. Given the values of the parameter set $\boldsymbol{\sigma}$, assuming independence of the prediction errors $e_r^{(k)}$ and using the Gaussian choice for the probability distribution of the prediction errors $e_{\omega_r}^{(k)}$ and $\boldsymbol{e}_{\phi_r}^{(k)}$, the probability $p(D \mid \boldsymbol{\theta}, \boldsymbol{\sigma})$ of observing the data from a model within the class of models M is estimated as follow:

$$p(D|\boldsymbol{\theta},\boldsymbol{\sigma}) = p(\hat{\omega}_1^{(k)},\ldots,\hat{\omega}_r^{(k)},\hat{\boldsymbol{\phi}}_1^{(k)},\ldots,\hat{\boldsymbol{\phi}}_m^{(k)}|\boldsymbol{\theta},\boldsymbol{\sigma}) = \prod_{k=1}^{N_D}\left[\prod_{r=1}^{m} p(\hat{\omega}_r^{(k)}|\boldsymbol{\theta},\boldsymbol{\sigma}) \cdot \prod_{r=1}^{m} p(\hat{\boldsymbol{\phi}}_r^{(k)}|\boldsymbol{\theta},\boldsymbol{\sigma})\right] \tag{3.6}$$

Since $\omega_r(\boldsymbol{\theta})$ in equation (3.3) is a deterministic quantity and the predictions errors $e_{\omega_r}^{(k)}$ are modeled by zero-mean Gaussian scalar variables so that

$e_{\omega_r}^{(k)} \sim N(0, \sigma_{\omega_r}^2 \hat{\omega}_r^{(k)2})$, the measured modal frequencies $\hat{\omega}_r^{(k)}$ are also implied to be Gaussian variables, that is, $\hat{\omega}_r^{(k)} \sim N\left(\omega_r(\boldsymbol{\theta}), \sigma_{\omega_r}^2 \hat{\omega}_r^{(k)2}\right)$ with mean $\omega_r(\boldsymbol{\theta})$ and variance $\sigma_{\omega_r}^2 \hat{\omega}_r^{(k)2}$. Therefore, the probability density function (PDF) of $\hat{\omega}_r^{(k)}$ involved in (3.6) given the values of $\boldsymbol{\theta}$ and $\boldsymbol{\sigma}$, is given by

$$p\left(\hat{\omega}_r^{(k)} \mid \boldsymbol{\theta}, \boldsymbol{\sigma}\right) = \frac{1}{\sqrt{2\pi}\ \sigma_{\omega_r} \hat{\omega}_r^{(k)}} \exp\left\{-\frac{1}{2} \frac{\left(\hat{\omega}_r^{(k)} - \omega_r(\boldsymbol{\theta})\right)^2}{\sigma_{\omega_r}^2 \hat{\omega}_r^{(k)2}}\right\} \tag{3.7}$$

Equivalently, since $\boldsymbol{\phi}_r(\boldsymbol{\theta})$ in equation (3.5) is a deterministic vector and the predictions errors $\boldsymbol{e}_{\boldsymbol{\phi}_r}^{(k)}$ are modeled by zero-mean Gaussian vector variables so that $\boldsymbol{e}_{\boldsymbol{\phi}_r}^{(k)} \sim N(\mathbf{0}, C_{\boldsymbol{\phi}_r}^{(k)})$, the measured modeshape components $\hat{\boldsymbol{\phi}}_r^{(k)}$ are also implied to be Gaussian vector variables, that is, $\hat{\boldsymbol{\phi}}_r^{(k)} \sim N\left(\boldsymbol{\phi}_r(\boldsymbol{\theta}), C_{\boldsymbol{\phi}_r}^{(k)}\right)$ with mean $\boldsymbol{\phi}_r(\boldsymbol{\theta})$ and covariance matrix $C_{\boldsymbol{\phi}_r}^{(k)}$. In that case, the probability density function (PDF) of $\hat{\boldsymbol{\phi}}_r^{(k)}$ involved in (3.6) given the values of $\boldsymbol{\theta}$ and $\boldsymbol{\sigma}$, is given by

$$p\left(\hat{\boldsymbol{\phi}}_r^{(k)} \mid \boldsymbol{\theta}, \boldsymbol{\sigma}\right) = \frac{1}{\left(\sqrt{2\pi}\right)^{N_0} \left|C_{\boldsymbol{\phi}_r}^{(k)}\right|^{1/2}} \exp\left\{-\frac{1}{2} \frac{\left\|\hat{\boldsymbol{\phi}}_r^{(k)} - \beta_r^{(k)} L_0 \boldsymbol{\phi}_r(\boldsymbol{\theta})\right\|^2}{\left\|C_{\boldsymbol{\phi}_r}^{(k)}\right\|}\right\} \tag{3.8}$$

Using the diagonal covariance matrix $C_{\boldsymbol{\phi}_r}^{(k)}$ with diagonal elements $\sigma_{\boldsymbol{\phi}_r}^2 \left\|\hat{\boldsymbol{\phi}}_r^{(k)}\right\|_{N_0}^2$ and substituting into (3.8), one has

$$p\left(\hat{\boldsymbol{\phi}}_r^{(k)} \mid \boldsymbol{\theta}, \boldsymbol{\sigma}\right) = \frac{1}{\left(\sqrt{2\pi}\right)^{N_0} \left(\sigma_{\boldsymbol{\phi}_r}\right)^{N_0} \left(\left\|\hat{\boldsymbol{\phi}}_r^{(k)}\right\|_{N_0}\right)^{N_0}} \exp\left\{-\frac{1}{2\sigma_{\boldsymbol{\phi}_r}^2} \frac{\left\|\hat{\boldsymbol{\phi}}_r^{(k)} - \beta_r^{(k)} L_0 \boldsymbol{\phi}_r(\boldsymbol{\theta})\right\|^2}{\left\|\hat{\boldsymbol{\phi}}_r^{(k)}\right\|_{N_0}^2}\right\} \tag{3.9}$$

By substituting equations (3.7) and (3.9) into equation (3.6), one derives that

$$\begin{aligned} p(D \mid \boldsymbol{\theta}, \boldsymbol{\sigma}) = {} & \prod_{k=1}^{N_D} \prod_{r=1}^{m} \frac{1}{\sqrt{2\pi}\ \sigma_{\omega_r} \hat{\omega}_r^{(k)}} \exp\left\{-\frac{1}{2\sigma_{\omega_r}^2} \frac{\left(\hat{\omega}_r^{(k)} - \omega_r(\boldsymbol{\theta})\right)^2}{\hat{\omega}_r^{(k)2}}\right\} \times \\ & \times \prod_{k=1}^{N_D} \prod_{r=1}^{m} \frac{1}{\left(\sqrt{2\pi}\right)^{N_0} \left(\sigma_{\boldsymbol{\phi}_r}\right)^{N_0} \left(\left\|\hat{\boldsymbol{\phi}}_r^{(k)}\right\|_{N_0}\right)^{N_0}} \exp\left\{-\frac{1}{2\sigma_{\boldsymbol{\phi}_r}^2} \frac{\left\|\hat{\boldsymbol{\phi}}_r^{(k)} - \beta_r^{(k)} L_0 \boldsymbol{\phi}_r(\boldsymbol{\theta})\right\|^2}{\left\|\hat{\boldsymbol{\phi}}_r^{(k)}\right\|_{N_0}^2}\right\} \end{aligned} \tag{3.10}$$

which successively simplifies to

$$p(D \mid \boldsymbol{\theta}, \boldsymbol{\sigma}) = \frac{1}{\left(\sqrt{2\pi}\right)^{mN_D} \prod_{k=1}^{N_D} \prod_{r=1}^{m} \hat{\omega}_r^{(k)} \prod_{r=1}^{m} \left(\sigma_{\omega_r}\right)^{N_D}} \exp\left\{-\frac{1}{2} \sum_{r=1}^{m} \frac{1}{\sigma_{\omega_r}^2} \sum_{k=1}^{N_D} \frac{\left(\hat{\omega}_r^{(k)} - \omega_r(\boldsymbol{\theta})\right)^2}{\hat{\omega}_r^{(k)2}}\right\} \times$$

$$\times \frac{1}{\left(\sqrt{2\pi}\right)^{mN_0N_D} \prod_{k=1}^{N_D}\prod_{r=1}^{m}\left(\left\|\hat{\boldsymbol{\phi}}_r^{(k)}\right\|_{N_0}\right)^{N_0} \prod_{r=1}^{m}\left(\sigma_{\phi_r}\right)^{N_0N_D}} \exp\left\{-\frac{1}{2}\sum_{r=1}^{m}\frac{1}{\sigma_{\phi_r}^2}\sum_{k=1}^{N_D}\frac{\left\|\hat{\boldsymbol{\phi}}_r^{(k)}-\beta_r L_0 \boldsymbol{\phi}_r(\boldsymbol{\theta})\right\|^2}{\left\|\hat{\boldsymbol{\phi}}_r^{(k)}\right\|_{N_0}^2}\right\}$$

and

$$p(D \mid \boldsymbol{\theta},\boldsymbol{\sigma}) = \frac{1}{\left(\sqrt{2\pi}\right)^{m(N_0+1)N_D} \prod_{k=1}^{N_D}\prod_{r=1}^{m}\left(\hat{\omega}_r^{(k)}\left\|\hat{\boldsymbol{\phi}}_r^{(k)}\right\|_{N_0}^{N_0}\right) \prod_{r=1}^{m}\left(\sigma_{\omega_r}^{N_D}\sigma_{\phi_r}^{N_0N_D}\right)} \times$$

$$\times \exp\left\{-\frac{1}{2}\sum_{r=1}^{m}\frac{1}{\sigma_{\omega_r}^2}\sum_{k=1}^{N_D}\frac{\left(\hat{\omega}_r^{(k)}-\omega_r(\boldsymbol{\theta})\right)^2}{\hat{\omega}_r^{(k)2}} + \frac{1}{\sigma_{\phi_r}^2}\sum_{k=1}^{N_D}\frac{\left\|\hat{\boldsymbol{\phi}}_r^{(k)}-\beta_r L_0 \boldsymbol{\phi}_r(\boldsymbol{\theta})\right\|^2}{\left\|\hat{\boldsymbol{\phi}}_r^{(k)}\right\|_{N_0}^2}\right\} \tag{3.11}$$

Equation (3.11) can be rewritten in the form

$$p(D|\boldsymbol{\theta},\boldsymbol{\sigma}) = \frac{1}{b\left(\sqrt{2\pi}\right)^{NN_D}\rho(\boldsymbol{\sigma})}\exp\left\{-\frac{NN_D}{2}J_D(\boldsymbol{\theta};\boldsymbol{\sigma})\right\} \tag{3.12}$$

where
$$J_D(\boldsymbol{\theta};\boldsymbol{\sigma}) = \sum_{i=1}^{n}\frac{\alpha_i}{\sigma_i^2}J_i(\boldsymbol{\theta}) \tag{3.13}$$

with $J_i(\boldsymbol{\theta}) = J_{\omega_i}(\boldsymbol{\theta})$, $J_{m+i}(\boldsymbol{\theta}) = J_{\phi_i}(\boldsymbol{\theta})$, $i = 1,\ldots,m$, $n = 2m$, represents the weighted measure of fit between the measured modal data and modal data predicted by a particular model within the selected model class, $J_{\omega_i}(\boldsymbol{\theta})$ and $J_{\phi_i}(\boldsymbol{\theta})$ are defined by

$$J_{\omega_r}(\boldsymbol{\theta}) = \frac{1}{N_D}\sum_{k=1}^{N_D}\frac{[\omega_r(\boldsymbol{\theta})-\hat{\omega}_r^{(k)}]^2}{[\hat{\omega}_r^{(k)}]^2} \tag{3.14}$$

and

$$J_{\phi_r}(\boldsymbol{\theta}) = \frac{1}{N_D}\sum_{k=1}^{N_D}\frac{\left\|\beta_r^{(k)}L_0\boldsymbol{\phi}_r(\boldsymbol{\theta})-\hat{\boldsymbol{\phi}}_r^{(k)}\right\|^2}{\left\|\hat{\boldsymbol{\phi}}_r^{(k)}\right\|^2} \tag{3.15}$$

respectively,

$$\rho(\boldsymbol{\sigma}) = \prod_{i=1}^{n}\left(\sigma_i\right)^{\alpha_i NN_D} \tag{3.16}$$

is a function of the prediction error parameters $\boldsymbol{\sigma}$, $N = m(N_0+1)$ is the number of measured data per modal set, $\alpha_r = 1/N$ and $\alpha_{m+r} = N_0/N$, $r = 1,\ldots,m$, satisfying $\sum_{i=1}^{n} \alpha_i = 1$, represent the number of data contained in each modal group in relation to the total number N of data, and

$$b = \prod_{k=1}^{N_D} \prod_{r=1}^{m} \left(\hat{\omega}_r^{(k)} \left\| \hat{\boldsymbol{\phi}}_r^{(k)} \right\|_{N_0}^{N_0} \right) = c = constant \tag{3.17}$$

Assuming that the prediction error parameters $\sigma_{\omega_r} = \sigma_1$, $r = 1,\ldots,m$, are the same for all the modal frequencies for each data set $k = 1,\cdots,N_D$ and that $\sigma_{\phi_r} = \sigma_2$, $r = 1,\ldots,m$, are the same for all modeshapes for each data set $k = 1,\cdots,N_D$. In this case, $n=2$, the prediction error parameters are $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$, the two measures of fit are given by

$$J_1(\boldsymbol{\theta}) = \frac{1}{m} \sum_{r=1}^{m} J_{\omega_r}(\boldsymbol{\theta}) \tag{3.18}$$

and

$$J_2(\boldsymbol{\theta}) = \frac{1}{m} \sum_{r=1}^{m} J_{\phi_r}(\boldsymbol{\theta}) \tag{3.19}$$

respectively, and the exponents α_i appearing in (3.16) are given by $\alpha_1 = m/N$ and $\alpha_2 = mN_0/N$.

3.4 Optimal Value of Structural Model Parameter given the Prediction Error Parameters

Given the values of the prediction error parameters $\boldsymbol{\sigma}$, the optimal value of the model parameter set $\boldsymbol{\theta}$ corresponds to the most probable model maximizing the updated PDF $p(\boldsymbol{\theta} \mid D, \boldsymbol{\sigma}, M)$ given in (3.2). In particular, using (3.12) and assuming a non-informative prior distribution $p(\boldsymbol{\theta} \mid \boldsymbol{\sigma}, M) = \pi(\boldsymbol{\theta}) = constant \ \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}$, where $\boldsymbol{\Theta}$ is the domain of definition of $\boldsymbol{\theta}$, the optimal values $\hat{\boldsymbol{\theta}}$ of the model parameters $\boldsymbol{\theta}$ are equivalently obtained by minimizing the measure of fit $J_D(\boldsymbol{\theta}; \boldsymbol{\sigma})$ defined in (3.13), i.e.

$$\hat{\boldsymbol{\theta}}(\boldsymbol{\sigma}) = \arg\min_{\boldsymbol{\theta}} J_D(\boldsymbol{\theta}; \boldsymbol{\sigma}) \tag{3.20}$$

The notation $\hat{\boldsymbol{\theta}}(\boldsymbol{\sigma})$ is used to show that the optimal value $\hat{\boldsymbol{\theta}}$ depends on the values of the prediction error parameter set $\boldsymbol{\sigma}$.

Hybrid algorithms based on evolution strategies and gradient methods are well-suited optimization tools for solving the resulting non-convex optimization problem and identifying the global optimum from multiple local ones.

3.5 Remarks

The methodology can readily be applied to alternative modal residual metrics that measure the fit between experimental and model predicted modal data. The formulation can be extended to identify the structural parameters of linear and non-linear models using measured acceleration time histories instead of modal properties.