1 Bayesian Estimation of Parameter of an Exponential Model

Consider a mathematical model of a physical process/system represented by the equation

$$\begin{aligned} Y_k &= x_0 \exp(-\alpha k \Delta t) + E_k \\ &= x_0 \beta^k + E_k \end{aligned} \tag{1}$$

where $\beta = \exp(-\alpha \Delta t)$ and E_k are independent identically distributed (iid) zero-mean Gaussian distributions, i.e. $E_k \sim N(0, \sigma^2)$. Note that equation (1) arises as the solution at time instances $t = k\Delta t$ of the first–order linear homogeneous differential equation

$$\dot{y} + \alpha y = 0 \tag{2}$$

with initial conditions $Y_0 = y(0) = x_0$. Given the observations $D \equiv (\hat{Y}_n, \hat{Y}_{2n}, \hat{Y}_{3n}, \ldots, \hat{Y}_{Nn}) \equiv \{\hat{Y}_{kn}\}_{1\to N}$ covering time instances that are multiple of n (n is given), we are interesting in updating the uncertainty in the parameter α of the system given the value of the variance σ^2.

Assume a uniform prior PDF for α and derive the expressions for the

1. Posterior PDF $p(\alpha \mid \{\hat{Y}_{kn}\}_{1\to N}, \sigma, I)$.
2. The MPV (or best estimate) $\hat{\alpha}$ of α
3. The uncertainty of α
4. Retain up to the quadratic terms in the Taylor series expansion of $L(\alpha)$ about the most probable value $\hat{\alpha}$ and derive the Gaussian asymptotic approximation for the posterior PDF of $p(\alpha \mid \{\hat{Y}_{kn}\}_{1\to N}, \sigma, I)$
5. Derive the expression for the posterior PDF for β, the MPV $\hat{\beta}$ and the uncertainty of β assuming a Gaussian prior PDF for β.

Perform the analysis for given values of $n\Delta t$, $x_0 = 1$, σ and number N of data.

Prior PDF: The uniform PDF for α is given by

$$p(\alpha \mid \sigma, I) = \begin{cases} 1/[\alpha_{\min} - \alpha_{\max}], & \alpha \in [\alpha_{\min}, \alpha_{\max}] \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Posterior PDF: Using Bayes' theorem, the inference about the value α given the data, the standard deviation σ and the information I (I includes the selection of the Gaussian model) is expressed by the posterior PDF

$$p(\alpha \mid D, \sigma, I) = \frac{p(D \mid \alpha, \sigma, I)\ p(\alpha \mid \sigma, I)}{p(D \mid \sigma, I)} \propto p(D \mid \alpha, \sigma, I)\ p(\alpha \mid \sigma, I) \tag{4}$$

Likelihood: Assuming that the measurements $\hat{Y}_{kn}$ are independent and E_k are distributed as Gaussian variables i.e. $E_k \sim N(0, \sigma^2)$, then using (1) the measurements $\hat{Y}_{kn}$ given the value of α

are also distributed as Gaussian variables with mean $x_0 \exp(-\alpha k\Delta t)$ and variance σ^2. Thus the likelihood is readily obtained in the form

$$\begin{aligned} p(D\,|\,\alpha,\sigma,I) &= p(\{\hat{Y}_{kn}\}_{1\to N}\,|\,\alpha,\sigma,I) = \prod_{k=1}^{N} p(\hat{Y}_{kn}\,|\,\alpha,\sigma,I) \\ &= \prod_{k=1}^{N} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2\sigma^2}\left[\hat{Y}_{kn} - x_0 \exp(-\alpha kn\Delta t)\right]^2\right] \\ &= \frac{1}{\left(\sqrt{2\pi}\right)^N \sigma^N} \exp\left[-\frac{1}{2\sigma^2}\sum_{k=1}^{N}\left[\hat{Y}_{kn} - x_0 \exp(-\alpha kn\Delta t)\right]^2\right] \end{aligned} \tag{5}$$

Estimation of Posterior PDF: Using (5) to replace the first factor in the right hand side (RHS) of (4) and the uniform prior PDF (3), the posterior PDF of the uncertain parameter α given the value of σ takes the form

$$p(\alpha\,|\,D,\sigma,I) \propto \frac{1}{\sigma^N} \exp\left[-\frac{1}{2\sigma^2}\sum_{k=1}^{N}\left[\hat{Y}_{kn} - x_0 \exp(-\alpha kn\Delta t)\right]^2\right]$$

Note that the distribution of the parameter α is not Gaussian.

Most Probable Value (MPV) or Best Estimate: The function $L(\alpha)$, defined in theory as the minus the logarithm of the posterior PDF of α, is given by

$$L(\alpha) = -\log p(\alpha\,|\,D,\sigma,I) = N\log\sigma + \frac{1}{2\sigma^2}\sum_{k=1}^{N}\left[\hat{Y}_{kn} - x_0 \exp(-\alpha kn\Delta t)\right]^2 + \text{constant} \tag{6}$$

The MPV of $\hat{\alpha}$ maximize the posterior PDF or, equivalently, minimize $L(\alpha)$. Specifically the derivative of $L(\alpha)$ with respect to α is

$$\begin{aligned} \frac{\partial L}{\partial \alpha} &= \frac{1}{\sigma^2}\sum_{k=1}^{N}\left[x_0 \exp(-\alpha kn\Delta t) - \hat{Y}_{kn}\right] x_0(-kn\Delta t)\exp(-\alpha kn\Delta t) \\ &= -\frac{x_0(n\Delta t)}{\sigma^2}\sum_{k=1}^{N}\left[x_0 \exp(-2\alpha kn\Delta t) - \hat{Y}_{kn}\exp(-\alpha kn\Delta t)\right]k \end{aligned} \tag{7}$$

The solution for the MPV $\hat{\alpha}$ is obtained by solving the equation

$$\left.\frac{\partial L}{\partial \alpha}\right|_{\alpha=\hat{\alpha}} = -\frac{x_0(n\Delta t)}{\sigma^2}\sum_{k=1}^{N}\left[x_0 \exp(-2\hat{\alpha} kn\Delta t) - \hat{Y}_{kn}\exp(-\hat{\alpha} kn\Delta t)\right]k = 0$$

However, it is clear that this equation cannot be solved analytically. Thus, the MPV $\hat{\alpha}$ can only be obtained numerically by minimizing the function $L(\alpha)$ given in (6). Any numerical optimization algorithm can be used to perform the optimization. Note that the MPV $\hat{\alpha}$ is independent of the value of σ.

Uncertainty in Model Parameters: The uncertainty in the value of the model parameter α is characterized by the Hessian of the function $L(\alpha)$ evaluated at the MPV $\hat{\alpha}$. Starting with (7) and differentiating once more with respect to α, the Hessian is given by

$$\left.\frac{\partial^2 L}{\partial \alpha^2}\right|_{\alpha=\hat{\alpha}} = -\frac{x_0(n\Delta t)}{\sigma^2}\sum_{k=1}^{N}\left[x_0(-2kn\Delta t)\exp(-2\hat{\alpha}kn\Delta t) - \hat{Y}_{kn}(-kn\Delta t)\exp(-\hat{\alpha}kn\Delta t)\right]k$$
$$= \frac{x_0(n\Delta t)^2}{\sigma^2}\sum_{k=1}^{N}\left[2x_0\exp(-2\hat{\alpha}kn\Delta t) - \hat{Y}_{kn}\exp(-\hat{\alpha}kn\Delta t)\right]k^2$$

The measure of the uncertainty, provided by the square root of the inverse of the Hessian of $L(\alpha)$ evaluated at the most probable value $\hat{\alpha}$, is given by

$$\sqrt{S} = \left(\left.\frac{d^2 L}{d\alpha^2}\right|_{\alpha=\hat{\alpha}}\right)^{-1/2}$$

Given the MPV $\hat{\alpha}$ and the uncertainty index $\sqrt{S}$ we can write a measure of the uncertainty interval of α in the form $\hat{\alpha} \pm \sqrt{S}$.

Asymptotic Posterior PDF: Following the theoretical result for the Bayesian Central Limit Theorem and using the MPV $\hat{\alpha}$ and the uncertainty index S, the posterior PDF of α follows asymptotically, for large number of data N, the Gaussian distribution

$$f(\alpha \mid D, \sigma, I) = \frac{1}{\sqrt{2\pi S}}\exp\left[-\frac{1}{2S}(\alpha - \hat{\alpha})^2\right]$$

Figure 1 shows the MPV $\hat{\alpha}$ and the uncertainty $\sqrt{S}$ in α for different values of the model parameters $n\Delta t$, $x_0 = 1$, σ and number N of data.

<u>Homework 2 (Deadline: 21 November 2013)</u>

<u>Exercise 2: Parameter Inference for Scalar Linear Difference Equation of 1st Order</u>

Consider a mathematical model of a physical process/system represented by the difference equation

$$Y_k = \alpha Y_{k-1} + E_k \tag{8}$$

where E_k are independent identically distributed (iid) zero-mean Gaussian distributions, i.e. $E_k \sim N(0,\sigma^2)$. Given the observations $D \equiv (\hat{Y}_0, \hat{Y}_1, \hat{Y}_2, \dots, \hat{Y}_N) \equiv \{\hat{Y}_k\}_{0\to N}$ covering all time instances, we are interesting in updating the uncertainty in the parameter α of the system given the value of the variance σ^2. Assume a Gaussian prior for α and derive the expressions for the

1. Posterior PDF $p(\alpha \mid \{\hat{Y}_k\}_{1\to N}, \sigma, I)$.
2. The MPV (or best estimate) $\hat{\alpha}$ of α
3. The uncertainty of α
4. Verify that the posterior PDF $p(\alpha \mid \{\hat{Y}_k\}_{1\to N}, \sigma, I)$ is a Gaussian distribution, that is, its asymptotic Gaussian approximation is exact.
5. Derive, as a special case of the previous analysis, the best estimate $\hat{\alpha}$ and the uncertainty of α for a uniform PDF with large enough bounds. Is the posterior PDF $p(\alpha \mid \{\hat{Y}_k\}_{1\to N}, \sigma, I)$ a Gaussian distribution?
6. Derive the posterior PDF $p(\alpha \mid \{\hat{Y}_k\}_{1\to N}, \sigma, I)$ for an inverse Gamma prior distribution. Without solving for the best estimate $\hat{\alpha}$ and the uncertainty of α, discuss the procedure for estimating the best estimate $\hat{\alpha}$ and the uncertainty of α. Can the MPV $\hat{\alpha}$ be estimated analytically? Is the posterior PDF $p(\alpha \mid D, \sigma, I)$ a Gaussian distribution?

<u>Exercise 3: Inference of Air Resistance Coefficient for a Falling Object</u>

Consider the mathematical model of a falling object with mass m, acceleration of gravity g and air resistance force $F_{res} = -m\alpha\upsilon^2$, where α is the air resistance coefficient. Using Newton's law, the equation of motion of the falling object is

$$m\frac{d\upsilon}{dt} = mg - m\alpha\upsilon^2 \tag{9}$$

Solving the nonlinear differential equation (9), the solution for the velocity can readily be obtained in the form

$$\upsilon(t) = \upsilon_\infty \tanh\left(\frac{g(t-t_0)}{\upsilon_\infty}\right)$$

where $\upsilon_\infty = \sqrt{g/\alpha}$ and t_0 is the initial time. Integrating the velocity $\upsilon = dx/dt$ with respect to time, the solution for the vertical displacement x of the falling object is finally obtained as

$$x(t) = \frac{1}{\alpha} \ln \cosh\left(\sqrt{g\alpha}\,(t - t_0)\right) \tag{10}$$

Measurements for the position of the falling object are obtained by a digital camera at regular time intervals $k\Delta t$. Given the observation data $D \equiv (\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_N) \equiv \{\hat{X}_k\}_{1\to N}$ of the location of the falling object at time instances $t = \Delta t, 2\Delta t, \ldots, N\Delta t$, respectively, we are interesting in estimating the uncertainty of the parameter α of the system given the value of the variance σ^2. Note that the measurements and the model predictions satisfy the model error equation

$$\hat{X}_k = x(k\Delta t) + E_k \tag{11}$$

where the measurement error terms E_k are independent identically distributed (iid) and follow a zero-mean Gaussian distribution $E_k \sim N(0, \sigma^2)$.

Assume a uniform prior for α and derive the expressions for the

1. Posterior PDF $p(\alpha \mid D, \sigma, I)$.
2. The function $L(\alpha) = -\ln p(\alpha \mid D, \sigma, I)$
3. The MPV (or best estimate) $\hat{\alpha}$ of α
4. The uncertainty of α
5. Retain up to the quadratic terms in the Taylor series expansion of $L(\alpha)$ about the most probable value $\hat{\alpha}$ and derive the Gaussian asymptotic approximation for the posterior PDF of $p(\alpha \mid D, \sigma, I)$

Perform the analysis for given values of $g = 9.81 m/s^2$, $t_0 = 0$, Δt and σ. Consider two cases:

(a) $\Delta t = ??$, $\sigma = ??$ and $N = ??$ with the data D given in file ????.dat.

(b) $\Delta t = ??$, $\sigma = ??$ and $N = ??$ with the data D given in file ????.dat.