Posterior System Analysis: Bayesian Inference

Observations or measurement data are collected from the system.

Let these observations be denoted by $\hat{Y}$.

The problem is now to update the uncertainty in the parameters using the information contained in the observations. This is achieved using the Bayes theorem.

BAYES THEOREM

Likelihood **Prior**

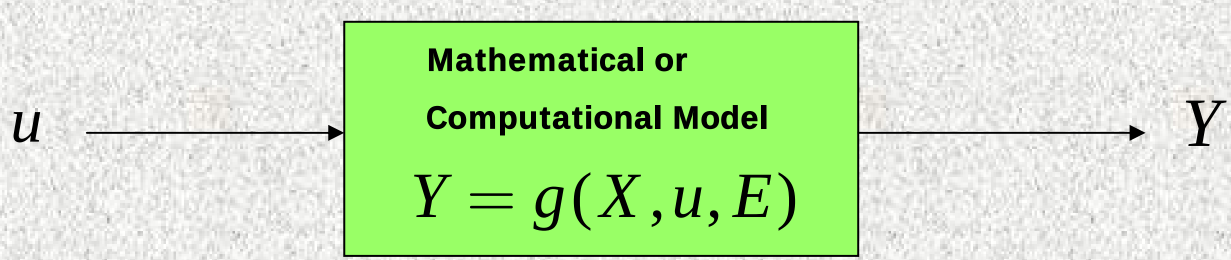
Posterior

$$f(x|\hat{Y},I) = \frac{f(\hat{Y}|x,I)\,f(x|I)}{f(\hat{Y}|I)}$$

Evidence

Bayes theorem gives the **posterior PDF (uncertainty)** of the model parameters which quantifies how plausible is each possible value of the parameter in light of the available observations from the system. This updated PDF of the uncertainty in the parameters is based on two quantities. The first one is called the **likelihood** and gives the probability to observe the data given a possible value of the model parameters. The likelihood is influenced by the data. The second one is the **prior probability** of the model parameters, which contains any information before data are utilized. The term in the denominator is called the **evidence** and for parameter estimation is just a normalization constant (does not depend on the parameters). For model selection, however, this term plays a crucial role.

Posterior System Analysis: Example 1



X is the uncertain model parameter; x is a possible value of X

Y is the uncertain output quantity of interest (QoI); y is a possible value of Y

E is the prediction error

u is the input; Assumed in this example to be known

Example 1 (Special Linear Case): $Y = X + E$

X and E are independent

$$f(x) = N(\mu, \sigma^2)$$

$$E \sim N(0, 1^2)$$

$\hat{Y} = 2$ A **single observation** (data) from the system

Posterior System Analysis: Example 1

Given the data $\hat{Y}$, the posterior PDF of the parameter x is obtained from Bayes theorem as

$$f(x|\hat{Y},I)=\frac{f(y=\hat{Y}|x,I)\,f(x|I)}{f(y=\hat{Y}|I)}$$

Using the mathematical model $Y = X + E$ we had obtained that

$$f(y|x,I)=N\left(x,\sigma^2\right)$$

which for $y=\hat{Y}=2$ gives

$$f(\hat{Y}|x,I)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\frac{(2-x)^2}{\sigma^2}\right]$$

Substituting in (1) along with the prior PDF

$$f(x|I)=N\left(\mu,\sigma^2\right)$$

one readily obtains that

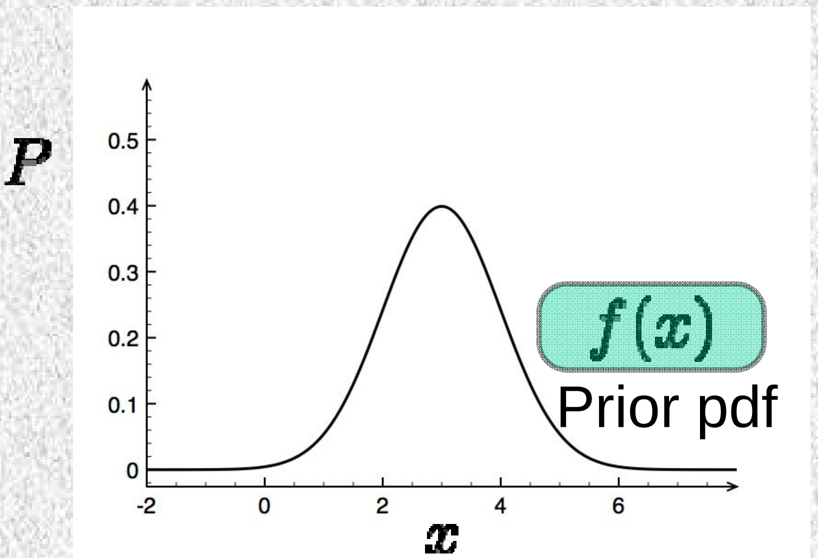
$$f(x|\hat{Y},I)\propto\exp\left[-\frac{1}{2}\frac{(2-x)^2+(x-\mu)^2}{\sigma^2}\right]$$

which can be shown to simplify to a normal distribution for the **posterior PDF** of the parameter

$$f(x|\hat{Y},I)=N\left(\frac{2+\mu}{2},\frac{1}{2}\sigma^2\right)$$

Posterior System Analysis: Example 1

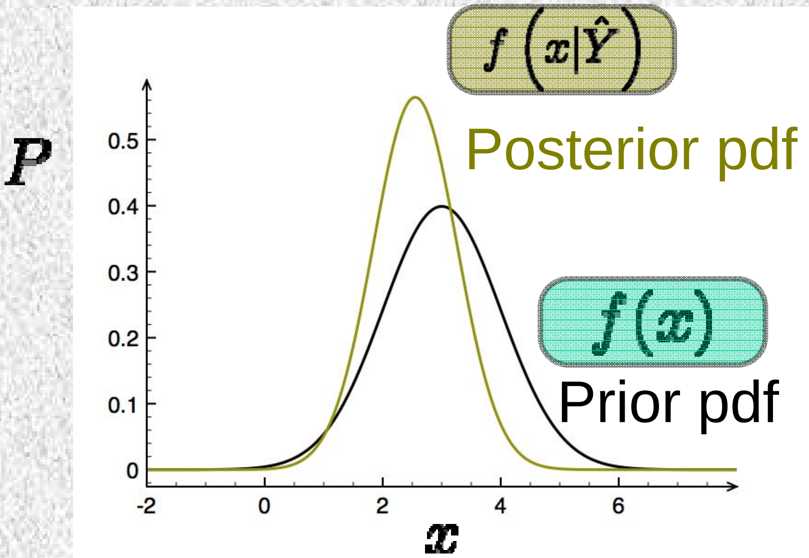
prior PDF $f(x) = N\left(\mu, \sigma^2\right)$



Posterior System Analysis: Example 1

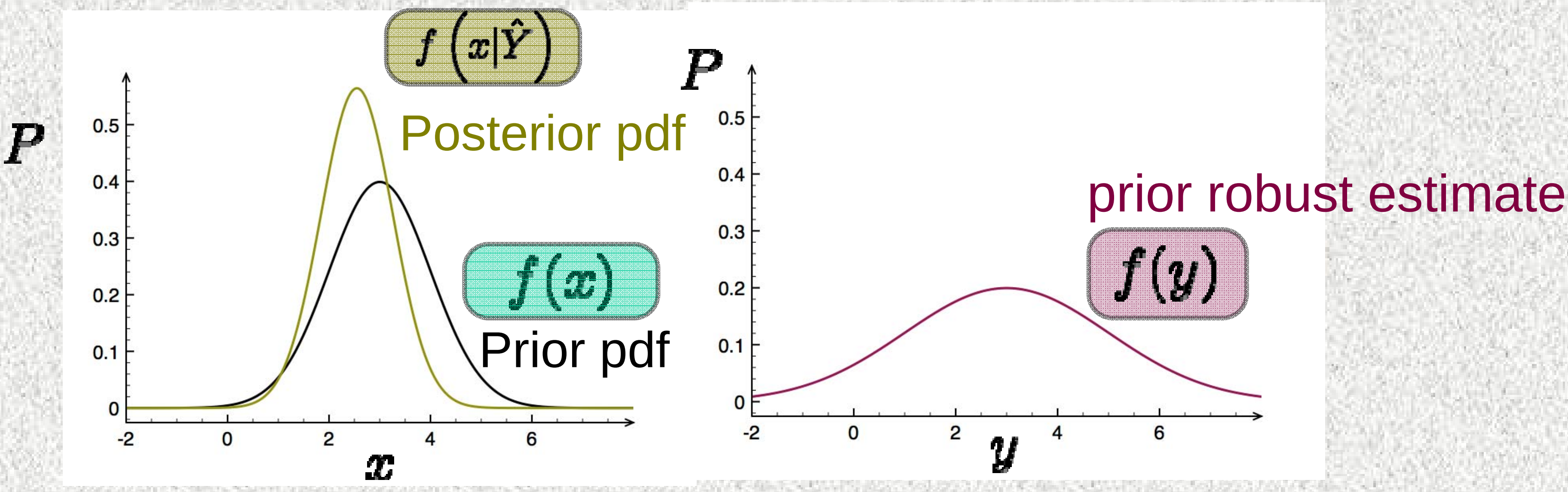
prior PDF $f(x) = N\left(\mu, \sigma^2\right)$

posterior PDF $f(x \mid \hat{Y}) = N\left(\frac{2+\mu}{2}, \frac{1}{2}\sigma^2\right)$



Posterior System Analysis: Example 1

The **prior robust prediction** for the QoI Y is $f(y|\hat{Y}) = N\left(\mu, \sigma^2 + 1\right)$

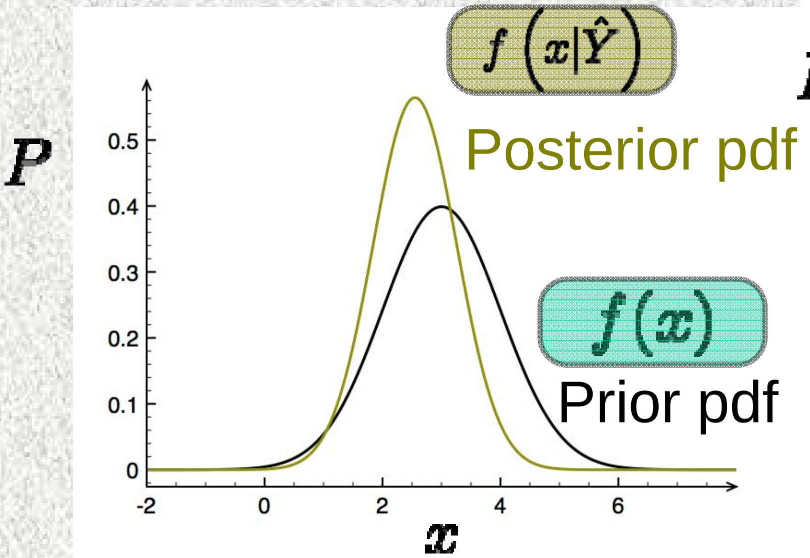


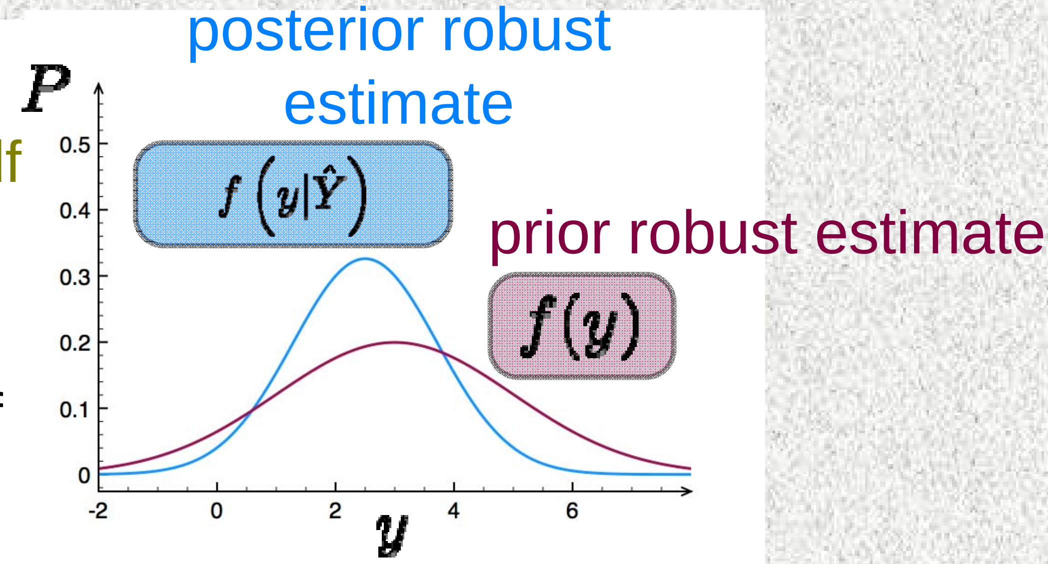
Posterior System Analysis: Example 1

The **prior robust prediction** for the QoI Y is $f(y\,|\,\hat{Y}) = N\left(\mu, \sigma^2 + 1\right)$

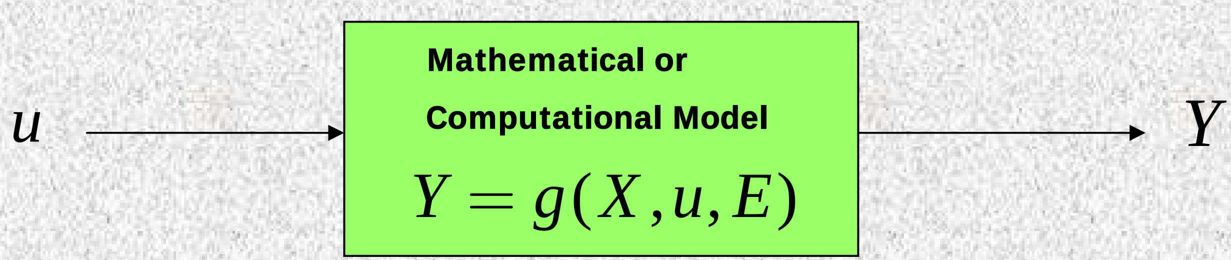
The **posterior robust prediction** for the QoI Y, taking into account the observations (measurements), is readily obtained form the fact that the **posterior PDF of the model parameter is normal**

$$f(y\,|\,\hat{Y}) = N\left(\frac{2+\mu}{2}, \frac{1}{2}\sigma^2 + 1\right)$$





Posterior System Analysis: Example 2



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Example 2 (Nonlinear Case): $Y_k = g_k(X, u) + E_k$

$\boldsymbol{X}$ and $\boldsymbol{E}$ are independent

$$f(x) = N(\mu, \sigma^2)$$

$E_k \sim N(0, 1^2)$ independent and identically distributed (i.i.d)

$\hat{Y} = \{\hat{Y}_1, \cdots, \hat{Y}_N\}$ **Observations** (data) from the system