Introductory Differential Equations using Sage

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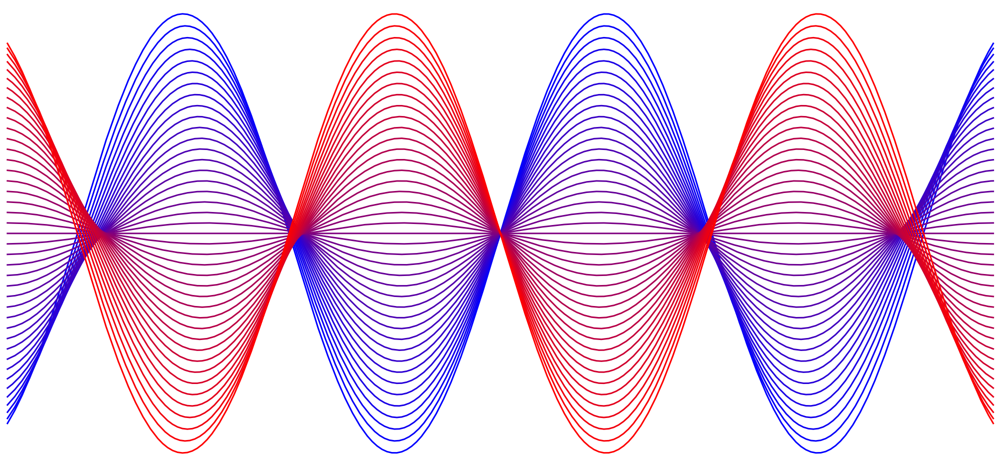


There are some things which cannot
be learned quickly, and time, which is all we have,
must be paid heavily for their acquiring.
They are the very simplest things,
and because it takes a man's life to know them
the little new that each man gets from life
is very costly and the only heritage he has to leave.

Ernest Hemingway
(From A. E. Hotchner, **Papa Hemingway**, Random House, NY, 1966)

Contents

Preface

The majority of this book came from lecture notes David Joyner (WDJ) typed up over the years for a course on differential equations with boundary value problems at the US Naval Academy (USNA). Though the USNA is a government institution and official work-related writing is in the public domain, so much of this was done at home during the night and weekends that he feels he has the right to claim copyright over this work. The DE course at the USNA has used various editions of the following three books (in order of most common use to least common use) at various times:

- Dennis G. Zill and Michael R. Cullen, **Differential equations with Boundary Value Problems**, 6th ed., Brooks/Cole, 2005.
- R. Nagle, E. Saff, and A. Snider, **Fundamentals of Differential Equations and Boundary Value Problems**, 4th ed., Addison/Wesley, 2003.
- W. Boyce and R. DiPrima, **Elementary Differential Equations and Boundary Value Problems**, 8th edition, John Wiley and Sons, 2005.

You may see some similarities but, for the most part, WDJ has taught things a bit differently and tried to impart this in these notes. Time will tell if there are any improvements.

After WDJ finished a draft of this book, he invited the second author, Marshall Hampton (MH), to revise and extend it. At the University of Minnesota Duluth, MH teaches a course on differential equations and linear algebra.

A new feature to this book is the fact that every section has at least one Sage exercise. Sage is FOSS (free and open source software), available on the most common computer platforms. Royalties for the sales of this book (if it ever makes it's way to a publisher) will go to further development of Sage .

The cover image was created with the following Sage code:

Sage

```
from math import cos, sin
def RK4(f, t_start, y_start, t_end, steps):
    '''
    fourth-order Runge-Kutta solver with fixed time steps.
    f must be a function of t,y.
    '''
    step_size = (t_end - t_start)/steps
    t_current = t_start
    argn = len(y_start)
    y_current = [x for x in y_start]
    answer_table = []
    answer_table.append([t_current,y_current])
    for j in range(0,steps):
        k1=f(t_current,y_current)
```

```
        k2=f(t_current+step_size/2,[y_current[i] + k1[i]*step_size/2 for i in range(argn)])
        k3=f(t_current+step_size/2,[y_current[i] + k2[i]*step_size/2 for i in range(argn)])
        k4=f(t_current+step_size,[y_current[i] + k3[i]*step_size for i in range(argn)])
        t_current += step_size
        y_current = [y_current[i] + (step_size/6)*(k1[i]+2*k2[i]+2*k3[i]+k4[i]) for i in range(len(k1))]
        answer_table.append([t_current,y_current])
    return answer_table

def e1(t, y):
    return [-y[1],sin(y[0])]

npi = N(pi)
sols = []
for v in srange(-1,1+.04,.05):
    p1 = RK4(e1,0.0,[0.0,v],-2.5*npi,200)[::-1]
    p1 = p1 + RK4(e1,0.0,[0.0,v],2.5*npi,200)
    sols.append(list_plot([[x[0],x[1][0]] for x in p1], plotjoined=True, rgbcolor = ((v+1)/2.01,0,(1.01-v)/2.01)))
f = 2
show(sum(sols), axes = True, figsize = [f*5*npi/3,f*2.5], xmin = -7, xmax = 7, ymin = -1, ymax = 1)
```

Acknowledgments

In a few cases we have made use of the *excellent* (public domain!) lecture notes by Sean Mauch, *Introduction to methods of Applied Mathematics*, available online at http://www.its.caltech.edu/~sean/book/unabridged.html (as of Fall, 2009).

In some cases, we have made use of the material on Wikipedia - this includes both discussion and in a few cases, diagrams or graphics. This material is licensed under the GFDL or the Attribution-ShareAlike Creative Commons license. In any case, the amount used here probably falls under the "fair use" clause.

Software used:

Most of the graphics in this text was created using Sage (http://www.sagemath.org/) and GIMP http://www.gimp.org/ by the authors. The most important components of Sage for our purposes are: Maxima, SymPy and Matplotlib. The circuit diagrams were created using Dia http://www.gnome.org/projects/dia/ and GIMP by the authors. A few diagrams were "taken" from Wikipedia http://www.wikipedia.org/ (and acknowledged in the appropriate placein the text). Of course, LaTeX was used for the typesetting. Many thanks to the developers of these programs for these free tools.

Chapter 1

First order differential equations

> But there is another reason for the high repute of mathematics: it is mathematics that offers the exact natural sciences a certain measure of security which, without mathematics, they could not attain.
>
> - *Albert Einstein*

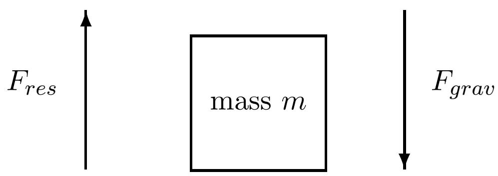
1.1 Introduction to DEs

Roughly speaking, a differential equation is an equation involving the derivatives of one or more unknown functions. Implicit in this vague definition is the assumption that the equation imposes a *constraint* on the unknown function (or functions). For example, we would not call the well-known product rule identity of differential calculus a differential equation.

In calculus (differential-, integral- and vector-), you've studied ways of analyzing functions. You might even have been convinced that functions you meet in applications arise naturally from physical principles. As we shall see, differential equations arise *naturally* from general physical principles. In many cases, the functions you met in calculus in applications to physics were actually solutions to a naturally-arising differential equation.

Example 1.1.1. Consider a falling body of mass m on which exactly three forces act:

- gravitation, F_{grav},
- air resistance, F_{res},
- an external force, $F_{ext} = f(t)$, where $f(t)$ is some given function.



Let $x(t)$ denote the distance fallen from some fixed initial position. The velocity is denoted by $v = x'$ and the acceleration by $a = x''$. We choose an orientation so that downwards is positive. In this case, $F_{grav} = mg$, where $g > 0$ is the gravitational constant. We assume that air resistance is proportional to velocity (a common assumption in physics), and write $F_{res} = -kv = -kx'$, where $k > 0$ is a "friction constant". The total force, F_{total}, is by hypothesis,

$$F_{total} = F_{grav} + F_{res} + F_{ext},$$

and, by Newton's 2nd Law[1],

$$F_{total} = ma = mx''.$$

Putting these together, we have

$$mx'' = ma = mg - kx' + f(t),$$

or

$$mx'' + mx' = f(t) + mg.$$

This is a differential equation in $x = x(t)$. It may also be rewritten as a differential equation in $v = v(t) = x'(t)$ as

$$mv' + kv = f(t) + mg.$$

This is an example of a "first order differential equation in v", which means that at most first order derivatives of the unknown function $v = v(t)$ occur.

In fact, you have probably seen solutions to this in your calculus classes, at least when $f(t) = 0$ and $k = 0$. In that case, $v'(t) = g$ and so $v(t) = \int g\, dt = gt + C$. Here the constant of integration C represents the initial velocity.

Differential equations occur in other areas as well: weather prediction (more generally, fluid-flow dynamics), electrical circuits, the temperature of a heated homogeneous wire, and many others (see the table below). They even arise in problems on Wall Street: the Black-Scholes equation is a PDE which models the pricing of derivatives [BS-intro]. Learning to solve differential equations helps you understand the behaviour of phenomenon present in these problems.

[1] "Force equals mass times acceleration." `http://en.wikipedia.org/wiki/Newtons_law`

phenomenon	description of DE
weather	Navier-Stokes equation [NS-intro] a non-linear vector-valued higher-order PDE
falling body	1st order linear ODE
motion of a mass attached to a spring	Hooke's spring equation 2nd order linear ODE [H-intro]
motion of a plucked guitar string	Wave equation 2nd order linear PDE [W-intro]
Battle of Trafalger	Lanchester's equations system of 2 1st order DEs [L-intro], [M-intro], [N-intro]
cooling cup of coffee in a room	Newton's Law of Cooling 1st order linear ODE
population growth	logistic equation non-linear, separable, 1st order ODE

Undefined terms and notation will be defined below, except for the equations themselves. For those, see the references or wait until later sections when they will be introduced[2].

Basic Concepts:

Here are some of the concepts to be introduced below:

- dependent variable(s),
- independent variable(s),
- ODEs,
- PDEs,
- order,
- linearity,
- solution.

It is really best to learn these concepts using examples. However, here are the general definitions anyway, with examples to follow.

The term "differential equation" is sometimes abbreviated DE, for brevity.

Dependent/independent variables: Put simply, a differential equation is an equation involving derivatives of one of more unknown functions. The variables you are differentiating with respect to are the **independent variables** of the DE. The variables (the "unknown functions") you are differentiating are the **dependent variables** of the DE. Other variables which might occur in the DE are sometimes called "parameters".

[2]Except for the important Navier-Stokes equation, which is relatively complicated and would take us too far afield, `http://en.wikipedia.org/wiki/Navier-stokes`.

ODE/PDE: If none of the derivatives which occur in the DE are partial derivatives (for example, if the dependent variable/unknown function is a function of a single variable) then the DE is called an **ordinary differential equation** or **ODE**. If some of the derivatives which occur in the DE are partial derivatives then the DE is a **partial differential equation** or **PDE**.

Order: The highest total number of derivatives you have to take in the DE is it's **order**.

Linearity: This can be described in a few different ways. First of all, a DE is *linear* if the only operations you perform on its terms are combinations of the following:

- differentiation with respect to independent variable(s),
- multiplication by a function of the independent variable(s).

Another way to define linearity is as follows. A **linear ODE** having independent variable t and the dependent variable is y is an ODE of the form

$$a_0(t)y^{(n)} + ... + a_{n-1}(t)y' + a_n(t)y = f(t),$$

for some given functions $a_0(t)$, ..., $a_n(t)$, and $f(t)$. Here

$$y^{(n)} = y^{(n)}(t) = \frac{d^n y(t)}{dt^n}$$

denotes the n-th derivative of $y = y(t)$ with respect to t. The terms $a_0(t)$, ..., $a_n(t)$ are called the **coefficients** of the DE and we will call the term $f(t)$ the **non-homogeneous term** or the **forcing function**. (In physical applications, this term usually represents an external force acting on the system. For instance, in the example above it represents the gravitational force, mg.)

Solution: An explicit **solution** to a DE having independent variable t and the dependent variable is x is simple a function $x(t)$ for which the DE is true for all values of t.

Here are some examples:

Example 1.1.2. Here is a table of examples. As an exercise, determine which of the following are ODEs and which are PDEs.

DE	indep vars	dep vars	order	linear?
$mx'' + kx' = mg$ falling body	t	x	2	yes
$mv' + kv = mg$ falling body	t	v	1	yes
$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ heat equation	t, x	u	2	yes
$mx'' + bx' + kx = f(t)$ spring equation	t	x	2	yes
$P' = k(1 - \frac{P}{K})P$ logistic population equation	t	P	1	no
$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial^2 t}$ wave equation	t, x	u	2	yes
$T' = k(T - T_{room})$ Newton's Law of Cooling	t	T	1	yes
$x' = -Ay$, $y' = -Bx$, Lanchester's equations	t	x, y	1	yes

Remark 1.1.1. Note that in many of these examples, the symbol used for the independent variable is not made explicit. For example, we are writing x' when we really mean $x'(t) = \frac{x(t)}{dt}$. This is very common shorthand notation and, in this situation, we shall usually use t as the independent variable whenever possible.

Example 1.1.3. Recall a linear ODE having independent variable t and the dependent variable is y is an ODE of the form

$$a_0(t)y^{(n)} + ... + a_{n-1}(t)y' + a_n(t)y = f(t),$$

for some given functions $a_0(t), \ldots, a_n(t)$, and $f(t)$. The order of this DE is n. In particular, a linear 1st order ODE having independent variable t and the dependent variable is y is an ODE of the form

$$a_0(t)y' + a_1(t)y = f(t),$$

for some $a_0(t)$, $a_1(t)$, and $f(t)$. We can divide both sides of this equation by the leading coefficient $a_0(t)$ without changing the solution y to this DE. Let's do that and rename the terms:

$$y' + p(t)y = q(t),$$

where $p(t) = a_1(t)/a_0(t)$ and $q(t) = f(t)/a_0(t)$. Every linear 1st order ODE can be put into this form, for some p and q. For example, the falling body equation $mv' + kv = f(t) + mg$ has this form after dividing by m and renaming v as y.

What does a differential equation like $mx'' + kx' = mg$ or $P' = k(1 - \frac{P}{K})P$ or $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial^2 t}$ really mean? In $mx'' + kx' = mg$, m and k and g are given constants. The only things that can vary are t and the unknown function $x = x(t)$.

Example 1.1.4. To be specific, let's consider $x'+x=1$. This means *for all* t, $x'(t)+x(t)=1$. In other words, a solution $x(t)$ is a function which, when added to its derivative you *always* get the constant 1. How many functions are there with that property? Try guessing a few "random" functions:

- Guess $x(t)=\sin(t)$. Compute $(\sin(t))'+\sin(t)=\cos(t)+\sin(t)=\sqrt{2}\sin(t+\frac{\pi}{4})$. $x'(t)+x(t)=1$ is *false*.
- Guess $x(t)=\exp(t)=e^t$. Compute $(e^t)'+e^t=2e^t$. $x'(t)+x(t)=1$ is *false*.
- Guess $x(t)=\exp(t)=t^2$. Compute $(t^2)'+t^2=2t+t^2$. $x'(t)+x(t)=1$ is *false*.
- Guess $x(t)=\exp(-t)=e^{-t}$. Compute $(e^{-t})'+e^{-t}=0$. $x'(t)+x(t)=1$ is *false*.
- Guess $x(t)=\exp(t)=1$. Compute $(1)'+1=0+1=1$. $x'(t)+x(t)=1$ is *true*.

We finally found a solution by considering the constant function $x(t)=1$. Here a way of doing this kind of computation with the aid of the computer algebra system Sage :

Sage

```
sage: t = var('t')
sage: de = lambda x: diff(x,t) + x - 1
sage: de(sin(t))
sin(t) + cos(t) - 1
sage: de(exp(t))
2*e^t - 1
sage: de(t^2)
t^2 + 2*t - 1
sage: de(exp(-t))
-1
sage: de(1)
0
```

Note we have rewritten $x'+x=1$ as $x'+x-1=0$ and then plugged various functions for x to see if we get 0 or not.

Obviously, we want a more systematic method for solving such equations than guessing all the types of functions we know one-by-one. We will get to those methods in time. First, we need some more terminology.

IVP: A first order **initial value problem** (abbreviated **IVP**) is a problem of the form

$$x'=f(t,x),\quad x(a)=c,$$

where $f(t,x)$ is a given function of two variables, and a, c are given constants. The equation $x(a)=c$ is the **initial condition**.

Under mild conditions of f, an IVP has a solution $x=x(t)$ which is unique. This means that if f and a are fixed but c is a parameter then the solution $x=x(t)$ will depend on c. This is stated more precisely in the following result.

Theorem 1.1.1. *(Existence and uniqueness) Fix a point* (t_0, x_0) *in the plane. Let* $f(t,x)$ *be a function of* t *and* x *for which both* $f(t,x)$ *and* $f_x(t,x) = \frac{\partial f(t,x)}{\partial x}$ *are continuous on some rectangle*

$$a < t < b, \quad c < x < d,$$

in the plane. Here a, b, c, d *are any numbers for which* $a < t_0 < b$ *and* $c < x_0 < d$*. Then there is an* $h > 0$ *and a unique solution* $x = x(t)$ *for which*

$$x' = f(t,x), \quad \text{for all } t \in (t_0 - h, t_0 + h),$$

and $x(t_0) = x_0$*.*

This is proven in §2.8 of Boyce and DiPrima [BD-intro], but we shall not prove this here (though we will return to it in more detail in §1.3 below). In most cases we shall run across, it is easier to construct the solution than to prove this general theorem.

Example 1.1.5. Let us try to solve

$$x' + x = 1, \quad x(0) = 1.$$

The solutions to the DE $x' + x = 1$ which we "guessed at" in the previous example, $x(t) = 1$, satisfies this IVP.

Here a way of finding this slution with the aid of the computer algebra system Sage :

Sage
```
sage: t = var('t')
sage: x = function('x', t)
sage: de = lambda y: diff(y,t) + y - 1
sage: desolve(de(x),[x,t],[0,1])
1
```

(The command `desolve` is a DE solver in Sage .) Just as an illustration, let's try another example. Let us try to solve

$$x' + x = 1, \quad x(0) = 2.$$

The Sage commands are similar:

Sage
```
sage: t = var('t')
sage: x = function('x', t)
sage: de = lambda y: diff(y,t) + y - 1
sage: x0 = 2 # this is forthe IC x(0) = 2
sage: soln = desolve(de(x),[x,t],[0,x0])
sage: solnx = lambda s: RR(soln.subs(t=s))
```

```
sage: P = plot(solnx,0,5)
sage: soln; show(P)
(e^t + 1)*e^(-t)
```

This gives the solution $x(t) = (e^t + 1)e^{-t} = 1 + e^{-t}$ and the plot given in Figure 1.1.

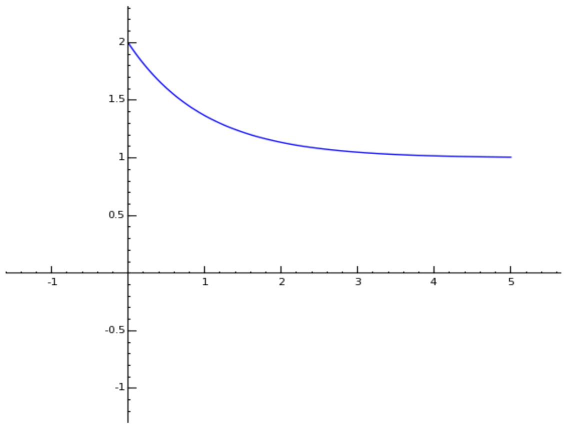


Figure 1.1: Solution to IVP $x' + x = 1$, $x(0) = 2$.

Example 1.1.6. Now let us consider an example which does not satisfy all of the assumptions of the existence and uniqueness theorem: $x' = x^{2/3}$. The function $x = (t/3 + C)^3$ is a solution to this differential equation for any choice of the constant C. If we consider only solutions with the initial value $x(0) = 0$, we see that we can choose $C = 0$ - i.e. $x = t^3/27$ satisfies the differential equation and has $x(0) = 0$. But there is another solution with that initial condition - the solution $x = 0$. So a solutions exist, but they are not necessarily unique.

Some conditions on the ODE which guarantee a unique solution will be presented in §1.3.

Exercises:

1. Verify that $x = t^3 + 5$ is a solution to the differential equation $x' = 3t^2$.

2. Subsitute $x = e^{rt}$ into the differential equation $x'' + x' - 6x = 0$ and determine all values of the parameter r which give a solution.

3. (a) Verify the, for any constant c, the function $x(t) = 1 + ce^{-t}$ solves $x' + x = 1$. Find the c for which this function solves the IVP $x' + x = 1$, $x(0) = 3$.

   (b) Solve

$$x' + x = 1, \quad x(0) = 3,$$

   using Sage .

1.2 Initial value problems

Recall, 1^{st} **order initial value problem**, or IVP, is simply a 1^{st} order ODE and an initial condition. For example,

$$x'(t) + p(t)x(t) = q(t), \qquad x(0) = x_0,$$

where $p(t)$, $q(t)$ and x_0 are given. The analog of this for 2^{nd} order linear DEs is this:

$$a(t)x''(t) + b(t)x'(t) + c(t)x(t) = f(t), \qquad x(0) = x_0,\ x'(0) = v_0,$$

where $a(t)$, $b(t)$, $c(t)$, x_0, and v_0 are given. This 2^{nd} order linear DE and initial conditions is an example of a 2^{nd} **order IVP**. In general, in an IVP, the number of initial conditions must match the order of the DE.

Example 1.2.1. Consider the 2^{nd} order DE

$$x'' + x = 0.$$

(We shall run across this DE many times later. As we will see, it represents the displacement of an undamped spring with a unit mass attached. The term **harmonic oscillator** is attached to this situation [O-ivp].) Suppose we know that the general solution to this DE is

$$x(t) = c_1 \cos(t) + c_2 \sin(t),$$

for any constants c_1, c_2. This means every solution to the DE must be of this form. (If you don't believe this, you can at least check it it is a solution by computing $x''(t) + x(t)$ and verifying that the terms cancel, as in the following Sage example. Later, we see how to derive this solution.) Note that there are two degrees of freedom (the constants c_1 and c_2), matching the order of the DE.

Sage

```
sage: t = var('t')
sage: c1 = var('c1')
sage: c2 = var('c2')
sage: de = lambda x: diff(x,t,t) + x
sage: de(c1*cos(t) + c2*sin(t))
0
sage: x = function('x', t)
sage: soln = desolve(de(x),[x,t]); soln
k1*sin(t) + k2*cos(t)
sage: solnx = lambda s: RR(soln.subs(k1=1, k2=0, t=s))
sage: P = plot(solnx,0,2*pi)
sage: show(P)
```

This is displayed in Figure 1.2.

Now, to solve the IVP

$$x'' + x = 0, \quad x(0) = 0,\ x'(0) = 1.$$

the problem is to solve for c_1 and c_2 for which the $x(t)$ satisfies the initial conditions. The two degrees of freedom in the general solution matching the number of initial conditions in the IVP. Plugging $t = 0$ into $x(t)$ and $x'(t)$, we obtain

$$0 = x(0) = c_1 \cos(0) + c_2 \sin(0) = c_1, \quad 1 = x'(0) = -c_1 \sin(0) + c_2 \cos(0) = c_2.$$

Therefore, $c_1 = 0$, $c_2 = 1$ and $x(t) = \sin(t)$ is the unique solution to the IVP.

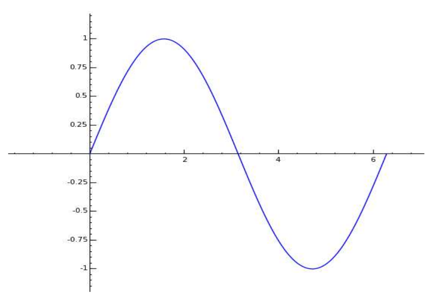


Figure 1.2: Solution to IVP $x'' + x = 0$, $x(0) = 0$, $x'(0) = 1$.

In Figure 1.2, you see the solution oscillates, as t increases.

Another example,

Example 1.2.2. Consider the 2^{nd} order DE

$$x'' + 4x' + 4x = 0.$$

(We shall run across this DE many times later as well. As we will see, it represents the displacement of a critially damped spring with a unit mass attached.) Suppose we know that the general solution to this DE is

$$x(t) = c_1 \exp(-2t) + c_2 t \exp(-2t) = c_1 e^{-2t} + c_2 t e^{-2t},$$

for any constants c_1, c_2. This means every solution to the DE must be of this form. (Again, you can at least check it is a solution by computing $x''(t)$, $4x'(t)$, $4x(t)$, adding them up and verifying that the terms cancel, as in the following Sage example.)

Sage

```
sage: t = var('t')
sage: c1 = var('c1')
sage: c2 = var('c2')
sage: de = lambda x: diff(x,t,t) + 4*diff(x,t) + 4*x
```

```
sage: de(c1*exp(-2*t) + c2*t*exp(-2*t))
0
sage: desolve(de(x),[x,t])
(k2*t + k1)*e^(-2*t)
sage: P = plot(t*exp(-2*t),0,pi)
sage: show(P)
```

The plot is displayed in Figure 1.3.

Now, to solve the IVP

$$x'' + 4x' + 4x = 0, \quad x(0) = 0,\ x'(0) = 1.$$

we solve for c_1 and c_2 using the initial conditions. Plugging $t = 0$ into $x(t)$ and $x'(t)$, we obtain

$$0 = x(0) = c_1 \exp(0) + c_2 \cdot 0 \cdot \exp(0) = c_1,$$
$$1 = x'(0) = c_1 \exp(0) + c_2 \exp(0) - 2c_2 \cdot 0 \cdot \exp(0) = c_1 + c_2.$$

Therefore, $c_1 = 0$, $c_1 + c_2 = 1$ and so $x(t) = t\exp(-2t)$ is the unique solution to the IVP. In Figure 1.3, you see the solution tends to 0, as t increases.

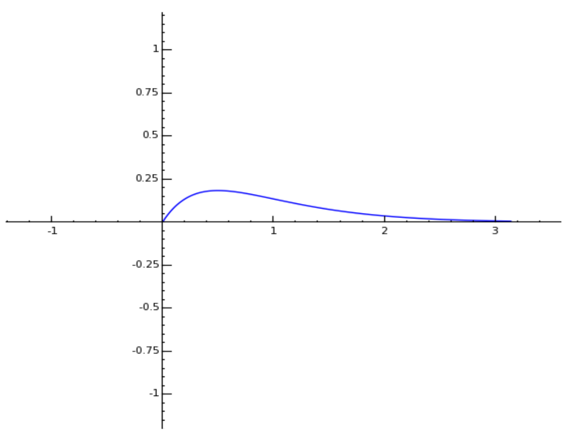


Figure 1.3: Solution to IVP $x'' + 4x' + 4x = 0$, $x(0) = 0$, $x'(0) = 1$.

Suppose, for the moment, that for some reason you mistakenly thought that the general solution to this DE was

$$x(t) = c_1 \exp(-2t) + c_2 \exp(-2t) = e^{-2t}(c_1 + c_2),$$

for arbitray constants c_1, c_2. (Note: the "extra t-factor" on the second term on the right is missing from this expression.) *Now*, if you try to solve for the constant c_1 and c_2 using the initial conditions $x(0) = 0$, $x'(0) = 1$ you will get the equations

$$c_1 + c_2 = 0$$
$$-2c_1 - 2c_2 = 1.$$

These equations are *impossible* to solve! The moral of the story is that you must have a correct general solution to insure that you can always solve your IVP.

One more quick example.

Example 1.2.3. Consider the 2^{nd} order DE

$$x'' - x = 0.$$

Suppose we know that the general solution to this DE is

$$x(t) = c_1 \exp(t) + c_2 \exp(-t) = c_1 e^t + c_2 e^{-t},$$

for any constants c_1, c_2. (Again, you can check it is a solution.)

The solution to the IVP

$$x'' - x = 0, \quad x(0) = 1,\ x'(0) = 0,$$

is $x(t) = \frac{e^t+e^{-t}}{2}$. (You can solve for c_1 and c_2 yourself, as in the examples above.) This particular function is also called a **hyperbolic cosine function**, denoted $\cosh(t)$ (pronounced "kosh"). The **hyperbolic sine function**, denoted $\sinh(t)$ (pronounced "sinch"), satisfies the IVP

$$x'' - x = 0, \quad x(0) = 0,\ x'(0) = 1.$$

The hyperbolic trig functions have many properties analogous to the usual trig functions and arise in many areas of applications [H-ivp]. For example, $\cosh(t)$ represents a catenary or hanging cable [C-ivp].

Sage

```
sage: t = var('t')
sage: c1 = var('c1')
sage: c2 = var('c2')
sage: de = lambda x: diff(x,t,t) - x
sage: de(c1*exp(-t) + c2*exp(-t))
0
sage: desolve(de(x)),[x,t])
k1*eˆt + k2*eˆ(-t)
sage: P = plot(eˆt/2-eˆ(-t)/2,0,3)
sage: show(P)
```

You see in Figure 1.4 that the solution tends to infinity, as t gets larger.

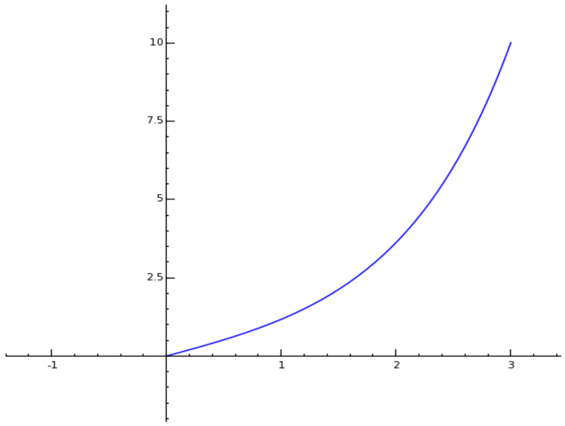


Figure 1.4: Solution to IVP $x'' - x = 0$, $x(0) = 0$, $x'(0) = 1$.

Exercises:

1. Find the value of the constant C that makes $x = Ce^{3t}$ a solution to the IVP $x' = 3x$, $x(0) = 4$.

2. Verify that $x = (C+t)\cos(t)$ satisfies the differential equation $x' + x\tan(t) - cos(t) = 0$ and find the value of C that gives the initial condition $x(2\pi) = 0$.

3. Use Sage to check that the general solution to the falling body problem

$$mv' + kv = mg,$$

is $v(t) = \frac{mg}{k} + ce^{-kt/m}$. If $v(0) = v_0$, you can solve for c in terms of v_0 to get $c = v_0 - \frac{mg}{k}$. Take $m = k = v_0 = 1$, $g = 9.8$ and use Sage to plot $v(t)$ for $0 < t < 1$.

1.3 Existence of solutions to ODEs

When do solutions to an ODE exist? When are they unique? This section gives some necessary conditions for determining existence and uniqueness.

1.3.1 First order ODEs

We begin by considering the first order initial value problem

$$x'(t) = f(t, x(t)), \quad x(a) = c. \tag{1.1}$$

What conditions on f (and a and c) guarantee that a solution $x = x(t)$ exists? If it exists, what (further) conditions guarantee that $x = x(t)$ is unique?

The following result addresses the first question.

Theorem 1.3.1. *("Peano's existence theorem" [P-intro]) Suppose f is bounded and continuous in x, and t. Then, for some value $\epsilon > 0$, there exists a solution $x = x(t)$ to the initial value problem within the range $[a - \epsilon, a + \epsilon]$.*

Giuseppe Peano (18581932) was an Italian mathematician, who is mostly known for his important work on the logical foundations of mathematics. For example, the common notations for union $\cup$ and intersections $\cap$ first appeared in his first book dealing with mathematical logic, written while he was teaching at the University of Turin.

Example 1.3.1. Take $f(x, t) = x^{2/3}$. This is continuous and bounded in x and t in $-1 < x < 1$, $t \in \mathbb{R}$. The IVP $x' = f(x, t)$, $x(0) = 0$ has **two** solutions, $x(t) = 0$ and $x(t) = t^3/27$.

You all know what continuity means but you may not be familiar with the slightly stronger notion of "Lipschitz continuity". This is defined next.

Definition 1.3.1. *Let $D \subset \mathbb{R}^2$ be a domain. A function $f : D \to \mathbb{R}$ is called Lipschitz continuous if there exists a real constant $K > 0$such that, for all $x_1, x_2 \in D$,*

$$|f(x_1) - f(x_2)| \leq K|x_1 - x_2|.$$

The smallest such K is called the Lipschitz constant of the function f on D.

For example,

- the function $f(x) = x^{2/3}$ defined on $[-1, 1]$ is not Lipschitz continuous;
- the function $f(x) = x^2$ defined on $[-3, 7]$ is Lipschitz continuous, with Lipschitz constant $K = 14$;
- the function f defined by $f(x) = x^{3/2}\sin(1/x)$ ($x \neq 0$) and $f(0) = 0$ restricted to $[0, 1]$, gives an example of a function that is differentiable on a compact set while not being Lipschitz.

Theorem 1.3.2. *("Picard's existence and uniqueness theorem" [PL-intro]) Suppose f is bounded, Lipschitz continuous in x, and continuous in t. Then, for some value $\epsilon > 0$, there exists a unique solution $x = x(t)$ to the initial value problem (1.1) within the range $[a - \epsilon, a + \epsilon]$.*

Charles Émile Picard (1856-1941) was a leading French mathematician. Picard made his most important contributions in the fields of analysis, function theory, differential equations, and analytic geometry. In 1885 Picard was appointed to the mathematics faculty at the Sorbonne in Paris. Picard was awarded the Poncelet Prize in 1886, the Grand Prix des Sciences Mathmatiques in 1888, the Grande Croix de la Légion d'Honneur in 1932, the Mittag-Leffler Gold Medal in 1937, and was made President of the International Congress of Mathematicians in 1920. He is the author of many books and his collected papers run to four volumes.

The proofs of Peano's theorem or Picard's theorem go *well* beyond the scope of this course. However, for the curious, a very brief indication of the main ideas will be given in the sketch below. For details, see an advanced text on differential equations.

sketch or the idea of the proof: A simple proof of existence of the solution is obtained by successive approximations. In this context, the method is known as Picard iteration.

Set $x_0(t) = c$ and

$$x_i(t) = c + \int_a^t f(s, x_{i-1}(s))\, ds.$$

It turns out that Lipschitz continuity implies that the mapping T defined by

$$T(y)(t) = c + \int_a^t f(s, y(s))\, ds,$$

is a contraction mapping on a certain Banach space. It can then be shown, by using the Banach fixed point theorem, that the sequence of "Picard iterates" x_i is convergent and that the limit is a solution to the problem. The proof of uniqueness uses a result called Grönwall's Lemma. $\square$

Example 1.3.2. Consider the IVP

$$x' = 1 - x, \qquad x(0) = 1,$$

with the constant solution $x(t) = 1$. Computing the Picard iterates by hand is easy: $x_0(t) = 1$, $x_1(t) = 1 + \int_0^t 1 - x_0(s))\, ds = 1$, $x_2(t) = 1 + \int_0^t 1 - x_1(s))\, ds = 1$, and so on. Since each $x_i(t) = 1$, we find the solution

$$x(t) = \lim_{i\to\infty} x_i(t) = \lim_{i\to\infty} 1 = 1.$$

We now try the Picard iteration method in Sage . Consider the IVP

$$x' = 1 - x, \qquad x(0) = 2,$$

which we considered earlier.

Sage

```
sage: var('t, s')
sage: f = lambda t,x: 1-x
sage: a = 0; c = 2
sage: x0 = lambda t: c; x0(t)
2
sage: x1 = lambda t: c + integral(f(s,x0(s)), s, a, t); x1(t)
2 - t
sage: x2 = lambda t: c + integral(f(s,x1(s)), s, a, t); x2(t)
t^2/2 - t + 2
sage: x3 = lambda t: c + integral(f(s,x2(s)), s, a, t); x3(t)
-t^3/6 + t^2/2 - t + 2
```

```
sage: x4 = lambda t: c + integral(f(s,x3(s)), s, a, t); x4(t)
t^4/24 - t^3/6 + t^2/2 - t + 2
sage: x5 = lambda t: c + integral(f(s,x4(s)), s, a, t); x5(t)
-t^5/120 + t^4/24 - t^3/6 + t^2/2 - t + 2
sage: x6 = lambda t: c + integral(f(s,x5(s)), s, a, t); x6(t)
t^6/720 - t^5/120 + t^4/24 - t^3/6 + t^2/2 - t + 2
sage: P1 = plot(x2(t), t, 0, 2, linestyle='--')
sage: P2 = plot(x4(t), t, 0, 2, linestyle='-.')
sage: P3 = plot(x6(t), t, 0, 2, linestyle=':')
sage: P4 = plot(1+exp(-t), t, 0, 2)
sage: (P1+P2+P3+P4).show()
```

From the graph in Figure 1.5 you can see how well these iterates are (or at least appear to be) converging to the true solution $x(t) = 1 + e^{-t}$.

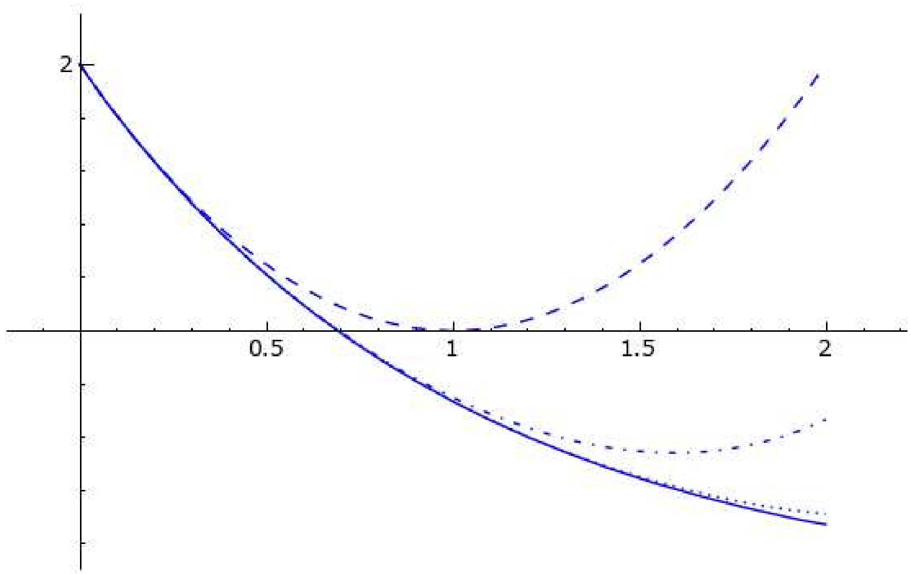


Figure 1.5: Picard iteration for $x' = 1 - x$, $x(0) = 2$.

More generally, here is some Sage code for Picard iteration.

Sage

```
def picard_iteration(f, a, c, N):
    '''
    Computes the N-th Picard iterate for the IVP

        x' = f(t,x), x(a) = c.

    EXAMPLES:
        sage: var('x t s')
```

```
        (x, t, s)
        sage: a = 0; c = 2
        sage: f = lambda t,x: 1-x
        sage: picard_iteration(f, a, c, 0)
        2
         sage: picard_iteration(f, a, c, 1)
        2 - t
        sage: picard_iteration(f, a, c, 2)
        t^2/2 - t + 2
        sage: picard_iteration(f, a, c, 3)
        -t^3/6 + t^2/2 - t + 2

    '''
    if N == 0:
        return c*t**0
    if N == 1:
        x0 = lambda t: c + integral(f(s,c*s**0), s, a, t)
        return expand(x0(t))
    for i in range(N):
        x_old = lambda s: picard_iteration(f, a, c, N-1).subs(t=s)
        x0 = lambda t: c + integral(f(s,x_old(s)), s, a, t)
    return expand(x0(t))
```

Exercise: Apply the Picard iteration method in Sage to the IVP

$$x' = (t + x)^2, \qquad x(0) = 2,$$

and find the first three iterates.

1.3.2 Higher order constant coefficient linear homogeneous ODEs

We begin by considering the second order[3] initial value problem

$$ax'' + bx' + cx = 0, \quad x(0) = d_0, \quad x'(0) = d_1, \tag{1.2}$$

where a, b, c, d_0, d_1 are constants and $a \neq 0$. What conditions guarantee that a solution $x = x(t)$ exists? If it exists, what (further) conditions guarantee that $x = x(t)$ is unique? It turns out that no conditions are needed - a solution to 1.2 always exists and is unique. As we will see later, we can construct distinct explicit solutions, denoted $x_1 = x_1(t)$ and $x_2 = x_2(t)$ and sometimes called **fundamental solutions**, to $ax'' + bx' + cx = 0$. If we let $x = c_1x_1 + c_2x_2$, for any constants c_1 and c_2, then we know that x is also a solution[4], sometimes called the **general solution** to $ax'' + bx' + cx = 0$. But how do we know there exist c_1 and c_2 for which this general solution also satisfies the initial conditions $x(0) = d_0$ and $x'(0) = d_1$? For this to hold, we need to be able to solve

[3]It turns out that the reasoning in the second order case is very similar to the general reasoning for n-th order DEs. For simplicity of presentation, we restrict to the 2-nd order case.

[4]This follows form the linearity assumption.

$$c_1x_1(0) + c_2x_2(0) = d_1, \quad c_1x_1'(0) + c_2x_2'(0) = d_2,$$

for c_1 and c_2. By Cramer's rule,

$$c_1 = \frac{\begin{vmatrix} d_1 & x_2(0) \\ d_2 & x_2'(0) \end{vmatrix}}{\begin{vmatrix} x_1(0) & x_2(0) \\ x_1'(0) & x_2'(0) \end{vmatrix}}, \qquad c_2 = \frac{\begin{vmatrix} x_1(0) & d_1 \\ x_1'(0) & d_2 \end{vmatrix}}{\begin{vmatrix} x_1(0) & x_2(0) \\ x_1'(0) & x_2'(0) \end{vmatrix}}.$$

For this solution to exist, the denominators in these quotients must be non-zero. This denominator is the value of the "Wronskian" [W-linear] at $t = 0$.

Definition 1.3.2. *For n functions $f_1, \ldots, f_n$, which are $n-1$ times differentiable on an interval I, the* **Wronskian** $W(f_1, ..., f_n)$ *as a function on I is defined by*

$$W(f_1, \ldots, f_n)(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix},$$

for $x \in I$.

The matrix constructed by placing the functions in the first row, the first derivative of each function in the second row, and so on through the $(n-1)$-st derivative, is a square matrix sometimes called a **fundamental matrix** of the functions. The Wronskian is the determinant of the fundamental matrix.

Theorem 1.3.3. *("Abel's identity") Consider a homogeneous linear second-order ordinary differential equation*

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y = 0$$

on the real line with a continuous function p. The Wronskian W of two solutions of the differential equation satisfies the relation

$$W(x) = W(0)\exp\left(-\int_0^x P(s)\,ds\right).$$

Example 1.3.3. Consider $x'' + 3x' + 2x = 0$.

Sage

```
sage: t = var("t")
sage: x = function("x",t)
sage: DE = diff(x,t,t)+3*diff(x,t)+2*x==0
sage: desolve(DE, [x,t])
k1*e^(-t) + k2*e^(-2*t)
```

```
sage: Phi = matrix([[e^(-t), e^(-2*t)],[-e^(-t), -2*e^(-2*t)]]); Phi

[     e^(-t)    e^(-2*t)]
[    -e^(-t) -2*e^(-2*t)]
sage: W = det(Phi); W
-e^(-3*t)
sage: Wt = e^(-integral(3,t)); Wt
e^(-3*t)
sage: W*W(t=0) == Wt
e^(-3*t) == e^(-3*t)
sage: bool(W*W(t=0) == Wt)
True
```

Definition 1.3.3. *We say n functions f_1, ..., f_n are* **linearly dependent** *over the interval I, if there are numbers $a_1, \cdots, a_n$ (not all of them zero) such that*

$$a_1 f_1(x) + \cdots + a_n f_n(x) = 0,$$

for $x \in I$. If the functions are not linearly dependent then they are called **linearly independent**.

Theorem 1.3.4. *If the Wronskian is non-zero at some point in an interval, then the associated functions are linearly independent on the interval.*

Example 1.3.4. If $f_1(t) = e^t$ and $f_2(t) = e^{-t}$ then

$$\begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = -2.$$

Indeed,

Sage

```
sage: var('t')
t
sage: f1 = exp(t); f2 = exp(-t)
sage: wronskian(f1,f2)
-2
```

Therefore, the fundamental solutions $x_1 = e^t$, $x_2 = e^{-t}$ are

Exercise: Using Sage , verify Abel's identity
(a) in the example $x'' - x = 0$,
(b) in the example $x'' + 2 * x' + 2x = 0$.

1.4 First order ODEs - separable and linear cases

Separable DEs:

We know how to solve any ODE of the form

$$y' = f(t),$$

at least in principle - just integrate both sides[5]. For a more general type of ODE, such as

$$y' = f(t, y),$$

this fails. For instance, if $y' = t + y$ then integrating both sides gives $y(t) = \int \frac{dy}{dt}\,dt = \int y'\,dt = \int t + y\,dt = \int t\,dt + \int y(t)\,dt = \frac{t^2}{2} + \int y(t)\,dt$. So, we have only succeeded in writing $y(t)$ in terms of its integral. Not very helpful.

However, there is a class of ODEs where this idea works, with some modification. If the ODE has the form

$$y' = \frac{g(t)}{h(y)}, \tag{1.3}$$

then it is called **separable**[6].

To solve a separable ODE:

(1) write the ODE (1.3) as $\frac{dy}{dt} = \frac{g(t)}{h(y)}$,

(2) "separate" the t's and the y's:

$$h(y)\,dy = g(t)\,dt,$$

(3) integrate both sides:

$$\boxed{\int h(y)\,dy = \int g(t)\,dt \;+\; C} \tag{1.4}$$

I've added a "$+C$" to emphasize that a constant of integration must be included in your answer (but only on one side of the equation).

[5]Recall y' really denotes $\frac{dy}{dt}$, so by the fundamental theorem of calculus, $y = \int \frac{dy}{dt}\,dt = \int y'\,dt = \int f(t)\,dt = F(t) + c$, where F is the "anti-derivative" of f and c is a constant of integration.

[6]It particular, any separable DE *must* be first order, ordinary differential equation.

The answer obtained in this manner is called an "implicit solution" of (1.3) since it expresses y *implicitly* as a function of t.

Why does this work? It is easiest to understand by working backwards from the formula (1.4). Recall that one form of the fundamental theorem of calculus is $\frac{d}{dy}\int h(y)dy = h(y)$. If we think of y as a being a function of t, and take the t-derivative, we can use the chain rule to get

$$g(t) = \frac{d}{dt}\int g(t)dt = \frac{d}{dt}\int h(y)dy = (\frac{d}{dy}\int h(y)dy)\frac{dy}{dt} = h(y)\frac{dy}{dt}.$$

So if we differentiate both sides of equation (1.4) with respect to t, we recover the original differential equation.

Example 1.4.1. Are the following ODEs separable? If so, solve them.

(a) $(t^2 + y^2)y' = -2ty$,

(b) $y' = -x/y$, $y(0) = -1$,

(c) $T' = k \cdot (T - T_{room})$, where $k < 0$ and T_{room} are constants,

(d) $ax' + bx = c$, where $a \neq 0$, $b \neq 0$, and c are constants

(e) $ax' + bx = c$, where $a \neq 0$, b, are constants and $c = c(t)$ is *not* a constant.

(f) $y' = (y - 1)(y + 1)$, $y(0) = 2$.

(g) $y' = y^2 + 1$, $y(0) = 1$.

Solutions:

(a) not separable,

(b) $y\,dy = -x\,dx$, so $y^2/2 = -x^2/2 + c$, so $x^2 + y^2 = 2c$. This is the general solution (note it does not give y explicitly as a function of x, you will have to solve for y algebraically to get that). The initial conditions say when $x = 0$, $y = 1$, so $2c = 0^2 + 1^2 = 1$, which gives $c = 1/2$. Therefore, $x^2 + y^2 = 1$, which is a circle. That is not a *function* so cannot be the solution we want. The solution is either $y = \sqrt{1 - x^2}$ or $y = -\sqrt{1 - x^2}$, but which one? Since $y(0) = -1$ (note the minus sign) it must be $y = -\sqrt{1 - x^2}$.

(c) $\frac{dT}{T - T_{room}} = kdt$, so $\ln|T - T_{room}| = kt + c$ (some constant c), so $T - T_{room} = Ce^{kt}$ (some constant C), so $T = T(t) = T_{room} + Ce^{kt}$.

(d) $\frac{dx}{dt} = (c - bx)/a = -\frac{b}{a}(x - \frac{c}{b})$, so $\frac{dx}{x - \frac{c}{b}} = -\frac{b}{a}\,dt$, so $\ln|x - \frac{c}{b}| = -\frac{b}{a}t + C$, where C is a constant of integration. This is the *implicit* general solution of the DE. The *explicit* general solution is $x = \frac{c}{b} + Be^{-\frac{b}{a}t}$, where B is a constant.

The explicit solution is easy find using Sage :

Sage

```
sage: a = var('a')
sage: b = var('b')
sage: c = var('c')
sage: t = var('t')
sage: x = function('x', t)
sage: de = lambda y: a*diff(y,t) + b*y - c
sage: desolve(de(x),[x,t])
(c*e^(b*t/a)/b + c)*e^(-b*t/a)
```

(e) If $c = c(t)$ is not constant then $ax' + bx = c$ is not separable.

(f) $\frac{dy}{(y-1)(y+1)} = dt$ so $\frac{1}{2}(\ln(y-1) - \ln(y+1)) = t + C$, where C is a constant of integration. This is the "general (implicit) solution" of the DE.

Note: the constant functions $y(t) = 1$ and $y(t) = -1$ are also solutions to this DE. These solutions cannot be obtained (in an obvious way) from the general solution.

The integral is easy to do using Sage :

Sage

```
sage: y = var('y')
sage: integral(1/((y-1)*(y+1)),y)
log(y - 1)/2 - (log(y + 1)/2)
```

Now, let's try to get Sage to solve for y in terms of t in $\frac{1}{2}(\ln(y-1) - \ln(y+1)) = t + C$:

Sage

```
sage: C = var('C')
sage: solve([log(y - 1)/2 - (log(y + 1)/2) == t+C],y)
[log(y + 1) == -2*C + log(y - 1) - 2*t]
```

This is not working. Let's try inputting the problem in a different form:

Sage

```
sage: C = var('C')
sage: solve([log((y - 1)/(y + 1)) == 2*t+2*C],y)
[y == (-e^(2*C + 2*t) - 1)/(e^(2*C + 2*t) - 1)]
```

This is what we want. Now let's assume the initial condition $y(0) = 2$ and solve for C and plot the function.

Sage

```
sage: solny=lambda t:(-e^(2*C+2*t)-1)/(e^(2*C+2*t)-1)
sage: solve([solny(0) == 2],C)
[C == log(-1/sqrt(3)), C == -log(3)/2]
sage: C = -log(3)/2
sage: solny(t)
(-e^(2*t)/3 - 1)/(e^(2*t)/3 - 1)
sage: P = plot(solny(t), 0, 1/2)
sage: show(P)
```

This plot is shown in Figure 1.6. The solution has a singularity at $t = ln(3)/2 = 0.5493....$

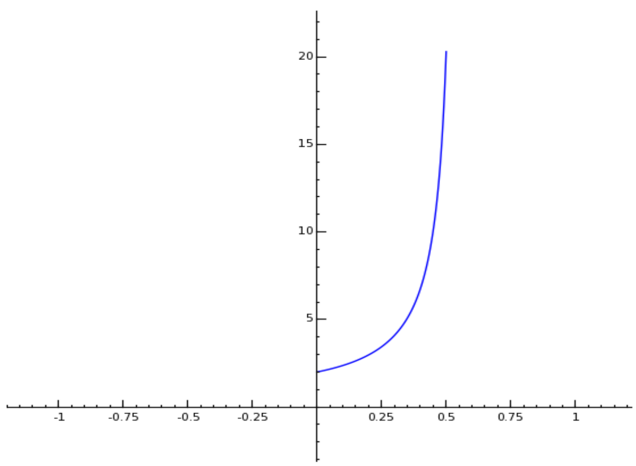


Figure 1.6: Plot of $y' = (y-1)(y+1)$, $y(0) = 2$, for $0 < t < 1/2$.

(g) $\frac{dy}{y^2+1} = dt$ so $\arctan(y) = t + C$, where C is a constant of integration. The initial condition $y(0) = 1$ says $\arctan(1) = C$, so $C = \frac{\pi}{4}$. Therefore $y = \tan(t + \frac{\pi}{4})$ is the solution.

1.4.1 Autonomous ODEs

A special subclass of separable ODEs is the class of **autonomous** ODEs, which have the form

$$y' = f(y),$$

where f is a given function (i.e., the slope y only depends on the value of the dependent variable y). The cases (c), (d), (f), and (g) above are examples.

One of the simplest examples of an autonomous ODE is $\frac{dy}{dt} = ky$, where k is a constant. We can divide by y and integrate to get

$$\int \frac{1}{y}\, dy = \log|y| = \int k\ dt = kt + C_1.$$

After exponentiating, we get

$$|y| = e^{kt+C_1} = e^{C_1}e^{kt}.$$

We can drop the absolute value if we allow positive and negative solutions:

$$y = \pm e^{C_1}e^{kt}.$$

Now note that $\pm e^{C_1}$ can be any nonzero number, but in fact $y = 0$ is also a solution to the ODE so we can write the general solution as $y = Ce^{kt}$ for an arbitrary constant C. If k is positive, solutions will grow in magnitude exponentially, and if k is negative solutions will decay to 0 exponentially.

Perhaps the most famous use of this type of ODE is in carbon dating.

Example 1.4.2. Carbon-14 has a half-life of about 5730 years, meaning that after that time one-half of a given amount will radioactively decay (into stable nitrogen-14). Prior to the nuclear tests of the 1950s, which raised the level of C-14 in the atmosphere, the ratio of C-14 to C-12 in the air, plants, and animals was 10^{-15}. If this ratio is measured in an archeological sample of bone and found to be $3.6 \cdot 10^{-17}$, how old is the sample?

Solution: Since a constant fraction of C-14 decays per unit time, the amount of C-14 satisfies a differential equation $y' = ky$ with solution $y = Ce^{kt}$. Since

$$y(5730) = Ce^{k5730} = y(0)/2 = C/2,$$

we can compute $k = -\log(2)/5730 \approx 1.21 \cdot 10^{-5}$.

We know that

$$y(0)/y(t_i) = \frac{10^{-17}}{10^{-15}} = 10^{-2} = \frac{C}{Ce^{kt_i}} = \frac{1}{e^{kt_i}},$$

where t_i is the time of death of whatever the sample is from. So $t_i = \frac{\log 10^2}{k} \approx -38069$ years before the present.

Here is a non-linear example.

Example 1.4.3. Consider

$$y' = (y-1)(y+1), \qquad y(0) = 1/2.$$

Here is one way to solve this using Sage :

Sage

```
sage: t = var('t')
sage: x = function('x', t)
sage: de = lambda y: diff(y,t) == y^2 - 1
sage: soln = desolve(de(x),[x,t]); soln
1/2*log(x(t) - 1) - 1/2*log(x(t) + 1) == c + t
sage:   # needs an abs. value ...
sage: c,xt = var("c,xt")
sage: solnxt = (1/2)*log(abs(xt - 1)) - (1/2)*log(abs(xt + 1))
                       == c + t
sage: solve(solnxt.subs(t=0, xt=1/2),c)
[c == -1/2*log(3/2) - 1/2*log(2)]
sage: c0 = solve(solnxt.subs(t=0, xt=1/2),c)[0].rhs(); c0
-1/2*log(3/2) - 1/2*log(2)
sage: soln0 = solnxt.subs(c=c0); soln0
1/2*log(abs(xt - 1)) - 1/2*log(abs(xt + 1))
     ==   t - 1/2*log(3/2) -1/2*log(2)
sage: implicit_plot(soln0,(t,-1/2,1/2),(xt,0,0.9))
```

Sage cannot solve this (implicit) solution for $x(t)$, though I'm sure you can do it by hand if you want. The (implicit) plot is given in Figure 1.7.

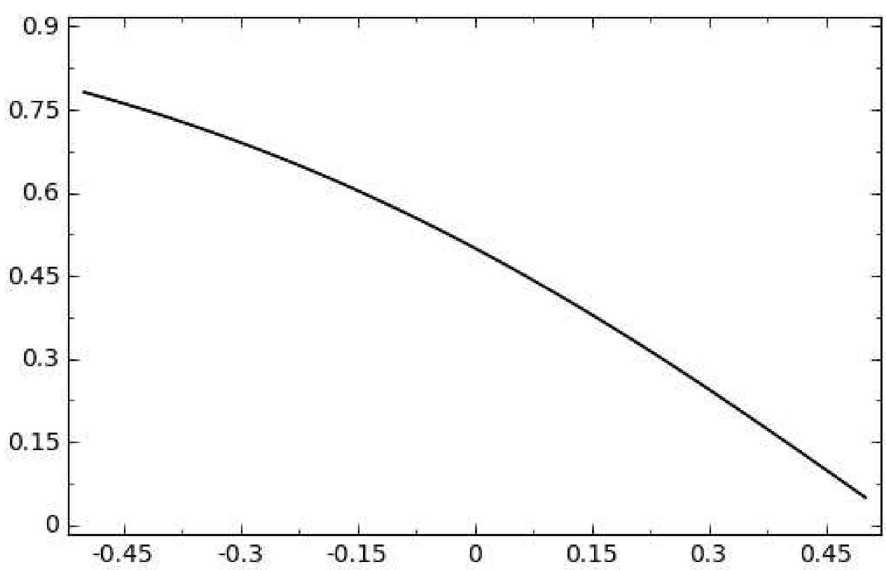


Figure 1.7: Plot of $y' = (y-1)(y+1)$, $y(0) = 1/2$, for $-1/2 < t < 1/2$.

A more complicated example is

$$y' = y(y-1)(y-2).$$

This has constant solutions $y(t) = 0$, $y(t) = 1$, and $y(t) = 2$. (Check this.) Several non-constant solutions are plotted in Figure 1.8.

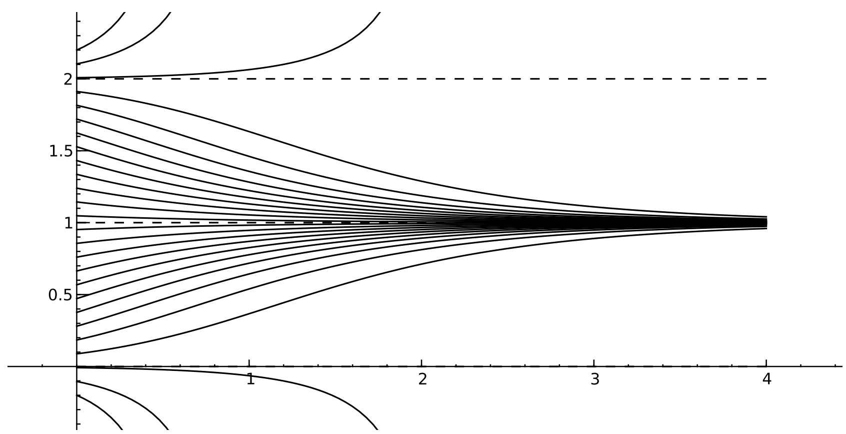


Figure 1.8: Plot of $y' = y(y-1)(y-2)$, $y(0) = y_0$, for $0 < t < 4$, and various values of y_0.

Exercise: Find the general solution to $y' = y(y-1)(y-2)$ either "by hand" or using Sage
.

1.4.2 Linear 1st order ODEs

The bottom line is that we want to solve any problem of the form

$$x' + p(t)x = q(t), \tag{1.5}$$

where $p(t)$ and $q(t)$ are given functions (which, let's assume, aren't "too horrible"). Every first order linear ODE can be written in this form. Examples of DEs which have this form: Falling Body problems, Newton's Law of Cooling problems, Mixing problems, certain simple Circuit problems, and so on.

There are two approaches

- "the formula",
- the method of integrating factors.

Both lead to the exact same solution.
"The Formula": The general solution to (1.5) is

$$x = \frac{\int e^{\int p(t)\,dt} q(t)\,dt + C}{e^{\int p(t)\,dt}}, \tag{1.6}$$

where C is a constant. The factor $e^{\int p(t)\,dt}$ is called the **integrating factor** and is often denoted by μ. This formula was apparently first discovered by Johann Bernoulli [F-1st].

Example 1.4.4. Solve

$$xy' + y = e^x.$$

We rewrite this as $y' + \frac{1}{x}y = \frac{e^x}{x}$. Now compute $\mu = e^{\int \frac{1}{x}\,dx} = e^{\ln(x)} = x$, so the formula gives

$$y = \frac{\int x\frac{e^x}{x}\,dx + C}{x} = \frac{\int e^x\,dx + C}{x} = \frac{e^x + C}{x}.$$

Here is one way to do this using Sage :

Sage

```
sage: t = var('t')
sage: x = function('x', t)
sage: de = lambda y: diff(y,t) + (1/t)*y - exp(t)/t
sage: desolve(de(x),[x,t])
(c + e^t)/t
```

"Integrating factor method": Let $\mu = e^{\int p(t)\,dt}$. Multiply both sides of (1.5) by μ:

$$\mu x' + p(t)\mu x = \mu q(t).$$

The product rule implies that

$$(\mu x)' = \mu x' + p(t)\mu x = \mu q(t).$$

(In response to a question you are probably thinking now: No, this is not obvious. This is Bernoulli's very clever idea.) Now just integrate both sides. By the fundamental theorem of calculus,

$$\mu x = \int (\mu x)'\,dt = \int \mu q(t)\,dt.$$

Dividing both side by μ gives (1.6).

Exercises: Find the general solution to the following seperable differential equations:

1. $x' = 2xt$.

2. $x' - x\sin(t) = 0$

3. $(1+t)x' = 2x$.

4. $x' - xt - x = 1 + t$.

   Solve the following linear equations. Find the general solution if no initial condition is given.

5. $x' + x = 1$, $x(0) = 0$.

6. $x' + 4x = 2te^{-4*t}$.

7. $tx' + 2x = 2t$, $x(1) = \frac{1}{2}$.

8. The function e^{-t^2} does not have an anti-derivative in terms of elementary functions, but this anti-derivative is important in probability. So we define a new function, $erf(t) := \frac{2}{\sqrt{\pi}} \int_0^u e^{-u^2} du$. Find the solution of $x' - 2xt = 1$ in terms of $erf(t)$.

9. (a) Use Sage 's `desolve` command to solve

$$tx' + 2x = e^t/t.$$

   (b) Use Sage to plot the solution to $y' = y^2 - 1$, $y(0) = 2$.

1.5 Isoclines and direction fields

Recall from vector calculus the notion of a two-dimensional vector field: $\vec{F}(x, y) = (g(x, y), h(x, y))$. To plot $\vec{F}$, you simply draw the vector $\vec{F}(x, y)$ at each point (x, y).

The idea of the **direction field** (or **slope field**) associated to the first order ODE

$$y' = f(x, y), \quad y(a) = c, \tag{1.7}$$

is similar. At each point (x, y) you plot a small vector having slope $f(x, y)$. For example, the vector field plot of $\vec{F}(x, y) = (1, f(x, y))$ or $\vec{F}(x, y) = (1, f(x, y))/\sqrt{1 + f(x, y)^2}$ (which is a unit vector).

How would you draw such a direction field plot *by hand*? You could compute the value of $f(x, y)$ for lots and lots of points (x, y) nd then plot a tiny arrow of slope $f(x, y)$ at each of these points. Unfortunately, this would be virtually impossible for a person to do if the number of points was large.

For this reason, the notion of the "isoclines" of the ODE is very useful. An **isocline** of (1.7) is a level curve of the function $z = f(x, y)$:

$$\{(x, y) \mid f(x, y) = m\},$$

where the given constant m is called the **slope** of the isocline. In terms of the ODE, this curve represents the collection of all points (x, y) at which the solution has slope m. In terms of the direction field of the ODE, it represents the collection of points where the vectors have slope m. This means that once you have draw a single isocline, you can sketch ten or more tiny vectors describing your direction field. Very useful indeed! This idea is recoded below more algorithmically.

How to draw the direction field of (1.7) by hand:

- Draw several isoclines, making sure to include one which contains the point (a, c). (You may want to draw these in pencil.)
- On each isocline, draw "hatch marks" or "arrows" along the line each having slope m.

This is a crude direction field plot. The plot of arrows form your direction field. The isoclines, having served their usefulness, can now safely be ignored.

Example 1.5.1. The direction field, with three isoclines, for

$$y' = 5x + y - 5, \quad y(0) = 1,$$

is given by the graph in Figure 1.9.

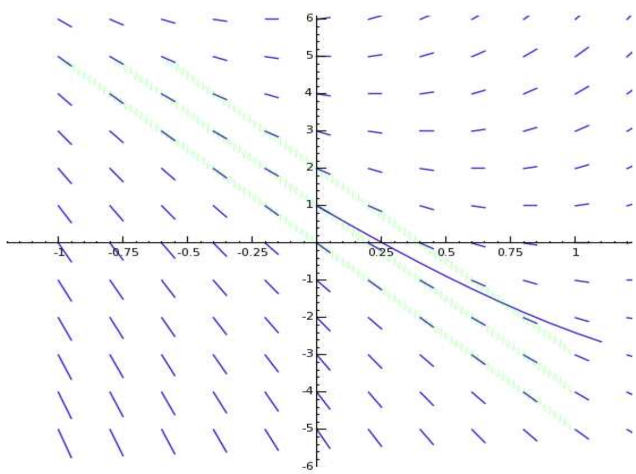


Figure 1.9: Plot of $y' = 5x + y - 5$, $y(0) = 1$, for $-1 < x < 1$.

The isoclines are the curves (coincidentally, lines) of the form $5x + y - 5 = m$. These are lines of slope -5, not to be confused with the fact that it represents an isocline of slope m.

The above example can be solved explicitly. (Indeed, $y = -5x + e^x$ solves $y' = 5x + y - 5$, $y(0) = 1$.) In the next example, such an explicit solution is not possible. Therefore, a numerical approximation plays a more important role.

Example 1.5.2. The direction field, with three isoclines, for

$$y' = x^2 + y^2, \quad y(0) = 3/2,$$

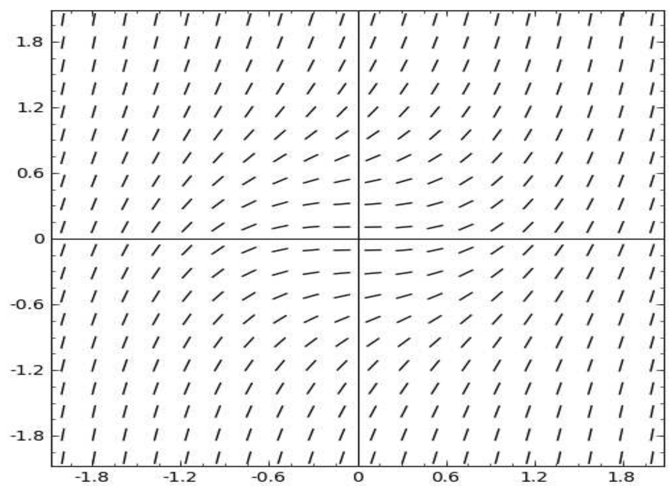


Figure 1.10: Direction field and solution plot of $y' = x^2 + y^2$, $y(0) = 3/2$, for $-2 < x < 2$.

is given by the in Figure 1.10.

The isoclines are the concentric circles $x^2 + y^2 = m$.

The plot in Figure 1.10 was obtaining using the Sage code below.

Sage

```
sage: x,y = var("x,y")
sage: f(x,y) = x^2 + y^2
sage: plot_slope_field(f(x,y), (x,-2,2),(y,-2,2)).show(aspect_ratio=1)
```

There is also a way to "manually draw" these direction fields using Sage .

Sage

```
sage: pts = [(-2+i/5,-2+j/5) for i in range(20) \
             for j in range(20)] # square [-2,2]x[-2,2]
sage: f = lambda p:p[0]^2+p[1]^2 # x = p[0] and y = p[1]
sage: arrows = [arrow(p, (p[0]+0.02,p[1]+(0.02)*f(p)), \
            width=1/100, rgbcolor=(0,0,1)) for p in pts]
sage: show(sum(arrows))
```

This gives the plot in Figure 1.11.

Exercises:

1. Match the solution curves in Figure 1.12 to the ODEs below.

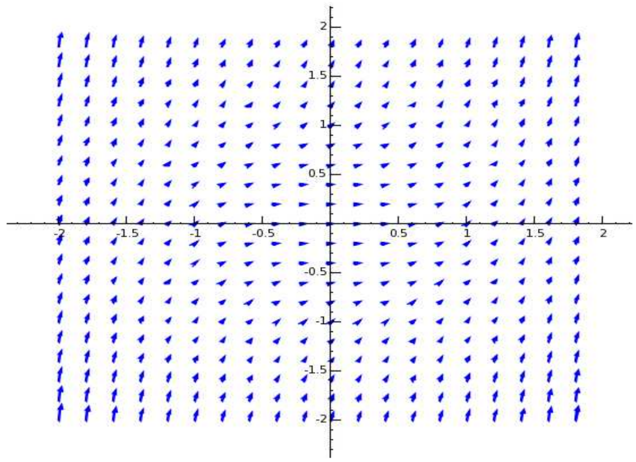


Figure 1.11: Direction field for $y' = x^2 + y^2$, $y(0) = 3/2$, for $-2 < x < 2$.

(a) $y' = y^2 - 1$

(b) $y' = \frac{y}{t^2-1}$

(c) $y' = \sin(t)\sin(y)$

(d) $y' = \sin(ty)$

(e) $y' = 2t + y$

(f) $y' = \sin(3t)$

2. Using Sage , plot the direction field for $y' = x^2 - y^2$.

1.6 Numerical solutions - Euler's method and improved Euler's method

Read Euler: he is our master in everything.

- *Pierre Simon de Laplace*

Leonhard Euler was a Swiss mathematician who made significant contributions to a wide range of mathematics and physics including calculus and celestial mechanics (see [Eu1-num] and [Eu2-num] for further details).

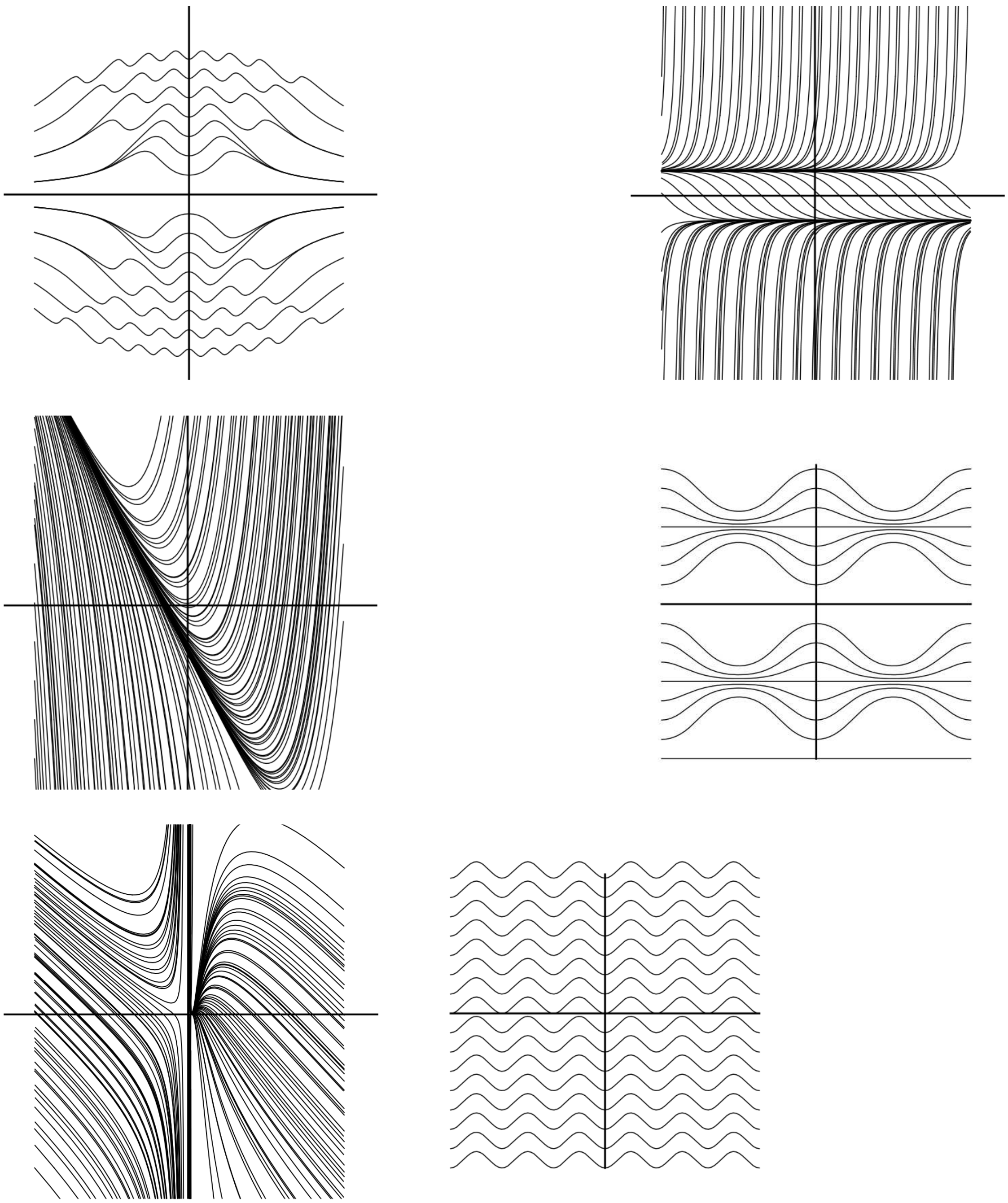


Figure 1.12: Solution curve plots for exercise 1

1.6.1 Euler's Method

The goal is to find an approximate solution to the problem

$$y' = f(x, y), \quad y(a) = c, \tag{1.8}$$

where $f(x, y)$ is some given function. We shall try to approximate the value of the solution at $x = b$, where $b > a$ is given. Sometimes such a method is called "numerically integrating (1.8)".

Note: the first order DE must be in the form (1.8) or the method described below does not work. A version of Euler's method for systems of 1-st order DEs and higher order DEs will also be described below.

Geometric idea: The basic idea can be easily expressed in geometric terms. We know the solution, whatever it is, must go through the point (a, c) and we know, at that point, its slope is $m = f(a, c)$. Using the point-slope form of a line, we conclude that the tangent line to the solution curve at (a, c) is (in (x, y)-coordinates, not to be confused with the dependent variable y and independent variable x of the DE)

$$y = c + (x - a)f(a, c).$$

In particular, if $h > 0$ is a given small number (called the **increment**) then taking $x = a+h$ the tangent-line approximation from calculus I gives us:

$$y(a + h) \cong c + h \cdot f(a, c).$$

Now we know the solution passes through a point which is "nearly" equal to $(a + h, c + h \cdot f(a, c)$. We now repeat this tangent-line approximation with (a, c) replaced by $(a + h, c + h \cdot f(a, c)$. Keep repeating this number-crunching at $x = a$, $x = a + h$, $x = a + 2h$, ..., until you get to $x = b$.

Algebraic idea: The basic idea can also be explained "algebraically". Recall from the definition of the derivative in calculus 1 that

$$y'(x) \cong \frac{y(x + h) - y(x)}{h},$$

$h > 0$ is a given and small. This an the DE together give $f(x, y(x)) \cong \frac{y(x+h)-y(x)}{h}$. Now solve for $y(x + h)$:

$$y(x + h) \cong y(x) + h \cdot f(x, y(x)).$$

If we call $h \cdot f(x, y(x))$ the "correction term" (for lack of anything better), call $y(x)$ the "old value of y", and call $y(x + h)$ the "new value of y", then this approximation can be re-expressed

$$y_{new} = y_{old} + h \cdot f(x, y_{old}).$$

Tabular idea: Let $n > 0$ be an integer, which we call the **step size**. This is related to the increment by

$$h = \frac{b-a}{n}.$$

This can be expressed simplest using a table.

x	y	$hf(x,y)$
a	c	$hf(a,c)$
$a+h$	$c+hf(a,c)$	$\vdots$
$a+2h$	$\vdots$	
$\vdots$		
b	???	xxx

The goal is to fill out all the blanks of the table but the xxx entry and find the ??? entry, which is the **Euler's method approximation for** $y(b)$.

Example 1.6.1. Use Euler's method with $h = 1/2$ to approximate $y(1)$, where

$$y' - y = 5x - 5, \quad y(0) = 1.$$

Putting the DE into the form (1.8), we see that here $f(x,y) = 5x + y - 5$, $a = 0$, $c = 1$.

x	y	$hf(x,y) = \frac{1}{2}(5x+y-5)$
0	1	-2
1/2	$1 + (-2) = -1$	$-7/4$
1	$-1 + (-7/4) = -11/4$	

so $y(1) \cong -\frac{11}{4} = -2.75$. This is the final answer.

Aside: For your information, $y = e^x - 5x$ solves the DE and $y(1) = e - 5 = -2.28....$

Here is one way to do this using Sage :

Sage

```
sage: x,y=PolynomialRing(QQ,2,"xy").gens()
sage: eulers_method(5*x+y-5,1,1,1/3,2)
         x                    y                  h*f(x,y)
         1                    1                     1/3
       4/3                  4/3                       1
       5/3                  7/3                    17/9
         2                 38/9                   83/27
sage: eulers_method(5*x+y-5,0,1,1/2,1,method="none")
[[0, 1], [1/2, -1], [1, -11/4], [3/2, -33/8]]
sage: pts = eulers_method(5*x+y-5,0,1,1/2,1,method="none")
sage: P = list_plot(pts)
sage: show(P)
sage: P = line(pts)
sage: show(P)
```

```
sage: P1 = list_plot(pts)
sage: P2 = line(pts)
sage: show(P1+P2)
```

The plot is given in Figure 1.13.

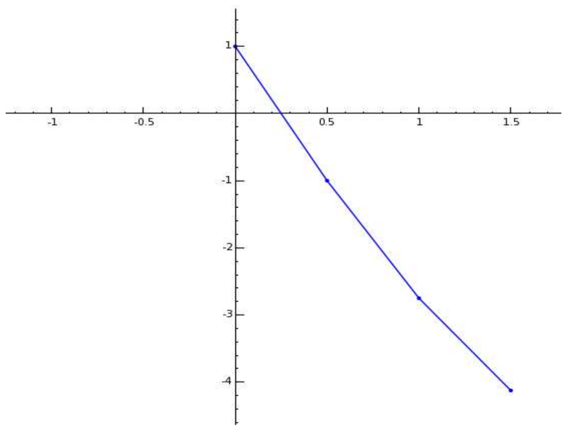


Figure 1.13: Euler's method with $h = 1/2$ for $x' + x = 1$, $x(0) = 2$.

1.6.2 Improved Euler's method

Geometric idea: The basic idea can be easily expressed in geometric terms. As in Euler's method, we know the solution must go through the point (a, c) and we know its slope there is $m = f(a, c)$. If we went out one step using the tangent line approximation to the solution curve, the approximate slope to the tangent line at $x = a + h, y = c + h \cdot f(a, c)$ would be $m' = f(a+h, c+h\cdot f(a, c))$. The idea is that instead of using $m = f(a, c)$ as the slope of the line to get our first approximation, use $\frac{m+m'}{2}$. The "improved" tangent-line approximation at (a, c) is:

$$y(a + h) \cong c + h \cdot \frac{m + m'}{2} = c + h \cdot \frac{f(a, c) + f(a + h, c + h \cdot f(a, c))}{2}.$$

(This turns out to be a better approximation than the tangent-line approximation $y(a+h) \cong c+h\cdot f(a, c)$ used in Euler's method.) Now we know the solution passes through a point which is "nearly" equal to $(a + h, c + h \cdot \frac{m+m'}{2})$. We now repeat this tangent-line approximation with (a, c) replaced by $(a+h, c+h\cdot f(a, c)$. Keep repeating this number-crunching at $x = a$, $x = a + h$, $x = a + 2h$, ..., until you get to $x = b$.

Tabular idea: The integer step size $n > 0$ is related to the increment by

$$h = \frac{b - a}{n},$$

as before.

The improved Euler method can be expressed simplest using a table.

x	y	$h\frac{m+m'}{2} = \frac{h}{2}(f(x,y)+f(x+h,y+h\cdot f(x,y)))$
a	c	$\frac{h}{2}(f(a,c)+f(a+h,c+h\cdot f(a,c)))$
$a+h$	$c+\frac{h}{2}(f(a,c)+f(a+h,c+h\cdot f(a,c)))$	$\vdots$
$a+2h$	$\vdots$	
$\vdots$		
b	???	xxx

The goal is to fill out all the blanks of the table but the xxx entry and find the ??? entry, which is the **improved Euler's method approximation for** $y(b)$.

Example 1.6.2. Use the improved Euler's method with $h = 1/2$ to approximate $y(1)$, where

$$y' - y = 5x - 5, \quad y(0) = 1.$$

Putting the DE into the form (1.8), we see that here $f(x,y) = 5x + y - 5$, $a = 0$, $c = 1$. We first compute the "correction term":

$$\begin{aligned} h\frac{f(x,y)+f(x+h,y+h\cdot f(x,y))}{2} &= \tfrac{1}{4}(5x+y-5+5(x+h)+(y+h\cdot f(x,y))-5) \\ &= \tfrac{1}{4}(5x+y-5+5(x+h)+(y+h\cdot(5x+y-5)-5) \\ &= (1+\tfrac{h}{2})5x+(1+\tfrac{h}{2})y-\tfrac{5}{2} \\ &= 25x/4+5y/4-5. \end{aligned}$$

x	y	$h\frac{m+m'}{2} = \frac{25x+5y-10}{4}$
0	1	$-15/8$
1/2	$1+(-15/8) = -7/8$	$-95/64$
1	$-7/8+(-95/64) = -151/64$	

so $y(1) \cong -\frac{151}{64} = -2.35...$ This is the final answer.

Aside: For your information, this is closer to the exact value $y(1) = e - 5 = -2.28...$ than the "usual" Euler's method approximation of -2.75 we obtained above.

1.6.3 Euler's method for systems and higher order DEs

We only sketch the idea in some simple cases. Consider the DE

$$y'' + p(x)y' + q(x)y = f(x), \quad y(a) = e_1, \quad y'(a) = e_2,$$

and the system

$$\begin{aligned} y_1' &= f_1(x, y_1, y_2), \quad y_1(a) = c_1, \\ y_2' &= f_2(x, y_1, y_2), \quad y_2(a) = c_2. \end{aligned}$$

We can treat both cases after first rewriting the DE as a system: create new variables $y_1 = y$ and let $y_2 = y'$. It is easy to see that

$$\begin{aligned} y_1' &= y_2, & y_1(a) &= e_1, \\ y_2' &= f(x) - q(x)y_1 - p(x)y_2, & y_2(a) &= e_2. \end{aligned}$$

Tabular idea: Let $n > 0$ be an integer, which we call the **step size**. This is related to the increment by

$$h = \frac{b - a}{n}.$$

This can be expressed simplest using a table.

x	y_1	$hf_1(x, y_1, y_2)$	y_2	$hf_2(x, y_1, y_2)$
a	e_1	$hf_1(a, e_1, e_2)$	e_2	$hf_2(a, e_1, e_2)$
$a + h$	$e_1 + hf_1(a, e_1, e_2)$	$\vdots$	$e_1 + hf_1(a, e_1, e_2)$	$\vdots$
$a + 2h$	$\vdots$			
$\vdots$				
b	???	xxx	xxx	xxx

The goal is to fill out all the blanks of the table but the xxx entry and find the ??? entries, which is the **Euler's method approximation for** $y(b)$.

Example 1.6.3. Using 3 steps of Euler's method, estimate $x(1)$, where $x'' - 3x' + 2x = 1$, $x(0) = 0$, $x'(0) = 1$

First, we rewrite $x'' - 3x' + 2x = 1$, $x(0) = 0$, $x'(0) = 1$, as a system of 1^{st} order DEs with ICs. Let $x_1 = x$, $x_2 = x'$, so

$$\begin{aligned} x_1' &= x_2, & x_1(0) &= 0, \\ x_2' &= 1 - 2x_1 + 3x_2, & x_2(0) &= 1. \end{aligned}$$

This is the DE rewritten as a system in standard form. (In general, the tabular method applies to any system but it must be in standard form.)

Taking $h = (1 - 0)/3 = 1/3$, we have

t	x_1	$x_2/3$	x_2	$(1 - 2x_1 + 3x_2)/3$
0	0	1/3	1	4/3
1/3	1/3	7/9	7/3	22/9
2/3	10/9	43/27	43/9	xxx
1	73/27	xxx	xxx	xxx

So $x(1) = x_1(1) \sim 73/27 = 2.7....$

Here is one way to do this using Sage :

Sage

```
sage: RR = RealField(sci_not=0, prec=4, rnd='RNDU')
sage: t, x, y = PolynomialRing(RR,3,"txy").gens()
sage: f = y; g = 1-2*x+3*y
sage: L = eulers_method_2x2(f,g,0,0,1,1/3,1,method="none")
sage: L
[[0, 0, 1], [1/3, 0.35, 2.5], [2/3, 1.3, 5.5],
 [1, 3.3, 12], [4/3, 8.0, 24]]
sage: eulers_method_2x2(f,g, 0, 0, 1, 1/3, 1)
 t        x         h*f(t,x,y)      y      h*g(t,x,y)
 0        0          0.35           1        1.4
 1/3      0.35       0.88           2.5      2.8
 2/3      1.3        2.0            5.5      6.5
 1        3.3        4.5            12       11
sage: P1 = list_plot([[p[0],p[1]] for p in L])
sage: P2 = line([[p[0],p[1]] for p in L])
sage: show(P1+P2)
```

The plot of the approximation to $x(t)$ is given in Figure 1.14.

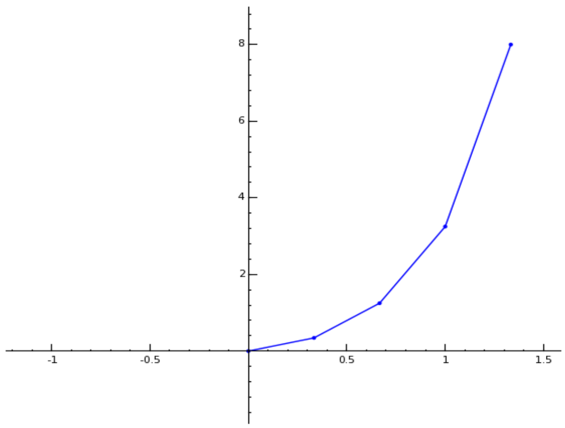


Figure 1.14: Euler's method with $h = 1/3$ for $x'' - 3x' + 2x = 1$, $x(0) = 0$, $x'(0) = 1$.

Exercise: Use Sage and Euler's method with $h = 1/3$ for the following problems:

1. (a) Use Euler's method to estimate $x(1)$ if $x(0) = 1$ and $\frac{dx}{dt} = x + t^2$, using 1,2, and 4 steps.
   (b) Find the exact value of $x(1)$ by solving the ODE (it is a linear ODE).
2. Find the approximate values of $x(1)$ and $y(1)$ where

$$\begin{cases} x' = x + y + t, & x(0) = 0, \\ y' = x - y, & y(0) = 0, \end{cases}$$

3. Find the approximate value of $x(1)$ where $x' = x^2 + t^2$, $x(0) = 1$.

1.7 Numerical solutions II - Runge-Kutta and other methods

The methods of 1.6 are sufficient for computing the solutions of many problems, but often we are given difficult cases that require more sophisticated methods. One class of methods are called the Runge-Kutta methods, particularly the fourth-order method of that class since it achieves a popular balance of efficiency and simplicity. Another class, the multistep methods, use information from some number m of previous steps. Within that class, we will briefly describe the Adams-Bashforth method. Finally, we will say a little bit about adaptive step sizes - i.e. changing h adaptively depending on some local estimate of the error.

1.7.1 Fourth-Order Runge Kutta method

To explain why we describe the method as "fourth-order" it is necessary to introduce a convenient (*big-O*) notation for the size of the errors in our approximations. We say that a function $f(x)$ is $O(g(x))$ as $x \to 0$ if there exist a positive constants M and x_0 such that for all $x \in [-x_0, x_0]$ we have $|f(x)| \leq M|g(x)|$. The constant M is called the *implied constant.* This notation is also commonly used for the case when $x \to \infty$ but in this text we will always be considering the behavior of $f(x)$ near $x = 0$. For example, $\sin(x) = O(x)$, as $x \to 0$, but $\sin(x) = O(1)$, as $x \to \infty$. More informally, a function is $O(g(x))$ if it approaches 0 at a rate equal to or faster than $g(x)$. As another example, the function $f(x) = 3x^2 + 6x^4$ is $O(x^2)$ (as $x \to 0$). As we approach $x = 0$, the higher order terms become less and less important, and eventually the $3x^2$ term dominates the value of the function. There are many concrete choices of x_0 and the implied constant M. One choice would be $x_0 = 1$ and $M = 9$.

For a numerical method we are interested in how fast the error decreases as we reduce the stepsize. By "the error" we mean the *global truncation error*: for the problem of approximating $y(x_f)$ if $y' = f(x, y)$ and $y(x_i) = y_i$ the global truncation error is defined as $E(h) = |y(x_f) - y_n|$. Here y_n is our approximate value for y at $x = x_f$ after taking n steps of stepsize $h = \frac{x_f - x_i}{n}$.

For Euler's method, $E(h) = O(h)$ and we say that it is a first-order method. This means that as we decrease the stepsize h, at some point our error will become linear in h. In other words, we expect that if we halve the stepsize our error will be reduced by half. The improved Euler method is a second-order method, so $E(h) = O(h^2)$. This is very good news, because while the improved Euler method involves roughly twice as much work per step as the Euler method, the error will eventually fall quadratically in h.

The **fourth-order Runge-Kutta method** involves computing four slopes and taking a weighted average of them. We denote these slopes as k_1, k_2, k_3, and k_4, and the formula are:

$$\begin{aligned} x_{n+1} &= x_n + h, \\ y_{n+1} &= y_n + h(k_1 + 2k_2 + 2k_3 + k_4)/6, \end{aligned}$$

where

$$k_1 = f(x_n, y_n),$$

$$k_2 = f(x_n + h/2, y_n + hk_1/2),$$

$$k_3 = f(x_n + h/2), y_n + hk_2/2),$$

and

$$k_4 = f(x_n + h, y_n + hk_3).$$

Example 1.7.1. Lets consider the IVP $y' = \frac{y(y-x)}{x(y+x)}$, $y(1) = 1$, and suppose we wish to approximate $y(2)$. The table below shows the Euler, improved Euler, and fourth-order Runge-Kutta (RK4) approximations for various numbers of steps from 1 to 512.

steps	Euler	imp. Euler	RK4
1	1.0	0.916666666667	0.878680484793
2	0.933333333333	0.889141488073	0.876938215214
4	0.90307164531	0.880183944727	0.876770226006
8	0.889320511452	0.877654079757	0.876757721415
16	0.882877913323	0.87698599324	0.87675688939
32	0.879775715551	0.876814710289	0.876756836198
64	0.878255683243	0.876771374145	0.876756832844
128	0.877503588678	0.876760476927	0.876756832634
256	0.877129540678	0.876757744807	0.876756832621
512	0.876943018826	0.876757060805	0.876756832620

The final Runge-Kutta value is correct to the number of digits shown. Note that even after 512 steps, Euler's method has not acheived the accuracy of 4 steps of Runge-Kutta or 8 steps of the improved Euler method.

1.7.2 Multistep methods - Adams-Bashforth

The **fourth-order Adams-Bashforth method** is:

$$x_{n+1} = x_n + h, \tag{1.9}$$

$$y_{n+1} = y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}), \tag{1.10}$$

where $f_i = f(x_i, y_i)$.

1.7.3 Adaptive step size

In our discussion of numerical methods we have only considered a fixed step size. In some applications this is sufficient but usually it is better to adaptively change the stepsize to keep the local error below some tolerance. One approach for doing this is to use two different methods for each step, and if the methods differ by more than the tolerance we decrease the stepsize. The Sage code below implements this for the improved Euler method and the fourth-order Runge-Kutta method.

Sage

```
def RK24(xstart, ystart, xfinish, f, nsteps = 10,  tol = 10^(-5.0)):
    '''
    Simple adaptive step-size routine. This compares the improved
```

```
    Euler method and the fourth-order Runge-Kutta method to
    estimate the error.

    EXAMPLE:
    The exact solution to this IVP is y(x) = exp(x), so y(1)
    should equal e = 2.718281828...
    Initially the stepsize is 1/10 but this is decreased during
    the calculation:
        sage: esol = RK24(0.0,1.0,1.0,lambda x,y: y)
        sage: print "Error is: ", N(esol[-1][1]-e)
        Error is:  -8.66619043193850e-9
    '''
    sol = [ystart]
    xvals = [xstart]
    h = N((xfinish-xstart)/nsteps)
    while xvals[-1] < xfinish:
        # Calculate slopes at various points:
        k1 = f(xvals[-1],sol[-1])
        rk2 = f(xvals[-1] + h/2,sol[-1] + k1*h/2)
        rk3 = f(xvals[-1] + h/2,sol[-1] + rk2*h/2)
        rk4 = f(xvals[-1] + h,sol[-1] + rk3*h)
        iek2 = f(xvals[-1] + h,sol[-1] + k1*h)
        # Runge-Kutta increment:
        rk_inc = h*(k1+2*rk2+2*rk3+rk4)/6
        # Improved Euler increment:
        ie_inc = h*(k1+iek2)/2
        #Check if the error is smaller than the tolerance:
        if abs(ie_inc - rk_inc) < tol:
            sol.append(sol[-1] + rk_inc)
            xvals.append(xvals[-1] + h)
        # If not, halve the stepsize and try again:
        else:
            h = h/2
    return zip(xvals,sol)
```

More sophisticated implementations will also increase the stepsize when the error stays small for several steps. A very popular scheme of this type is the Runge-Kutta-Fehlberg method, which combines fourth- and fifth-order Runge-Kutta methods in a particularly efficient way [A-ode].

Exercises:

1. For the initial value problem $y(1) = 1$ and $\frac{dy}{dx} = \frac{y}{x}\left(\frac{y-x}{x+y}\right)$, approximate $y(2)$ by using:

    (a) a 2-step improved Euler method

    (b) a 1-step 4th-order Runge-Kutta method.

2. Compare the results of using the fourth-order Runge-Kutta and Adams-Bashforth methods to approximate $y(3)$ for the IVP $\frac{dy}{dx} = sinc(x) - y$, $y(0) = 1$ for 5, 10, and 100 steps. Use the Runge-Kutta values to prime the Adams-Bashforth method.

3. Modify the adaptive stepsize code so that the stepsize is increased if the estimated error is below some threshold. Pick a problem and time your code compared to the original version.

4. Sometimes we wish to model quantities that are subjected to random influences, as in Brownian motion or in financial markets. If the random influences change on arbitrarily small timescales, the result is usually nondifferentiable. In these cases, it is better to express the evolution in terms of integrals, but they are often called *stochastic differential equations* nonetheless. One example from financial markets is a model of the value of a stock, which in integral form is:

$$S(t) = S_0 + \int_0^t \mu S \, ds + \int_0^t \sigma S \, dW,$$

where W is a Brownian motion. For such a model there is a stochastic analogue of Euler's method called the **Euler-Maruyama method** (see [H-sde] for more details and references). The main subtlety is correctly scaling the increment of the Brownian motion with the stepsize: $dW = \sqrt{dt} * w_{0,1}$, where $w(0,1)$ is a sample from a normal distribution with mean 0 and standard deviation 1. So for this example, the Euler-Maruyama method gives:

$$S_{i+1} = S_i + 2S_i h + \sqrt{h} S_i w_{0,1}.$$

Implement the Euler-Maruyama method to simulate an IVP of this example with $\mu = 2$ and $\sigma = 1$ from $t = 0$ to $t = 1$ with 100 steps, with $S(0) = S_0 = 0$. To generate the $w(0,1)$ in Sage you can use the `normalvariate` command. Compute the average trajectory of 100 simulation runs - is it equal to the deterministic ODE $S' = \mu S$? (This is a reasonable guess since the expected value of the Brownian motion W is 0.)

1.8 Newtonian mechanics

We briefly recall how the physics of the falling body problem leads naturally to a differential equation (this was already mentioned in the introduction and forms a part of Newtonian mechanics [M-mech]). Consider a mass m falling due to gravity. We orient coordinates to that downward is positive. Let $x(t)$ denote the distance the mass has fallen at time t and $v(t)$ its velocity at time t. We assume only two forces act: the force due to gravity, F_{grav}, and the force due to air resistence, F_{res}. In other words, we assume that the total force is given by

$$F_{total} = F_{grav} + F_{res}.$$

We know that $F_{grav} = mg$, where $g > 0$ is the gravitational constant, from high school physics. We assume, as is common in physics, that air resistance is proportional to velocity: $F_{res} = -kv = -kx'(t)$, where $k \geq 0$ is a constant. Newton's second law [N-mech] tells us that $F_{total} = ma = mx''(t)$. Putting these all together gives $mx''(t) = mg - kx'(t)$, or

$$v'(t) + \frac{k}{m}v(t) = g. \tag{1.11}$$

This is the differential equation governing the motion of a falling body. Equation (1.11) can be solved by various methods: separation of variables or by integrating factors. If we assume $v(0) = v_0$ is given and if we assume $k > 0$ then the solution is

$$v(t) = \frac{mg}{k} + (v_0 - \frac{mg}{k})e^{-kt/m}. \tag{1.12}$$

In particular, we see that the limiting velocity is $v_{limit} = \frac{mg}{k}$.

Example 1.8.1. Wile E. Coyote (see [W-mech] if you haven't seen him before) has mass 100 kgs (with chute). The chute is released 30 seconds after the jump from a height of 2000 m. The force due to air resistence is given by $\vec{F}_{res} = -k\vec{v}$, where

$$k = \begin{cases} 15, & \text{chute closed,} \\ 100, & \text{chute open.} \end{cases}$$

Find

(a) the distance and velocity functions during the time when the chute is closed (i.e., $0 \leq t \leq 30$ seconds),

(b) the distance and velocity functions during the time when the chute is open (i.e., $30 \leq t$ seconds),

(c) the time of landing,

(d) the velocity of landing. (Does Wile E. Coyote survive the impact?)

soln: Taking $m = 100$, $g = 9.8$, $k = 15$ and $v(0) = 0$ in (1.12), we find

$$v_1(t) = \frac{196}{3} - \frac{196}{3} e^{-\frac{3}{20}t}.$$

This is the velocity with the time t starting the moment the parachutist jumps. After $t = 30$ seconds, this reaches the velocity $v_0 = \frac{196}{3} - \frac{196}{3}e^{-9/2} = 64.607....$ The distance fallen is

$$\begin{aligned} x_1(t) &= \int_0^t v_1(u)\, du \\ &= \tfrac{196}{3}t + \tfrac{3920}{9}e^{-\frac{3}{20}t} - \tfrac{3920}{9}. \end{aligned}$$

After 30 seconds, it has fallen $x_1(30) = \frac{13720}{9} + \frac{3920}{9}e^{-9/2} = 1529.283...$ meters.

Taking $m = 100$, $g = 9.8$, $k = 100$ and $v(0) = v_0$, we find

$$v_2(t) = \frac{49}{5} + e^{-t}\left(\frac{833}{15} - \frac{196}{3}\,e^{-9/2}\right).$$

This is the velocity with the time t starting the moment Wile E. Coyote opens his chute (i.e., 30 seconds after jumping). The distance fallen is

$$\begin{aligned} x_2(t) &= \int_0^t v_2(u)\,du + x_1(30) \\ &= \tfrac{49}{5}\,t - \tfrac{833}{15}\,e^{-t} + \tfrac{196}{3}\,e^{-t}e^{-9/2} + \tfrac{71099}{45} + \tfrac{3332}{9}\,e^{-9/2}. \end{aligned}$$

Now let us solve this using Sage .

Sage

```
sage: RR = RealField(sci_not=0, prec=50, rnd='RNDU')
sage: t = var('t')
sage: v = function('v', t)
sage: m = 100; g = 98/10; k = 15
sage: de = lambda v: m*diff(v,t) + k*v - m*g
sage: desolve(de(v),[v,t],[0,0])
196/3*(e^(3/20*t) - 1)*e^(-3/20*t)
sage: soln1 = lambda t: 196/3-196*exp(-3*t/20)/3
sage: P1 = plot(soln1(t),0,30,plot_points=1000)
sage: RR(soln1(30))
64.607545559502
```

This solves for the velocity before the coyote's chute is opened, $0 < t < 30$. The last number is the velocity Wile E. Coyote is traveling at the moment he opens his chute.

Sage

```
sage: t = var('t')
sage: v = function('v', t)
sage: m = 100; g = 98/10; k = 100
sage: de = lambda v: m*diff(v,t) + k*v - m*g
sage: desolve(de(v),[v,t],[0,RR(soln1(30))])
1/10470*(102606*e^t + 573835)*e^(-t)
sage: soln2 = lambda t: 49/5+(631931/11530)*exp(-(t-30))\
              + soln1(30) - (631931/11530) - 49/5
sage: RR(soln2(30))
64.607545559502
sage: RR(soln1(30))
64.607545559502
sage: P2 = plot(soln2(t),30,50,plot_points=1000)
sage: show(P1+P2)
```

This solves for the velocity after the coyote's chute is opened, $t > 30$. The last command plots the velocity functions together as a single plot. (You would see a break in the graph if you omitted the Sage 's `plot` option `,plot_points=1000`. That is because the number of samples taken of the function by default is not sufficient to capture the jump the function takes at $t = 30$.) The terms at the end of `soln2` were added to insure $x_2(30) = x_1(30)$.

Next, we find the distance traveled at time t:

Sage

```
sage: integral(soln1(t),t)
3920*e^(-(3*t/20))/9 + 196*t/3
sage: x1 = lambda t: 3920*e^(-(3*t/20))/9 + 196*t/3
sage: RR(x1(30))
1964.8385851589
```

This solves for the distance the coyote traveled before the chute was open, $0 < t < 30$. The last number says that he has gone about 1965 meters when he opens his chute.

Sage

```
sage: integral(soln2(t),t)
49*t/5 - (631931*e^(30 - t)/11530)
sage: x2 = lambda t: 49*t/5 - (631931*e^(30 - t)/11530)
          + x1(30) + (631931/11530) - 49*30/5
sage: RR(x2(30.7))
1999.2895090436
sage: P4 = plot(x2(t),30,50)
sage: show(P3+P4)
```

(Again, you see a break in the graph because of the round-off error.) The terms at the end of `x2` were added to insure $x_2(30) = x_1(30)$. You know he is close to the ground at $t = 30$, and going quite fast (about 65 m/s!). It makes sense that he will hit the ground soon afterwards (with a large puff of smoke, if you've seen the cartoons), even though his chute will have slowed him down somewhat.

The graph of the velocity $0 < t < 50$ is in Figure 1.15. Notice how it drops at $t = 30$ when the chute is opened. The graph of the distance fallen $0 < t < 50$ is in Figure 1.16. Notice how it slows down at $t = 30$ when the chute is opened.

The time of impact is $t_{impact} = 30.7....$ This was found numerically by solving $x_2(t) = 2000$.

The velocity of impact is $v_2(t_{impact}) \approx 37$ m/s.

Exercise: Drop an object with mass 10 kgs from a height of 2000 m. Suppose the force due to air resistence is given by $\vec{F}_{res} = -10\vec{v}$. Find the velocity after 10 seconds using Sage . Plot this velocity function for $0 < t < 10$.

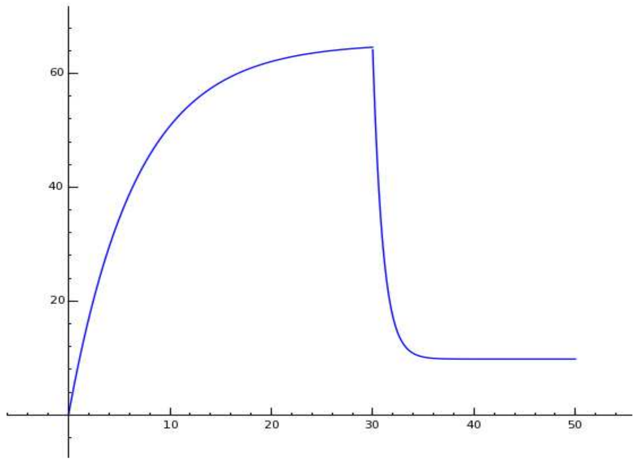


Figure 1.15: Velocity of falling parachutist.

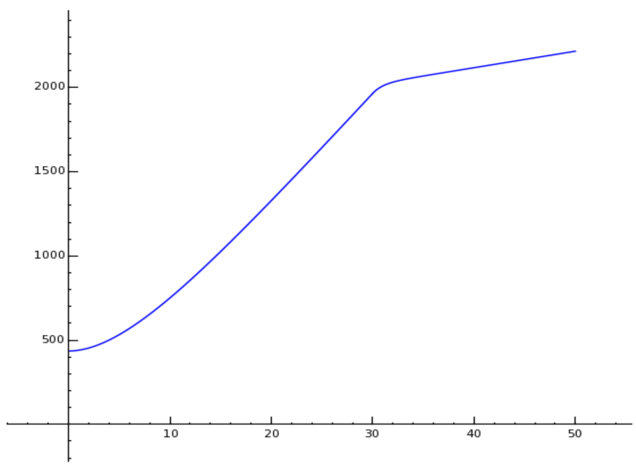


Figure 1.16: Distance fallen by a parachutist.

1.9 Application to mixing problems

Suppose that we have two chemical substances where one is soluable in the other, such as salt and water. Suppose that we have a tank containing a mixture of these substances, and the mixture of them is poured in and the resulting "well-mixed" solution pours out through a value at the bottom. (The term "well-mixed" is used to indicate that the fluid being poured in is assumed to instantly dissolve into a homogeneous mixture the moment it goes into the tank.) The rough idea is depicted in Figure 1.17.

Assume for concreteness that the chemical substances are salt and water. Let

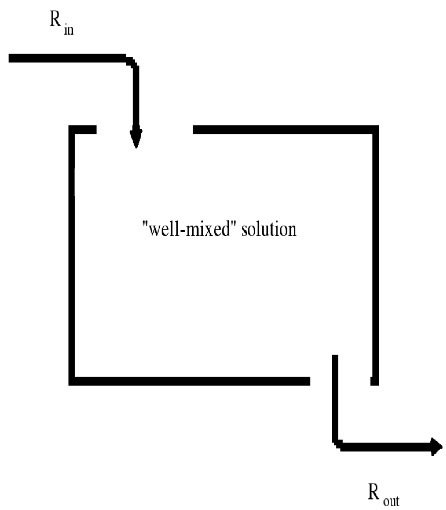


Figure 1.17: Solution pours into a tank, mixes with another type of solution. and then pours out.

- $A(t)$ denote the amount of salt at time t,
- FlowRateIn = the rate at which the solution pours into the tank,
- FlowRateOut = the rate at which the mixture pours out of the tank,
- C_{in} = "concentration in" = the concentration of salt in the solution being poured into the tank,
- C_{out} = "concentration out" = the concentration of salt in the solution being poured out of the tank,
- R_{in} = rate at which the salt is being poured into the tank = (FlowRateIn)(C_{in}),
- R_{out} = rate at which the salt is being poured out of the tank = (FlowRateOut)(C_{out}).

Remark 1.9.1. *Some things to make note of:*

- *If FlowRateIn = FlowRateOut then the "water level" of the tank stays the same.*
- *We can determine C_{out} as a function of other quantities:*

$$C_{out} = \frac{A(t)}{T(t)},$$

where $T(t)$ denotes the volume of solution in the tank at time t.

- *The rate of change of the amount of salt in the tank, $A'(t)$, more properly could be called the "net rate of change". If you think if it this way then you see $A'(t) = R_{in} - R_{out}$.*

Now the differential equation for the amount of salt arises from the above equations:

$$A'(t) = (\text{FlowRateIn})C_{in} - (\text{FlowRateOut})\frac{A(t)}{T(t)}.$$

Example 1.9.1. Consider a tank with 200 liters of salt-water solution, 30 grams of which is salt. Pouring into the tank is a brine solution at a rate of 4 liters/minute and with a concentration of 1 grams per liter. The "well-mixed" solution pours out at a rate of 5 liters/minute. Find the amount at time t.

We know

$$A'(t) = (\text{FlowRateIn})C_{in} - (\text{FlowRateOut})\frac{A(t)}{T(t)} = 4 - 5\frac{A(t)}{200-t}, \quad A(0) = 30.$$

Writing this in the standard form $A' + pA = q$, we have

$$A(t) = \frac{\int \mu(t)q(t)\,dt + C}{\mu(t)},$$

where $\mu = e^{\int p(t)\,dt} = e^{-5\int \frac{1}{200-t}\,dt} = (200-t)^{-5}$ is the "integrating factor". This gives $A(t) = 200 - t + C \cdot (200-t)^5$, where the initial condition implies $C = -170 \cdot 200^{-5}$.

Here is one way to do this using Sage :

Sage

```
sage: t = var('t')
sage: A = function('A', t)
sage: de = lambda A: diff(A,t) + (5/(200-t))*A - 4
sage: desolve(de(A),[A,t])
(t - 200)^5*(c - 1/(t - 200)^4)
```

This is the form of the general solution. (Sage uses Maxima and %c is Maxima's notation for an arbitrary constant.) Let us now solve this general solution for c, using the initial conditions.

Sage

```
sage: c,t = var('c,t')
sage: tank = lambda t: 200-t
sage: solnA = lambda t: (c + 1/tank(t)^4)*tank(t)^5
sage: solnA(t)
(c - (1/(t - 200)^4))*(t - 200)^5
sage: solnA(0)
```

```
-320000000000*(c - 1/1600000000)
sage: solve([solnA(0) == 30],c)
[c == 17/32000000000]
sage: c = 17/32000000000
sage: solnA(t)
(17/32000000000 - (1/(t - 200)^4))*(t - 200)^5
sage: P = plot(solnA(t),0,200)
sage: show(P)
```

This plot is given in Figure 1.18.

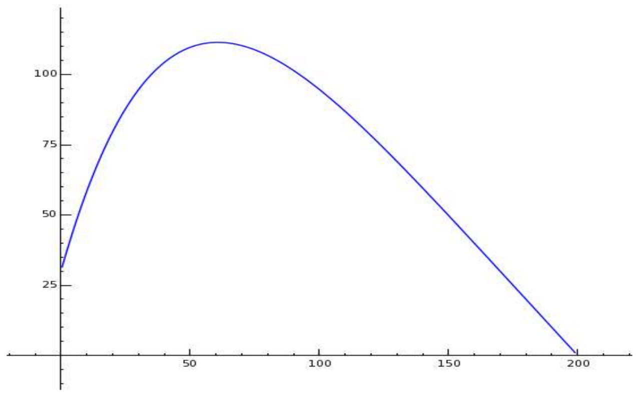


Figure 1.18: $A(t)$, $0 < t < 200$, $A' = 4 - 5A(t)/(200 - t)$, $A(0) = 30$.

Exercises:

1. Now use Sage to solve the same problem but with the same flow rate out as 4 liters/min (note: in this case, the "water level" in the tank is constant). Find and plot the solution $A(t)$, $0 < t < 200$.

2. Consider two tanks that are linked in a cascade - i.e. the first tank empties into the second. Suppose the first tank has 100 liters of water in it, and the second has 300 liters of water. Each tank initially has 50 kilograms of salt dissolved in the water. Suppose that pure water flows into the first tank at 5 liters per minute, well-mixed water flows from the first tank to the second at the same rate (5 liters/minute), and well-mixed water also flows out of the second tank at 5 liters/minute.

   (a) Find the amount of salt in the first tank $x_1(t)$. Note that this does not depend on what is happening in the second tank.
   (b) Find the amount of salt in the second tank $x_2(t)$.
   (c) Find the time when there is the maximum amount of salt in the second tank.

Chapter 2

Second order differential equations

> If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.
> - *John von Neumann*

2.1 Linear differential equations

To begin, we want to describe the general form a solution to a linear ODE can take. We want to describe the solution as a sum of terms which can be computed explicitly in some way.

Before doing this, we introduce two pieces of terminology.

- Suppose $f_1(t)$, $f_2(t)$, $\ldots$, $f_n(t)$ are given functions. A **linear combination** of these functions is another function of the form

  $$c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t),$$

  for some constants c_1, ..., c_n. For example, $3\cos(t) - 2\sin(t)$ is a linear combination of $\cos(t)$, $\sin(t)$. An *arbitrary* linear combination of $\cos(t)$, $\sin(t)$ would be written as $c_1\cos(t) + c_2\sin(t)$.

- A linear ODE of the form

  $$y^{(n)} + b_1(t)y^{(n-1)} + ... + b_{n-1}(t)y' + b_n(t)y = f(t), \tag{2.1}$$

  is called **homogeneous** if $f(t) = 0$ (i.e., f is the 0 function) and otherwise it is called **non-homogeneous**.

Consider the n-th order ODE

$$y^{(n)} + b_1(t)y^{(n-1)} + ... + b_{n-1}(t)y' + b_n(t)y = 0. \tag{2.2}$$

Suppose there are n functions $y_1(t)$, $\ldots$, $y_n(t)$ such that

- each $y = y_i(t)$ $(1 \le i \le n)$ satisfies this homogeneous ODE (2.2),
- *every* solution y to (2.2) is a linear combination of these functions $y_1, \ldots, y_n$:

  $$y = c_1y_1 + \cdots + c_ny_n,$$

  for some (unique) constants c_1, ... , c_n.

In this case, the y_i's are called **fundamental solutions**.

Remark 2.1.1. *If you are worried that this definition is not very practical, then don't. We shall give a condition later (the "Wronskian test") which will make it much easier to see if a collection of n functions form a set of fundamental solutions.*

The following result describes the general solution to a linear ODE.

Theorem 2.1.1. Consider a linear ODE of the above form (2.1), for some given continuous functions $b_1(t)$, $\ldots$, $b_n(t)$, and $f(t)$. Then the following hold.

- There are n functions $y_1(t)$, $\ldots$, $y_n(t)$ (above-mentioned fundamental solutions), each $y = y_i(t)$ $(1 \le i \le n)$ satisfying the homogeneous ODE, such that every solution y_h to (2.2) can be written

  $$y_h = c_1y_1 + \cdots + c_ny_n,$$

  for some (unique) constants c_1, ... , c_n.

- Suppose you know a solution $y_p(t)$ (a **particular solution**) to (2.1). Then every solution $y = y(t)$ (the **general solution**) to the DE (2.1) has the form

  $$y(t) = y_h(t) + y_p(t), \tag{2.3}$$

  where y_h (the "homogeneous part" of the general solution) is a linear combination

  $$y_h(t) = c_1y_1(t) + y_2(t) + ... + c_ny_n(t),$$

  for some constants c_i, $1 \le i \le n$.

- Conversely, every function of the form (2.3), for any constants c_i for $1 \le i \le n$, is a solution to (2.1).

Example 2.1.1. Recall Example 1.1.4 in the introduction where we looked for functions solving $x' + x = 1$ by "guessing". We found that the function $x_p(t) = 1$ is a particular solution to $x' + x = 1$. The function $x_1(t) = e^{-t}$ is a fundamental solution to $x' + x = 0$. The general solution is therefore $x(t) = 1 + c_1 e^{-t}$, for a constant c_1.

Example 2.1.2. Let's look for functions solving $x'' - x = 1$ by "guessing". Motivated by the above example, we find that the function $x_p(t) = -1$ is a particular solution to $x'' - x = 1$. The functions $x_1(t) = e^t$, $x_2(t) = e^{-t}$ are fundamental solutions to $x'' - x = 0$. The general solution is therefore $x(t) = 1 + c_1 e^{-t}$, for a constant c_1.

Example 2.1.3. The charge on the capacitor of an RLC electrical circuit is modeled by a 2-nd order linear DE [C-linear].

Series RLC Circuit notations:

- $E = E(t)$ - the voltage of the power source (a battery or other "electromotive force", measured in volts, V)
- $q = q(t)$ - the current in the circuit (measured in coulombs, C)
- $i = i(t)$ - the current in the circuit (measured in amperes, A)
- L - the inductance of the inductor (measured in henrys, H)
- R - the resistance of the resistor (measured in ohms, Ω);
- C - the capacitance of the capacitor (measured in farads, F)

The charge q on the capacitor satisfies the linear IPV:

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

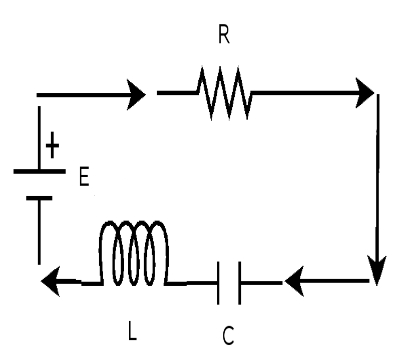


Figure 2.1: RLC circuit.

Example 2.1.4. The displacement from equilibrium of a mass attached to a spring suspended from a ceiling as in Figure 2.2 is modeled by a 2-nd order linear DE [O-ivp].

Spring-mass notations:

- $f(t)$ - the external force acting on the spring (if any)
- $x = x(t)$ - the displacement from equilibrium of a mass attached to a spring
- m - the mass
- b - the damping constant (if, say, the spring is immersed in a fluid)
- k - the spring constant.

The displacement x satisfies the linear IPV:

$$mx'' + bx' + kx = f(t), \quad x(0) = x_0, \quad x'(0) = v_0.$$

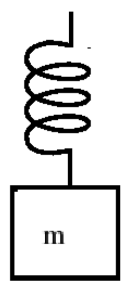


Figure 2.2: spring-mass model.

Notice that each general solution to an n-th order ODE has n "degrees of freedom" (the arbitrary constants c_i). According to this theorem, to find the general solution of a linear ODE, we need only find a particular solution y_p and n fundamental solutions $y_1(t)$, ..., $y_n(t)$.

Example 2.1.5. Let us try to solve

$$x' + x = 1, \quad x(0) = c,$$

where $c = 1$, $c = 2$, and $c = 3$. (Three different IVP's, three different solutions, find each one.)

The first problem, $x' + x = 1$ and $x(0) = 1$, is easy. The solutions to the DE $x' + x = 1$ which we "guessed at" in the previous example, $x(t) = 1$, satisfies this.

The second problem, $x' + x = 1$ and $x(0) = 2$, is not so simple. To solve this (and the third problem), we really need to know what the form is of the "general solution".

According to the theorem above, the general solution x has the form $x = x_p + x_h$. In this case, $x_p(t) = 1$ and $x_h(t) = c_1 x_1(t) = c_1 e^{-t}$, by an earlier example. Therefore, every solution to the DE above is of the form $x(t) = 1 + c_1 e^{-t}$, for some constant c_1. We use the initial condition to solve for c_1:

- $x(0) = 1$: $1 = x(0) = 1 + c_1 e^0 = 1 + c_1$ so $c_1 = 0$ and $x(t) = 1$.
- $x(0) = 2$: $2 = x(0) = 1 + c_1 e^0 = 1 + c_1$ so $c_1 = 1$ and $x(t) = 1 + e^{-t}$.
- $x(0) = 3$: $3 = x(0) = 1 + c_1 e^0 = 1 + c_1$ so $c_1 = 2$ and $x(t) = 1 + 2e^{-t}$.

Here is one way to use Sage to solve for c_1. (Of course, you can do this yourself, but this shows you the Sage syntax for solving equations. Type `solve?` in Sage to get more details.) We use Sage to solve the last IVP discussed above and then to plot the solution.

Sage

```
sage: t = var('t')
sage: x = function('x',t)
sage: desolve(diff(x,t)+x==1,[x,t])
(c + e^t)*e^(-t)
sage: c = var('c')
sage: solnx = lambda t: 1+c*exp(-t)  # the soln from desolve
sage: solnx(0)
c + 1
sage: solve([solnx(0) == 3],c)
[c == 2]
sage: c0 = solve([solnx(0) == 3], c)[0].rhs()
sage: solnx1 = lambda t: 1+c0*exp(-t); solnx1(t)
sage: P = plot(solnx1(t),0,5)
sage: show(P)
```

This plot is shown in Figure 2.3.

Exercise: Use Sage to solve and plot the solution to $x' + x = 1$ and $x(0) = 2$.

2.2 Linear differential equations, continued

To better describe the form a solution to a linear ODE can take, we need to better understand the nature of fundamental solutions and particular solutions.

Recall that the general solution to

$$y^{(n)} + b_1(t)y^{(n-1)} + ... + b_{n-1}(t)y' + b_n(t)y = f(t),$$

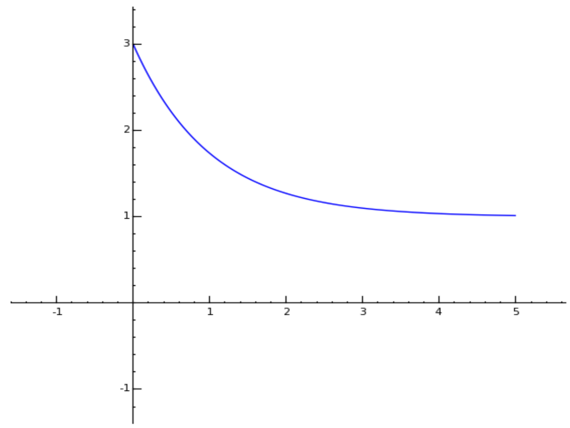


Figure 2.3: Solution to IVP $x' + x = 1$, $x(0) = 3$.

has the form $y = y_p + y_h$, where y_h is a linear combination of fundamental solutions.

Example 2.2.1. The general solution to the differential equation equation

$$x'' - 5x' + 6x = 0$$

has the form $x = x(t) = c_1 \exp(2t) + c_2 \exp(3t)$, for arbitrary constants c_1 and c_2. Suppose we are also given n initial conditions $y(x_0) = a_0$, $y'(x_0) = a_1$, ..., $y^{(n-1)}(x_0) = a_{n-1}$. For example, we could impose the initial conditions: $x(0) = 3$ and $x'(0) = 4$. Of course, no matter what x_0 and v_0 are are given, we want to be able to solve for the coefficients c_1, c_2 in $x(t) = c_1 \exp(2t) + c_2 \exp(3t)$ to obtain a unique solution. More generally, we want to be able to solve an n-th order IVP and obtain a unique solution.

A few questions arise.

- How do we know this can be done?
- How do we know that (for some c_1, c_2) some linear combination $x(t) = c_1 \exp(2t) + c_2 \exp(3t)$ isn't identically 0 (which, if true, would imply that $x = x(t)$ couldn't possibly satisfy $x(0) = 3$ and $x'(0) = 4$)?

We shall answer this question below.

The complete answer to the questions mentioned in the above example actually involves methods from linear algebra which go beyond this course. The basic idea though is not hard to understand and it involves what is called "the Wronskian[1]" [W-linear].

Before we motivate the idea of the Wronskian by returning to the above example, we need to recall a basic fact from linear algebra.

[1] Josef Wronski was a Polish-born French mathematician who worked in many different areas of applied mathematics and mechanical engineering [Wr-linear].

Lemma 2.2.1. (Cramer's rule) Consider the system of two equations in two unknowns x, y:

$$ax + by = s_1, \qquad cx + dy = s_2.$$

The solution to this system is

$$x = \frac{\det\begin{pmatrix} s_1 & b \\ s_2 & d \end{pmatrix}}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}}, \qquad y = \frac{\det\begin{pmatrix} a & s_1 \\ c & s_2 \end{pmatrix}}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}}.$$

Note the determinant $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ is in the denominator of both expressions. In particular, if the determinant is 0 then the formula is invalid (and in that case, the solution either does not exist or is not unique).

Example 2.2.2. Write the general solution to $x'' - 5x' + 6x = 0$ as $x(t) = c_1x_1(t) + c_2x_2(t)$ (we know $x_1(t) = \exp(2t)$, $x_2(t) = \exp(3t)$, but we leave it in this more abstract notation to make a point). Assume the initial conditions $x(0) = 3$ and $x'(0) = 4$ hold. We can try solve for c_1, c_2 but plugging $t = 0$ into the general solution:

$$3 = x(0) = c_1e^0 + c_2e^0 = c_1 + c_2, \qquad 4 = x'(0) = c_12e^0 + c_23e^0 = 2c_1 + 3c_2.$$

You can solve these "by hand" for c_1, c_2 (and you encouraged to do so). However, to motivate Wronskian's we shall use the initial conditions in the more abstract form of the general solution:

$$3 = x(0) = c_1x_1(0) + c_2x_2(0), \qquad 4 = x'(0) = c_1x_1'(0) + c_2x_2'(0).$$

Cramers' rule gives us the solution for this system of two equations in two unknowns c_1, c_2:

$$c_1 = \frac{\det\begin{pmatrix} 3 & x_2(0) \\ 4 & x_2'(0) \end{pmatrix}}{\det\begin{pmatrix} x_1(0) & x_2(0) \\ x_1'(0) & x_2'(0) \end{pmatrix}}, \qquad y = \frac{\det\begin{pmatrix} x_1(0) & 3 \\ x_1'(0) & 4 \end{pmatrix}}{\det\begin{pmatrix} x_1(0) & x_2(0) \\ x_1'(0) & x_2'(0) \end{pmatrix}}.$$

In the denominator of these expressions, you see the "Wronskian" of the fundamental solutions x_1, x_2 evaluated at $t = 0$.

From the example above we see "Wronskians" arise "naturally" in the process of solving for c_1 and c_2.

In general terms, what is a Wronskian? It is best to explain what this means not just for two functions (say, fundamental solutions x_1, x_2 of a second-order ODE, as we did above) but for any finite number of functions. This more general case would be useful in case we wanted to try to solve a higher order ODE by the same method. If $f_1(t)$, $f_2(t)$, $\dots$, $f_n(t)$

are given n-times differentiable functions then their **fundamental matrix** is the $n \times n$ matrix

$$\Phi = \Phi(f_1, ..., f_n) = \begin{pmatrix} f_1(t) & f_2(t) & \dots & f_n(t) \\ f_1'(t) & f_2'(t) & \dots & f_n'(t) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \dots & f_n^{(n-1)}(t) \end{pmatrix}.$$

The determinant of the fundamental matrix is called the **Wronskian**, denoted $W(f_1, ..., f_n)$. The Wronskian actually helps us answer both questions above simultaneously.

Example 2.2.3. Take $f_1(t) = \sin^2(t)$, $f_2(t) = \cos^2(t)$, and $f_3(t) = 1$. Sage allows us to easily compute the Wronskian:

Sage

```
sage: t = var('t')
sage: Phi = matrix([[sin(t)^2,cos(t)^2,1],
                [diff(sin(t)^2,t),diff(cos(t)^2,t),0],
                [diff(sin(t)^2,t,t),diff(cos(t)^2,t,t),0]])
sage: Phi

[                 sin(t)^2                 cos(t)^2                        1]
[          2*sin(t)*cos(t)         -2*sin(t)*cos(t)                        0]
[-2*sin(t)^2 + 2*cos(t)^2  2*sin(t)^2 - 2*cos(t)^2                        0]
sage: det(Phi)
0
```

Here `det(Phi)` is the determinant of the fundamental matrix `Phi`. Since it is zero, this means

$$W(sin(t)^2, cos(t)^2, 1) = 0.$$

Let's try another example using Sage .

Sage

```
sage: t = var('t')
sage: Phi = matrix([[sin(t)^2,cos(t)^2],  [diff(sin(t)^2,t),diff(cos(t)^2,t)]])
sage: Phi

[       sin(t)^2        cos(t)^2]
[ 2*cos(t)*sin(t) -2*cos(t)*sin(t)]
sage: Phi.det()
-2*cos(t)*sin(t)^3 - 2*cos(t)^3*sin(t)
```

This means $W(sin(t)^2, cos(t)^2) = -2\cos(t)\sin(t)^3 - 2\cos(t)^3\sin(t)$, which is non-zero.

If there are constants c_1, ..., c_n, not all zero, for which

$$c_1 f_1(t) + c_2 f_2(t) \cdots + c_n f_n(t) = 0, \quad \text{for all } t, \tag{2.4}$$

then the functions f_i $(1 \leq i \leq n)$ are called **linearly dependent**. If the functions f_i $(1 \leq i \leq n)$ are not linearly dependent then they are called **linearly independent** (this definition is frequently seen for linearly independent vectors [L-linear] but holds for functions as well). This condition (2.4) can be interpreted geometrically as follows. Just as $c_1x+c_2y = 0$ is a line through the origin in the plane and $c_1x + c_2y + c_3z = 0$ is a plane containing the origin in 3-space, the equation

$$c_1x_1 + c_2x_2 \cdots + c_nx_n = 0,$$

is a "hyperplane" containing the origin in n-space with coordinates $(x_1, ..., x_n)$. This condition (2.4) says geometrically that the graph of the space curve $\vec{r}(t) = (f_1(t), \ldots, f_n(t))$ lies entirely in this hyperplane. If you pick n functions "at random" then they are "probably" linearly independent, because "random" space curves don't lie in a hyperplane. But certainly not all collections of functions are linearly independent.

Example 2.2.4. Consider just the two functions $f_1(t) = \sin^2(t)$, $f_2(t) = \cos^2(t)$. We know from the Sage computation in the example above that these functions are linearly independent.

Sage

```
sage: P = parametric_plot((sin(t)^2,cos(t)^2),0,5)
sage: show(P)
```

The Sage plot of this space curve $\vec{r}(t) = (\sin(t)^2, \cos(t)^2)$ is given in Figure 2.4. It is obviously not contained in a line through the origin, therefore making it geometrically clear that these functions are linearly independent.

The following two results answer the above questions.

Theorem 2.2.1. *(***Wronskian test***) If $f_1(t)$, $f_2(t)$, $\ldots$, $f_n(t)$ are given n-times differentiable functions with a non-zero Wronskian then they are linearly independent.*

As a consequence of this theorem, and the Sage computation in the example above, $f_1(t) = \sin^2(t)$, $f_2(t) = \cos^2(t)$, are linearly independent.

Theorem 2.2.2. *Given any homogeneous n-th linear ODE*

$$y^{(n)} + b_1(t)y^{(n-1)} + ... + b_{n-1}(t)y' + b_n(t)y = 0,$$

with differentiable coefficients, there always exists n solutions $y_1(t)$, ..., $y_n(t)$ which have a non-zero Wronskian.

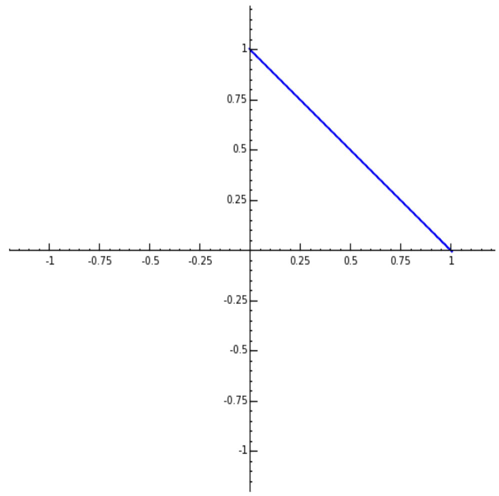


Figure 2.4: Parametric plot of $(\sin(t)^2, \cos(t)^2)$.

The functions $y_1(t)$, ..., $y_n(t)$ in the above theorem are called **fundamental solutions**. We shall not prove either of these theorems here. Please see [BD-intro] for further details.

Exercises:

1. Use Sage to compute the Wronskian of

   (a) $f_1(t) = \sin(t),\ f_2(t) = \cos(t)$,

   (b) $f_1(t) = 1,\ f_2(t) = t,\ f_3(t) = t^2,\ f_4(t) = t^3$.

2. Use the Wronskian test to check that

   (a) $y_1(t) = \sin(t),\ y_2(t) = \cos(t)$ are fundamental solutions for $y'' + y = 0$,

   (b) $y_1(t) = 1,\ y_2(t) = t,\ y_3(t) = t^2,\ y_4(t) = t^3$ are fundamental solutions for $y^{(4)} = y'''' = 0$.

2.3 Undetermined coefficients method

The method of undetermined coefficients [U-uc] can be used to solve the following type of problem.

PROBLEM: Solve

$$ay'' + by' + cy = f(x), \tag{2.5}$$

where $a \neq 0$, b and c are constants, and $f(x)$ is a special type of function. (Even the case $a = 0$ can be handled similarly, though some of the discussion below might need to be slightly modified.) Thet assumption that $f(x)$ is of a "special form" will be explained in moe detail later.

More-or-less equivalent is the method of annihilating operators [A-uc] (they solve the same class of DEs), but that method will be discussed separately.

2.3.1 Simple case

For the moment, let us assume $f(x)$ has the "simple" form $a_1 \cdot p(x) \cdot e^{a_2 x} \cdot \cos(a_3 x)$, or $a_1 \cdot p(x) \cdot e^{a_2 x} \cdot \sin(a_3 x)$, where a_1, a_2, a_3 are constants and $p(x)$ is a polynomial.

Solution:

- Solve the *homogeneous* DE $ay'' + by' + cy = 0$ as follows. Let r_1 and r_2 denote the roots of the characteristic polynomial $aD^2 + bD + c = 0$.

  - $r_1 \neq r_2$ real: the solution is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.
  - $r_1 = r_2$ real: if $r = r_1 = r_2$ then the solution is $y = c_1 e^{rx} + c_2 x e^{rx}$.
  - r_1, r_2 complex: if $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, where α and β are real, then the solution is $y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$.

  Denote this solution y_h (some texts use y_c) and call this the **homogeneous part** of the solution. (Some texts call this the complementary part of the solution.)

- Compute $f(x)$, $f'(x)$, $f''(x)$, ... Write down the list of all the different terms which arise (via the product rule), ignoring constant factors, plus signs, and minus signs:

$$t_1(x),\ t_2(x),\ ...,\ \ t_k(x).$$

  If any one of these agrees with y_1 or y_2 then multiply them all by x. (If, after this, any of them *still* agrees with y_1 or y_2 then multiply them all again by x.)

- Let y_p be a linear combination of these functions (your "guess"):

$$y_p = A_1 t_1(x) + ... + A_k t_k(x).$$

  This is called the **general form of the particular solution** (when you have *not* solved for the constants A_i). The A_i's are called **undetermined coefficients**.

- Plug y_p into (2.5) and solve for A_1, ..., A_k.

- Let $y = y_h + y_p = y_p + c_1 y_1 + c_2 y_2$. This is the **general solution** to (2.5). If there are any initial conditions for (2.5), solve for then c_1, c_2 now.

Diagramatically:

Factor characteristic polynomial

↓

Compute y_h

↓

Compute the general form of the particular, y_p

↓

Compute the undetermined coefficients

↓

Answer: $y = y_h + y_p$.

Examples

Example 2.3.1. Solve

$$y'' - y = \cos(2x).$$

- The characteristic polynomial is $r^2 - 1 = 0$, which has ± 1 for roots. The "homogeneous solution" is therefore $y_h = c_1 e^x + c_2 e^{-x}$.
- We compute $f(x) = \cos(2x)$, $f'(x) = -2\sin(2x)$, $f''(x) = -4\cos(2x)$, ... They are all linear combinations of

  $$f_1(x) = \cos(2x), \ \ f_2(x) = \sin(2x).$$

  None of these agrees with $y_1 = e^x$ or $y_2 = e^{-x}$, so we do not multiply by x.
- Let y_p be a linear combination of these functions:

  $$y_p = A_1 \cos(2x) + A_2 \sin(2x).$$

- You can compute both sides of $y_p'' - y_p = \cos(2x)$:

  $$(-4A_1 \cos(2x) - 4A_2 \sin(2x)) - (A_1 \cos(2x) + A_2 \sin(2x)) = \cos(2x).$$

  Equating the coefficients of $\cos(2x)$, $\sin(2x)$ on both sides gives 2 equations in 2 unknowns: $-5A_1 = 1$ and $-5A_2 = 0$. Solving, we get $A_1 = -1/5$ and $A_2 = 0$.
- The general solution: $y = y_h + y_p = c_1 e^x + c_2 e^{-x} - \frac{1}{5}\cos(2x)$.

Example 2.3.2. Solve

$$y'' - y = x\cos(2x).$$

- The characteristic polynomial is $r^2 - 1 = 0$, which has ± 1 for roots. The "homogeneous solution" is therefore $y_h = c_1 e^x + c_2 e^{-x}$.

- We compute $f(x) = x\cos(2x)$, $f'(x) = \cos(2x) - 2x\sin(2x)$, $f''(x) = -2\sin(2x) - 2\sin(2x) - 2x\cos(2x)$, ... They are all linear combinations of

$$f_1(x) = \cos(2x),\ f_2(x) = \sin(2x),\ f_3(x) = x\cos(2x),\ .f_4(x) = x\sin(2x).$$

None of these agrees with $y_1 = e^x$ or $y_2 = e^{-x}$, so we do not multiply by x.

- Let y_p be a linear combination of these functions:

$$y_p = A_1\cos(2x) + A_2\sin(2x) + A_3x\cos(2x) + A_4x\sin(2x).$$

- In principle, you can compute both sides of $y_p'' - y_p = x\cos(2x)$ and solve for the A_i's. (Equate coefficients of $x\cos(2x)$ on both sides, equate coefficients of $\cos(2x)$ on both sides, equate coefficients of $x\sin(2x)$ on both sides, and equate coefficients of $\sin(2x)$ on both sides. This gives 4 equations in 4 unknowns, which can be solved.) You will not be asked to solve for the A_i's for a problem this hard.

Example 2.3.3. Solve

$$y'' + 4y = x\cos(2x).$$

- The characteristic polynomial is $r^2 + 4 = 0$, which has $\pm 2i$ for roots. The "homogeneous solution" is therefore $y_h = c_1\cos(2x) + c_2\sin(2x)$.
- We compute $f(x) = x\cos(2x)$, $f'(x) = \cos(2x) - 2x\sin(2x)$, $f''(x) = -2\sin(2x) - 2\sin(2x) - 2x\cos(2x)$, ... They are all linear combinations of

$$f_1(x) = \cos(2x),\ f_2(x) = \sin(2x),\ f_3(x) = x\cos(2x),\ .f_4(x) = x\sin(2x).$$

Two of these agree with $y_1 = \cos(2x)$ or $y_2 = \sin(2x)$, so we <u>do</u> multiply by x:

$$f_1(x) = x\cos(2x),\ f_2(x) = x\sin(2x),\ f_3(x) = x^2\cos(2x),\ .f_4(x) = x^2\sin(2x).$$

- Let y_p be a linear combination of these functions:

$$y_p = A_1x\cos(2x) + A_2x\sin(2x) + A_3x^2\cos(2x) + A_4x^2\sin(2x).$$

- In principle, you can compute both sides of $y_p'' + 4y_p = x\cos(2x)$ and solve for the A_i's. You will not be asked to solve for the A_i's for a problem this hard.

2.3.2 Non-simple case

More generally, suppose that you want to solve $ay'' + by' + cy = f(x)$, where $f(x)$ is a *sum* of functions of the "simple" functions in the previous subsection. In other words, $f(x) = f_1(x) + f_2(x) + ... + f_k(x)$, where each $f_j(x)$ is of the form $c \cdot p(x) \cdot e^{ax} \cdot \cos(bx)$, or $c \cdot p(x) \cdot e^{ax} \cdot \sin(bx)$, where a, b, c are constants and $p(x)$ is a polynomial. You can proceed in either one of the following ways.

1. Split up the problem by solving each of the k problems $ay'' + by' + cy = f_j(x)$, $1 \le j \le k$, obtaining the solution $y = y_j(x)$, say. The solution to $ay'' + by' + cy = f(x)$ is then $y = y_1 + y_2 + .. + y_k$ (the superposition principle).

2. Proceed as in the examples above but with the following slight revision:

- Find the "homogeneous solution" y_h to $ay'' + by' = cy = 0$, $y_h = c_1y_1 + c_2y_2$.
- Compute $f(x)$, $f'(x)$, $f''(x)$, ... Write down the list of all the different terms which arise, ignoring constant factors, plus signs, and minus signs:

$$t_1(x),\ t_2(x),\ ...,\ \ t_k(x).$$

- Group these terms into their *families*. Each family is determined from its parent(s) - which are the terms in $f(x) = f_1(x) + f_2(x) + ... + f_k(x)$ which they arose form by differentiation. For example, if $f(x) = x\cos(2x) + e^{-x}\sin(x) + \sin(2x)$ then the terms you get from differentiating and ignoring constants, plus signs and minus signs, are

$$x\cos(2x), x\sin(2x), \cos(2x), \sin(2x), \qquad (\text{from } x\cos(2x)),$$

$$e^{-x}\sin(x), e^{-x}\cos(x), \qquad (\text{from } e^{-x}\sin(x)),$$

and

$$\sin(2x), \cos(2x), \qquad (\text{from } \sin(2x)).$$

The first group absorbes the last group, since you can only count the *different* terms. Therefore, there are only two families in this example:

$$\{x\cos(2x), x\sin(2x), \cos(2x), \sin(2x)\}$$

is a "family" (with "parent" $x\cos(2x)$ and the other terms as its "children") and

$$\{e^{-x}\sin(x), e^{-x}\cos(x)\}$$

is a "family" (with "parent" $e^{-x}\sin(x)$ and the other term as its "child").
If any one of these terms agrees with y_1 or y_2 then multiply the *entire family* by x. In other words, if any child or parent is "bad" then the entire family is "bad". (If, after this, any of them *still* agrees with y_1 or y_2 then multiply them all again by x.)

- Let y_p be a linear combination of these functions (your "guess"):

$$y_p = A_1t_1(x) + ... + A_kt_k(x).$$

This is called the **general form of the particular solution**. The A_i's are called **undetermined coefficients**.

- Plug y_p into (2.5) and solve for A_1, ..., A_k.
- Let $y = y_h + y_p = y_p + c_1y_1 + c_2y_2$. This is the **general solution** to (2.5). If there are any initial conditions for (2.5), solve for then c_1, c_2 *last - after the undetermined coefficients.*

Example 2.3.4. Solve

$$y''' - y'' - y' + y = 12xe^x.$$

We use Sage for this.

Sage

```
sage: x = var("x")
sage: y = function("y",x)
sage: R.<D> = PolynomialRing(QQ[I], "D")
sage: f = D^3 - D^2 - D + 1
sage: f.factor()
 (D + 1) * (D - 1)^2
sage: f.roots()
 [(-1, 1), (1, 2)]
```

So the roots of the characteristic polynomial are $1, 1, -1$, which means that the homogeneous part of the solution is

$$y_h = c_1e^x + c_2xe^x + c_3e^{-x}.$$

Sage

```
sage: de = lambda y: diff(y,x,3) - diff(y,x,2) - diff(y,x,1) + y
sage: c1 = var("c1"); c2 = var("c2"); c3 = var("c3")
sage: yh = c1*e^x + c2*x*e^x + c3*e^(-x)
sage: de(yh)
 0
sage: de(x^3*e^x-(3/2)*x^2*e^x)
 12*x*e^x
```

This just confirmed that y_h solves $y'''-y''-y'+1 = 0$. Using the derivatives of $F(x) = 12xe^x$, we generate the general form of the particular:

Sage

```
sage: F = 12*x*e^x
sage: diff(F,x,1); diff(F,x,2); diff(F,x,3)
 12*x*e^x + 12*e^x
 12*x*e^x + 24*e^x
 12*x*e^x + 36*e^x
sage: A1 = var("A1"); A2 = var("A2")
sage: yp = A1*x^2*e^x + A2*x^3*e^x
```

Now plug this into the DE and compare coefficients of like terms to solve for the undertermined coefficients:

Sage

```
sage: de(yp)
 12*x*e^x*A2 + 6*e^x*A2 + 4*e^x*A1
sage: solve([12*A2 == 12, 6*A2+4*A1 == 0],A1,A2)
 [[A1 == -3/2, A2 == 1]]
```

Finally, lets check if this is correct:

Sage

```
sage: y = yh + (-3/2)*x^2*e^x + (1)*x^3*e^x
sage: de(y)
 12*x*e^x
```

Exercise: Using Sage , solve

$$y''' - y'' + y' - y = 12xe^x.$$

2.3.3 Annihilator method

We consider again the same type of differential equation as in the above subsection, but take a slightly different approach here.

PROBLEM: Solve

$$ay'' + by' + cy = f(x). \tag{2.6}$$

We *assume* that $f(x)$ is of the form $c \cdot p(x) \cdot e^{ax} \cdot \cos(bx)$, or $c \cdot p(x) \cdot e^{ax} \cdot \sin(bx)$, where a, b, c are constants and $p(x)$ is a polynomial.

soln:

- Write the ODE in symbolic form $(aD^2 + bD + c)y = f(x)$.
- Find the "homogeneous solution" y_h to $ay'' + by' = cy = 0$, $y_h = c_1y_1 + c_2y_2$.
- Find the differential operator L which annihilates $f(x)$: $Lf(x) = 0$. The following **annihilator table** may help.

function	annihilator
x^k	D^{k+1}
$x^k e^{ax}$	$(D-a)^{k+1}$
$x^k e^{\alpha x}\cos(\beta x)$	$(D^2-2\alpha D+\alpha^2+\beta^2)^{k+1}$
$x^k e^{\alpha x}\sin(\beta x)$	$(D^2-2\alpha D+\alpha^2+\beta^2)^{k+1}$

- Find the general solution to the homogeneous ODE, $L\cdot(aD^2+bD+c)y=0$.
- Let y_p be the function you get by taking the solution you just found and subtracting from it any terms in y_h.
- Solve for the undetermined coefficients in y_p as in the method of undetermined coefficients.

Example

Example 2.3.5. Solve

$$y''-y=\cos(2x).$$

- The DE is $(D^2-1)y=\cos(2x)$.
- The characteristic polynomial is $r^2-1=0$, which has ±1 for roots. The "homogeneous solution" is therefore $y_h=c_1e^x+c_2e^{-x}$.
- We find $L=D^2+4$ annihilates $\cos(2x)$.
- We solve $(D^2+4)(D^2-1)y=0$. The roots of the characteristic polynomial $(r^2+4)(r^2-1)$ are $\pm2i,\pm1$. The solution is

$$y=A_1\cos(2x)+A_2\sin(2x)+A_3e^x+A_4e^{-x}.$$

- This solution agrees with y_h in the last two terms, so we guess

$$y_p=A_1\cos(2x)+A_2\sin(2x).$$

- Now solve for A_1 and A_2 as before: Compute both sides of $y_p''-y_p=\cos(2x)$,

$$(-4A_1\cos(2x)-4A_2\sin(2x))-(A_1\cos(2x)+A_2\sin(2x))=\cos(2x).$$

  Next, equate the coefficients of $\cos(2x)$, $\sin(2x)$ on both sides to get 2 equations in 2 unknowns. Solving, we get $A_1=-1/5$ and $A_2=0$.
- The general solution: $y=y_h+y_p=c_1e^x+c_2e^{-x}-\frac{1}{5}\cos(2x)$.

Exercises: Solve the following problems using the method of undetermined coefficients:

1. (a) Find the general solution to $x'' - x' - 20x = 0$.
   (b) Use the general solution to solve the initial value problem $x'' - x' - 20x = 0$, $x(0) = 1$, $x'(0) = 1$.

2. Find the general solution to $x'' - x' = t$.

3. Find the general solution to $y'' + 4y = \frac{2}{\cos(2x)}$.

2.4 Variation of parameters

The **method of variation of parameters** is originally attributed to Joseph Louis Lagrange (1736-1813), an Italian-born mathematician and astronomer, who worked much of his professional life in Berlin and Paris [L-var]. It involves, at one step, a prepeated differentiation which is theoretically described by the so-called Leibniz rule, described next.

2.4.1 The Leibniz rule

In general, the **Leibniz rule** (or generalized product rule) is

$$(fg)' = f'g + fg',$$

$$(fg)'' = f''g + 2f'g' + fg'',$$

$$(fg)''' = f'''g + 3f''g' + 3f'g'' + fg''',$$

and so on, where the coefficients occurring in $(fg)^{(n)}$ are **binomial coefficients** $\begin{pmatrix} n \\ i \end{pmatrix}$ computed for example using **Pascal's triangle**,

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1, & \\ 1 & & 4 & & 6 & & 4 & & 1, \end{array}$$

and so on. (For second order ODEs, we only need the first two Leibntiz rules.)

Using Sage , this can be checked as follows:

Sage

```
sage: t = var('t')
sage: x = function('x', t)
sage: y = function('y', t)
sage: diff(x(t)*y(t),t)
x(t)*diff(y(t), t, 1) + y(t)*diff(x(t), t, 1)
sage: diff(x(t)*y(t),t,t)
x(t)*diff(y(t), t, 2) + 2*diff(x(t), t, 1)*diff(y(t), t, 1)
 + y(t)*diff(x(t), t, 2)
sage: diff(x(t)*y(t),t,t,t)
x(t)*diff(y(t), t, 3) + 3*diff(x(t), t, 1)*diff(y(t), t, 2)
 + 3*diff(x(t), t, 2)*diff(y(t), t, 1) + y(t)*diff(x(t), t, 3)
```

2.4.2 The method

Consider an ordinary constant coefficient non-homogeneous 2nd order linear differential equation,

$$ay'' + by' + cy = F(x)$$

where $F(x)$ is a given function and a, b, and c are constants. (The varation of parameters method works even if a, b, and c depend on the independent variable x. However, for simplicity, we assume that they are constants here.)

Let $y_1(x)$, $y_2(x)$ be fundamental solutions of the corresponding homogeneous equation

$$ay'' + by' + cy = 0.$$

Starts by *assuming* that there is a particular solution in the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x), \tag{2.7}$$

where $u_1(x)$, $u_2(x)$ are unknown functions [V-var]. We want to solve for u_1 and u_2.

By assumption, y_p solves the ODE, so

$$ay_p'' + by_p' + cy_p = F(x).$$

After some algebra, this becomes:

$$a(u_1'y_1 + u_2'y_2)' + a(u_1'y_1' + u_2'y_2') + b(u_1'y_1 + u_2'y_2) = F.$$

If we *assume*

$$u_1'y_1 + u_2'y_2 = 0$$

then we get massive simplification:

$$a(u_1'y_1' + u_2'y_2') = F.$$

Cramer's rule (Lemma 2.2.1) implies that the solution to this system is

$$u_1' = \frac{\det\begin{pmatrix} 0 & y_2 \\ F(x) & y_2' \end{pmatrix}}{\det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}}, \qquad u_2' = \frac{\det\begin{pmatrix} y_1 & 0 \\ y_1' & F(x) \end{pmatrix}}{\det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}}. \tag{2.8}$$

(Note that the Wronskian $W(y_1, y_2)$ of the fundamental solutions is in the denominator.) Solve these for u_1 and u_2 by integration and then plug them back into (2.7) to get your particular solution.

Example 2.4.1. Solve

$$y'' + y = \tan(x).$$

soln: The functions $y_1 = \cos(x)$ and $y_2 = \sin(x)$ are fundamental solutions with Wronskian $W(\cos(x), \sin(x)) = 1$. The formulas (2.8) become:

$$u_1' = \frac{\det\begin{pmatrix} 0 & \sin(x) \\ \tan(x) & \cos(x) \end{pmatrix}}{1}, \qquad u_2' = \frac{\det\begin{pmatrix} \cos(x) & 0 \\ -\sin(x) & \tan(x) \end{pmatrix}}{1}.$$

Therefore,

$$u_1' = -\frac{\sin^2(x)}{\cos(x)}, \quad u_2' = \sin(x).$$

Therefore, using methods from integral calculus, $u_1 = -\ln|\tan(x) + \sec(x)| + \sin(x)$ and $u_2 = -\cos(x)$. Using Sage , this can be check as follows:

Sage

```
sage: integral(-sin(t)^2/cos(t),t)
-log(sin(t) + 1)/2 + log(sin(t) - 1)/2 + sin(t)
sage: integral(cos(t)-sec(t),t)
sin(t) - log(tan(t) + sec(t))
sage: integral(sin(t),t)
-cos(t)
```

As you can see, there are other forms the answer can take. The particular solution is

$$y_p = (-\ln|\tan(x) + \sec(x)| + \sin(x))\cos(x) + (-\cos(x))\sin(x).$$

The homogeneous (or complementary) part of the solution is

$$y_h = c_1 \cos(x) + c_2 \sin(x),$$

so the general solution is

$$\begin{aligned} y &= y_h + y_p = c_1\cos(x) + c_2\sin(x) \\ &+(-\ln|\tan(x)+\sec(x)| + \sin(x))\cos(x) + (-\cos(x))\sin(x). \end{aligned}$$

Using Sage , this can be carried out as follows:

Sage

```
sage: SR = SymbolicExpressionRing()
sage: MS = MatrixSpace(SR, 2, 2)
sage: W = MS([[cos(t),sin(t)],[diff(cos(t), t),diff(sin(t), t)]])
sage: W

[ cos(t)  sin(t)]
[-sin(t)  cos(t)]
sage: det(W)
sin(t)^2 + cos(t)^2
sage: U1 = MS([[0,sin(t)],[tan(t),diff(sin(t), t)]])
sage: U2 = MS([[cos(t),0],[diff(cos(t), t),tan(t)]])
sage: integral(det(U1)/det(W),t)
-log(sin(t) + 1)/2 + log(sin(t) - 1)/2 + sin(t)
sage: integral(det(U2)/det(W),t)
-cos(t)
```

Exercises:

1. Find the general solution to $y'' + 4y = \frac{2}{\cos(2x)}$ using variation of parameters.

2. Use Sage to solve $y'' + y = \cot(x)$.

2.5 Applications of DEs: Spring problems

Ut tensio, sic vis[2].

- Robert Hooke, 1678

2.5.1 Part 1

One of the ways DEs arise is by means of modeling physical phenomenon, such as spring equations. For these problems, consider a spring suspended from a ceiling. We shall consider three cases: (1) no mass is attached at the end of the spring, (2) a mass is attached and the system is in the rest position, (3) a mass is attached and the mass has been displaced from the rest position.

[2] "As the extension, so the force."

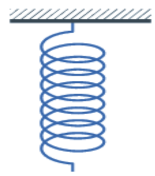


Figure 2.5: A spring at rest, without mass attached.

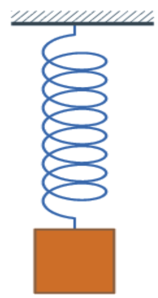


Figure 2.6: A spring at rest, with mass attached.

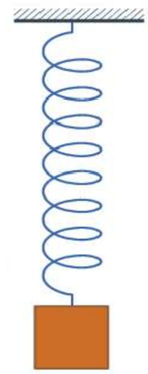


Figure 2.7: A spring in motion.

One can also align the springs left-to-right instead of top-to-bottom, without changing the discussion below.

Notation: Consider the first two situations above: (a) a spring at rest, without mass attached and (b) a spring at rest, with mass attached. The distance the mass pulls the spring down is sometimes called the "stretch", and denoted s. (A formula for s will be given later.)

Now place the mass in motion by imparting some initial velocity (tapping it upwards with a hammer, say, and start your timer). Consider the second two situations above: (a) a spring at rest, with mass attached and (b) a spring in motion. The difference between these two positions at time t is called the **displacement** and is denoted $x(t)$. Signs here will be choosen so that down is positive.

Assume exactly three forces act:

1. the restoring force of the spring, F_{spring},
2. an external force (driving the ceiling up and down, but may be 0), F_{ext},
3. a damping force (imagining the spring immersed in oil or that it is in fact a shock absorber on a car), F_{damp}.

In other words, the total force is given by

$$F_{total} = F_{spring} + F_{ext} + F_{damp}.$$

Physics tells us that the following are approximately true:

1. (**Hooke's law** [H-intro]): $F_{spring} = -kx$, for some "spring constant" $k > 0$,
2. $F_{ext} = F(t)$, for some (possibly zero) function F,
3. $F_{damp} = -bv$, for some "damping constant" $b \geq 0$ (where v denotes velocity),
4. (**Newton's 2nd law** [N-mech]): $F_{total} = ma$ (where a denotes acceleration).

Putting this all together, we obtain $mx'' = ma = -kx + F(t) - bv = -kx + F(t) - bx'$, or

$$\boxed{mx'' + bx' + kx = F(t).}$$

This is the **spring equation**. When $b = F(t) = 0$ this is also called the equation for simple harmonic motion. The solution in the case of simple harmonic motion has the form

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t),$$

where $\omega = \sqrt{k/m}$. There is a more compact and useful form of the solution, $A \sin(\omega t + \phi)$, useful for graphing. This compact form is obtained using the formulas

$$c_1 \cos(\omega t) + c_2 \sin(\omega t) = A \sin(\omega t + \phi), \tag{2.9}$$

where $A = \sqrt{c_1^2 + c_2^2}$ denotes the **amplitude** and $\phi = 2\arctan(\frac{c_1}{c_2+A})$ is the **phase shift**.

Consider again first two figures above: (a) a spring at rest, without mass attached and (b) a spring at rest, with mass attached. The mass in the second figure is at rest, so the gravitational force on the mass, mg, is balanced by the restoring force of the spring: $mg = ks$, where s is the stretch.In particular, the spring constant can be computed from the stretch:

$$\boxed{k = \frac{mg}{s}.}$$

Example 2.5.1. A spring at rest is suspended from the ceiling without mass. A 2 kg weight is then attached to this spring, stretching it 9.8 cm. From a position 2/3 m above equilibrium the weight is give a downward velocity of 5 m/s.

(a) Find the equation of motion.

(b) What is the amplitude and period of motion?

(c) At what time does the mass first cross equilibrium?

(d) At what time is the mass first exactly 1/2 m below equilibrium?

We shall solve this problem using Sage below. Note $m = 2$, $b = F(t) = 0$ (since no damping or external force is even mentioned), and the stretch is $s = 9.8$ cm $= 0.098$ m. Therefore, the spring constant is given by $k = mg/s = 2 \cdot 9.8/(0.098) = 200$. Therefore, the DE is $2x'' + 200x = 0$. This has general solution $x(t) = c_1 \cos(10t) + c_2 \sin(10t)$. The constants c_1 and c_2 can be computed from the initial conditions $x(0) = -2/3$ (down is positive, up is negative), $x'(0) = 5$.

Using Sage , the displacement can be computed as follows:

Sage

```
sage: t = var('t')
sage: x = function('x', t)
sage: m = var('m')
sage: b = var('b')
sage: k = var('k')
sage: F = var('F')
sage: de = lambda y: m*diff(y,t,t) + b*diff(y,t) + k*y - F
sage: de(x)
b*D[0](x)(t) + k*x(t) + m*D[0, 0](x)(t) - F
sage: m = 2; b = 0; k = 200; F = 0
sage: de(x)
200.000000000000*x(t) + 2*D[0, 0](x)(t)
sage: desolve(de(x),[x,t])
k1*sin(10*t)+k2*cos(10*t)
sage: print desolve_laplace(de(x(t)),["t","x"],[0,-2/3,5])
sin(10*t)/2-2*cos(10*t)/3
```

Now we write this in the more compact and useful form $A \sin(\omega t + \phi)$ using formula (2.9) and Sage .

Sage

```
sage: c1 = -2/3; c2 = 1/2
sage: A = sqrt(c1^2 + c2^2)
sage: A
5/6
sage: phi = 2*atan(c1/(c2 + A))
sage: phi
-2*atan(1/2)
sage: RR(phi)
-0.927295218001612
sage: sol = lambda t: c1*cos(10*t) + c2*sin(10*t)
sage: sol2 = lambda t: A*sin(10*t + phi)
sage: P = plot(sol(t),0,2)
sage: show(P)
```

This plot is displayed in Figure 2.8.

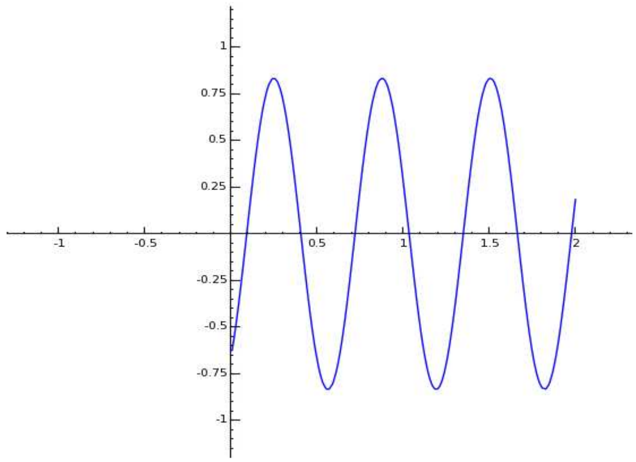


Figure 2.8: Plot of $2x'' + 200x = 0$, $x(0) = -2/3$, $x'(0) = 5$, for $0 < t < 2$.

(You can also, if you want, type `show(plot(sol2(t),0,2))` to check that these two functions are indeed the same.) Of course, the period is $2\pi/10 = \pi/5 \approx 0.628$.

To answer (c) and (d), we solve $x(t) = 0$ and $x(t) = 1/2$:

Sage

```
sage: solve(A*sin(10*t + phi) == 0,t)
[t == atan(1/2)/5]
sage: RR(atan(1/2)/5)
0.0927295218001612
sage: solve(A*sin(10*t + phi) == 1/2,t)
[t == (asin(3/5) + 2*atan(1/2))/10]
sage: RR((asin(3/5) + 2*atan(1/2))/10)
0.157079632679490
```

In other words, $x(0.0927...) \approx 0$, $x(0.157...) \approx 1/2$.

Exercise: Using the problem above and Sage , answer the following questions.

(a) At what time does the weight pass through the equilibrium position heading down for the 2nd time?

(b) At what time is the weight exactly 5/12 m below equilibrium and heading up?

2.5.2 Part 2

Recall from the previos subsection, the spring equation

$$mx'' + bx' + kx = F(t)$$

where $x(t)$ denotes the displacement at time t.

Until otherwise stated, we assume there is no external force: $F(t) = 0$.

The roots of the characteristic polynomial $mD^2 + bD = k = 0$ are

$$\frac{-b \pm \sqrt{b^2 - 4mk}}{2m},$$

by the quadratic formula. There are three cases:

(a) real distinct roots: in this case the discriminant $b^2 - 4mk$ is positive, so $b^2 > 4mk$. In other words, b is "large". This case is referred to as **overdamped**. In this case, the roots are negative,

$$r_1 = \frac{-b - \sqrt{b^2 - 4mk}}{2m} < 0, \quad \text{and} \quad r_1 = \frac{-b + \sqrt{b^2 - 4mk}}{2m} < 0,$$

so the solution $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ is exponentially decreasing.

(b) real repeated roots: in this case the discriminant $b^2 - 4mk$ is zero, so $b = \sqrt{4mk}$. This case is referred to as **critically damped**. This case is said to model new suspension systems in cars [D-spr].

(c) Complex roots: in this case the discriminant $b^2 - 4mk$ is negative, so $b^2 < 4mk$. In other words, b is "small". This case is referred to as **underdamped** (or **simple harmonic** when $b = 0$).

Example 2.5.2. An 8 lb weight stretches a spring 2 ft. Assume a damping force numerically equal to 2 times the instantaneous velocity acts. Find the displacement at time t, provided that it is released from the equilibrium position with an upward velocity of 3 ft/s. Find the equation of motion and classify the behaviour.

We know $m = 8/32 = 1/4$, $b = 2$, $k = mg/s = 8/2 = 4$, $x(0) = 0$, and $x'(0) = -3$. This means we must solve

$$\frac{1}{4}x'' + 2x' + 4x = 0, \quad x(0) = 0, \quad x'(0) = -3.$$

The roots of the characteristic polynomial are -4 and -4 (so we are in the repeated real roots case), so the general solution is $x(t) = c_1 e^{-4t} + c_2 t e^{-4t}$. The initial conditions imply $c_1 = 0$, $c_2 = -3$, so

$$x(t) = -3te^{-4t}.$$

Using Sage , we can compute this as well:

Sage

```
sage: t = var(''t'')
sage: x = function(''x'')
sage: de = lambda y: (1/4)*diff(y,t,t) + 2*diff(y,t) + 4*y
sage: de(x(t))
diff(x(t), t, 2)/4 + 2*diff(x(t), t, 1) + 4*x(t)
sage: desolve(de(x(t)),[x,t])
'(%k2*t+%k1)*%e^-(4*t)'
sage: desolve_laplace(de(x(t)),[''t'',''x''],[0,0,-3])
'-3*t*%e^-(4*t)'
sage: f = lambda t : -3*t*e^(-4*t)
sage: P = plot(f,0,2)
sage: show(P)
```

The graph is shown in Figure 2.9.

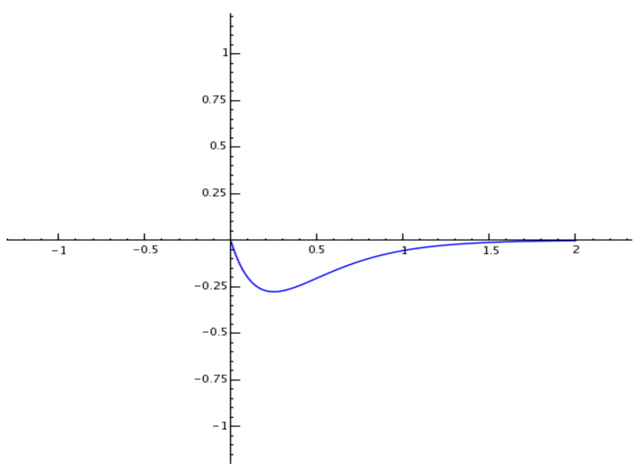


Figure 2.9: Plot of $(1/4)x'' + 2x' + 4x = 0$, $x(0) = 0$, $x'(0) = -3$, for $0 < t < 2$.

Example 2.5.3. An 2 kg weight is attached to a spring having spring constant 10. Assume a damping force numerically equal to 4 times the instantaneous velocity acts. Find the displacement at time t, provided that it is released from 1 m below equilibrium with an upward velocity of 1 ft/s. Find the equation of motion and classify the behaviour.

Using Sage , we can compute this as well:

Sage

```
sage: t = var(''t'')
sage: x = function(''x'')
sage: de = lambda y: 2*diff(y,t,t) + 4*diff(y,t) + 10*y
sage: desolve_laplace(de(x(t)),["t","x"],[0,1,1])
'%e^-t*(sin(2*t)+cos(2*t))'
sage: desolve_laplace(de(x(t)),["t","x"],[0,1,-1])
'%e^-t*cos(2*t)'
sage: sol = lambda t: e^(-t)*cos(2*t)
sage: P = plot(sol(t),0,2)
sage: show(P)
sage: P = plot(sol(t),0,4)
sage: show(P)
```

The graph is shown in Figure 2.10.

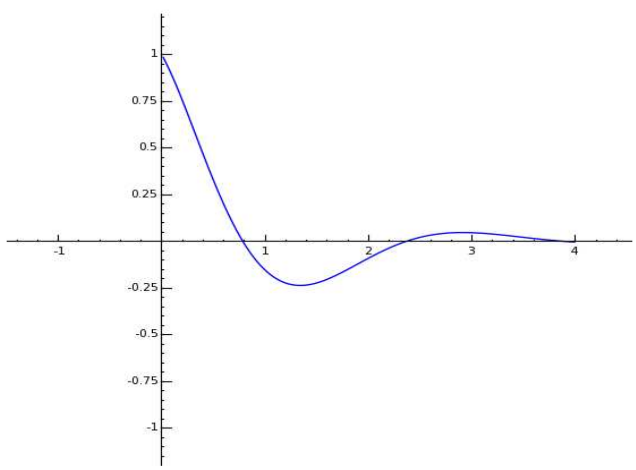


Figure 2.10: Plot of $2x'' + 4x' + 10x = 0$, $x(0) = 1$, $x'(0) = -1$, for $0 < t < 4$.

Exercise: Refer to Example 2.5.3 above. Use Sage to find what time the weight passes through the equilibrium position heading down for the 2nd time.

Exercise: An 2 kg weight is attached to a spring having spring constant 10. Assume a damping force numerically equal to 4 times the instantaneous velocity acts. Use Sage to find the displacement at time t, provided that it is released from 1 m below equilibrium (with no initial velocity).

2.5.3 Part 3

If the frequency of the driving force of the spring matches the frequency of the homogeneous part $x_h(t)$, in other words if

$$x'' + \omega^2 x = F_0 \cos(\gamma t),$$

satisfies $\omega = \gamma$ then we say that the spring-mass system is in **(pure or mechanical) resonance** and γ is called the **resonance frequency**. This notion models a mechanical system when the frequency of its oscillations matches the system's natural frequency of vibration. It may cause violent vibrations in certain structures, such as small airplanes (a phenomenon known as resonance "disaster").

Example 2.5.4. Solve

$$x'' + \omega^2 x = F_0 \cos(\gamma t), \quad x(0) = 0, \quad x'(0) = 0,$$

where $\omega = \gamma = 2$ (ie, mechanical resonance). We use Sage for this:

Sage

```
sage: t = var('t')
sage: x = function('x', t)
sage: (m,b,k,w,F0) = var("m,b,k,w,F0")
sage: de = lambda y: diff(y,t,t) + w^2*y - F0*cos(w*t)
sage: m = 1; b = 0; k = 4; F0 = 1; w = 2
sage: desolve(de(x),[x,t])
k1*sin(2*t) + k2*cos(2*t) + 1/4*t*sin(2*t) + 1/8*cos(2*t)
sage: soln = lambda t : t*sin(2*t)/4   # this is the soln satisfying the ICs
sage: P = plot(soln(t),0,10)
sage: show(P)
```

This is displayed in Figure 2.5.4.

Example 2.5.5. Solve

$$x'' + \omega^2 x = F_0 \cos(\gamma t), \quad x(0) = 0, \quad x'(0) = 0,$$

where $\omega = 2$ and $\gamma = 3$ (ie, mechanical resonance). We use Sage for this:

Sage

```
sage: t = var('t')
sage: x = function('x', t)
sage: (m,b,k,w,g,F0) = var("m,b,k,w,g,F0")
sage: de = lambda y: diff(y,t,t) + w^2*y - F0*cos(g*t)
sage: m = 1; b = 0; k = 4; F0 = 1; w = 2; g = 3
```

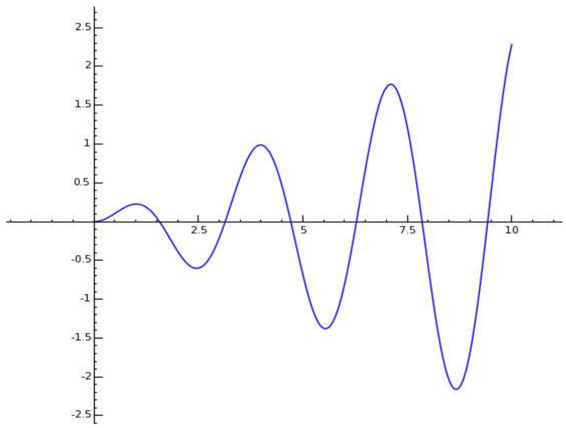


Figure 2.11: A forced undamped spring, with resonance.

```
sage: desolve(de(x),[x,t])
k1*sin(2*t) + k2*cos(2*t) - 1/5*cos(3*t)
sage: soln = lambda t : cos(2*t)/5-cos(3*t)/5   # this is the soln satisfying the ICs
sage: P = plot(soln(t),0,10)
sage: show(P)
```

This is displayed in Figure 2.5.5.

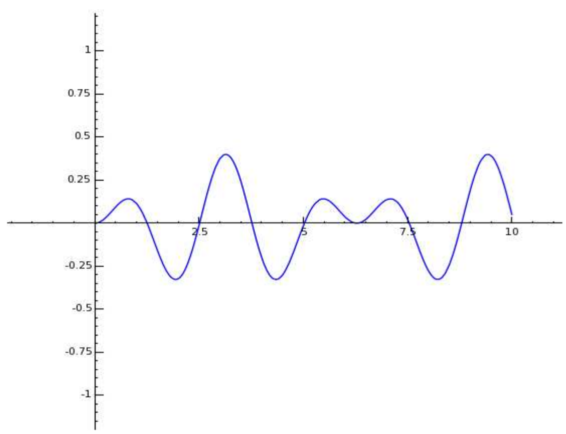


Figure 2.12: A forced undamped spring, no resonance.

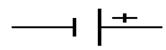
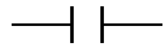
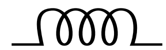
EE object	term in DE (the voltage drop)	units	symbol
charge	$q = \int i(t)\,dt$	coulombs	
current	$i = q'$	amps	
emf	$e = e(t)$	volts V	
resistor	$Rq' = Ri$	ohms Ω	
capacitor	$C^{-1}q$	farads	
inductor	$Lq'' = Li'$	henries	

Figure 2.13: Dictionary for electrical circuits

2.6 Applications to simple LRC circuits

An LRC circuit is a closed loop containing an inductor of L henries, a resistor of R ohms, a capacitor of C farads, and an EMF (electro-motive force), or battery, of $E(t)$ volts, all connected in series.

They arise in several engineering applications. For example, AM/FM radios with analog tuners typically use an LRC circuit to tune a radio frequency. Most commonly a variable capacitor is attached to the tuning knob, which allows you to change the value of C in the circuit and tune to stations on different frequencies [R-cir].

We use the following "dictionary" to translate between the diagram and the DEs.

Next, we recall the circuit laws of Gustav Kirchoff (also spelled Kirchhoff), a German physicist who lived from 1824 to 1887. He was born in Königsberg, which was part of Germany but is now part of the Kaliningrad Oblast, which is an an exclave of Russia surrounded by Lithuania, Poland, and the Baltic Sea.

Kirchoff's First Law: The algebraic sum of the currents travelling into any node is zero.

Kirchoff's Second Law: The algebraic sum of the voltage drops around any closed loop is zero.

Generally, the charge at time t on the capacitor, $q(t)$, satisfies the DE

$$Lq'' + Rq' + \frac{1}{C}q = E(t). \tag{2.10}$$

Example 2.6.1. In this example, we model a very simple type of radio tuner, using a variable capacitor to represent the tuning dial. Consider the simple LC circuit given by the diagram in Figure 2.14.

According to Kirchoff's 2^{nd} Law and the above "dictionary", this circuit corresponds to the DE

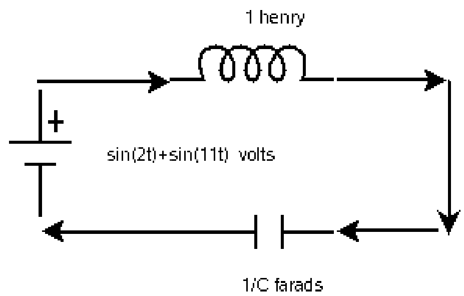


Figure 2.14: A simple LC circuit.

$$q'' + \frac{1}{C}q = \sin(2t) + \sin(11t).$$

The homogeneous part of the solution is

$$q_h(t) = c_1\cos(t/\sqrt{C}) + c_1\sin(t/\sqrt{C}).$$

If $C \neq 1/4$ and $C \neq 1/121$ then

$$q_p(t) = \frac{1}{C^{-1}-4}\sin(2t) + \frac{1}{C^{-1}-121}\sin(11t).$$

When $C = 1/4$ and the initial charge and current are both zero, the solution is

$$q(t) = -\frac{1}{117}\sin(11t) + \frac{161}{936}\sin(2t) - \frac{1}{4}t\cos(2t).$$

Sage

```
sage: t = var("t")
sage: q = function("q",t)
sage: L,R,C = var("L,R,C")
sage: E = lambda t:sin(2*t)+sin(11*t)
sage: de = lambda y: L*diff(y,t,t) + R*diff(y,t) + (1/C)*y-E(t)
sage: L,R,C=1,0,1/4
sage: de(q)
-sin(2*t) - sin(11*t) + 4*q(t) + D[0, 0](q)(t)
sage: print desolve_laplace(de(q(t)),["t","q"],[0,0,0])
-sin(11*t)/117+161*sin(2*t)/936-t*cos(2*t)/4
sage: soln = lambda t: -sin(11*t)/117+161*sin(2*t)/936-t*cos(2*t)/4
sage: P = plot(soln,0,10)
sage: show(P)
```

This is displayed in Figure 2.6.1.

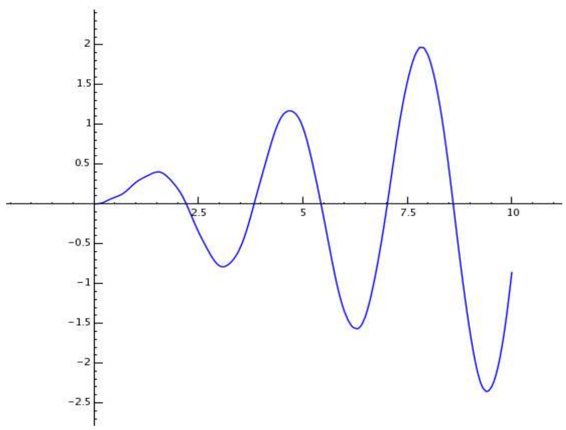


Figure 2.15: A LC circuit, with resonance.

You can see how the frequency $\omega = 2$ dominates the other terms.

When $0 < R < 2\sqrt{L/C}$ the homogeneous form of the charge in (2.10) has the form

$$q_h(t) = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t),$$

where $\alpha = -R/2L < 0$ and $\beta = \sqrt{4L/C - R^2}/(2L)$. This is sometimes called the **transient part** of the solution. The remaining terms in the charge are called the **steady state terms**.

Example 2.6.2. An LRC circuit has a 1 henry inductor, a 2 ohm resistor, 1/5 farad capacitor, and an EMF of $50\cos(t)$. If the initial charge and current is 0, since the charge at time t.

The IVP describing the charge $q(t)$ is

$$q'' + 2q' + 5q = 50\cos(t), \quad q(0) = q'(0) = 0.$$

The homogeneous part of the solution is

$$q_h(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t).$$

The general form of the particular solution using the method of undetermined coefficients is

$$q_p(t) = A_1\cos(t) + A_2\sin(t).$$

Solving for A_1 and A_2 gives

$$q_p(t) = -10e^{-t}\cos(2t) - \frac{15}{2}e^{-t}\sin(2t).$$

Sage

```
sage: t = var("t")
sage: q = function("q",t)
sage: L,R,C = var("L,R,C")
sage: E = lambda t: 50*cos(t)
sage: de = lambda y: L*diff(y,t,t) + R*diff(y,t) + (1/C)*y-E(t)
sage: L,R,C = 1,2,1/5
sage: desolve(de(q),[q,t])
(k1*sin(2*t) + k2*cos(2*t))*e^(-t) + 5*sin(t) + 10*cos(t)
sage: soln = lambda t: e^(-t)*(-15*sin(2*t)/2-10*cos(2*t))\
   +5*sin(t)+10*cos(t)  # the soln to the above ODE+ICs
sage: P = plot(soln,0,10)
sage: soln_ss = lambda t: 5*sin(t)+10*cos(t)
sage: P_ss = plot(soln_ss,0,10,linestyle=":")
sage: soln_tr = lambda t: e^(-t)*(-15*sin(2*t)/2-10*cos(2*t))
sage: P_tr = plot(soln_tr,0,10,linestyle="--")
sage: show(P+P_ss+P_tr)
```

This plot (the solution superimposed with the transient part of the solution) is displayed in Figure 2.6.2.

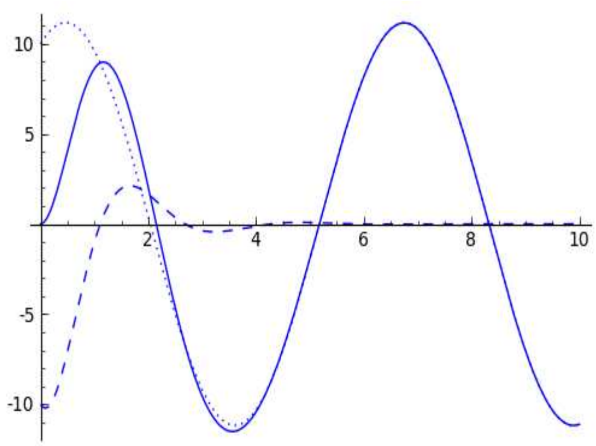


Figure 2.16: A LRC circuit, with damping, and the transient part (dashed) of the solution.

Exercise: Use Sage to solve

$$q'' + \frac{1}{C}q = \sin(2t) + \sin(11t), \quad q(0) = q'(0) = 0,$$

in the case $C = 1/121$.

2.7 The power series method

2.7.1 Part 1

In this part, we recall some basic facts about power series and Taylor series. We will turn to solving ODEs in part 2.

Roughly speaking, **power series** are simply infinite degree polynomials

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... = \sum_{k=0}^{\infty} a_k x^k, \tag{2.11}$$

for some real or complex numbers $a_0, a_1, \dots$. A power series is a way of expressing a "complicated" function $f(x)$ as a sum of "simple" functions like x, x^2, … . The number a_k is called the **coefficient** of x^k, for $k = 0, 1, ...$ Let us ignore for the moment the precise meaning of this infinite sum. (How do you associate a value to an infinite sum? Does the sum converge for some values of x? If so, for which values? ...) We will return to that issue later.

First, some motivation. Why study these? This type of function is convenient for several reasons

- it is easy to differentiate a power series (term-by-term):

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + ... = \sum_{k=0}^{\infty} k a_k x^{k-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k,$$

- it is easy to integrate such a series (term-by-term):

$$\int f(x)\, dx = a_0 x + \frac{1}{2} a_1 x^2 + \frac{1}{3} a_2 x^3 + ... = \sum_{k=0}^{\infty} \frac{1}{k+1} a_k x^{k+1} = \sum_{k=1}^{\infty} \frac{1}{k} a_{k+1} x^k,$$

- if (as is often the case) the summands $a_k x^k$'s tend to zero very quickly, then the sum of the first few terms of the series are often a good numerical approximation for the function itself,

- power series enable one to reduce the solution of certain differential equations down to (often the much easier problem of) solving certain recurrence relations.

- Power series expansions arise naturally in Taylor's theorem from differential calculus.

Theorem 2.7.1. (**Taylor's Theorem**) If $f(x)$ is $n+1$ times continuously differentiable in (a, x) then there exists a point $\xi \in (a, x)$ such that

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi). \tag{2.12}$$

The sum

$$T_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a),$$

is called the **n-th degree Taylor polynomial of f centered at a**. For the case $n = 0$, the formula is

$$f(x) = f(a) + (x-a)f'(\xi),$$

which is just a rearrangement of the terms in the **mean value theorem** from differential calculus,

$$f'(\xi) = \frac{f(x) - f(a)}{x - a}.$$

The **Taylor series of f centered at** a is the series

$$f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots \ .$$

When this series converges to $f(x)$ at each point x in some inverval centered about a then we say f has a **Taylor series expansion** (or Taylor series representation) at a. A Taylor series is basically just a power series but using powers of $x-a$ instead of powers of x.

As the examples below indicate, many of the functions you are used to seeing from calculus have a Taylor series representation.

- Geometric series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n \tag{2.13}$$

To see this, assume $|x| < 1$ and let $n \to \infty$ in the polynomial identity

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1 - x^{n+1}}{1-x}.$$

For $x \geq 1$, the series does not converge.

- The exponential function:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned} \tag{2.14}$$

To see this, take $f(x) = e^x$ and $a = 0$ in Taylor's theorem (2.12), using the fact that $\frac{d}{dx}e^x = e^x$ and $e^0 = 1$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \frac{\xi^{n+1}}{(n+1)!},$$

for some ξ between 0 and x. Perhaps it is not clear to everyone that as n becomes larger and larger (x fixed), the last ("remainder") term in this sum goes to 0. However, Stirling's formula tells us how large the factorial function grows,

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + O(\frac{1}{n})),$$

so we may indeed take the limit as $n \to \infty$ to get (2.14).

`Wikipedia`'s entry on "Power series" [P1-ps] has a nice animation showing how more and more terms in the Taylor polynomials approximate e^x better and better. This animation can also be constructed using `Sage` (`http://wiki.sagemath.org/interact/calculus#TaylorSeri`

- The cosine function:

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \end{aligned} \tag{2.15}$$

This too follows from Taylor's theorem (take $f(x) = \cos x$ and $a = 0$). However, there is another trick: Replace x in (2.14) by ix and use the fact ("Euler's formula") that $e^{ix} = \cos(x) + i\sin(x)$. Taking real parts gives (2.15). Taking imaginary parts gives (2.16), below.

- The sine function:

$$\begin{aligned}\sin x &= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \\ &= 1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad (2.16)\end{aligned}$$

Indeed, you can formally check (using formal term-by-term differentiation) that

$$-\frac{d}{dx}\cos(x) = \sin(x).$$

(Alternatively, you can use this fact to deduce (2.16) from (2.15).)

- The logarithm function:

$$\begin{aligned}\log(1-x) &= -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \\ &= -\sum_{n=0}^{\infty} \frac{1}{n}x^n \qquad (2.17)\end{aligned}$$

This follows from (2.13) since (using formal term-by-term integration)

$$\int_0^x \frac{1}{1-t} = -\log(1-x).$$

Sage

```
sage: taylor(sin(x), x, 0, 5)
 x - x^3/6 + x^5/120
sage: P1 = plot(sin(x),0,pi)
sage: P2 = plot(x,0,pi,linestyle="--")
sage: P3 = plot(x-x^3/6,0,pi,linestyle="-.")
sage: P4 = plot(x-x^3/6+x^5/120,0,pi,linestyle=":")
sage: T1 = text("x",(3,2.5))
sage: T2 = text("x-x^3/3!",(3.5,-1))
sage: T3 = text("x-x^3/3!+x^5/5!",(3.7,0.8))
sage: T4 = text("sin(x)",(3.4,0.1))
sage: show(P1+P2+P3+P4+T1+T2+T3+T4)
```

This is displayed in Figure 2.17.

Exercise: Use Sage to plot successive Taylor polynomial approximations for $\cos(x)$.

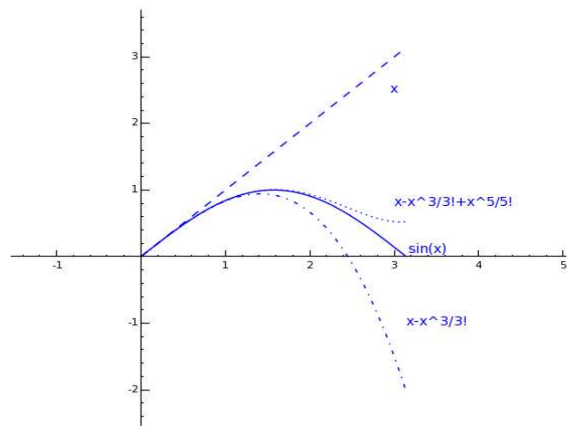


Figure 2.17: Taylor polynomial approximations for $\sin(x)$.

Finally, we turn to the meaning of these sums. How do you associate a value to an infinite sum? Does the sum converge for some values of x? If so, for which values? . We will (for the most part) answer all of these.

First, consider our infinite power series $f(x)$ in (2.11), where the a_k are all given and x is fixed for the momemnt. The **partial sums** of this series are

$$f_0(x) = a_0, \quad f_1(x) = a_0 + a_1 x, \quad f_2(x) = a_0 + a_1 x + a_2 x^2, \cdots .$$

We say that the series in (2.11) **converges at** x if the limit of partial sums

$$\lim_{n\to\infty} f_n(x)$$

exists. There are several tests for determining whether or not a series converges. One of the most commonly used tests is the

Root test: Assume

$$L = \lim_{k\to\infty} |a_k x^k|^{1/k} = |x| \lim_{k\to\infty} |a_k|^{1/k}$$

exists. If $L < 1$ then the infinite power series $f(x)$ in (2.11) converges at x. In general, (2.11) converges for all x satisfying

$$-\lim_{k\to\infty} |a_k|^{-1/k} < x < \lim_{k\to\infty} |a_k|^{-1/k}.$$

The number $\lim_{k\to\infty} |a_k|^{-1/k}$ (if it exists, though it can be ∞) is called the **radius of convergence**.

Example 2.7.1. The radius of convergence of e^x (and $\cos(x)$ and $\sin(x)$) is ∞. The radius of convergence of $1/(1-x)$ (and $\log(1+x)$) is 1.

Example 2.7.2. The radius of convergence of

$$f(x) = \sum_{k=0}^{\infty} \frac{k^7 + k + 1}{2^k + k^2} x^k$$

can be determined with the help of Sage . We want to compute

$$\lim_{k \to \infty} |\frac{k^7 + k + 1}{2^k + k^2}|^{-1/k}.$$

Sage

```
sage: k = var('k')
sage: limit(((k^7+k+1)/(2^k+k^2))^(-1/k),k=infinity)
2
```

In other words, the series converges for all x satisfying $-2 < x < 2$.

Exercise: Use Sage to find the radius of convergence of

$$f(x) = \sum_{k=0}^{\infty} \frac{k^3 + 1}{3^k + 1} x^{2k}$$

2.7.2 Part 2

In this part, we solve some DEs using power series.

We want to solve a problem of the form

$$y''(x) + p(x)y'(x) + y(x) = f(x), \tag{2.18}$$

in the case where $p(x)$, $q(x)$ and $f(x)$ have a power series expansion. We will call a **power series solution** a series expansion for $y(x)$ where we have produced some algorithm or rule which enables us to compute as many of its coefficients as we like.

Solution strategy: Write $y(x) = a_0 + a_1 x + a_2 x^2 + ... = \sum_{k=0}^{\infty} a_k x^k$, for some real or complex numbers $a_0, a_1, ...$

- Plug the power series expansions for y, p, q, and f into the DE (2.18).

- Comparing coefficients of like powers of x, derive relations between the a_j's.
- Using these recurrence relations [R-ps] and the ICs, solve for the coefficients of the power series of $y(x)$.

Example 2.7.3. Solve $y' - y = 5$, $y(0) = -4$, using the power series method.

This is easy to solve by undetermined coefficients: $y_h(x) = c_1 e^x$ and $y_p(x) = A_1$. Solving for A_1 gives $A_1 = -5$ and then solving for c_1 gives $-4 = y(0) = -5 + c_1 e^0$ so $c_1 = 1$ so $y = e^x - 5$.

Solving this using power series, we compute

$$\begin{array}{rcl} y'(x) & = & a_1 + 2a_2 x + 3a_3 x^2 + ... = \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k \\ -y(x) & = & -a_0 - a_1 x - a_2 x^2 - ... = \sum_{k=0}^{\infty} -a_k x^k \\ ---- & -- & ------------------------ \\ 5 & = & (-a_0 + a_1) + (-a_1 + 2a_2)x + ... = \sum_{k=0}^{\infty}(-a_k + (k+1)a_{k+1})x^k \end{array}$$

Comparing coefficients,

- for $k = 0$: $5 = -a_0 + a_1$,
- for $k = 1$: $0 = -a_1 + 2a_2$,
- for general k: $0 = -a_k + (k+1)a_{k+1}$ for $k > 0$.

The IC gives us $-4 = y(0) = a_0$, so

$$a_0 = -4, \quad a_1 = 1, \quad a_2 = 1/2, \quad a_3 = 1/6, \quad \cdots \quad , a_k = 1/k!.$$

This implies

$$y(x) = -4 + x + x/2 + \ \cdots \ + x^k/k! + \ \cdots \ = -5 + e^x,$$

which is in agreement from the previous discussion.

Example 2.7.4. Solve **Bessel's equation** [B-ps] of the 0-th order,

$$x^2 y'' + xy' + x^2 y = 0, \quad y(0) = 1, \quad y'(0) = 0,$$

using the power series method.

This DE is so well-known (it has important applications to physics and engineering) that the series expansion has already been worked out (most texts on special functions or differential equations have this but an online reference is [B-ps]). Its Taylor series expansion around 0 is:

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!^2} \Big(\frac{x}{2}\Big)^{2m}$$

for all x. We shall see below that $y(x) = J_0(x)$.

Let us try solving this ourselves using the power series method. We compute

$$\begin{array}{rcl}
x^2y''(x) & = & 0 + 0\cdot x + 2a_2x^2 + 6a_3x^3 + 12a_4x^4 + ... = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k \\
xy'(x) & = & 0 + a_1x + 2a_2x^2 + 3a_3x^3 + ... = \sum_{k=0}^{\infty} ka_kx^k \\
x^2y(x) & = & 0 + 0\cdot x + a_0x^2 + a_1x^3 + ... = \sum_{k=2}^{\infty} a_{k-2}x^k \\
---- & -- & ------------------------ \\
0 & = & 0 + a_1x + (a_0 + 4a_2)x^2 + ... = a_1x + \sum_{k=2}^{\infty}(a_{k-2} + k^2a_k)x^k.
\end{array}$$

By the ICs, $a_0 = 1$, $a_1 = 0$. Comparing coefficients,

$$k^2a_k = -a_{k-2}, \quad k \geq 2,$$

which implies

$$a_2 = -(\frac{1}{2})^2, \quad a_3 = 0, \quad a_4 = (\frac{1}{2}\cdot\frac{1}{4})^2, \quad a_5 = 0, \quad a_6 = -(\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{6})^2, \cdots .$$

In general,

$$a_{2k} = (-1)^k 2^{-2k}\frac{1}{k!^2}, \quad a_{2k+1} = 0,$$

for $k \geq 1$. This is in agreement with the series above for J_0.

Some of this computation can be formally done in Sage using power series rings.

Sage

```
sage: R6.<a0,a1,a2,a3,a4,a5,a6> = PolynomialRing(QQ,7)
sage: R.<x> = PowerSeriesRing(R6)
sage: y = a0 + a1*x + a2*x^2 + a3*x^3 + a4*x^4 + a5*x^5 +\
   a6*x^6 + O(x^7)
sage: y1 = y.derivative()
sage: y2 = y1.derivative()
sage: x^2*y2 + x*y1 + x^2*y
a1*x + (a0 + 4*a2)*x^2 + (a1 + 9*a3)*x^3 + (a2 + 16*a4)*x^4 +\
 (a3 + 25*a5)*x^5 + (a4 + 36*a6)*x^6 + O(x^7)
```

This is consistent with our “paper and pencil” computations above.

Sage knows quite a few special functions, such as the various types of Bessel functions.

Sage

```
sage: b = lambda x:bessel_J(x,0)
```

```
sage: P = plot(b,0,20,thickness=1)
sage: show(P)
sage: y = lambda x: 1 - (1/2)^2*x^2 + (1/8)^2*x^4 - (1/48)^2*x^6
sage: P1 = plot(y,0,4,thickness=1)
sage: P2 = plot(b,0,4,linestyle="--")
sage: show(P1+P2)
```

This is displayed in Figure 2.18-2.19.

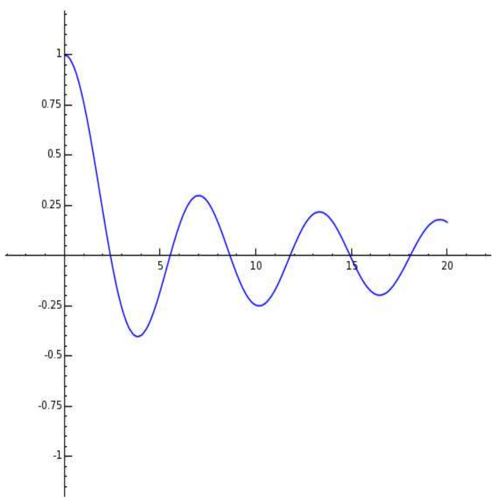


Figure 2.18: The Bessel function $J_0(x)$, for $0 < x < 20$.

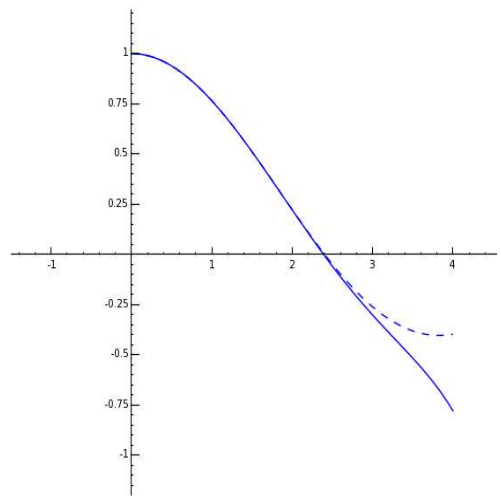


Figure 2.19: A Taylor polynomial approximation for $J_0(x)$.

Exercises:

1. Using a power series around $t = 0$ of the form $y(t) = \sum_{n=0}^{\infty} c_n t^n$, find the recurrence relation for the c_i if y satisfies $y'' + ty' + y = 0$.

2. (a) Find the recurrence relation of the coefficients of the power series solution to the ODE $x' = 4x^3 x$ around $x = 0$, i.e. for solutions of the form $x = \sum_{n=0}^{\infty} c_n t^n$.

   (b) Find an explicit formula for c_n in terms of n.

3. Use Sage to find the first 5 terms in the power series solution to $x'' + x = 0$, $x(0) = 1$, $x'(0) = 0$. Plot this Taylor polynomial approximation over $-\pi < t < \pi$.

4. Find two linearly independent solutions of **Airy's equation** $x'' - tx = 0$ using power series.

2.8 The Laplace transform method

What we know is not much. What we do not know is immense.
- *Pierre Simon Laplace*

2.8.1 Part 1

Pierre Simon Laplace (1749 1827) was a French mathematician and astronomer who is regarded as one of the greatest scientists of all time. His work was pivotal to the development of both celestial mechanics and probability, [L-lt], [LT-lt].

The **Laplace transform** (abbreviated LT) of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F(s)$, defined by:

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt.$$

The LT sends "nice" functions of t (we will be more precise later) to functions of another variable s. It has the wonderful property that it transforms constant-coefficient differential equations in t to *algebraic equations* in s.

The LT has two very familiar properties: Just as the integral of a sum is the sum of the integrals, the Laplace transform of a sum is the sum of Laplace transforms:

$$\mathcal{L}[f(t) + g(t)] = \mathcal{L}[f(t)] + \mathcal{L}[g(t)]$$

Just as constant factor can be taken outside of an integral, the LT of a constant times a function is that constant times the LT of that function:

$$\mathcal{L}[af(t)] = a\mathcal{L}[f(t)]$$

In other words, the LT is **linear**.

For which functions f is the LT actually defined on? We want the indefinite integral to converge, of course. A function $f(t)$ is of **exponential order** α if there exist constants t_0 and M such that

$$|f(t)| < Me^{\alpha t}, \quad \text{for all } t > t_0.$$

If $\int_0^{t_0} f(t)\, dt$ exists and $f(t)$ is of exponential order α then the Laplace transform $\mathcal{L}[f](s)$ exists for $s > \alpha$.

Example 2.8.1. Consider the Laplace transform of $f(t) = 1$. The LT integral converges for $s > 0$.

$$\begin{aligned}\mathcal{L}[f](s) &= \int_0^\infty e^{-st}\, dt \\ &= \left[-\frac{1}{s}e^{-st}\right]_0^\infty \\ &= \frac{1}{s}\end{aligned}$$

Example 2.8.2. Consider the Laplace transform of $f(t) = e^{at}$. The LT integral converges for $s > a$.

$$\begin{aligned}\mathcal{L}[f](s) &= \int_0^\infty e^{(a-s)t}\,dt \\ &= \left[-\frac{1}{s-a}e^{(a-s)t}\right]_0^\infty \\ &= \frac{1}{s-a}\end{aligned}$$

Example 2.8.3. Consider the Laplace transform of the translated **unit step** (or Heaviside) function,

$$u(t-c) = \begin{cases} 0 & \text{for } t < c \\ 1 & \text{for } t > c, \end{cases}$$

where $c > 0$. (this is sometimes also denoted $H(t-c)$.) This function is "off" (i.e., equal to 0) until you get to $t = c$, at which time it turns "on". The LT of it is

$$\begin{aligned}\mathcal{L}[u(t-c)] &= \int_0^\infty e^{-st} H(t-c)\,dt \\ &= \int_c^\infty e^{-st}\,dt \\ &= \left[\frac{e^{-st}}{-s}\right]_c^\infty \\ &= \frac{e^{-cs}}{s} \quad \text{for } s > 0\end{aligned}$$

The **inverse Laplace transform** in denoted

$$f(t) = \mathcal{L}^{-1}[F(s)](t),$$

where $F(s) = \mathcal{L}[f(t)](s)$.

Example 2.8.4. Consider

$$f(t) = \begin{cases} 1, & \text{for } t < 2, \\ 0, & \text{on } t \geq 2. \end{cases}$$

(Incidently, this can also be written $1 - u(t-2)$.) We show how Sage can be used to compute the LT of this.

Sage

```
sage: t = var('t')
sage: s = var('s')
sage: f = Piecewise([[(0,2),1],[(2,infinity),0]])
```

```
sage: f.laplace(t, s)
1/s - e^(-(2*s))/s
sage: f1 = lambda t: 1
sage: f2 = lambda t: 0
sage: f = Piecewise([[(0,2),f1],[(2,10),f2]])
sage: P = f.plot(rgbcolor=(0.7,0.1,0.5),thickness=3)
sage: show(P)
```

According to Sage , $\mathcal{L}[f](s) = 1/s - e^{-2s}/s$. Note the function f was redefined for plotting purposes only (the fact that it was redefined over $0 < t < 10$ means that Sage will plot it over that range.) The plot of this function is displayed in Figure 2.20.

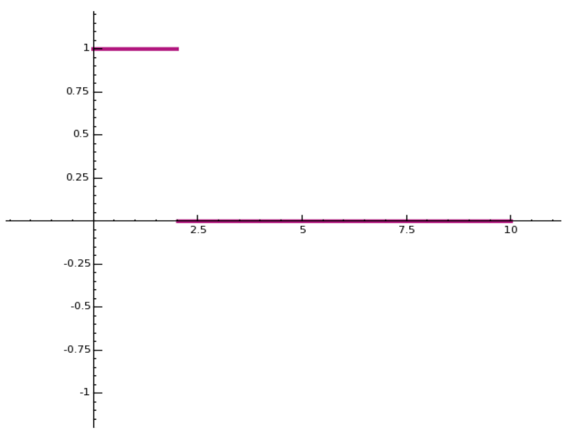


Figure 2.20: The piecewise constant function $1 - u(t-2)$.

Next, some properties of the LT.

- Differentiate the definition of the LT with respect to s:

$$F'(s) = -\int_0^\infty e^{-st} t f(t)\, dt.$$

  Repeating this:

$$\frac{d^n}{ds^n} F(s) = (-1)^n \int_0^\infty e^{-st} t^n f(t)\, dt. \tag{2.19}$$

- In the definition of the LT, replace $f(t)$ by it's derivative $f'(t)$:

$$\mathcal{L}\left[f'(t)\right](s) = \int_0^\infty e^{-st} f'(t)\, dt.$$

Now integrate by parts ($u = e^{-st}$, $dv = f'(t)\,dt$):

$$\int_0^\infty e^{-st} f'(t)\,dt = f(t)e^{-st}|_0^\infty - \int_0^\infty f(t)\cdot(-s)\cdot e^{-st}\,dt = -f(0) + s\mathcal{L}\left[f(t)\right](s).$$

Therefore, if $F(s)$ is the LT of $f(t)$ then $sF(s) - f(0)$ is the LT of $f'(t)$:

$$\mathcal{L}\left[f'(t)\right](s) = s\mathcal{L}\left[f(t)\right](s) - f(0). \tag{2.20}$$

- Replace f by f' in (2.20),

$$\mathcal{L}\left[f''(t)\right](s) = s\mathcal{L}\left[f'(t)\right](s) - f'(0), \tag{2.21}$$

and apply (2.20) again:

$$\mathcal{L}\left[f''(t)\right](s) = s^2\mathcal{L}\left[f(t)\right](s) - sf(0) - f'(0), \tag{2.22}$$

- Using (2.20) and (2.22), the LT of any constant coefficient ODE

$$ax''(t) + bx'(t) + cx(t) = f(t)$$

is

$$a(s^2\mathcal{L}\left[x(t)\right](s) - sx(0) - x'(0)) + b(s\mathcal{L}\left[x(t)\right](s) - x(0)) + c\mathcal{L}\left[x(t)\right](s) = F(s),$$

where $F(s) = \mathcal{L}\left[f(t)\right](s)$. In particular, the LT of the solution, $X(s) = \mathcal{L}\left[x(t)\right](s)$, satisfies

$$X(s) = (F(s) + asx(0) + ax'(0) + bx(0))/(as^2 + bs + c).$$

Note that the denominator is the characteristic polynomial of the DE.

Moral of the story: it is generally very easy to compute *the Laplace transform* $X(s)$ of the solution to any constant coefficient non-homogeneous linear ODE. Computing the actual solution $x(t)$ is usually *much* more work.

Example 2.8.5. We know now how to compute not only the LT of $f(t) = e^{at}$ (it's $F(s) = (s-a)^{-1}$) but also the LT of any function of the form $t^n e^{at}$ by differentiating it:

$$\mathcal{L}\left[te^{at}\right] = -F'(s) = (s-a)^{-2},\ \mathcal{L}\left[t^2e^{at}\right] = F''(s) = 2\cdot(s-a)^{-3},\ \mathcal{L}\left[t^3e^{at}\right] = -F'(s) = 2\cdot3\cdot(s-a)^{-4},\ \dots,$$

and in general

$$\mathcal{L}\left[t^n e^{at}\right] = -F'(s) = n!\cdot(s-a)^{-n-1}. \tag{2.23}$$

Let us now solve a first order ODE using Laplace transforms.

Example 2.8.6. Let us solve the DE

$$x' + x = t^{100}e^{-t}, \quad x(0) = 0.$$

using LTs. Note this would be highly impractical to solve using undetermined coefficients. (You would have 101 undetermined coefficients to solve for!)

First, we compute the LT of the solution to the DE. The LT of the LHS: by (2.23),

$$\mathcal{L}\left[x' + x\right] = sX(s) + X(s),$$

where $F(s) = \mathcal{L}\left[f(t)\right](s)$. For the LT of the RHS, let

$$F(s) = \mathcal{L}\left[e^{-t}\right] = \frac{1}{s+1}.$$

By (2.19),

$$\frac{d^{100}}{ds^{100}}F(s) = \mathcal{L}\left[t^{100}e^{-t}\right] = \frac{d^{100}}{ds^{100}}\frac{1}{s+1}.$$

The first several derivatives of $\frac{1}{s+1}$ are as follows:

$$\frac{d}{ds}\frac{1}{s+1} = -\frac{1}{(s+1)^2}, \quad \frac{d^2}{ds^2}\frac{1}{s+1} = 2\frac{1}{(s+1)^3}, \quad \frac{d^3}{ds^3}\frac{1}{s+1} = -62\frac{1}{(s+1)^4},$$

and so on. Therefore, the LT of the RHS is:

$$\frac{d^{100}}{ds^{100}}\frac{1}{s+1} = 100!\frac{1}{(s+1)^{101}}.$$

Consequently,

$$X(s) = 100!\frac{1}{(s+1)^{102}}.$$

Using (2.23), we can compute the ILT of this:

$$x(t) = \mathcal{L}^{-1}\left[X(s)\right] = \mathcal{L}^{-1}\left[100!\frac{1}{(s+1)^{102}}\right] = \frac{1}{101}\mathcal{L}^{-1}\left[101!\frac{1}{(s+1)^{102}}\right] = \frac{1}{101}t^{101}e^{-t}.$$

Let us now solve a second order ODE using Laplace transforms.

Example 2.8.7. Let us solve the DE

$$x'' + 2x' + 2x = e^{-2t}, \quad x(0) = x'(0) = 0,$$

using LTs.

The LT of the LHS: by (2.23) and (2.21),

$$\mathcal{L}\left[x'' + 2x' + 2x\right] = (s^2 + 2s + 2)X(s),$$

as in the previous example. The LT of the RHS is:

$$\mathcal{L}\left[e^{-2t}\right] = \frac{1}{s+2}.$$

Solving for the LT of the solution algebraically:

$$X(s) = \frac{1}{(s+2)((s+1)^2+1)}.$$

The inverse LT of this can be obtained from LT tables after rewriting this using partial fractions:

$$X(s) = \frac{1}{2}\cdot\frac{1}{s+2} - \frac{1}{2}\frac{s}{(s+1)^2+1} = \frac{1}{2}\cdot\frac{1}{s+2} - \frac{1}{2}\frac{s+1}{(s+1)^2+1} + \frac{1}{2}\frac{1}{(s+1)^2+1}.$$

The inverse LT is:

$$x(t) = \mathcal{L}^{-1}\left[X(s)\right] = \frac{1}{2}\cdot e^{-2t} - \frac{1}{2}\cdot e^{-t}\cos(t) + \frac{1}{2}\cdot e^{-t}\sin(t).$$

We show how Sage can be used to do some of this. We break the Sage solution into steps.

Step 1: First, we type in the ODE and take its Laplace transform.

Sage

```
sage: s,t,X = var('s,t,X')
sage: x = function("x",t)
sage: de = diff(x,t,t)+2*diff(x,t)+2*x==e^(-2*t)
sage: laplace(de,t,s)
s^2*laplace(x(t), t, s) + 2*s*laplace(x(t), t, s) - s*x(0) + 2*laplace(x(t), t, s) - 2*x(0) - D[0](x
sage: LTde = laplace(de,t,s)
```

Step 2: Now we solve this equation for `X = laplace(x(t), t, s)`. For this, we use Python to do some string replacements. Python is the underlying language for Sage and has very powerful string manipulation functions.

Sage

```
sage: strLTde = str(LTde).replace("laplace(x(t), t, s)",  "X")
sage: strLTde0 = strLTde.replace("x(0)","0")
sage: strLTde00 = strLTde0.replace("D[0](x)(0)","0")
sage: LTde00 = sage_eval(strLTde00,locals={"s":s,"X":X})
sage: soln = solve(LTde00,X)
sage: Xs = soln[0].rhs(); Xs
1/(s^3 + 4*s^2 + 6*s + 4)
```

Step 3: Now that we have solved for the Laplace transform of the solution, we take inverse Laplace transforms to get the solution to the original ODE. There are various ways to do this. One way which is convenient if you also want to check the answer using tables, if to compute the partial fraction decomposition then take inverse Laplace transforms.

Sage

```
sage: factor(s^3 + 4*s^2 + 6*s + 4)
(s + 2)*(s^2 + 2*s + 2)
sage: f = 1/((s+2)*((s+1)^2+1))
sage: f.partial_fraction()
 1/(2*(s + 2)) - s/(2*(s^2 + 2*s + 2))
sage: f.inverse_laplace(s,t)
 e^(-t)*(sin(t)/2 - cos(t)/2) + e^(-(2*t))/2
```

Exercise: Use Sage to solve the DE

$$x'' + 2x' + 5x = e^{-t}, \quad x(0) = x'(0) = 0.$$

2.8.2 Part 2

In this part, we shall focus on two other aspects of Laplace transforms:

- solving differential equations involving unit step (Heaviside) functions,
- convolutions and applications.

It follows from the definition of the Laplace transform that if

$$f(t) \stackrel{\mathcal{L}}{\longmapsto} F(s) = \mathcal{L}[f(t)](s),$$

then

$$f(t)u(t-c) \stackrel{\mathcal{L}}{\longmapsto} e^{-cs}\mathcal{L}[f(t+c)](s), \tag{2.24}$$

and

$$f(t-c)u(t-c) \stackrel{\mathcal{L}}{\longmapsto} e^{-cs}F(s). \tag{2.25}$$

These two properties are called **translation theorems**.

Example 2.8.8. First, consider the Laplace transform of the piecewise-defined function $f(t) = (t-1)^2 u(t-1)$. Using (2.25), this is

$$\mathcal{L}[f(t)] = e^{-s}\mathcal{L}[t^2](s) = 2\frac{1}{s^3}e^{-s}.$$

Second, consider the Laplace transform of the piecewise-constant function

$$f(t) = \begin{cases} 0 & \text{for } t < 0, \\ -1 & \text{for } 0 \le t \le 2, \\ 1 & \text{for } t > 2. \end{cases}$$

This can be expressed as $f(t) = -u(t) + 2u(t-2)$, so

$$\begin{aligned} \mathcal{L}[f(t)] &= -\mathcal{L}[u(t)] + 2\mathcal{L}[u(t-2)] \\ &= -\frac{1}{s} + 2\frac{1}{s}e^{-2s}. \end{aligned}$$

Finally, consider the Laplace transform of $f(t) = \sin(t)u(t-\pi)$. Using (2.24), this is

$$\mathcal{L}[f(t)] = e^{-\pi s}\mathcal{L}[\sin(t+\pi)](s) = e^{-\pi s}\mathcal{L}[-\sin(t)](s) = -e^{-\pi s}\frac{1}{s^2+1}.$$

The plot of this function $f(t) = \sin(t)u(t-\pi)$ is displayed in Figure 2.21.

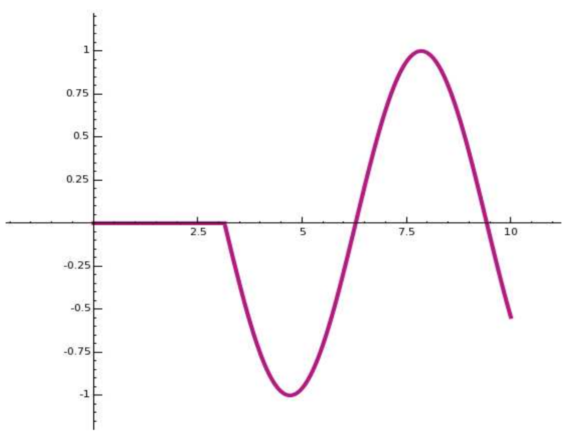


Figure 2.21: The piecewise continuous function $u(t-\pi)\sin(t)$.

We show how Sage can be used to compute these Laplace transforms.

Sage

```
sage: t = var('t')
sage: s = var('s')
sage: assume(s>0)
```

```
sage: f = Piecewise([[(0,1),0],[(1,infinity),(t-1)^2]])
sage: f.laplace(t, s)
2*e^(-s)/s^3
sage: f = Piecewise([[(0,2),-1],[(2,infinity),2]])
sage: f.laplace(t, s)
3*e^(-(2*s))/s - 1/s
sage: f = Piecewise([[(0,pi),0],[(pi,infinity),sin(t)]])
sage: f.laplace(t, s)
-e^(-(pi*s))/(s^2 + 1)
sage: f1 = lambda t: 0
sage: f2 = lambda t: sin(t)
sage: f = Piecewise([[(0,pi),f1],[(pi,10),f2]])
sage: P = f.plot(rgbcolor=(0.7,0.1,0.5),thickness=3)
sage: show(P)
```

The plot given by these last few commands is displayed in Figure 2.21.

Before turning to differential equations, let us introduce convolutions.

Let $f(t)$ and $g(t)$ be continuous (for $t \geq 0$ - for $t < 0$, we assume $f(t) = g(t) = 0$). The *convolution* of $f(t)$ and $g(t)$ is defined by

$$(f * g) = \int_0^t f(u)g(t-u)\,du = \int_0^t f(t-u)g(u)\,du.$$

The **convolution theorem** states

$$\mathcal{L}[f * g(t)](s) = F(s)G(s) = \mathcal{L}[f](s)\mathcal{L}[g](s).$$

The Laplace transform of the convolution is the product of the Laplace transforms. (Or, equivalently, the inverse Laplace transform of the product is the convolution of the inverse Laplace transforms.)

To show this, do a change-of-variables in the following double integral:

$$\begin{aligned}
\mathcal{L}[f * g(t)](s) &= \int_0^\infty e^{-st} \int_0^t f(u)g(t-u)\,du\,dt \\
&= \int_0^\infty \int_u^\infty e^{-st} f(u)g(t-u)\,dt\,du \\
&= \int_0^\infty e^{-su} f(u) \int_u^\infty e^{-s(t-u)} g(t-u)\,dt\,du \\
&= \int_0^\infty e^{-su} f(u)\,du \int_0^\infty e^{-sv} g(v)\,dv \\
&= \mathcal{L}[f](s)\mathcal{L}[g](s).
\end{aligned}$$

Example 2.8.9. Consider the inverse Laplace transform of $\frac{1}{s^3-s^2}$. This can be computed using partial fractions and Laplace transform tables. However, it can also be computed using convolutions.

First we factor the denominator, as follows

$$\frac{1}{s^3 - s^2} = \frac{1}{s^2}\frac{1}{s-1}.$$

We know the inverse Laplace transforms of each term:

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t, \qquad \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] = e^t$$

We apply the convolution theorem:

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{s^2}\frac{1}{s-1}\right] &= \int_0^t ue^{t-u}\,du \\
&= e^t\left[-ue^{-u}\right]_0^t - e^t\int_0^t -e^{-u}\,du \\
&= -t - 1 + e^t
\end{aligned}$$

Therefore,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\frac{1}{s-1}\right](t) = e^t - t - 1.$$

Example 2.8.10. Here is a neat application of the convolution theorem. Consider the convolution

$$f(t) = 1 * 1 * 1 * 1 * 1.$$

What is it? First, take the Laplace transform. Since the Laplace transform of the convolution is the product of the Laplace transforms:

$$\mathcal{L}[1 * 1 * 1 * 1 * 1](s) = (1/s)^5 = \frac{1}{s^5} = F(s).$$

We know from Laplace transform tables that $\mathcal{L}^{-1}\left[\frac{4!}{s^5}\right](t) = t^4$, so

$$f(t) = \mathcal{L}^{-1}\left[F(s)\right](t) = \frac{1}{4!}\mathcal{L}^{-1}\left[\frac{4!}{s^5}\right](t) = \frac{1}{4!}t^4.$$

You can also compute $1 * 1 * 1 * 1 * 1$ directly in Sage :

Sage

```
sage: t,z = var("t,z")
sage: f(t) = 1
sage: ff = integral(f(t-z)*f(z),z,0,t); ff
t
sage: fff = integral(f(t-z)*ff(z),z,0,t); fff
1/2*t^2
sage: ffff = integral(f(t-z)*fff(z),z,0,t); ffff
```

```
1/6*t^3
sage: fffff = integral(f(t-z)*ffff(z),z,0,t); fffff
1/24*t^4
sage: s = var("s")
sage: (1/s^5).inverse_laplace(s,t)
1/24*t^4
```

Now let us turn to solving a DE of the form

$$ay'' + by' + cy = f(t), \quad y(0) = y_0, \quad y'(0) = y_1. \tag{2.26}$$

First, take Laplace transforms of both sides:

$$as^2Y(s) - asy_0 - ay_1 + bsY(s) - by_0 + cY(s) = F(s),$$

so

$$Y(s) = \frac{1}{as^2 + bs + c}F(s) + \frac{asy_0 + ay_1 + by_0}{as^2 + bs + c}. \tag{2.27}$$

The function $\frac{1}{as^2+bs+c}$ is sometimes called the **transfer function** (this is an engineering term) and its inverse Laplace transform,

$$w(t) = \mathcal{L}^{-1}\left[\frac{1}{as^2 + bs + c}\right](t),$$

the **weight function** for the DE.

Lemma 2.8.1. *If $a \neq 0$ then $w(t) = 0$.*

(The only proof we have of this is a case-by-case proof using Laplace transform tables. Case 1 is when the roots of $as^2 + bs + c = 0$ are real and distinct, case 2 is when the roots are real and repeated, and case 3 is when the roots are complex. In each case, $w(0) = 0$. The verification of this is left to the reader, if he or she is interested.)

By the above lemma and the first derivative theorem,

$$w'(t) = \mathcal{L}^{-1}\left[\frac{s}{as^2 + bs + c}\right](t).$$

Using this and the convolution theorem, the inverse Laplace transform of (2.27) is

$$y(t) = (w * f)(t) + ay_0 \cdot w'(t) + (ay_1 + by_0) \cdot w(t). \tag{2.28}$$

This proves the following fact.

Theorem 2.8.1. *The unique solution to the DE (2.26) is (2.28).*

Example 2.8.11. Consider the DE $y'' + y = 1$, $y(0) = y'(0) = 1$.

The weight function is the inverse Laplace transform of $\frac{1}{s^2+1}$, so $w(t) = \sin(t)$. By (2.28),

$$y(t) = 1 * \sin(t) = \int_0^t \sin(u)\,du = -\cos(u)|_0^t = 1 - \cos(t).$$

(Yes, it is just that easy!)

If the "impulse" $f(t)$ is piecewise-defined, sometimes the convolution term in the formula (2.28) is awkward to compute.

Example 2.8.12. Consider the DE $y'' - y' = u(t-1)$, $y(0) = y'(0) = 0$.

Taking Laplace transforms gives $s^2 Y(s) - sY(s) = \frac{1}{s}e^{-s}$, so

$$Y(s) = \frac{1}{s^3 - s^2}e^{-s}.$$

We know from a previous example that

$$\mathcal{L}^{-1}\left[\frac{1}{s^3 - s^2}\right](t) = e^t - t - 1,$$

so by the translation theorem (2.25), we have

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s^3 - s^2}e^{-s}\right](t) = (e^{t-1} - (t-1) - 1) \cdot u(t-1) = (e^{t-1} - t) \cdot u(t-1).$$

At this stage, Sage lacks the functionality to solve this differential equation as easily as others but we can still use Sage to help witht he solution.

First, we initialize some variables and take the Laplace transform of $f(t) = u(t-1)$.

Sage

```
sage: s,t,X = var('s,t,X')
sage: x = function("x",t)
sage: f1 = 0
sage: f2 = 1
sage: f = Piecewise([[(0,1),f1],[(1,Infinity),f2]])
sage: F = f.laplace(t,s); F
e^(-s)/s
```

Next, we take the Laplace transform of the differential equation with an arbitrary function in place of $u(t-1)$.

Sage

```
sage: ft = function("ft",t)
sage: de = diff(x,t,t) - x==ft
sage: LTde = laplace(de,t,s); LTde
s^2*laplace(x(t), t, s) - s*x(0) - laplace(x(t), t, s) - D[0](x)(0) ==
```

```
laplace(ft(t), t, s)
```

Next,we take this equation and solve it for $X(s)$ using Python's string manipulation functionality:

Sage

```
sage: strLTde = str(LTde).replace("laplace(x(t), t, s)",  "X")
sage: strLTde0 = strLTde.replace("x(0)","0")
sage: strLTde00 = strLTde0.replace("D[0](x)(0)","0")
sage: strLTde00F = strLTde00.replace("laplace(ft(t), t, s)",  "F")
sage: strLTde00F
's^2*X - s*0 - X - 0 == F'
sage: LTde00F = sage_eval(strLTde00F,locals={"s":s,"X":X,"F":F})
sage: LTde00F
X*s^2 - X == e^(-s)/s
sage: soln = solve(LTde00F,X)
sage: Xs = soln[0].rhs(); Xs
e^(-s)/(s^3 - s)
sage: Xs.partial_fraction()
1/2*e^(-s)/(s - 1) + 1/2*e^(-s)/(s + 1) - e^(-s)/s
```

Unfortunately, at this time, Sage cannot take the inverse Laplace transform of this.

Exercise: Use Sage to solve the following problems.

(a) Find the Laplace transform of $u(t-\pi/4)\cos(t)$.

(b) Compute the convolution $\sin(t) * \cos(t)$. Do this directly and using the convolution theorem.

Chapter 3

Matrix theory and systems of DEs

...there is no study in the world which brings into more harmonious action all the faculties of the mind than [mathematics] ...
- *James Joseph Sylvester*

In order to handle systems of differential equations, in which there is more than one dependent variable, it is necessary to learn some linear algebra. It is best to take a full course in that subject, but for those cases where that is not possible we aim to provide the required background in this chapter. In contrast to a linear algebra textbook *per se* we will omit many proofs. Linear algebra is a tremendously useful subject to understand, and we encourage the reader to take a more complete course in it if at all possible. An excellent free reference for linear algebra is the text by Robert Beezer [B-rref].

3.1 Row reduction and solving systems of equations

Row reduction is the engine that drives a lot of the machinary of matrix theory. What we call *row reduction* others call *computing the reduced row echelon form* or *Gauss-Jordan reduction* or *Gauss elimination*.

3.1.1 The Gauss elimination game

This is actually a discussion of solving systems of equations using the method of *row reduction*, but it's more fun to formulate it in terms of a game.

To be specific, let's focus on a 2×2 system (by "2×2" I mean 2 equations in the 2 unknowns x, y):

$$\left\{ \begin{array}{rcl} ax + by & = & r_1 \\ cx + dy & = & r_2 \end{array} \right. \tag{3.1}$$

Here a, b, c, d, r_1, r_2 are given constants. Putting these two equations down together means to solve them simultaneously, not individually. In geometric terms, you may think of each equation above as a line the the plane. To solve them simultaneously, you are to find the point of intersection (if it exists) of these two lines. Since a, b, c, d, r_1, r_2 have not been specified, it is conceivable that there are

- no solutions (the lines are parallel but distinct),
- infinitely many solutions (the lines are the same),
- exactly one solution (the lines are distinct and not parallel).

"Usually" there is exactly one solution. Of course, you can solve this by simply manipulating equations since it is such a low-dimensional system but the object of this lecture is to show you a method of solution which is "scalable" to "industrial-sized" problems (say 1000×1000 or larger).

Strategy:

Step 1: Write down the *augmented matrix* of (3.1):

$$A = \begin{pmatrix} a & b & r_1 \\ c & d & r_2 \end{pmatrix}$$

This is simply a matter of stripping off the unknowns and recording the coefficients in an array. Of course, the system must be written in "standard form" (all the terms with "x" get aligned together, ...) to do this correctly.

Step 2: Play the Gauss elimination game (described below) to computing the row reduced echelon form of A, call it B say.

Step 3: Read off the solution from the right-most column of B.

The Gauss Elimination Game

Legal moves: These actually apply to any $m \times n$ matrix A with $m < n$.

1. $R_i \leftrightarrow R_j$: You can swap row i with row j.
2. $cR_i \to R_i$ ($c \neq 0$): You can replace row i with row i multiplied by any non-zero constant c. (Don't confuse this c with the c in (3.1)).
3. $cR_i + R_j \to R_i$ ($c \neq 0$): You can replace row i with row i multiplied by any non-zero constant c plus row j, $j \neq i$.

Note that move 1 simply corresponds to reordering the system of equations (3.1)). Likewise, move 2 simply corresponds to scaling equation i in (3.1)). In general, these "legal moves" correspond to algebraic operations you would perform on (3.1)) to solve it. However, there are fewer symbols to push around when the augmented matrix is used.

Goal: You *win* the game when you can achieve the following situation. Your goal is to find a sequence of legal moves leading to a matrix B satisfying the following criteria:

1. all rows of B have leaading non-zero term equal to 1 (the position where this leading term in B occurs is called a *pivot position*),

2. B contains as many 0's as possible

3. all entries above and below a pivot position must be 0,

4. the pivot position of the i^{th} row is to the left and above the pivot position of the $(i+1)^{st}$ row (therefore, all entries below the diagonal of B are 0).

This matrix B is unique (this is a theorem which you can find in any text on elementary matrix theory or linear algebra[1]) and is called the *row reduced echelon form* of A, sometimes written $rref(A)$.

Two comments: (1) If you are your friend both start out playing this game, it is likely your choice of legal moves will differ. That is to be expected. However, you must get the same result in the end. (2) Often if someone is to get "stuck" it is becuase they forget that one of the goals is to "kill as many terms as possible (i.e., you need B to have as many 0's as possible). If you forget this you might create non-zero terms in the matrix while killing others. You should try to think of each move as being made in order to to kill a term. The exception is at the very end where you can't kill any more terms but you want to do row swaps to put it in diagonal form.

Now it's time for an example.

Example 3.1.1. Solve

$$\left\{ \begin{array}{rcl} x+2y &=& 3 \\ 4x+5y &=& 6 \end{array} \right. \tag{3.2}$$

using row reduction.

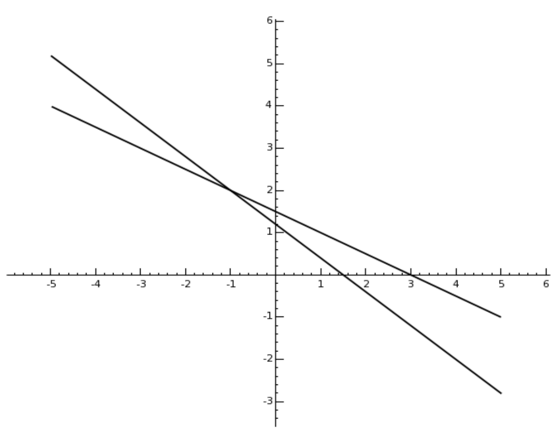


Figure 3.1: lines $x+2y=3$, $4x+5y=6$ in the plane.

The augmented matrix is

[1] For example, [B-rref] or [H-rref].

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

One sequence of legal moves is the following:

$-4R_1 + R_2 \to R_2$, which leads to $\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{pmatrix}$

$-(1/3)R_2 \to R_2$, which leads to $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$

$-2R_2 + R_1 \to R_1$, which leads to $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$

Now we are done (we won!) since this matrix satisfies all the goals for a row reduced echelon form.

The latter matrix corresponds to the system of equations

$$\begin{cases} x + 0y & = & -1 \\ 0x + y & = & 2 \end{cases} \tag{3.3}$$

Since the "legal moves" were simply matrix analogs of algebraic manipulations you'd appy to the system (3.2), the solution to (3.2) is the same as the solution to (3.3), whihc is obviously $x = -1, y = 2$. You can visually check this from the graph given above.

To find the row reduced echelon form of

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

using Sage , just type the following:

Sage

```
sage: MS = MatrixSpace(QQ,2,3)
sage: A = MS([[1,2,3],[4,5,6]])
sage: A
[1 2 3]
[4 5 6]
sage: A.echelon_form()
[ 1  0 -1]
[ 0  1  2]
```

3.1.2 Solving systems using inverses

There is another method of solving "square" systems of linear equations which we discuss next.

One can rewrite the system (3.1) as a single matrix equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix},$$

or more compactly as

$$A\vec{X} = \vec{r}, \tag{3.4}$$

where $\vec{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\vec{r} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$. How do you solve (3.4)? The obvious this to do ("divide by A") is the right idea:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \vec{X} = A^{-1}\vec{r}.$$

Here A^{-1} is a matrix with the property that $A^{-1}A = I$, the identity matrix (which satisfies $I\vec{X} = \vec{X}$).

If A^{-1} exists (and it usually does), how do we compute it? There are a few ways. One, if using a formula. In the 2×2 case, the inverse is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

There is a similar formula for larger sized matrices but it is so unwieldy that is is usually not used to compute the inverse.

Example 3.1.2. In the 2×2 case, the formula above for the inverse is easy to use and we see for example,

$$\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}^{-1} = \frac{1}{-3} \begin{pmatrix} 5 & -2 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} -5/3 & 2/3 \\ 4/3 & -1/3 \end{pmatrix}.$$

To find the inverse of

$$\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$$

using Sage , just type the following:

Sage

```
sage: MS = MatrixSpace(QQ,2,2)
sage: A = MS([[1,2],[4,5]])
sage: A
[1 2]
[4 5]
sage: A^(-1)
[-5/3  2/3]
[ 4/3 -1/3]
```

A better way to compute A^{-1} is the following. Compute the row reduced echelon form of the matrix (A, I), where I is the identity matrix of the same size as A. This new matrix will be (if the inverse exists) (I, A^{-1}). You can read off the inverse matrix from this.

In other words, the following result holds.

Lemma 3.1.1. Let A be an invertible $n \times n$ matrix. The row reduced echelon form of the $n \times 2n$ matrix (A, I), where I is the identity matrix of the same size as A, is the $n \times 2n$ matrix (I, A^{-1}).

Here is an example.

Example 3.1.3. Solve

$$\begin{cases} x + 2y &= 3 \\ 4x + 5y &= 6 \end{cases}$$

using (a) row reduction, (b) matrix inverses.

In fact, this system was solved using row-reduction in Example 3.1.1 above. The idea was that the result of the Sage commands

Sage

```
sage: MS = MatrixSpace(QQ,2,3)
sage: A = MS([[1,2,3],[4,5,6]])
sage: A
[1 2 3]
[4 5 6]
sage: A.echelon_form()
[ 1  0 -1]
[ 0  1  2]
```

told us that $x = -1$ and $y = 2$. You just read off the last column of the row-reduced echelon form of the matrix.

This is

$$\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix},$$

so

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 6 \end{pmatrix}.$$

To compute the inverse matrix, apply the Gauss elimination game to

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{pmatrix}$$

Using the same sequence of legal moves as in the previous example, we get

$-4R_1 + R_2 \to R_2$, which leads to $\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -3 & -4 & 1 \end{pmatrix}$

$-(1/3)R_2 \to R_2$, which leads to $\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 4/3 & -1/3 \end{pmatrix}$

$-2R_2 + R_1 \to R_1$, which leads to $\begin{pmatrix} 1 & 0 & -5/3 & 2/3 \\ 0 & 1 & 4/3 & -1/3 \end{pmatrix}$.

Therefore the inverse is

$$A^{-1} = \begin{pmatrix} -5/3 & 2/3 \\ 4/3 & -1/3 \end{pmatrix}.$$

Now, to solve the system, compute

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} -5/3 & 2/3 \\ 4/3 & -1/3 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

To make Sage do the above computation, just type the following:

Sage

```
sage: MS = MatrixSpace(QQ,2,2)
sage: A = MS([[1,2],[4,5]])
sage: V = VectorSpace(QQ,2)
sage: v = V([3,6])
sage: A^(-1)*v
 (-1, 2)
```

Of course, this again tells us that $x = -1$ and $y = 2$ is the solution to the original system.

Exercise: Using Sage , solve

$$\begin{cases} x + 2y + z &= 1 \\ -x + 2y - z &= 2 \\ y + 2z &= 3 \end{cases}$$

using (a) row reduction, (b) matrix inverses.

3.1.3 Solving higher-dimensional linear systems

Gauss-Jordan reduction, revisited.

Example 3.1.4. Solve the linear system

$$\begin{aligned} x + 2y + 3z &= 0, \\ 4x + 5y + 6z &= 3, \\ 7x + 8y + 9z &= 6 \end{aligned}$$

We form the augmented coefficient matrix and row-reduce until we obtain the reduced row-echelon form:

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 3 \\ 7 & 8 & 9 & 6 \end{pmatrix} \xrightarrow{-4R_1+R_2} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 3 \\ 7 & 8 & 9 & 6 \end{pmatrix}$$

$$\xrightarrow{-7R_1+R_3} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 3 \\ 0 & -6 & -12 & 6 \end{pmatrix} \xrightarrow{-R_2/3} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & -6 & -12 & 6 \end{pmatrix}$$

$$\xrightarrow{-R_3/6} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \end{pmatrix} \xrightarrow{-R_2+R_3} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-2R_2+R_1} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Reinterpreting these rows as equations we have $x - z = 2$ and $y + 2z = -1$. The pivot variables are x and y, and z is the only free variable. Finally, we write the solutions in parametric vector form:

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 2+t \\ -1-2t \\ t \end{pmatrix}.$$

Exercises:

1. Write the following system as a matrix equation, and find all the solutions.

$$\begin{aligned} 3x + 2y + z &= 2 \\ x - y + 2z &= 2 \\ 3x + 7y - 4z &= -2 \end{aligned}$$

3.2 Quick survey of linear algebra

3.2.1 Matrix arithmetic

Matrix multiplication. Noncommutativity. Failure of cancellation. Zero and identity matrices. Matrix inverses.

Lemma 3.2.1. *If A and B are invertible $n \times n$ matrices, then AB is invertible with inverse $B^{-1}A^{-1}$.*

We can use row-reduction to compute a matrix inverse if it exists.

Example 3.2.1. To invert

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

we first augment it with the 3×3 identity matrix, and then row-reduce the lefthand block until we obtain the identity matrix:

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{-R_1+R_2} \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1+R_3} \left(\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 & 1 \end{array}\right) \xrightarrow{Swap(R_2,R_3)} \left(\begin{array}{cccccc} 1 & 0 & 0 & 4 & -5 & -2 \\ 0 & 1 & 0 & -3 & 4 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right).$$

So $A^{-1} = \left(\begin{array}{ccc} 4 & -5 & -2 \\ -3 & 4 & 1 \\ 1 & -1 & 0 \end{array}\right)$.

Exercises:

1. For $A = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$, compute the product $A * A$.
2. Find as many 2 by 2 matrices A such that $A * A = I$ as you can. Do you think you have found all of them?
3. Find the inverse A^{-1} of the matrix $A = \left[\begin{array}{ccc} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right]$ by using row operations (multiplying rows, adding a multiple of one row to another, and interchanging rows) on the matrix A adjoined to the 3×3 identity matrix.
4. For the previous exercise, write down the elementary matrices that perform each of the row operations used.

3.2.2 Determinants

Definition 3.2.1. *The* **$(i,\ j)$-th minor** *of a matrix A is the determinant of the of the submatrix obtained by deleting row i and column j from A.*

Example 3.2.2. The 2,2 minor of $\left(\begin{array}{ccc} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 2 & 3 & 4 \end{array}\right)$ is equal to $\det \left(\begin{array}{cc} 1 & -1 \\ 2 & 4 \end{array}\right) = 6$.

Exercises:

1. Find a three by three matrix with strictly positive entries ($a_{ij} > 0$ for each entry a_{ij}) whose determinant is equal to 1. Try to find such a matrix with the smallest sum of entries, that is, such that $\sum_{i=1,j=1}^{i=3,j=3} a_{ij} = a_{11} + a_{12} + \ldots + a_{33}$ is as low as possible.
2. The $n \times n$ Vandermonde determinant is defined as:

$$V(x_1, x_2, \ldots, x_n) = \left|\begin{array}{ccccc} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{array}\right|$$

   Show that the 2×2 Vandermonde determinant $V(a, b) = b - a$. Then show that the 3×3 Vandermonde determinant $V(1, a, b)$ can be factored into $(a - 1)(b - 1)(b - a)$.

3.2.3 Vector spaces

So far we have worked with vectors which are n-tuples of numbers. But the essential operations we perform on such tuples can be generalized to a much wider class of objects, which we also call vectors. A collection of vectors that are compatible in the sense that they can be added together and multiplied by some sort of scalar is called a vector space. Here is a more formal definition.

Definition 3.2.2. *A* **vector space** V *is a set* S *on which there is an operation of addition which take pairs of elements of* S *to elements of* S*, together with a field of numbers* K *for which there is an operation* $*$ *of scalar multiplication. These operations must have the following properties:*

1. *There exists a unique* $\mathbf{0} \in S$ *such that* $\mathbf{0} + v = v$ *for all* $v \in S$*.*
2. $v + w = w + v$ *for all* $v, w \in S$*.*
3. $v + (w + x) = (v + w) + x$ *for all* $v, w, x \in S$*.*
4. $(-1) * v + v = \mathbf{0}$ *for all* $v \in S$*.*
5. $0 * v = \mathbf{0}$*.*
6. $1 * v = v$ *for all* $v \in S$*.*
7. $(s + t) * (v + w) = s * v + t * v + s * w + t * w$ *for all* $s, t \in K$ *and all* $v, w \in S$*.*
8. $s * (t * v) = (s * t) * v$ *for all* $s, t \in K$ *and all* $v \in S$*.*

For convenience we do not usually explicitly indicate scalar multiplication with a $*$, so we write $5 * v$ simply as $5v$.

For our purposes the field K will always be either the field of real numbers, denoted $\mathbb{R}$, or the field of complex numbers, denoted by $\mathbb{C}$. There are many other fields of numbers used in mathematics but they will not be addressed here so we will not formally define the concept of a field. Unless indicated otherwise, we will use $\mathbb{R}$ as the field in all of our vector spaces.

Here are some vector space examples:

1. For each positive integer n, the set of lists of n real numbers, $\mathbb{R}^n$, is a vector space. For $n = 2$ and $n = 3$ these are the familiar real planar vectors and real spatial vectors.
2. The set of continuous real-valued functions on an interval $[a, b]$ forms a vector space, denoted $C([a, b])$. The addition is the usual addition of functions.
3. For each positive integer n, the set of polynomials of degree at most n forms a vector space. We will denote this space by P_n.

Definition 3.2.3. *A* **subspace** *of a vector space* V *is a subset of elements of* V *that is itself a vector space.*

The important thing to understand about subspaces is that they must be closed under the operations of scalar multiplication and vector addition - not every subset of a vector space is a subspace. In particular, every subspace W must contain the 0 vector since if $w \in W$, then $-w \in W$, and then so is $w + -w = 0$.

Here are some vector subspace examples:

1. The set $W = \{(x, x) | x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$. If we think of $\mathbb{R}^2$ as the x, y plane, then W is simply the line $y = x$.

2. The set $W = \{(x, -2x) | x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$. If we think of $\mathbb{R}^2$ as the x, y plane, then W is simply the line $y = -2x$. In fact every line through the origin is a subspace.

Exercises:

1. Are the following sets subspaces of $\mathbb{R}^3$ or not? If not, explain why.

   (a) $\{(x_1, x_2, x_3) |\ x_1 * x_2 * x_3 = 0\}$
   (b) $\{(x_1, x_2, 0) |\ x_1 = 5 * x_2\}$
   (c) The span of the vectors $(1, 2, 3)$, $(4, 5, 6)$ and $(7, 8, 9)$.

3.2.4 Bases, dimension, linear independence and span

This section briefly recalls some basic notions of linear algebra.

Definition 3.2.4. *A set of vectors* $\{v_1, \ldots, v_m\}$ *is* **linearly dependent** *if there are constants* $c_1, \ldots, c_m$ *which are not all zero for which* $c_1 v_1 + \ldots + c_m v_m = 0$.

An expression of the form $c_1 v_1 + \ldots + c_m v_m$, for constants $c_1, \ldots, c_m$, is called a **span** or **linear combination** of the vectors in $\{v_1, \ldots, v_m\}$.

Example 3.2.3. The vectors $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$, and $v_3 = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$ are linearly dependent. To see this, we can write the linear dependence condition $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ as a matrix-vector equation:

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and solve the system with row-reduction. Since it is a homogeneous system (i.e. the left-hand side is the zero vector), we only need to reduce the coefficient matrix:

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \xrightarrow{-2R_1+R_2} \begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 3 & 6 & 9 \end{pmatrix} \xrightarrow{-3R_1+R_3} \begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix}$$

$$\xrightarrow{-2R_2+R_3} \begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{-R_2/3} \begin{pmatrix} 1 & 4 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{-4R_2+R_1} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

which tells us that c_3 is a free variable, so it can be chosen to be nonzero, and the vectors are linearly dependent. As a particular example, we could choose $c_3 = 1$, and then $c_1 = 5$ and $c_2 = -3$, so $5v_1 - 3v_2 + v_3 = 0$.

Definition 3.2.5. *A* **basis** *of a vector space V is a set of linearly independent vectors that span V.*

Example 3.2.4.

Theorem 3.2.1. *Every basis of a vector space has the same number of elements, if finite.*

Example 3.2.5.

Definition 3.2.6. *The* **dimension** *of a vector space V is the number of elements in a basis of V. If the bases of V are not finite, we say V is infinite-dimensional.*

Example 3.2.6. Compute the dimension of the subspace W of $\mathbb{R}^4$ spanned by the vectors $v_1 = (1, 1, 1, 1)^T$, $v_2 = (1, 2, 1, 2)^T$, and $v_3 = (1, 3, 1, 3)^T$.

To compute the dimension, we need to know if v_1, v_2, and v_3 form a basis of W. Since W is defined as their span, we only need to check if they are linearly dependent or not. As before, we do this by row-reducing a matrix whose columns consist of the v_i:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \xrightarrow{-R_1+R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \xrightarrow{-R_1+R_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\xrightarrow{-R_1+R_4} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow{-R_2+R_4} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This shows that only two of the three vectors are linearly independent, so the dimension of W is equal to two.

Exercises:

1. Find a basis for the subspace defined by the following equations for $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$:

$$\begin{aligned} 2x_1 + x_3 - 2x_4 - 2x_5 &= 0 \\ x_1 + 2x_3 - x_4 + 2x_5 &= 0 \\ -3x_1 - 4x_3 + 3x_4 - 2x_5 &= 0 \end{aligned}$$

2. Consider the triple of vectors $v_1 = (0, 2, 3, -2)$, $v_2 = (3, -1, 4, 1)$, and $v_3 = (6, -8, -1, 8)$.

   (a) Is the set $\{v_1, v_2, v_3\}$ linearly independent or dependent?

   (b) What is the dimension of their span?

   (c) If the vectors are linearly independent, find an additional vector v_4 that makes $\{v_1, v_2, v_3, v_4\}$ a basis for $\mathbb{R}^4$. If they are linearly dependent, write v_1 as a linear combination of v_2 and v_3.

3.2.5 The Wronskian

We have run into the Wronskian already in connection with the definition of the fundamental solution of a linear second order DE. This section introduces a clever formula for computing the Wronskian (up to a constant factor) which avoids determinants.

Definition 3.2.7. *If $F = (f_1, f_2, \ldots, f_n)$ is a list of functions then the* **Wronskian** *of F is*

$$W(F) = \det \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{pmatrix}.$$

It is somewhat surprising that the Wronskian of two solutions of a linear, homogeneous second-order ODE can be computed without knowing the solutions, a result known as **Abel's theorem**.

Theorem 3.2.2. *(Abel's Theorem) If y_1 and y_2 are two linearly independent solutions of $y'' + py' + qy = 0$, then $W(y_1, y_2) = Ce^{-\int p\, dx}$.*

Proof. $W(y_1, y_2) = Ce^{-\int pdx}$ if and only if $W(y_1, y_2)$ satisfies the differential equation $W' = -pW$. Since $W = y_1y_2' - y_1'y_2$, $W' = y_1y_2'' - y_1''y_2$. We also know that since the y_i satisfy $y'' + py' + qy = 0$, $y_i = -py_i' - qy_i$ for $i = 1$ and $i = 2$. If we use those relations to substitute for y_2'' and y_1'', we find that

$$W' = y_1(-py_2' - qy_2) - y_2(-py_1' - qy_1) = p(y_1'y_2 - y_1y_2') = -pW.$$

□

Exercises:

1. Compute the Wronskian of the set of functions $\{1, e^x, e^{-x}\}$.

2. Compute the dimension of the vector space spanned by the set of functions $\{1, e^x, e^{-x}\}$.

3.3 Application: Solving systems of DEs

Suppose we have a system of DEs in "standard form"

$$\left\{ \begin{array}{rcl} x' & = & ax + by + f(t), \quad x(0) = x_0, \\ y' & = & cx + dy + g(t), \quad y(0) = y_0, \end{array} \right. \tag{3.5}$$

where a, b, c, d, x_0, y_0 are given constants and $f(t), g(t)$ are given "nice" functions. (Here "nice" will be left vague but basically we don't want these functions to annoy us with any bad behaviour while trying to solve the DEs by the method of Laplace transforms.)

One way to solve this system if to take Laplace transforms of both sides. If we let

$$X(s) = \mathcal{L}[x(t)](s), Y(s) = \mathcal{L}[y(t)](s), F(s) = \mathcal{L}[f(t)](s), G(s) = \mathcal{L}[g(t)](s),$$

then (3.5) becomes

$$\left\{ \begin{array}{rcl} sX(s) - x_0 & = & aX(s) + bY(s) + F(s), \\ sY(s) - y_0 & = & cX(s) + dY(s) + G(s). \end{array} \right. \tag{3.6}$$

This is now a 2×2 system of linear equations in the unknowns $X(s), Y(s)$ with augmented matrix

$$A = \left(\begin{array}{ccc} s-a & -b & F(s) + x_0 \\ -c & s-d & G(s) + y_0 \end{array} \right).$$

Example 3.3.1. Solve

$$\left\{ \begin{array}{rcl} x' & = & -y + 1, \quad x(0) = 0, \\ y' & = & -x + t, \quad y(0) = 0, \end{array} \right.$$

The augmented matrix is

$$A = \left(\begin{array}{ccc} s & 1 & 1/s \\ 1 & s & 1/s^2 \end{array} \right).$$

The row reduced echelon form of this is

$$\left(\begin{array}{ccc} 1 & 0 & 1/s^2 \\ 0 & 1 & 0 \end{array} \right).$$

Therefore, $X(s) = 1/s^2$ and $Y(s) = 0$. Taking inverse Laplace transforms, we see that the solution to the system is $x(t) = t$ and $y(t) = 0$. It is easy to check that this is indeed the solution.

To make Sage compute the row reduced echelon form, just type the following:

Sage

```
sage: R = PolynomialRing(QQ,"s")
sage: F = FractionField(R)
sage: s = F.gen()
```

```
sage: MS = MatrixSpace(F,2,3)
sage: A = MS([[s,1,1/s],[1,s,1/s^2]])
sage: A.echelon_form()
[    1     0 1/s^2]
[    0     1     0]
```

To make Sage compute the Laplace transform, just type the following:

Sage

```
sage: s,t = var('s,t')
sage: f(t) = 1
sage: laplace(f(t),t,s)
1/s
sage: f(t) = t
sage: laplace(f(t),t,s)
s^(-2)
```

To make Sage compute the inverse Laplace transform, just type the following:

Sage

```
sage: s,t = var('s,t')
sage: F(s) = 1/s^2
sage: inverse_laplace(F(s),s,t)
t
sage: F(s) = 1/(s^2+1)
sage: inverse_laplace(F(s),s,t)
sin(t)
```

Example 3.3.2. The displacement from equilibrium (respectively) for coupled springs attached to a wall on the left

coupled springs

```
|------\/\/\/\/\---|mass1|----\/\/\/\/\/----|mass2|
         spring1              spring2
```

is modeled by the system of 2nd order ODEs

$$m_1 x_1'' + (k_1 + k_2)x_1 - k_2 x_2 = 0, \quad m_2 x_2'' + k_2(x_2 - x_1) = 0,$$

where x_1 denotes the displacement from equilibrium of mass 1, denoted m_1, x_2 denotes the displacement from equilibrium of mass 2, denoted m_2, and k_1, k_2 are the respective spring constants [CS-rref].

As another illustration of solving linear systems of equations to solving systems of linear 1st order DEs, we use Sage to solve the above problem with $m_1 = 2$, $m_2 = 1$, $k_1 = 4$, $k_2 = 2$, $x_1(0) = 3$, $x_1'(0) = 0$, $x_2(0) = 3$, $x_2'(0) = 0$.

Soln: Take Laplace transforms of the first DE, $2 * x_1''(t) + 6 * x_1(t) - 2 * x_2(t) = 0$. This says $-2x_1'(0) + 2s^2 * X_1(s) - 2sx_1(0) - 2X_2(s) + 2X_1(s) = 0$ (where the Laplace transform of a lower case function is the upper case function). Take Laplace transforms of the second DE, $2 * x_2''(t) + 2 * x_2(t) - 2 * x_1(t) = 0$. This says $s^2X_2(s) + 2X_2(s) - 2X_1(s) - 3s = 0$. Solve these two equations:

Sage

```
sage: s,X,Y = var('s X Y')
sage: eqns = [(2*s^2+6)*X-2*Y == 6*s, -2*X +(s^2+2)*Y == 3*s]
sage: solve(eqns, X,Y)
[[X == (3*s^3 + 9*s)/(s^4 + 5*s^2 + 4),
  Y == (3*s^3 + 15*s)/(s^4 + 5*s^2 + 4)]]
```

This says $X_1(s) = (3s^3+9s)/(s^4+5s^2+4)$, $X_2(s) = (3s^3+15s)/(s^4+5s^2+4)$. Take inverse Laplace transforms to get the answer:

Sage

```
sage: s,t = var('s t')
sage: inverse_laplace((3*s^3 + 9*s)/(s^4 + 5*s^2 + 4),s,t)
cos(2*t) + 2*cos(t)
sage: inverse_laplace((3*s^3 + 15*s)/(s^4 + 5*s^2 + 4),s,t)
4*cos(t) - cos(2*t)
```

Therefore, $x_1(t) = \cos(2t) + 2\cos(t)$, $x_2(t) = 4\cos(t) - \cos(2t)$. Using Sage , this can be plotted parametrically using

Sage

```
sage: P = parametric_plot([cos(2*t) + 2*cos(t),4*cos(t) - cos(2*t)],0,3)
sage: show(P)
```

3.3.1 Modeling battles using Lanchester's equations

The goal of military analysis is a means of reliably predicting the outcome of military encounters, given some basic information about the forces' status. The case of two combatants in an "aimed fire" battle was solved during World War I by Frederick William Lanchester, a British engineer in the Royal Air Force, who discovered a way to model battle-field casualties using systems of differential equations. He assumed that if two armies fight, with

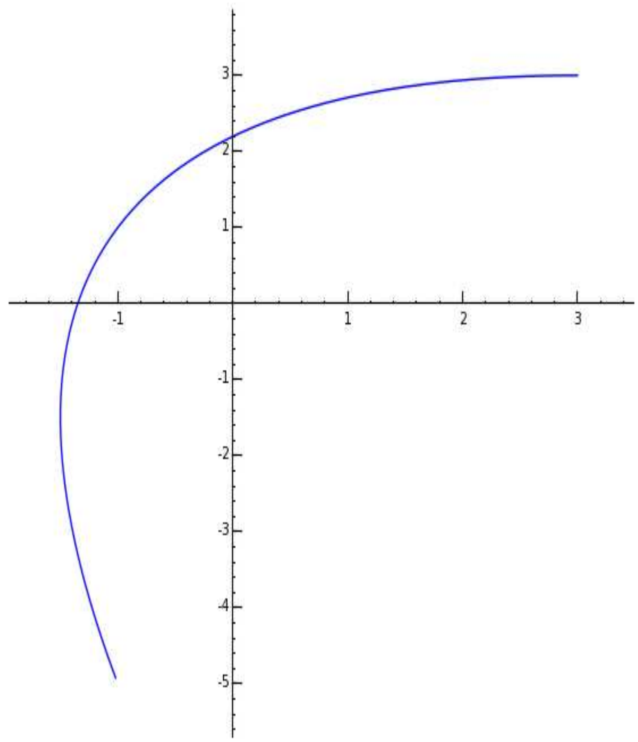


Figure 3.2: curves $x(t) = \cos(2t) + 2\cos(t)$, $y(t) = 4\cos(t) - \cos(2t)$ along the t-axis.

$x(t)$ troops on one side and $y(t)$ on the other, the rate at which soldiers in one army are put out of action is proportional to the troop strength of their enemy. This give rise to the system of differential equations

$$\begin{cases} x'(t) = -Ay(t), & x(0) = x_0, \\ y'(t) = -Bx(t), & y(0) = y_0, \end{cases}$$

where $A > 0$ and $B > 0$ are constants (called their **fighting effectiveness coefficients**) and x_0 and y_0 are the intial troop strengths. For some historical examples of actual battles modeled using Lanchester's equations, please see references in the paper by McKay [M-intro].

We show here how to solve these using Laplace transforms.

Example 3.3.3. Solve

$$\begin{cases} x' &= &-4y, & x(0) = 400, \\ y' &= &-x, & y(0) = 100, \end{cases}$$

This models a battle between "x-men" and "y-men", where the "x-men" die off at a higher rate than the "y-men" (but there are more of them to begin with too).

The augmented matrix is

$$A = \begin{pmatrix} s & 4 & 400 \\ 1 & s & 100 \end{pmatrix}.$$

The row reduced echelon form of this is

$$\begin{pmatrix} 1 & 0 & \frac{400(s-1)}{s^2-4} \\ 0 & 1 & \frac{100(s-4)}{s^2-4} \end{pmatrix}.$$

Therefore,

$$X(s) = 400\frac{s}{s^2-4} - 200\frac{2}{s^2-4}, \quad Y(s) = 100\frac{s}{s^2-4} - 200\frac{2}{s^2-4}.$$

Taking inverse Laplace transforms, we see that the solution to the system is $x(t) = 400\cosh(2t) - 200\sinh(2t)$ and $y(t) = 100\cosh(2t) - 200\sinh(2t)$. The "$x$-men" win and, in fact,

$$x(0.275) = 346.4102..., \quad y(0.275) = -0.1201... .$$

Question: What is $x(t)^2 - 4y(t)^2$? (Hint: It's a constant. Can you explain this?)

To make Sage plot this just type the following:

Sage

```
sage: f = lambda x: 400*cosh(2*x)-200*sinh(2*x)
sage: g = lambda x: 100*cosh(2*x)-200*sinh(2*x)
sage: P = plot(f,0,1)
sage: Q = plot(g,0,1)
sage: show(P+Q)
sage: g(0.275)
 -0.12017933629675781
sage: f(0.275)
 346.41024490088557
```

Here is a similar battle but with different initial conditions.

Example 3.3.4. A battle is modeled by

$$\begin{cases} x' = -4y, & x(0) = 150, \\ y' = -x, & y(0) = 90. \end{cases}$$

(a) Write the solutions in parameteric form. (b) Who wins? When? State the losses for each side.

Solution: Take Laplace transforms of both sides:

$$sL\left[x\left(t\right)\right]\left(s\right) - x\left(0\right) = -4\,L\left[y\left(t\right)\right]\left(s\right),$$

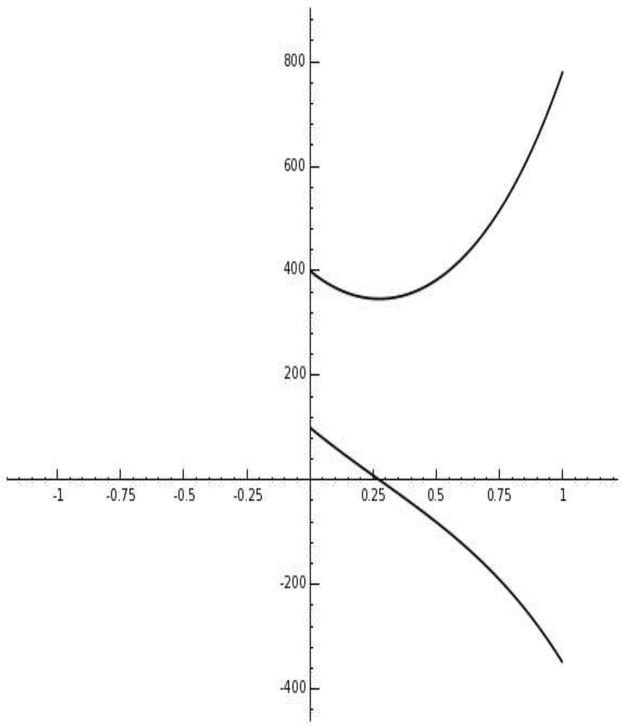


Figure 3.3: curves $x(t) = 400\cosh(2t) - 200\sinh(2t)$, $y(t) = 100\cosh(2t) - 200\sinh(2t)$ along the t-axis.

$$sL\left[x\left(t\right)\right]\left(s\right) - x\left(0\right) = -4\,L\left[y\left(t\right)\right]\left(s\right).$$

Solving these equations gives

$$L\left[x\left(t\right)\right]\left(s\right) = \frac{sx\left(0\right) - 4\,y\left(0\right)}{s^2 - 4} = \frac{150\,s - 360}{s^2 - 4},$$

$$L\left[y\left(t\right)\right]\left(s\right) = -\frac{-sy\left(0\right) + x\left(0\right)}{s^2 - 4} = -\frac{-90\,s + 150}{s^2 - 4}.$$

Inverting using Laplace transform tables gives:

$$x(t) = -15\,e^{2\,t} + 165\,e^{-2\,t}$$

$$y(t) = 90\,\cosh\left(2\,t\right) - 75\,\sinh\left(2\,t\right)$$

Their graph is given in Figure 3.4.

The "y-army" wins. Solving for $x(t) = 0$ gives $t_{win} = \log(11)/4 = .5994738182...$, so the number of survivors is $y(t_{win}) = 49.7493718$, so 49 survive.

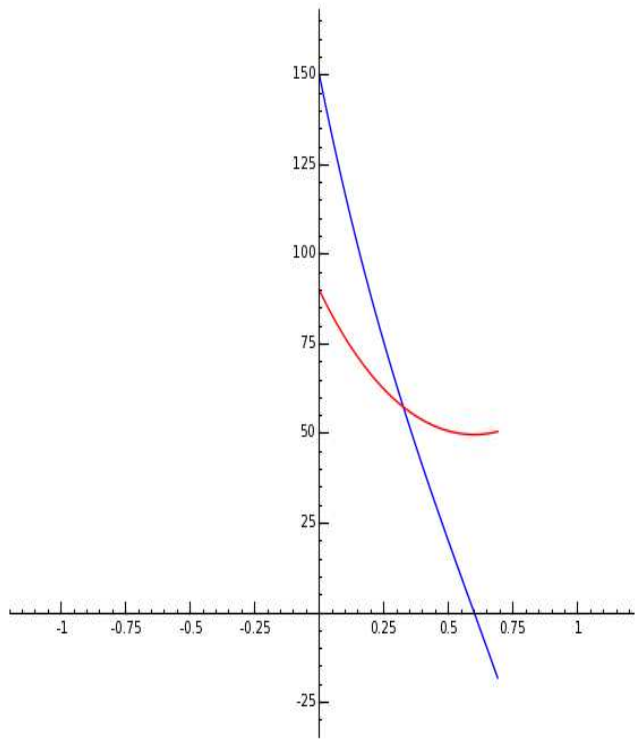


Figure 3.4: Lanchester's model for the x vs. y battle.

Lanchester's square law: Suppose that if you are more interested in y as a function of x, instead of x and y as functions of t. One can use the chain rule form calculus to derive from the system $x'(t) = -Ay(t), y'(t) = -Bx(t)$ the single equation

$$\frac{dy}{dx} = \frac{B}{A}\frac{x}{y}.$$

This differential equation can be solved by the method of separation of variables: $Aydy = Bxdx$, so

$$Ay^2 = Bx^2 + C,$$

where C is an unknown constant. (To solve for C you must be given some initial conditions.) The quantity Bx^2 is called the **fighting strength of the X-men** and the quantity Ay^2 is called the **fighting strength of the Y-men** ("fighting strength" is not to be confused with "troop strength"). This relationship between the troop strengths is sometimes called **Lanchester's square law** and is sometimes expressed as saying the relative fight strength is a constant:

$$Ay^2 - Bx^2 = \text{constant}.$$

Suppose your total number of troops is some number T, where $x(0)$ are initially placed in a fighting capacity and $T - x(0)$ are in a support role. If your troops outnumber the enemy then you want to choose the number of support units to be the smallest number such that the fighting effectiveness is not decreasing (therefore is roughly constant). The remainder should be engaged with the enemy in battle [M-intro].

A battle between three forces gives rise to the differential equations

$$\begin{cases} x'(t) = -A_1 y(t) - A_2 z(t), & x(0) = x_0, \\ y'(t) = -B_1 x(t) - B_2 z(t), & y(0) = y_0, \\ z'(t) = -C_1 x(t) - C_2 y(t), & z(0) = z_0, \end{cases}$$

where $A_i > 0$, $B_i > 0$, and $C_i > 0$ are constants and x_0, y_0 and z_0 are the intial troop strengths.

Example 3.3.5. Consider the battle modeled by

$$\begin{cases} x'(t) = -y(t) - z(t), & x(0) = 100, \\ y'(t) = -2x(t) - 3z(t), & y(0) = 100, \\ z'(t) = -2x(t) - 3y(t), & z(0) = 100. \end{cases}$$

The Y-men and Z-men are better fighters than the X-men, in the sense that the coefficient of z in 2nd DE (describing their battle with y) is higher than that coefficient of x, and the coefficient of y in 3rd DE is also higher than that coefficient of x. However, as we will see, the worst fighter wins! (The X-men do have the best defensive abilities, in the sense that A_1 and A_2 are small.)

Taking Laplace transforms, we obtain the system

$$\begin{cases} sX(s) + Y(s) + Z(s) = 100 \\ 2X(s) + sY(s) + 3Z(s) = 100, \\ 2X(s) + 3Y(s) + sZ(s) = 100, \end{cases}$$

which we solve by row reduction using the augmented matrix

$$\begin{pmatrix} s & 1 & 1 & 100 \\ 2 & s & 3 & 100 \\ 2 & 3 & s & 100 \end{pmatrix}$$

This has row-reduced echelon form

$$\begin{pmatrix} 1 & 0 & 0 & \frac{100s+100}{s^2+3s-4} \\ 0 & 1 & 0 & \frac{100s-200}{s^2+3s-4} \\ 0 & 0 & 1 & \frac{100s-200}{s^2+3s-4} \end{pmatrix}$$

This means $X(s) = \frac{100s+100}{s^2+3s-4}$ and $Y(s) = Z(s) = \frac{100s-200}{s^2+3s-4}$. Taking inverse LTs, we get the solution: $x(t) = 40e^t + 60e^{-4t}$ and $y(t) = z(t) = -20e^t + 120e^{-4t}$. In other words, the worst fighter wins!

In fact, the battle is over at $t = \log(6)/5 = 0.35...$ and at this time, $x(t) = 71.54....$ Therefore, the worst fighters, the X-men, not only won but have lost less than 30% of their men!

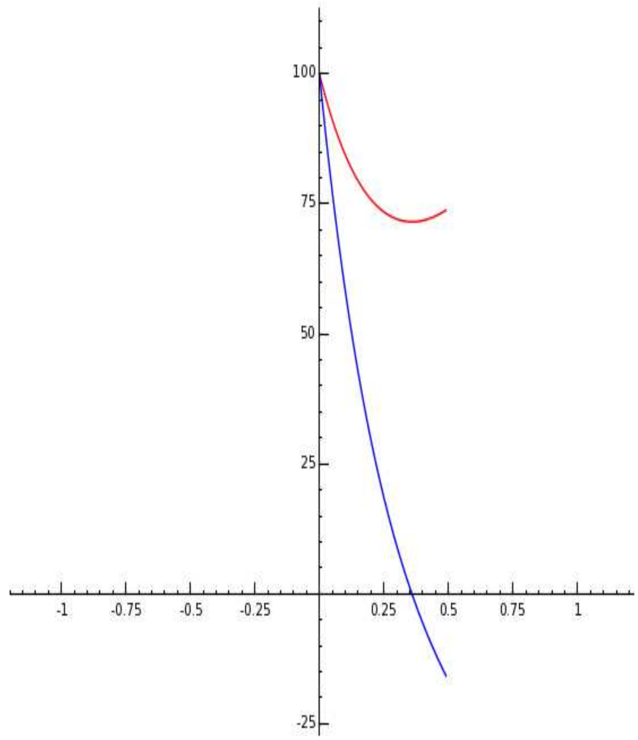


Figure 3.5: Lanchester's model for the x vs. y vs z battle.

Exercise: Use Sage to solve this: A battle is modeled by

$$\begin{cases} x' = -4y, & x(0) = 150, \\ y' = -x, & y(0) = 40. \end{cases}$$

(a) Write the solutions in parameteric form. (b) Who wins? When? State the losses for each side.

3.3.2 Romeo and Juliet

> Romeo: If I profane with my unworthiest hand This holy shrine, the gentle sin is this: My lips, two blushing pilgrims, ready stand To smooth that rough touch with a tender kiss.

> Juliet: Good pilgrim, you do wrong your hand too much, Which mannerly devotion shows in this; For saints have hands that pilgrims' hands do touch, And palm to palm is holy palmers' kiss.
> - *Romeo and Juliet*, Act I, Scene V

After solving these equations involving battles, we can't resist ending this section with the Romeo and Juliet system of equations.

William Shakespeare's play *Romeo and Juliet* about two young "star-cross'd lovers" was one of his most popular. Though written in the late 1590's, its ideas resonate even today, as the characters Romeo and Juliet have become emblematic of doomed young lovers.

Let $r = r(t)$ denote the love Romeo has for Juliet at time t and let $j = j(t)$ denote the love Juliet has for Romeo at time t.

$$\left\{ \begin{array}{lcrl} r' & = & Aj, & r(0) = r_0, \\ j' & = & -Br + Cj, & j(0) = j_0, \end{array} \right. \tag{3.7}$$

where $A > 0$, $B > 0$, $C > 0$, r_0, j_0 are given constants. This indicates how Romeo is madly in love with Juliet. Pure and simple. The more she loves him, the more he loves her. Juliet is more of a complex character. She has eyes for Romeo and her love for him makes her feel good about herself, which makes her love him even more. However, if she senses Romeo seems to love her too much, she reacts negatively and her love wanes.

A few examples illustrate the cyclical nature of the relationship that can arise.

Example 3.3.6. Solve

$$\left\{ \begin{array}{ll} r' = 5j, & r(0) =, \\ j' = -r + 2j, & j(0) = . \end{array} \right.$$

Sage

```
sage: t = var("t")
sage: r = function("r",t)
sage: j = function("j",t)
sage: de1 = diff(r,t) == 5*j
sage: de2 = diff(j,t) == -r+2*j
sage: soln = desolve_system([de1, de2], [r,j],ics=[0,4,6])
sage: rt = soln[0].rhs(); rt
(13*sin(2*t) + 4*cos(2*t))*e^t
sage: jt = soln[1].rhs(); jt
(sin(2*t) + 6*cos(2*t))*e^t
```

To solve this using Laplace transforms, take Laplace transforms of both sides, to obtain

$$sR(s) - 4 = 5J(s), \quad sJ(s) - 6 = -R(s) + 2J(s),$$

where $R(s) = \mathcal{L}[r(t)](s)$, $J(s) = \mathcal{L}[j(t)](s)$. This gives rise to the augmented matrix

$$\begin{pmatrix} s & -5 & 4 \\ 1 & s-2 & 6 \end{pmatrix}.$$

Computing the row-reduced echelon form gives

$$R(s) = -10\frac{2-3s}{s(s^2-2s+5)} + \frac{4}{s}, \quad J(s) = -2\frac{2-3s}{(s^2-2s+5}.$$

Taking inverse Laplace transforms gives

$$r(t) = (13\,\sin(2\,t) + 4\,\cos(2\,t))e^t, \quad ,j(t) = (\sin(2\,t) + 6\,\cos(2\,t))e^t.$$

The parametric plot of $x = r(t)$, $y = j(t)$ is given in Figure 3.6.

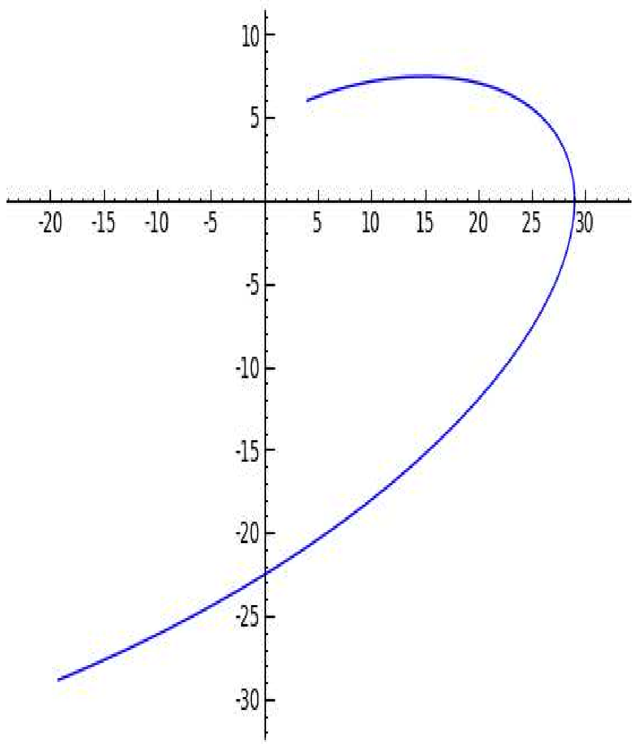


Figure 3.6: Romeo and Juliet plots.

Example 3.3.7. Solve

$$\begin{cases} r' = 10j, & r(0) = 4, \\ j' = -r + j/10, & j(0) = 6. \end{cases}$$

To solve this using Laplace transforms, take Laplace transforms of both sides, to obtain

$$sR(s) - 4 = 10J(s), \quad sJ(s) - 6 = -R(s) + \frac{1}{10}J(s),$$

where $R(s) = \mathcal{L}[r(t)](s)$, $J(s) = \mathcal{L}[j(t)](s)$. Solving this as above gives

$$r(t) = (13\sin(2t) + 4\cos(2t))e^t, \qquad j(t) = (\sin(2t) + 6\cos(2t))e^t.$$

Sage

```
sage: t = var("t")
sage: r = function("r",t)
sage: j = function("j",t)
sage: de1 = diff(r,t) == 10*j
sage: de2 = diff(j,t) == -r+(1/10)*j
sage: soln = desolve_system([de1, de2], [r,j],ics=[0,4,6])
sage: rt = soln[0].rhs(); rt
(13*sin(2*t) + 4*cos(2*t))*e^t
sage: jt = soln[1].rhs(); jt
(sin(2*t) + 6*cos(2*t))*e^t
```

The parametric plot of $x = r(t)$, $y = j(t)$ is given in Figure 3.7.

Exercise: Use Sage to analyze the problem

$$\begin{cases} r' = 2j, & r(0) = 4, \\ j' = -r + 3j, & j(0) = 6. \end{cases}$$

Exercise: (Much harder) Use Sage to analyze the Lotka-Volterra/Predator-Prey model

$$\begin{cases} x' = x(-1 + 2y), & x(0) = 4, \\ y' = y(-3x + 4y), & y(0) = 6. \end{cases}$$

Use Euler's method and separation of variables applied to $dy/dx = \frac{y(-3x+4y)}{x(-1+2y)}$.

3.3.3 Electrical networks using Laplace transforms

Suppose we have an electrical network (i.e., a series of electrical circuits) involving emfs (electromotive forces or batteries), resistors, capacitors and inductors. We use the "dictionary" from §2.6 to translate between the network diagram and the DEs.

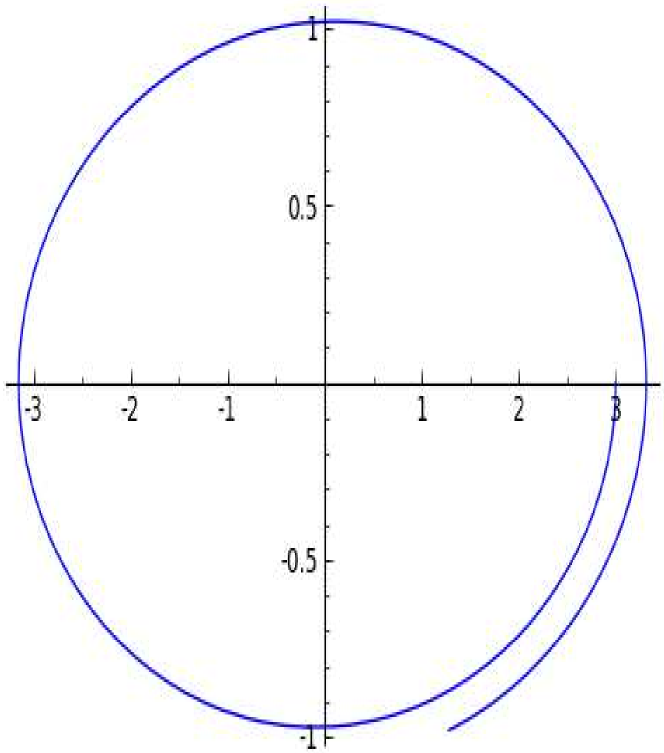
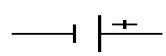
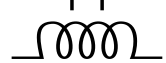


Figure 3.7: Romeo and Juliet plots.

EE object	term in DE (the voltage drop)	units	symbol
charge	$q = \int i(t)\,dt$	coulombs	
current	$i = q'$	amps	
emf	$e = e(t)$	volts V	
resistor	$Rq' = Ri$	ohms Ω	
capacitor	$C^{-1}q$	farads	
inductor	$Lq'' = Li'$	henries	

Also, recall §from §2.6 Kirchoff's Laws.

Example 3.3.8. Consider the simple RC circuit given by the diagram in Figuer 3.8.

According to Kirchoff's 2^{nd} Law and the above "dictionary", this circuit corresponds to the DE

$$q' + 5q = 2.$$

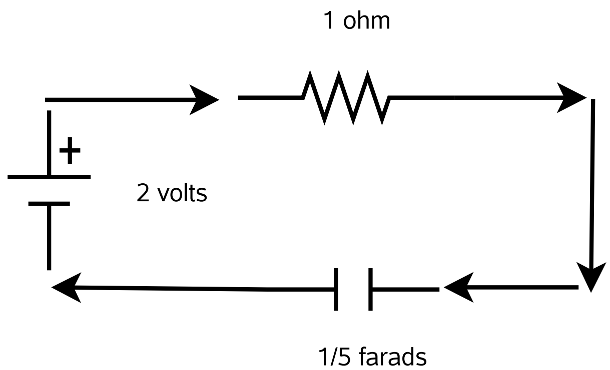


Figure 3.8: A simple circuit.

The general solution to this is $q(t) = 1 + ce^{-2t}$, where c is a constant which depends on the initial charge on the capacitor.

Aside: The convention of assuming that electricity flows from positive to negative on the terminals of a battery is referred to as "conventional flow". The physically-correct but opposite assumption is referred to as "electron flow". We shall assume the "electron flow" convention.

Example 3.3.9. Consider the network given by the following diagram.

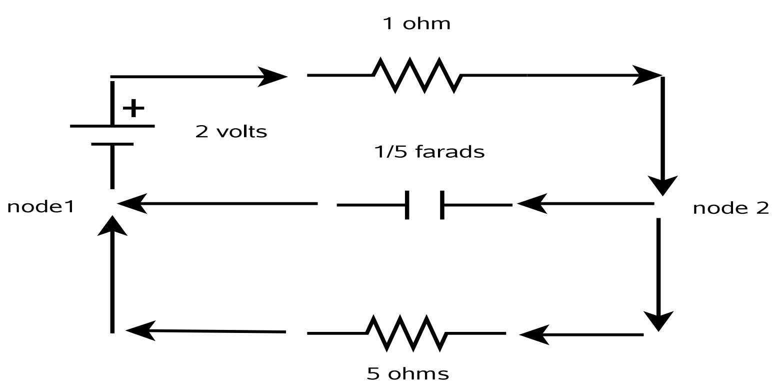


Figure 3.9: A network.

Assume the initial charges are 0.

One difference between this circuit and the one above is that the charges on the three paths between the two nodes (labeled node 1 and node 2 for convenience) must be labeled. The charge passing through the 5 ohm resistor we label q_1, the charge on the capacitor we denote by q_2, and the charge passing through the 1 ohm resistor we label q_3.

There are three closed loops in the above diagram: the "top loop", the "bottom loop", and the "big loop". The loops will be traversed in the "clockwise" direction. Note the "top loop" looks like the simple circuit given in Example 1 but it cannot be solved in the same way, since the current passing through the 5 ohm resistor will affect the charge on the capacitor. This current is not present in the circuit of Example 1 but it does occur in the network above.

Kirchoff's Laws and the above "dictionary" give

$$\begin{cases} q_3' + 5q_2 = 2, & q_1(0) = 0, \\ 5q_1' - 5q_2 = 0, & q_2(0) = 0, \\ 5q_1' + q_3' = 2, & q_3(0) = 0. \end{cases}$$

Notice the minus sign in front of the term associated to the capacitor ($-5q_2$). This is because we are going clockwise, against the "direction of the current". Kirchoff's 1^{st} law says $q_3' = q_1' + q_2'$. Since $q_1(0) = q_2(0) = q_3(0) = 0$, this implies $q_3 = q_1 + q_2$. After taking Laplace transforms of the 3 differential equations above, we get

$$sQ_3(s) + 5Q_2(s) = 2/s, \qquad 5sQ_1(s) - 5Q_2(s) = 0.$$

Note you don't need to take th eLT of the 3^{rd} equation since it is the sum of the first two equations. The LT of the above $q_1+q_2 = q_3$ (Kirchoff's law) gives $Q_1(s)+Q_2(s)-Q_3(s) = 0$. We therefore have this matrix equation

$$\begin{pmatrix} 0 & 5 & s \\ 5s & 0 & s \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} Q_1(s) \\ Q_2(s) \\ Q_3(s) \end{pmatrix} = \begin{pmatrix} 2/s \\ 2/s \\ 0 \end{pmatrix}.$$

The augmented matrix describing this system is

$$\begin{pmatrix} 0 & 5 & s & 2/s \\ 5s & 0 & s & 2/s \\ 1 & 1 & -1 & 0 \end{pmatrix}$$

The row-reduced echelon form is

$$\begin{pmatrix} 1 & 0 & 0 & 2/(s^3+6s^2) \\ 0 & 1 & 0 & 2/(s^2+6s) \\ 0 & 0 & 1 & 2(s+1)/(s^3+6s^2) \end{pmatrix}$$

Therefore

$$Q_1(s) = \frac{2}{s^3+6s^2}, \qquad Q_2(s) = \frac{2}{s^2+6s}, \qquad Q_3(s) = \frac{2(s+1)}{s^2(s+6)}.$$

This implies

$$q_1(t) = -1/18 + e^{-6t}/18 + t/3, \qquad q_2(t) = 1/3 - e^{-6t}/3, \qquad q_3(t) = q_2(t) + q_1(t).$$

This computation can be done in Sage as well:

Sage

```
sage: s = var("s")
sage: MS = MatrixSpace(SymbolicExpressionRing(), 3, 4)
sage: A = MS([[0,5,s,2/s],[5*s,0,s,2/s],[1,1,-1,0]])
sage: B = A.echelon_form(); B

[   1          0          0    2/(5*s^2) - (-2/(5*s) - 2/(5*s^2))/(5*(-s/5 - 6/5))]
[   0          1          0   2/(5*s) - (-2/(5*s) - 2/(5*s^2))*s/(5*(-s/5 - 6/5)) ]
[   0          0          1               (-2/(5*s) - 2/(5*s^2))/(-s/5 - 6/5)     ]

sage: Q1 = B[0,3]
sage: t = var("t")
sage: Q1.inverse_laplace(s,t)
e^(-(6*t))/18 + t/3 - 1/18
sage: Q2 = B[1,3]
sage: Q2.inverse_laplace(s,t)
1/3 - e^(-(6*t))/3
sage: Q3 = B[2,3]
sage: Q3.inverse_laplace(s,t)
-5*e^(-(6*t))/18 + t/3 + 5/18
```

Example 3.3.10. Consider the network given by the following diagram.

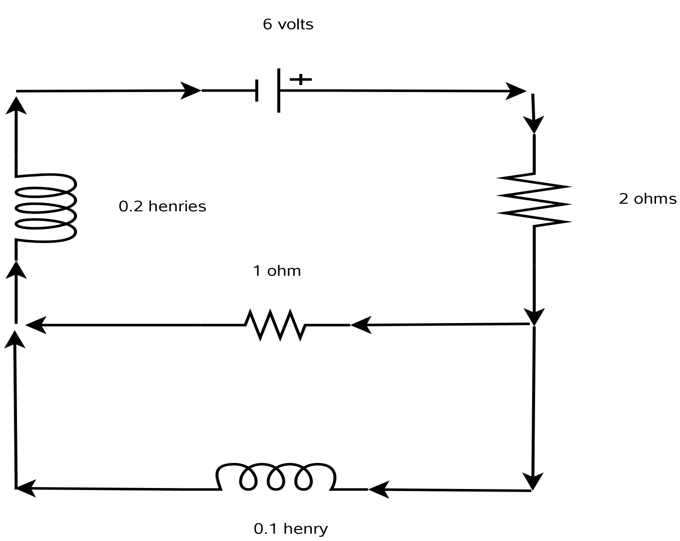


Figure 3.10: Another network.

Assume the initial charges are 0.

Using Kirchoff's Laws, you get a system

$$\begin{cases} i_1 - i_2 - i_3 = 0, \\ 2i_1 + i_2 + (0.2)i_1' = 6, \\ (0.1)i_3' - i_2 = 0. \end{cases}$$

Take LTs of these three DEs. You get a 3×3 system in the unknowns $I_1(s) = \mathcal{L}[i_1(t)](s)$, $I_2(s) = \mathcal{L}[i_2(t)](s)$, and $I_3(s) = \mathcal{L}[i_3(t)](s)$. The augmented matrix of this system is

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ 2 + s/5 & 1 & 0 & 6/s \\ 0 & -1 & s/10 & 0 \end{pmatrix}$$

(Check this yourself!) The row-reduced echelon form is

$$\begin{pmatrix} 1 & 0 & 0 & \frac{30(s+10)}{s(s^2+25s+100)} \\ 0 & 1 & 0 & \frac{30}{s^2+25s+100} \\ 0 & 0 & 1 & \frac{300}{s(s^2+25s+100)} \end{pmatrix}$$

Therefore

$$I_1(s) = -\frac{1}{s+20} - \frac{2}{s+5} + \frac{3}{s}, \quad I_2(s) = -\frac{2}{s+20} + \frac{2}{s+5}, \quad I_3(s) = \frac{1}{s+20} - \frac{4}{s+5} + \frac{3}{s}.$$

This implies

$$i_1(t) = 3 - 2e^{-5t} - e^{-20t}, \quad i_2(t) = 2e^{-5t} - 2e^{-20t}, \quad i_3(t) = 3 - 4e^{-5t} + e^{-20t}.$$

Exercise: Use Sage to solve for $i_1(t)$, $i_2(t)$, and $i_3(t)$ in the above problem.

3.4 Eigenvalue method for systems of DEs

Motivation

First, we shall try to motivate the study of eigenvalues and eigenvectors. This section hopefully will convince you that

- if our goal in life is to discover the "simplest" matrices, then diagonal matrices are wonderful,
- if our goal in lifeis to find the "best" coordinate system to work with, then conjugation is very natural,
- if our goal in life is to conjugate a given square matrix matrix into a diagonal one, then eigenvalues and eigenvectors are also natural.

Diagonal matrices are wonderful: We'll focus for simplicity on the 2×2 case, but everything applies to the general case.

- Addition is easy:

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} + \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} a_1+b_1 & 0 \\ 0 & a_2+b_2 \end{pmatrix}.$$

- Multiplication is easy:

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \cdot \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} a_1\cdot b_1 & 0 \\ 0 & a_2\cdot b_2 \end{pmatrix}.$$

- Powers are easy:

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}^n = \begin{pmatrix} a_1^n & 0 \\ 0 & a_2^n \end{pmatrix}.$$

- You can even exponentiate them:

$$\begin{aligned} exp(t\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \\ &\quad + \tfrac{1}{2!}t^2\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}^2 + \tfrac{1}{3!}t^3\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}^3 + \ldots \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} ta_1 & 0 \\ 0 & ta_2 \end{pmatrix} \\ &\quad + \begin{pmatrix} \frac{1}{2!}t^2a_1^2 & 0 \\ 0 & \frac{1}{2!}t^2a_2^2 \end{pmatrix} + \begin{pmatrix} \frac{1}{3!}t^3a_1^3 & 0 \\ 0 & \frac{1}{3!}t^3a_2^3 \end{pmatrix} + \ldots \\ &= \begin{pmatrix} \exp(ta_1) & 0 \\ 0 & \exp(ta_2) \end{pmatrix}. \end{aligned}$$

So, diagonal matrices are wonderful.

Conjugation is natural. You and your friend are piloting a rocket in space. You handle the controls, your friend handles the map. To communicate, you have to "change coordinates". Your coordinates are those of the rocketship (straight ahead is one direction, to the right is another). Your friends coordinates are those of the map (north and east are map directions). Changing coordinates corresponds algebraically to conjugating by a suitable matrix. Using an example, we'll see how this arises in a specific case.

Your basis vectors are

$$v_1 = (1,0), \quad v_2 = (0,1),$$

which we call the "v-space coordinates", and the map's basis vectors are

$$w_1 = (1, 1), \qquad w_2 = (1, -1),$$

which we call the "w-space coordinates".

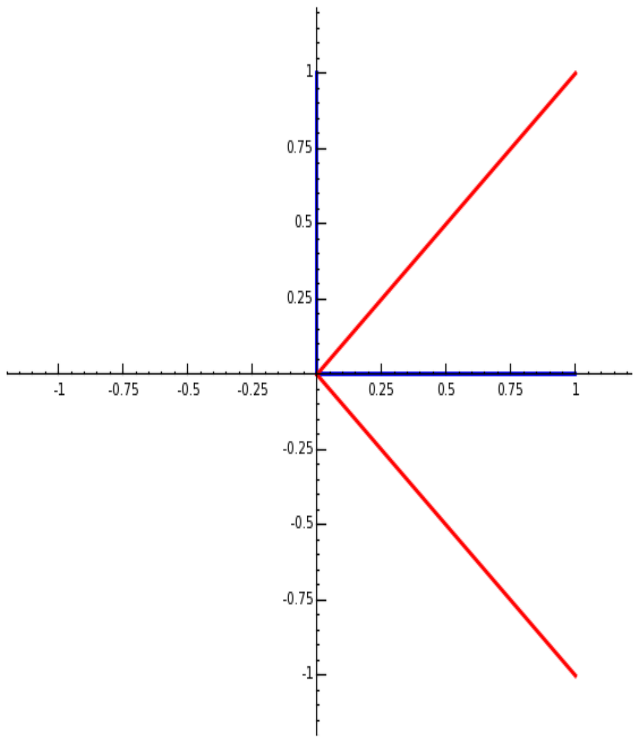


Figure 3.11: basis vectors v_1, v_2 and w_1, w_2.

For example, the point $(7, 3)$ is, in v-space coordinates of course $(7, 3)$ but in the w-space coordinates, $(5, 2)$ since $5w_1 + 2w_2 = 7v_1 + 3v_2$. Indeed, the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ sends $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ to $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$.

Suppose we flip about the 45^o line (the "diagonal") in each coordinate system. In the v-space:

$$av_1 + bv_2 \longmapsto bv_1 + av_2,$$
$$\begin{pmatrix} a \\ b \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

In other words, in v-space, the "flip map" is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

In the w-space:

$$wv_1 + wv_2 \longmapsto aw_1 - bw_2,$$
$$\begin{pmatrix} a \\ b \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

In other words, in w-space, the "flip map" is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Conjugating by the matrix A converts the "flip map" in w-space to the the "flip map" in v-space:

$$A \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Eigenvalues are natural too

At first glance, matrices can seem to be complicated objects. The eigenvalues of a matrix are

- relatively easy to compute,
- tell us something about the "behavior" of the corresponding matrix.

This is a bit vague since we really don't need to know about matrices in detail (take a course in matrix theory for that), just enough to help us solve equations. For us, as you will see, eigenvalues are very useful for solving equations associated to a matrix.

Each $n \times n$ matrix A has exactly n (counted according to multiplicity) eigenvalues λ_1, λ_2, ..., λ_n. These numbers are the roots of the **characteristic polynomial**

$$p(\lambda) = p_A(\lambda) = \det(A - \lambda I). \tag{3.8}$$

The corresponding **eigenvectors** are non-zero vectors $\vec{v}$ which satisfy the **eigenvector equation**

$$A\vec{v} = \lambda\vec{v}, \tag{3.9}$$

where λ is one of the eigenvalues. In this case, we say $\vec{v}$ **corresponds** to λ.

Example 3.4.1. Consider an $n \times n$ diagonal matrix. The standard basis elements ($e_1 = (1, 0, ..., 0)$, ..., $e_n = (0, ..., 0, 1)$) are the eigenvectors and the diagonal elements are the eigenvalues.

Example 3.4.2. Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 0 & -1 & 1 \\ -4 & 0 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

We compute

$$p(\lambda) = \det \begin{pmatrix} 0-\lambda & -1 & 1 \\ -4 & -\lambda & 2 \\ 0 & 0 & 3-\lambda \end{pmatrix} = -(\lambda-2)(\lambda+2)(\lambda-3).$$

Therefore, $\lambda_1 = -2$, $\lambda_2 = 2$, $\lambda_3 = 3$. We solve for the corresponding eigenvectors, $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$. For example, let

$$\vec{v}_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We solve for x, y, z in the eigenvector equation

$$\begin{pmatrix} 0-\lambda & -1 & 1 \\ -4 & -\lambda & 2 \\ 0 & 0 & 3-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3 \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

This gives rise to the system of linear equations

$$\begin{array}{cccc} & -y & +z & = 3x \\ -4x & & +2z & = 3y \\ & & 3z & = 3z. \end{array}$$

You can find some non-zero solvution to these using row-reduction/Gauss elimination, for example. The row-reduced echelon form of

$$\begin{pmatrix} -3 & -1 & 1 & 0 \\ -4 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is

$$A = \begin{pmatrix} 1 & 0 & -1/5 & 0 \\ 0 & 1 & -2/5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so z is anything (non-zero, that is), $y = 2z/5$, and $x = z/5$. One non-zero solution is $x = 1$, $y = 2$, $z = 5$, so

$$\vec{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}.$$

However, any other scalar multiply of this will also satisfy the eigenvector equation. The other eigenvectors can be computed similarly. We obtain in this way,

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}.$$

Since this section is only intended to be motivation, we shall not prove this here (see any text on linear algebra, for example [B-rref] or [H-rref]).

Sage

```
sage: MS = MatrixSpace(CC,2,2)
sage: A = MS([[0,1],[1,0]])
sage: A.eigenspaces()

[
(1.00000000000000, [
(1.00000000000000, 1.00000000000000)
]),
(-1.00000000000000, [
(1.00000000000000, -1.00000000000000)
])
]
```

Solution strategy

PROBLEM: Solve

$$\begin{cases} x' = ax + by, & x(0) = x_0, \\ y' = cx + dy, & y(0) = y_0. \end{cases}$$

Solution: Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

In matrix notation, the system of DEs becomes

$$\vec{X}' = A\vec{X}, \quad \vec{X}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

where $\vec{X} = \vec{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. In a similar manner to how we solved homogeneous constant coefficient 2nd order ODEs $ax'' + bx' + cx = 0$ by using "Euler's guess" $x = Ce^{rt}$, we try to guess an exponential: $\vec{X}(t) = \vec{c}e^{\lambda t}$ (λ is used instead of r to stick with notational convention; $\vec{c}$ in place of C since we need a constant *vector*). Plugging this guess into the matrix DE $\vec{X}' = A\vec{X}$ gives $\lambda\vec{c}e^{\lambda t} = A\vec{c}e^{\lambda t}$, or (cancelling $e^{\lambda t}$)

$$A\vec{c} = \lambda\vec{c}.$$

This means that λ is an eigenvalue of A with eigenvector $\vec{c}$.

- Find the eigenvalues. These are the roots of the characteristic polynomial

$$p(\lambda) = \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = \lambda^2 - (a+d)\lambda + (ad-bc).$$

Call them λ_1, λ_2 (in any order you like).

You can use the quadratic formula, for example to get them:

$$\lambda_1 = \frac{a+d}{2} + \frac{\sqrt{(a+d)^2 - 4(ad-bc)}}{2}, \quad \lambda_2 = \frac{a+d}{2} - \frac{\sqrt{(a+d)^2 - 4(ad-bc)}}{2}.$$

- Find the eigenvectors. If $b \neq 0$ then you can use the formulas

$$\vec{v}_1 = \begin{pmatrix} b \\ \lambda_1 - a \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} b \\ \lambda_2 - a \end{pmatrix}.$$

In general, you can get them by solving the **eigenvector equation** $A\vec{v} = \lambda\vec{v}$.

Example 3.4.3. The eigenvalues and eigenvectors can be computed using Sage numericaly as follows.

Sage

```
sage: A = matrix([[1,2],[3,4]])
sage: A.eigenvalues()
[-0.3722813232690144?, 5.372281323269015?]
sage: A.eigenvectors_right()

[(-0.3722813232690144?, [(1, -0.6861406616345072?)], 1),
 (5.372281323269015?, [(1, 2.186140661634508?)], 1)]
```

In some cases, they can be computed "exactly" or "symbolicaly", as follows.

Sage

```
sage: A = matrix(QQ[I],[[1,1],[-5,-1]])
sage: A.eigenvalues()
[2*I, -2*I]
sage: A.eigenvectors_right()

[(2*I, [
(1, 2*I - 1)
], 1), (-2*I, [
(1, -2*I - 1)
], 1)]
```

- Plug these into the following formulas:

(a) $\lambda_1 \neq \lambda_2$, real:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1\vec{v}_1\exp(\lambda_1 t) + c_2\vec{v}_2\exp(\lambda_2 t).$$

(b) $\lambda_1 = \lambda_2 = \lambda$, real:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \vec{v}_1 \exp(\lambda t) + c_2(\vec{v}_1 t + \vec{p}) \exp(\lambda t),$$

where $\vec{p}$ is any non-zero vector satisfying $(A - \lambda I)\vec{p} = \vec{v}_1$.

(c) $\lambda_1 = \alpha + i\beta$, complex: write $\vec{v}_1 = \vec{u}_1 + i\vec{u}_2$, where $\vec{u}_1$ and $\vec{u}_2$ are both *real vectors*.

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= c_1[\exp(\alpha t)\cos(\beta t)\vec{u}_1 - \exp(\alpha t)\sin(\beta t)\vec{u}_2] \\ &+ c_2[-\exp(\alpha t)\cos(\beta t)\vec{u}_2 - \exp(\alpha t)\sin(\beta t)\vec{u}_1]. \end{aligned} \quad (3.10)$$

Examples

Example 3.4.4. Solve

$$x'(t)) = x(t) - y(t), \quad y'(t) = 4x(t) + y(t), \quad x(0) = -1, \quad y(0) = 1.$$

Let

$$A = \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}$$

and so the characteristc polynomial is

$$p(x) = \det(A - xI) = x^2 - 2x + 5.$$

The eigenvalues are

$$\lambda_1 = 1 + 2i, \quad \lambda_2 = 1 - 2i,$$

so $\alpha = 1$ and $\beta = 2$. Eigenvectors $\vec{v}_1, \vec{v}_2$ are given by

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 2i \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} -1 \\ -2i \end{pmatrix},$$

though we actually only need to know $\vec{v}_1$. The real and imaginary parts of $\vec{v}_1$ are

$$\vec{u}_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \qquad \vec{u}_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

The solution is then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -c_1 \exp(t)\cos(2t) + c_2 \exp(t)\sin(2t) \\ -2c_1 \exp(t)\sin(2t) - 2c_2 \exp(t)\cos(2t), \end{pmatrix}$$

so $x(t) = -c_1 \exp(t)\cos(2t) + c_2 \exp(t)\sin(2t)$ and $y(t) = -2c_1 \exp(t)\sin(2t) - 2c_2 \exp(t)\cos(2t)$.

Since $x(0) = -1$, we solve to get $c_1 = 1$. Since $y(0) = 1$, we get $c_2 = -1/2$. The solution is: $x(t) = -\exp(t)\cos(2t) - \frac{1}{2}\exp(t)\sin(2t)$ and $y(t) = -2\exp(t)\sin(2t) + \exp(t)\cos(2t)$.

Example 3.4.5. Solve

$$x'(t) = -2x(t) + 3y(t), \quad y'(t) = -3x(t) + 4y(t).$$

Let

$$A = \begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix}$$

and so the characteristc polynomial is

$$p(x) = \det(A - xI) = x^2 - 2x + 1.$$

The eigenvalues are

$$\lambda_1 = \lambda_2 = 1.$$

An eigenvector $\vec{v}_1$ is given by

$$\vec{v}_1 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Since we can multiply any eigenvector by a non-zero scalar and get another eigenvector, we shall use instead

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Let $\vec{p} = \begin{pmatrix} r \\ s \end{pmatrix}$ be any non-zero vector satisfying $(A - \lambda I)\vec{p} = \vec{v}_1$. This means

$$\begin{pmatrix} -2-1 & 3 \\ -3 & 4-1 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

There are infinitely many possibly solutions but we simply take $r = 0$ and $s = 1/3$, so

$$\vec{p} = \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}.$$

The solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \exp(t) + c_2 (\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}) \exp(t),$$

or $x(t) = c_1 \exp(t) + c_2 t \exp(t)$ and $y(t) = c_1 \exp(t) + \frac{1}{3} c_2 \exp(t) + c_2 t \exp(t)$.

Exercises: Use Sage to find eigenvalues and eigenvectors of the following matrices

$$\begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix},$$

$$\begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & -1 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & -13 \end{pmatrix}.$$

3.5 Introduction to variation of parameters for systems

The method called variation of parameters for *systems* of ordinary differential equations has no relation to the method variation of parameters for 2nd order ordinary differential equations discussed in an earlier lecture except for their name.

3.5.1 Motivation

Recall that when we solved the 1st order ordinary differential equation

$$y' = ay, \qquad y(0) = y_0, \tag{3.11}$$

for $y = y(t)$ using the method of separation of variables, we got the formula

$$y = ce^{at} = e^{at}c, \tag{3.12}$$

where c is a constant depending on the initial condition (in fact, $c = y(0)$).
Consider a 2×2 system of linear 1st order ordinary differential equations in the form

$$\begin{cases} x' = ax + by, & x(0) = x_0, \\ y' = cx + dy, & y(0) = y_0. \end{cases}$$

This can be rewritten in the form

$$\vec{X}' = A\vec{X}, \tag{3.13}$$

where $\vec{X} = \vec{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, and A is the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We can solve (3.13) analogously to (3.11), to obtain

$$\vec{X} = e^{tA}\vec{c}, \tag{3.14}$$

$\vec{c} = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$ is a constant depending on the initial conditions and e^{tA} is a "matrix exponential" defined by

$$e^B = \sum_{n=0}^{\infty} \frac{1}{n!} B^n,$$

for any square matrix B.

You might be thinking to yourself: I can't compute the matrix exponential so what good is this formula (3.14)? Good question! The answer is that the eigenvalues and eigenvectors of the matrix A enable you to compute e^{tA}. This is the basis for the formulas for the solution of a system of ordinary differential equations using the eigenvalue method discussed in §3.4.

The eigenvalue method in §3.4 shows us how to write every solution to (3.13) in the form

$$\vec{X} = c_1\vec{X}_1(t) + c_2\vec{X}_2(t),$$

for some vector-valued solutions $\vec{X}_1(t)$ and $\vec{X}_2(t)$ called **fundamental solutions**. Frequently, we call the matrix of fundamental solutions,

$$\Phi = \left(\vec{X}_1(t), \vec{X}_2(t)\right),$$

the **fundamental matrix**. The fundamental matrix is, roughly speaking, e^{tA}. It is analogous to the Wronskian of two fundamental solutions to a second order ordinary differential equation.

See examples below for more details.

3.5.2 The method

Recall that we we solved the 1st order ordinary differential equation

$$y' + p(t)y = q(t) \tag{3.15}$$

for $y = y(t)$ using the method of integrating factors, we got the formula

$$y = (e^{\int p(t)\,dt})^{-1}(\int e^{\int p(t)\,dt} q(t)\,dt \ + \ c). \tag{3.16}$$

Consider a 2×2 system of linear 1st order ordinary differential equations in the form

$$\begin{cases} x' = ax + by + f(t), & x(0) = x_0, \\ y' = cx + dy + g(t), & y(0) = y_0. \end{cases}$$

This can be rewritten in the form

$$\vec{X}' = A\vec{X} + \vec{F}, \tag{3.17}$$

where $\vec{F} = \vec{F}(t) = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$. Equation (3.17) can be seen to be in a form analogous to (3.15) by replacing $\vec{X}$ by y, A by $-p$ and $\vec{F}$ by q. It turns out that (3.17) can be solved in a way analogous to (3.15) as well. Here is the **variation of parameters formula for systems**:

$$\vec{X} = \Phi(\int \Phi^{-1}\vec{F}(t)\,dt \ + \ \vec{c}), \tag{3.18}$$

where $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ is a constant vector determined by the initial conditions and Φ is the fundamental matrix.

Example 3.5.1. A battle between X-men and Y-men is modeled by

$$\begin{cases} x' = -y + 1, & x(0) = 100, \\ y' = -4x + e^t, & y(0) = 50. \end{cases}$$

The non-homogeneous terms 1 and e^t represent reinforcements. Find out who wins, when, and the number of survivers.

Here A is the matrix

$$A = \begin{pmatrix} 0 & -1 \\ -4 & 0 \end{pmatrix}$$

and $\vec{F} = \vec{F}(t) = \begin{pmatrix} 1 \\ e^t \end{pmatrix}$.

In the method of variation of parameters, you must solve the homogeneous system first.

The eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = -2$, with associated eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, resp.. The general solution to the homogeneous system

$$\begin{cases} x' = -y, \\ y' = -4x, \end{cases}$$

is

$$\vec{X} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t} = c_1 \vec{X}_1(t) + c_2 \vec{X}_2(t),$$

where

$$\vec{X}_1(t) = \begin{pmatrix} e^{2t} \\ -2e^{2t} \end{pmatrix}, \qquad \vec{X}_2(t) = \begin{pmatrix} e^{-2t} \\ 2e^{-2t} \end{pmatrix}.$$

For the solution of the non-homogeneous equation, we must compute the fundamental matrix:

$$\Phi = \begin{pmatrix} e^{2t} & e^{-2t} \\ -2e^{2t} & 2e^{-2t} \end{pmatrix}, \quad \text{so} \quad \Phi^{-1} = \frac{1}{4} \begin{pmatrix} 2e^{-2t} & -e^{-2t} \\ 2e^{2t} & e^{2t} \end{pmatrix}.$$

Next, we compute the product,

$$\Phi^{-1}\vec{F} = \frac{1}{4} \begin{pmatrix} 2e^{-2t} & -e^{-2t} \\ 2e^{2t} & e^{2t} \end{pmatrix} \begin{pmatrix} 1 \\ e^t \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{-2t} - \frac{1}{4}e^{-t} \\ \frac{1}{2}e^{2t} + \frac{1}{4}e^{3t} \end{pmatrix}$$

and its integral,

$$\int \Phi^{-1}\vec{F}\,dt = \begin{pmatrix} -\frac{1}{4}e^{-2t}+\frac{1}{4}e^{-t} \\ \frac{1}{4}e^{2t}+\frac{1}{12}e^{3t} \end{pmatrix}.$$

Finally, to finish (3.18), we compute

$$\begin{aligned}\Phi(\textstyle\int \Phi^{-1}\vec{F}(t)\,dt \;+\; \vec{c}) &= \begin{pmatrix} e^{-2t} & e^{-2t} \\ -2e^{2t} & 2e^{2t} \end{pmatrix}\begin{pmatrix} -\frac{1}{4}e^{-2t}+\frac{1}{4}e^{-t}+c1 \\ \frac{1}{4}e^{2t}+\frac{1}{12}e^{3t}+c_2 \end{pmatrix} \\ &= \begin{pmatrix} c_1e^{2t}+\frac{1}{3}e^{t}+c_2e^{-2t} \\ 1-\frac{1}{3}e^{t}-2c_1e^{2t}+2c_2e^{-2t} \end{pmatrix}.\end{aligned}$$

This gives the general solution to the original system

$$x(t) = c_1e^{2t}+\frac{1}{3}e^{t}+c_2e^{-2t},$$

and

$$y(t) = 1-\frac{1}{3}e^{t}-2c_1e^{2t}+2c_2e^{-2t}.$$

We aren't done! It remains to compute c_1, c_2 using the ICs. For this, solve

$$\frac{1}{3}+c_1+c_2=100, \qquad \frac{2}{3}-2c_1+2c_2=50.$$

We get

$$c_1 = 75/2, \qquad c_2 = 373/6,$$

so

$$x(t) = \frac{75}{2}e^{2t}+\frac{1}{3}e^{t}+\frac{373}{6}e^{-2t},$$

and

$$y(t) = 1-\frac{1}{3}e^{t}-75e^{2t}+\frac{373}{3}e^{-2t}.$$

As you can see from Figure 3.12, the X-men win. The solution to $y(t) = 0$ is about $t_0 = 0.1279774...$ and $x(t_0) = 96.9458...$ "survive".

Example 3.5.2. Solve

$$\begin{cases} x' = -y+1, \\ y' = x+\cot(t). \end{cases}$$

Here A is the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $\vec{F} = \vec{F}(t) = \begin{pmatrix} 0 \\ \cot(t) \end{pmatrix}$.

In the method of variation of parameters, you must solve the homogeneous system first.

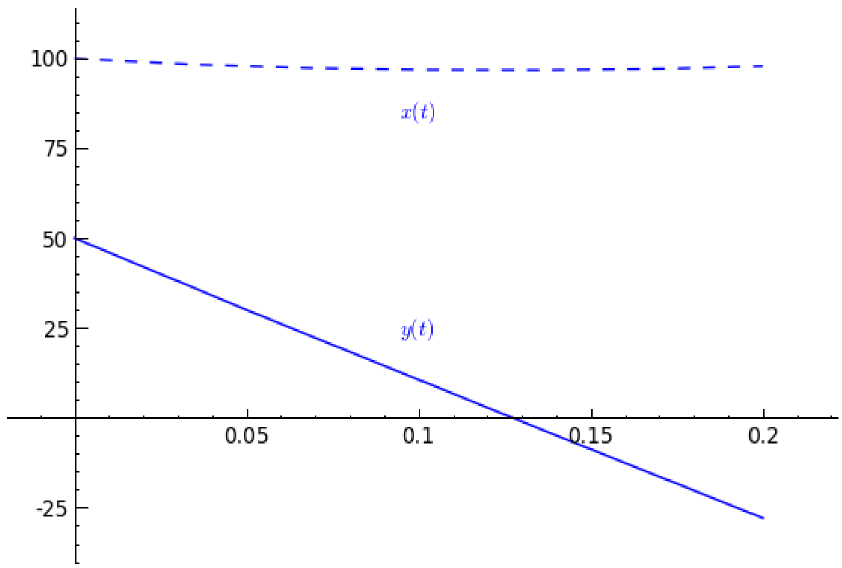


Figure 3.12: Solution to system $x' = -y + 1$, $x(0) = 100$, $y' = -4x + e^t$, $y(0) = 50$.

The eigenvalues of A are $\lambda_1 = i$, $\lambda_2 = -i$, with associated eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$, resp.. Therefore, in the notation of (3.10), we have $\alpha = 0$, $\beta = 1$, $\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $\vec{u}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

The general solution to the homogeneous system

$$\begin{cases} x' = -y, \\ y' = x, \end{cases}$$

is

$$\vec{X} = c_1[\cos(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin(t)\begin{pmatrix} 0 \\ -1 \end{pmatrix}] + c_2[\cos(t)\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}] = c_1\vec{X}_1(t) + c_2\vec{X}_2(t),$$

where

$$\vec{X}_1(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}, \qquad \vec{X}_2(t) = \begin{pmatrix} \sin(t) \\ -\cos(t) \end{pmatrix}.$$

For the solution of the non-homogeneous equation, we must compute the fundamental matrix:

$$\Phi = \begin{pmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{pmatrix}, \quad \text{so} \quad \Phi^{-1} = \begin{pmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{pmatrix} = \Phi,$$

since $\cos(t)^2 + \sin(t)^2 = 1$. Next, we compute the product,

$$\Phi^{-1}\vec{F} = \begin{pmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{pmatrix} \begin{pmatrix} 0 \\ \cot(t) \end{pmatrix} = \begin{pmatrix} \cos(t) \\ -\cos(t)^2/\sin(t) \end{pmatrix} = \begin{pmatrix} \cos(t) \\ \sin(t) - 1/\sin(t) \end{pmatrix}$$

and its integral,

$$\int \Phi^{-1}\vec{F}\,dt = \begin{pmatrix} \sin(t) \\ -\cos(t) - \frac{1}{2}\frac{\cos(t)-1}{\cos(t)+1} \end{pmatrix}.$$

Finally, we compute

$$\begin{aligned}\Phi(\textstyle\int \Phi^{-1}\vec{F}(t)\,dt \ + \ \vec{c}) &= \begin{pmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{pmatrix} [\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \sin(t) \\ -\cos(t) - \frac{1}{2}\frac{\cos(t)-1}{\cos(t)+1} \end{pmatrix}] \\ &= \begin{pmatrix} c_1\cos(t) + c_2\sin(t) - \frac{1}{2}\frac{\sin(t)\cos(t)}{(\cos(t)+1)} + \frac{1}{2}\frac{\sin(t)}{(\cos(t)+1)} \\ c_1\sin(t) - c_2\cos(t) + 1 + \frac{1}{2}\frac{\cos(t)^2}{(\cos(t)+1)} - \frac{1}{2}\frac{\cos(t)}{(\cos(t)+1)} \end{pmatrix}.\end{aligned}$$

Therefore,

$$x(t) = c_1\cos(t) + c_2\sin(t) - \frac{1}{2}\frac{\sin(t)\cos(t)}{(\cos(t)+1)} + \frac{1}{2}\frac{\sin(t)}{(\cos(t)+1)},$$

and

$$y(t) = c_1\sin(t) - c_2\cos(t) + 1 + \frac{1}{2}\frac{\cos(t)^2}{(\cos(t)+1)} - \frac{1}{2}\frac{\cos(t)}{(\cos(t)+1)}.$$

Chapter 4

Introduction to partial differential equations

> The deep study of nature is the most fruitful source of mathematical discoveries.
>
> - *Jean-Baptist-Joseph Fourier*

4.1 Introduction to separation of variables

Recall, a *partial differential equation* (PDE) is an equation satisfied by an unknown function (called the dependent variable) and its partial derivatives. The variables you differentiate with respect to are called the independent variables. If there is only one independent variable then it is called an *ordinary differential equation*.

Examples include

- the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, where u is the dependent variable and x, y are the independent variables,
- the heat equation $u_t = \alpha u_{xx}$,
- and the wave equation $u_{tt} = c^2 u_{xx}$.

All these PDEs are of second order (you have to differentiate twice to express the equation). Here, we consider a first order PDE which arises in applications and use it to introduce the method of solution called *separation of variables*.

The transport or advection equation

Advection is the transport of a some conserved scalar quantity in a vector field. A good example is the transport of pollutants or silt in a river (the motion of the water carries these impurities downstream) or traffic flow.

The advection equation is the PDE governing the motion of a conserved quantity as it is advected by a given velocity field. The advection equation expressed mathematically is:

$$\frac{\partial u}{\partial t} + \nabla \cdot (u\mathbf{a}) = 0$$

where $\nabla\cdot$ is the divergence and $\mathbf{a}$ is the velocity field of the fluid. Frequently, it is assumed that $\nabla \cdot \mathbf{a} = 0$ (this is expressed by saying that *the velocity field is solenoidal*). In this case, the above equation reduces to

$$\frac{\partial u}{\partial t} + \mathbf{a} \cdot \nabla u = 0.$$

Assume we have horizontal pipe in which water is flowing at a constant rate c in the positive x direction. Add some salt to this water and let $u(x,t)$ denote the concentration (say in lbs/gallon) at time t. Note that the amount of salt in an interval I of the pipe is $\int_I u(x,t)\,dx$. This concentration satisfies the *transport* (or *advection*) equation:

$$u_t + cu_x = 0.$$

(For a derivation of this, see for example Strauss [S-pde], §1.3.) How do we solve this?

Solution 1: D'Alembert noticed that the directional derivative of $u(x,t)$ in the direction $\vec{v} = \frac{1}{\sqrt{1+c^2}}\langle c, 1\rangle$ is $D_{\vec{v}}(u) = \frac{1}{\sqrt{1+c^2}}(cu_x + u_t) = 0$. Therefore, $u(x,t)$ is constant along the lines in the direction of $\vec{v}$, and so $u(x,t) = f(x-ct)$, for some function f. We will not use this method of solution in the example below but it does help visualize the shape of the solution. For instance, imagine the plot of $z = f(x-ct)$ in (x,t,z) space. The contour lying above the line $x = ct + k$ (k fixed) is the line of constant height $z = f(k)$. $\square$

Solution 2: The method of *separation of variables* indicates that we start by assuming that $u(x,t)$ can be factored:

$$u(x,t) = X(x)T(t),$$

for some (unknown) functions X and T. (We shall work on removing this assumption later. This assumption "works" because partial differentiation of functions like x^2t^3 is so much simpler that partial differentiation of "mixed" functions like $\sin(x^2+t^3)$.) Substituting this into the PDE gives

$$X(x)T'(t) + cX'(x)T(t) = 0.$$

Now separate all the x's on one side and the t's on the other (divide by $X(x)T(t)$):

$$\frac{T'(t)}{T(t)} = -c\frac{X'(x)}{X(x)}.$$

(This same "trick" works when you apply the separation of variables method to other linear PDE's, such as the heat equation or wave equation, as we will see in later lessons.) It is impossible for a function of an independent variable x to be identically equal to a function of an independent variable t unless both are constant. (Indeed, try taking the partial derivative of $\frac{T'(t)}{T(t)}$ with respect to x. You get 0 since it doesn't depend on x. Therefore, the partial derivative of $-c\frac{X'(x)}{X(x)}$ is akso 0, so $\frac{X'(x)}{X(x)}$ is a constant!) Therefore, $\frac{T'(t)}{T(t)} = -c\frac{X'(x)}{X(x)} = K$, for some (unknown) constant K. So, we have two ODEs:

$$\frac{T'(t)}{T(t)} = K, \quad \frac{X'(x)}{X(x)} = -K/c.$$

Therefore, we have converted the PDE into two ODEs. Solving, we get

$$T(t) = c_1 e^{Kt}, \quad X(x) = c_2 e^{-Kx/c},$$

so, $u(x,t) = Ae^{Kt-Kx/c} = Ae^{-\frac{K}{c}(x-ct)}$, for some constants K and A (where A is shorthand for c_1c_2; in terms of D'Alembert's solution, $f(y) = Ae^{-\frac{K}{c}(y)}$). The "general solution" is a sum of these (for various A's and K's). □

This can also be done in Sage :

Sage

```
sage: t = var("t")
sage: T = function("T",t)
sage: K = var("K")
sage: T0 = var("T0")
sage: sage: desolve(diff(T,t) ==  K*T, [T,t], [0,T0])
T0*e^(K*t)
sage: x = var("x")
sage: X = function("X",x)
sage: c = var("c")
sage: X0 = var("X0")
sage: desolve(diff(X,x) == -c^(-1)*K*X, [X,x], [0,X0])
X0*e^(-K*x/c)
sage: solnX =desolve(diff(X,x) == -c^(-1)*K*X, [X,x], [0,X0])
sage: solnX
X0*e^(-K*x/c)
sage: solnT = desolve(diff(T,t) ==  K*T, [T,t], [0,T0])
sage: solnT
T0*e^(K*t)
sage: solnT*solnX
T0*X0*e^(K*t - K*x/c)
```

Example 4.1.1. Assume water is flowing along a horizontal pipe at 3 gal/min in the x direction and that there is an initial concentration of salt distributed in the water with concentration of $u(x,0) = e^{-x}$. Using separation of variables, find the concentration at time t. Plot this for various values of t.

Solution: The method of separation of variables gives the "separated form" of the solution to the transport PDE as $u(x,t) = Ae^{Kt-Kx/c}$, where $c = 3$. The initial condition implies

$$e^{-x} = u(x,0) = Ae^{K\cdot 0-Kx/c} = Ae^{-Kx/3},$$

so $A = 1$ and $K = 3$. Therefore, $u(x,t) = e^{3t-x}$. In other words, the salt concentration is increasing in time. This makes sense if you think about it this way: "freeze" the water motion at time $t = 0$. There is a lot of salt at the beginning of the pipe and less and less salt as you move along the pipe. Now go down the pipe in the x-direction some amount where you can barely tell there is any salt in the water. Now "unfreeze" the water motion. Looking along the pipe, you see the concentration is increasing since the saltier water is now moving toward you.

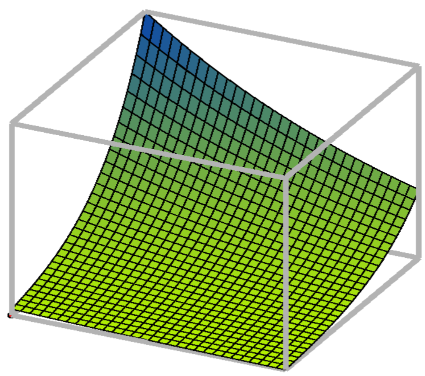


Figure 4.1: Transport with velocity $c = 3$.

An interactive version of this plot can be produced in Sage with:

Sage
```
sage: t,x = var("t,x")
sage: plot3d(exp(3*t-x),[x,0,2],[t,0,2])
```

□

What if the initial concentration was not $u(x,0) = e^{-x}$ but instead $u(x,0) = e^{-x} + 3e^{-5x}$? How does the solution to

$$u_t + 3u_x = 0, \quad u(x,0) = e^{-x} + 3e^{-5x}, \tag{4.1}$$

differ from the method of solution used above? In this case, we must use the fact that (by superposition) "the general solution" is of the form

$$u(x,t) = A_1e^{K_1(t-x/3)} + A_2e^{K_2(t-x/3)} + A_3e^{K_3(t-x/3)} + \ldots\ , \tag{4.2}$$

for some constants $A_1, K_1, ...$ To solve this PDE (4.1), we must answer the following questions: (1) How many terms from (4.2) are needed? (2) What are the constants $A_1, K_1, ...$? There are two terms in $u(x, 0)$, so we can hope that we only need to use two terms and solve

$$e^{-x} + 3e^{-5x} = u(x, 0) = A_1 e^{K_1(0-x/3)} + A_2 e^{K_2(0-x/3)}$$

for A_1, K_1, A_2, K_2. Indeed, this is possible to solve: $A_1 = 1$, $K_1 = 3$, $A_2 = 3$, $K_1 = 15$. This gives

$$u(x, t) = e^{3(t-x/3)} + 3e^{15(t-x/3)}.$$

Exercise: Using Sage , solve and plot the solution to the following problem. Assume water is flowing along a horizontal pipe at 3 gal/min in the x direction and that there is an initial concentration of salt distributed in the water with concentration of $u(x, 0) = e^x$.

4.2 Fourier series, sine series, cosine series

History: Fourier series were discovered by J. Fourier, a Frenchman who was a mathematician among other things. In fact, Fourier was Napolean's scientific advisor during France's invasion of Egypt in the late 1800's. When Napolean returned to France, he "elected" (i.e., appointed) Fourier to be a Prefect - basically an important administrative post where he oversaw some large construction projects, such as highway constructions. It was during this time when Fourier worked on the theory of heat on the side. His solution to the heat equation is basically what undergraduates often learn in a DEs with BVPs class. The exception being that our understanding of Fourier series now is much better than what was known in the early 1800's and some of these facts, like Dirichlet's theorem, are covered as well.

Motivation: Fourier series, since series, and cosine series are all expansions for a function $f(x)$, much in the same way that a Taylor series $a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + ...$ is an expansion. Both Fourier and Taylor series can be used to approximate $f(x)$. There are at least three important differences between the two types of series. (1) For a function to have a Taylor series it must be differentiable[1], whereas for a Fourier series it does not even have to be continuous. (2) Another difference is that the Taylor series is typically not periodic (though it can be in some cases), whereas a Fourier series is *always* periodic. (3) Finally, the Taylor series (when it converges) always converges to the function $f(x)$, but the Fourier series may not (see Dirichlet's theorem below for a more precise description of what happens).

Definition 4.2.1. Let $f(x)$ be a function defined on an interval of the real line. We allow $f(x)$ to be discontinuous but the points in this interval where $f(x)$ is discontinuous must be finite in number and must be jump discontinuities.

[1]Remember the formula for the n-th Taylor series coefficient centered at $x = x_0$ - $a_n = \frac{f^{(n)}(x_0)}{n!}$?

- First, we discuss Fourier series. To have a Fourier series you must be given two things: (1) a "period" $P = 2L$, (2) a function $f(x)$ defined on an interval of length $2L$, usually we take $-L < x < L$ (but sometimes $0 < x < 2L$ is used instead). The **Fourier series of $f(x)$ with period $2L$** is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})],$$

where a_n and b_n are given by the formulas[2],

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\,dx, \tag{4.3}$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})\,dx. \tag{4.4}$$

- Next, we discuss cosine series. To have a cosine series you must be given two things: (1) a "period" $P = 2L$, (2) a function $f(x)$ defined on the interval of length L, $0 < x < L$. The **cosine series of $f(x)$ with period $2L$** is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}),$$

where a_n is given by

$$a_n = \frac{2}{L}\int_{0}^{L} \cos(\frac{n\pi x}{L}) f(x)\ dx.$$

The cosine series of $f(x)$ is exactly the same as the Fourier series of the **even extension** of $f(x)$, defined by

$$f_{even}(x) = \begin{cases} f(x), & 0 < x < L, \\ f(-x), & -L < x < 0. \end{cases}$$

- Finally, we define sine series. To have a sine series you must be given two things: (1) a "period" $P = 2L$, (2) a function $f(x)$ defined on the interval of length L, $0 < x < L$. The **sine series of $f(x)$ with period $2L$** is

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}),$$

where b_n is given by

[2] These formulas were not known to Fourier. To compute the Fourier coefficients a_n, b_n he used sometimes ingenious round-about methods using large systems of equations.

$$b_n = \frac{2}{L}\int_0^L \sin(\frac{n\pi x}{L})f(x)\,dx.$$

The sine series of $f(x)$ is exactly the same as the Fourier series of the **odd extension** of $f(x)$, defined by

$$f_{odd}(x) = \begin{cases} f(x), & 0 < x < L, \\ -f(-x), & -L < x < 0. \end{cases}$$

One last remark: the symbol $\sim$ is used above instead of $=$ because of the fact that the Fourier series may not converge to $f(x)$ (see "Dirichlet's theorem" below). Do you remember right-hand and left-hand limits from calculus 1? Recall they are denoted $f(x+) = \lim_{\epsilon\to 0,\epsilon>0} f(x+\epsilon)$ and $f(x-) = \lim_{\epsilon\to 0,\epsilon>0} f(x-\epsilon)$, resp.. The meaning of $\sim$ is that the series does necessarily not converge to the value of $f(x)$ at every point[3]. The convergence proprties are given by the theorem below.

Theorem 4.2.1. (**Dirichlet's theorem**[4]) Let $f(x)$ be a function as above and let $-L < x < L$. The Fourier series of $f(x)$,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})],$$

(where a_n and b_n are as in the formulas (4.3), (4.4)) converges to

$$\frac{f(x+) + f(x-)}{2}.$$

In other words, the Fourier series of $f(x)$ converges to $f(x)$ only if $f(x)$ is continuous at x. If $f(x)$ is not continuous at x then then Fourier series of $f(x)$ converges to the "midpoint of the jump".

Example 4.2.1. If $f(x) = 2 + x$, $-2 < x < 2$, then the definition of L implies $L = 2$. Without even computing the Fourier series, we can evaluate it using Dirichlet's theorem.

Question: Using periodicity and Dirichlet's theorem, find the value that the Fourier series of $f(x)$ converges to at $x = 1, 2, 3$. (Ans: $f(x)$ is continuous at 1, so the FS at $x = 1$ converges to $f(1) = 3$ by Dirichlet's theorem. $f(x)$ is not defined at 2. It's FS is periodic with period 4, so at $x = 2$ the FS converges to $\frac{f(2+)+f(2-)}{2} = \frac{0+4}{2} = 2$. $f(x)$ is not defined at 3. It's FS is periodic with period 4, so at $x = 3$ the FS converges to $\frac{f(-1)+f(-1+)}{2} = \frac{1+1}{2} = 1$.)

The formulas (4.3) and (4.4) enable us to compute the Fourier series coefficients a_0, a_n and b_n. (We skip the details.) These formulas give that the Fourier series of $f(x)$ is

[3]Fourier believed his series converged to the function in the early 1800's but we now know this is not always true.

[4]Pronounced "Dear-ish-lay".

$$f(x) \sim \frac{4}{2} + \sum_{n=1}^{\infty} -4\,\frac{n\pi\,\cos(n\pi)}{n^2\pi^2}\,\sin\left(\frac{n\pi x}{2}\right).$$

The Fourier series approximations to $f(x)$ are

$$S_0 = 2,\ S_1 = 2 + \frac{4}{\pi}\sin\left(\frac{\pi x}{2}\right),\ S_2 = 2 + 4\,\frac{\sin\left(\frac{1}{2}\pi x\right)}{\pi} - 2\,\frac{\sin(\pi x)}{\pi},\ \ldots$$

The graphs of each of these functions get closer and closer to the graph of $f(x)$ on the interval $-2 < x < 2$. For instance, the graph of $f(x)$ and of S_8 are given below:

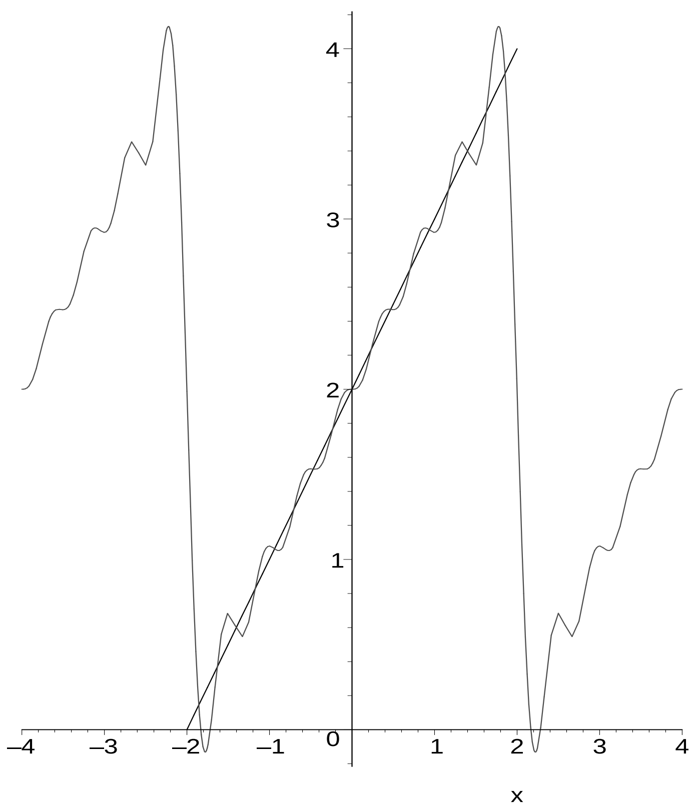


Figure 4.2: Graph of $f(x)$ and a Fourier series approximation of $f(x)$.

Notice that $f(x)$ is only defined from $-2 < x < 2$ yet the Fourier series is not only defined everywhere but is periodic with period $P = 2L = 4$. Also, notice that S_8 is not a bad approximation to $f(x)$.

This can also be done in Sage . First, we define the function.

Sage

```
sage: f = lambda x:x+2
```

```
sage: f = Piecewise([[(-2,2),f]])
```

This can be plotted using the command `f.plot().show()`. Next, we compute the Fourier series coefficients:

Sage

```
sage: f.fourier_series_cosine_coefficient(0,2) # a_0
4
sage: f.fourier_series_cosine_coefficient(1,2) # a_1
0
sage: f.fourier_series_cosine_coefficient(2,2) # a_2
0
sage: f.fourier_series_cosine_coefficient(3,)  # a_3
0
sage: f.fourier_series_sine_coefficient(1,2) # b_1
4/pi
sage: f.fourier_series_sine_coefficient(2,) # b_2
-2/pi
sage: f.fourier_series_sine_coefficient(3,2) # b_3
4/(3*pi)
```

Finally, the partial Fourier series and it's plot verses the function can be computed using the following Sage commands.

Sage

```
sage: f.fourier_series_partial_sum(3,2)
-2*sin(pi*x)/pi + 4*sin(pi*x/2)/pi + 2
sage: P1 = f.plot_fourier_series_partial_sum(15,2,-5,5,linestyle=":")
sage: P2 = f.plot(rgbcolor=(1,1/4,1/2))
sage: (P1+P2).show()
```

The plot (which takes 15 terms of the Fourier series) is given below.

Example 4.2.2. This time, let's consider an example of a cosine series. In this case, we take the piecewise constant function $f(x)$ defined on $0 < x < 3$ by

$$f(x) = \begin{cases} 1, & 0 < x < 2, \\ -1, & 2 \le x < 3. \end{cases}$$

We see therefore $L = 3$. The formula above for the cosine series coefficients gives that

$$f(x) \sim \frac{1}{3} + \sum_{n=1}^{\infty} 4\,\frac{\sin\left(\frac{2}{3}\,n\pi\right)}{n\pi}\cos(\frac{n\pi x}{3}).$$

The first few partial sums are

$$S_2 = 1/3 + 2\,\frac{\sqrt{3}\cos\left(\frac{1}{3}\,\pi\,x\right)}{\pi},$$

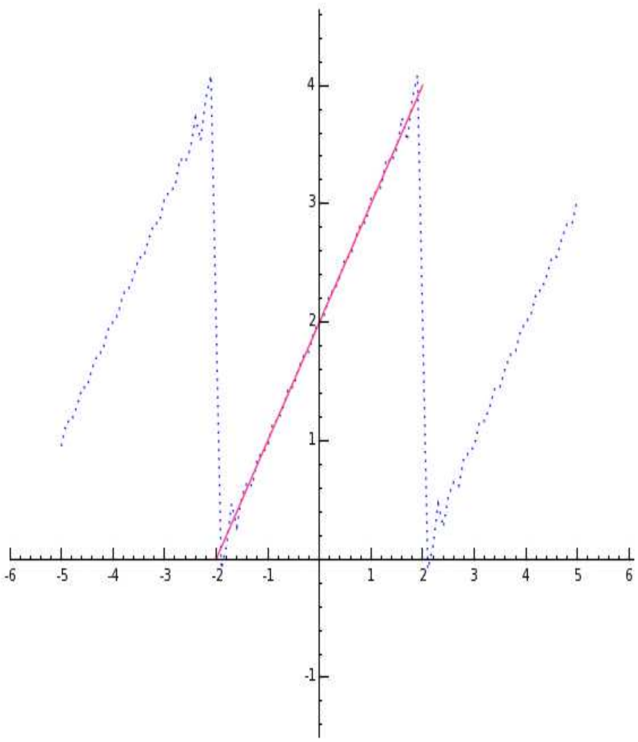


Figure 4.3: Graph of $f(x) = x + 2$ and a Fourier series approximation, $L = 2$.

$$S_3 = 1/3 + 2\,\frac{\sqrt{3}\cos\left(\frac{1}{3}\,\pi\,x\right)}{\pi} - \frac{\sqrt{3}\cos\left(\frac{2}{3}\,\pi\,x\right)}{\pi}, \ldots$$

As before, the more terms in the cosine series we take, the better the approximation is, for $0 < x < 3$. Comparing the picture below with the picture above, note that even with more terms, this approximation is not as good as the previous example. The precise reason for this is rather technical but basically boils down to the following: roughly speaking, the more differentiable the function is, the faster the Fourier series converges (and therefore the better the partial sums of the Fourier series will approximate $f(x)$). Also, notice that the cosine series approximation S_{10} is an even function but $f(x)$ is not (it's only defined from $0 < x < 3$).

For instance, the graph of $f(x)$ and of S_{10} are given below:

Example 4.2.3. Finally, let's consider an example of a sine series. In this case, we take the piecewise constant function $f(x)$ defined on $0 < x < 3$ by the same expression we used in the cosine series example above.

Question: Using periodicity and Dirichlet's theorem, find the value that the sine series of $f(x)$ converges to at $x = 1, 2, 3$. (Ans: $f(x)$ is continuous at 1, so the FS at $x = 1$ converges to $f(1) = 1$. $f(x)$ is not continuous at 2, so at $x = 2$ the SS converges to $\frac{f(2+)+f(2-)}{2} = \frac{f(-2+)+f(2-)}{2} =$

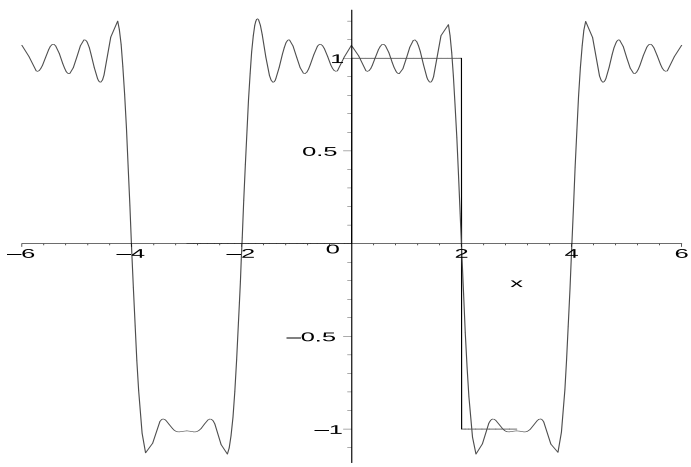


Figure 4.4: Graph of $f(x)$ and a cosine series approximation of $f(x)$.

$\frac{-1+1}{2} = 0$. $f(x)$ is not defined at 3. It's SS is periodic with period 6, so at $x = 3$ the SS converges to $\frac{f_{odd}(3-)+f_{odd}(3+)}{2} = \frac{-1+1}{2} = 0$.)

The formula above for the sine series coefficients give that

$$f(x) = \sum_{n=1}^{\infty} 2\,\frac{\cos(n\pi) - 2\,\cos\left(\frac{2}{3}\,n\pi\right) + 1}{n\pi}\,\sin\left(\frac{n\pi x}{3}\right).$$

The partial sums are

$$S_2 = 2\,\frac{\sin(1/3\,\pi\,x)}{\pi} + 3\,\frac{\sin\left(\frac{2}{3}\,\pi\,x\right)}{\pi},$$

$$S_3 = 2\,\frac{\sin\left(\frac{1}{3}\,\pi\,x\right)}{\pi} + 3\,\frac{\sin\left(\frac{2}{3}\,\pi\,x\right)}{\pi} - 4/3\,\frac{\sin(\pi\,x)}{\pi}, \ldots$$

These partial sums S_n, as $n \to \infty$, converge to their limit about as fast as those in the previous example. Instead of taking only 10 terms, this time we take 40. Observe from the graph below that the value of the sine series at $x = 2$ does seem to be approaching 0, as Dirichlet's Theorem predicts. The graph of $f(x)$ with S_{40} is

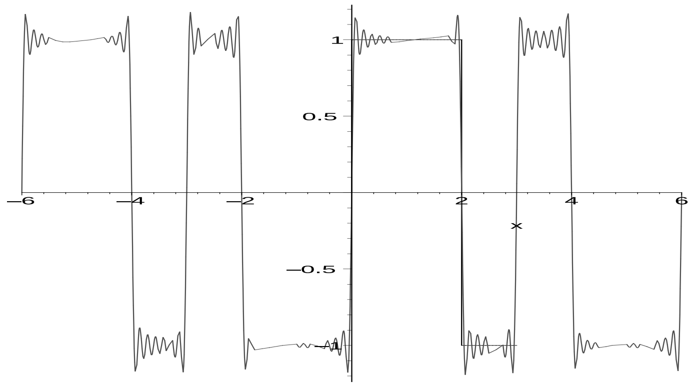


Figure 4.5: Graph of $f(x)$ and a sine series approximation of $f(x)$.

Exercise: Let $f(x) = x^2$, $-2 < x < 2$ and $L = 2$. Use Sage to compute the first 10 terms of the Fourier series, and plot the corresponding partial sum. Next plot the partial sum of the first 50 terms and compare them.

Exercise: What mathematical results do the following Sage commands give you? In other words, if you can seen someone typing these commands into a computer, explain what problem they were trying to solve.

Sage

```
sage: x = var("x")
sage: f0(x) = 0
sage: f1(x) = -1
sage: f2(x) = 1
sage: f = Piecewise([[(-2,0),f1],[(0,3/2),f0],[(3/2,2),f2]])
sage: P1 = f.plot()
sage: a10 = [f.fourier_series_cosine_coefficient(n,2) for n in range(10)]
sage: b10 = [f.fourier_series_sine_coefficient(n,2) for n in range(10)]
sage: fs10 = a10[0]/2 + sum([a10[i]*cos(i*pi*x/2) for i in range(1,10)]) + sum([b10[i]*sin(i*pi*x/2) for i in range(10)])
sage: P2 = fs10.plot(-4,4,linestyle=":")
sage: (P1+P2).show()
sage: ### these commands below are more time-consuming:
sage: a50 = [f.fourier_series_cosine_coefficient(n,2) for n in range(50)]
sage: b50 = [f.fourier_series_sine_coefficient(n,2) for n in range(50)]
sage: fs50 = a50[0]/2 + sum([a50[i]*cos(i*pi*x/2) for i in range(1,50)]) + sum([b50[i]*sin(i*pi*x/2) for i in range(50)])
sage: P3 = fs50.plot(-4,4,linestyle="--")
sage: (P1+P2+P3).show()
sage: a100 = [f.fourier_series_cosine_coefficient(n,2) for n in range(100)]
sage: b100 = [f.fourier_series_sine_coefficient(n,2) for n in range(100)]
sage: fs100 = a100[0]/2 + sum([a100[i]*cos(i*pi*x/2) for i in range(1,100)]) + sum([b100[i]*sin(i*pi*x/2) for i in range(100)])
sage: P3 = fs100.plot(-4,4,linestyle="--")
sage: (P1+P2+P3).show()
sage:
```

4.3 The heat equation

The differential equations of the propagation of heat express the most general conditions, and reduce the physical questions to problems of pure analysis, and this is the proper object of theory.

- *Jean-Baptist-Joseph Fourier*

The heat equation with **zero ends** boundary conditions models the temperature of an (insulated) wire of length L:

$$\begin{cases} k\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \\ u(0,t) = u(L,t) = 0. \end{cases}$$

Here $u(x,t)$ denotes the temperature at a point x on the wire at time t. The initial temperature $f(x)$ is specified by the equation

$$u(x,0) = f(x).$$

In this model, it is assumed no heat escapes out the middle of the wire (which, say, is coated with some kind of insulating plastic). However, due to the boundary conditions, $u(0,t) = u(L,t) = 0$, heat can escape out the ends.

4.3.1 Method for zero ends

- Find the sine series of $f(x)$:

$$f(x) \sim \sum_{n=1}^{\infty} b_n(f)\sin(\frac{n\pi x}{L}),$$

- The solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n(f)\sin(\frac{n\pi x}{L})\exp(-k(\frac{n\pi}{L})^2 t).$$

Example 4.3.1. Let

$$f(x) = \begin{cases} -1, & 0 \le x \le \pi/2, \\ 2, & \pi/2 < x < \pi. \end{cases}$$

Then $L = \pi$ and

$$b_n(f) = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx = -2\,\frac{2\cos(n\pi) - 3\cos(\frac{1}{2}n\pi) + 1}{n\pi}.$$

Thus

$$f(x) \sim b_1(f)\sin(x) + b_2(f)\sin(2x) + ... = \frac{2}{\pi}\sin(x) - \frac{6}{\pi}\sin(2x) + \frac{2}{3\pi}\sin(3x) + ...$$

This can also be done in Sage :

Sage

```
sage: x = var("x")
sage: f1(x) = -1
sage: f2(x) = 2
sage: f = Piecewise([[(0,pi/2),f1],[(pi/2,pi),f2]])
sage: P1 = f.plot()
sage: b10 = [f.sine_series_coefficient(n,pi) for n in range(1,10)]
sage: b10
[2/pi, -6/pi, 2/(3*pi), 0, 2/(5*pi), -2/pi, 2/(7*pi), 0, 2/(9*pi)]
sage: ss10 = sum([b10[n]*sin((n+1)*x) for n in range(len(b50))])
sage: ss10
2*sin(9*x)/(9*pi) + 2*sin(7*x)/(7*pi) - 2*sin(6*x)/pi
+ 2*sin(5*x)/(5*pi) + 2*sin(3*x)/(3*pi) - 6*sin(2*x)/pi + 2*sin(x)/pi
sage: b50 = [f.sine_series_coefficient(n,pi) for n in range(1,50)]
sage: ss50 = sum([b50[n]*sin((n+1)*x) for n in range(len(b))])
sage: P2 = ss10.plot(-5,5,linestyle="--")
sage: P3 = ss50.plot(-5,5,linestyle=":")
sage: (P1+P2+P3).show()
```

This illustrates how the series converges to the function. The function $f(x)$, and some of the partial sums of its sine series, looks like Figure 4.6.

As you can see, taking more and more terms gives functions which better and better approximate $f(x)$.

The solution to the heat equation, therefore, is

$$u(x,t) = \sum_{n=1}^{\infty} b_n(f)\sin(\frac{n\pi x}{L})\exp(-k(\frac{n\pi}{L})^2 t).$$

Next, we see how Sage can plot the solution to the heat equation (we use $k = 1$):

Sage

```
sage: t = var("t")
sage: soln50 = sum([b[n]*sin((n+1)*x)*e^(-(n+1)^2*t) for n in range(len(b50))])
sage: soln50a = sum([b[n]*sin((n+1)*x)*e^(-(n+1)^2*(1/10)) for n in range(len(b50))])
sage: P4 = soln50a.plot(0,pi,linestyle=":")
sage: soln50b = sum([b[n]*sin((n+1)*x)*e^(-(n+1)^2*(1/2)) for n in range(len(b50))])
sage: P5 = soln50b.plot(0,pi)
sage: soln50c = sum([b[n]*sin((n+1)*x)*e^(-(n+1)^2*(1/1)) for n in range(len(b50))])
sage: P6 = soln50c.plot(0,pi,linestyle="--")
sage: (P1+P4+P5+P6).show()
```

Taking 50 terms of this series, the graph of the solution at $t = 0$, $t = 0.5$, $t = 1$, looks approximately like Figure 4.7.

4.3.2 Method for insulated ends

The heat equation with **insulated ends** boundary conditions models the temperature of an (insulated) wire of length L:

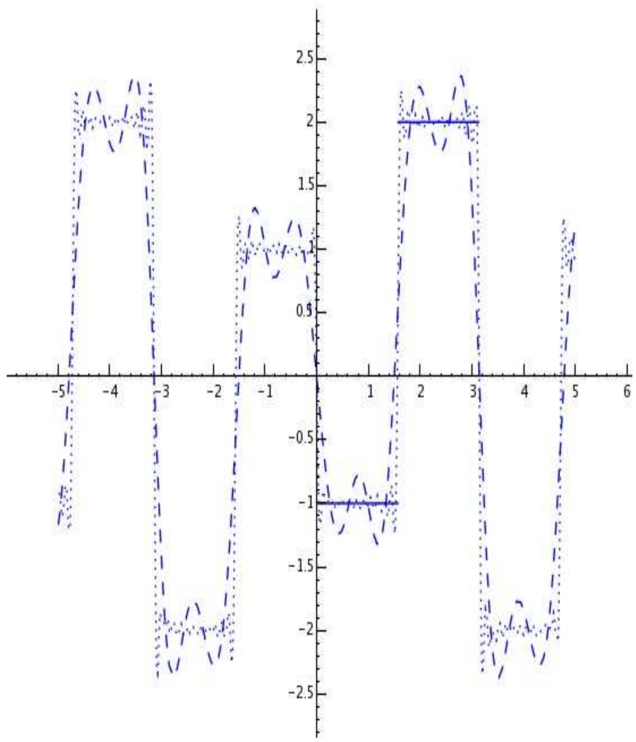


Figure 4.6: $f(x)$ and two sine series approximations.

$$\begin{cases} k\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \\ u_x(0,t) = u_x(L,t) = 0. \end{cases}$$

Here $u_x(x,t)$ denotes the partial derivative of the temperature at a point x on the wire at time t. The initial temperature $f(x)$ is specified by the equation $u(x,0) = f(x)$.

- Find the cosine series of $f(x)$:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n(f)\cos(\frac{n\pi x}{L}),$$

- The solution is

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n(f)\cos(\frac{n\pi x}{L}))\exp(-k(\frac{n\pi}{L})^2 t).$$

Example 4.3.2. Let

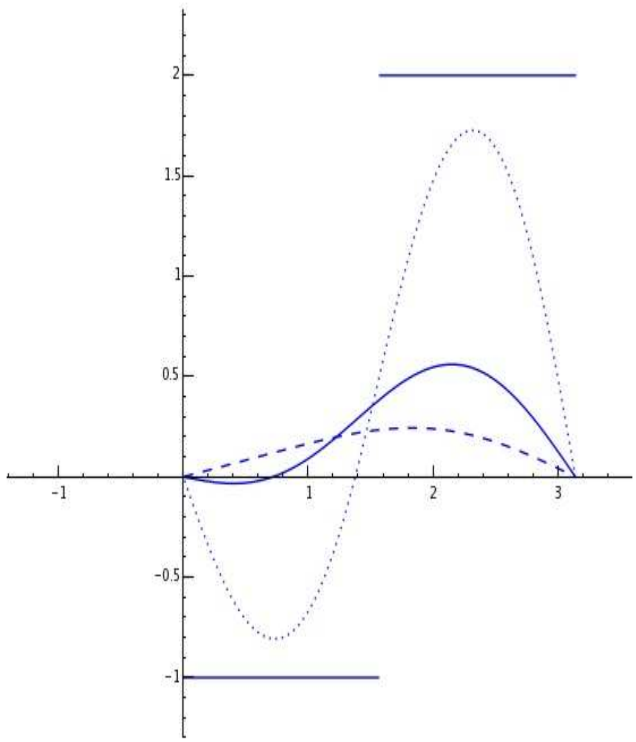


Figure 4.7: $f(x)$, $u(x,0.1)$, $u(x,0.5)$, $u(x,1.0)$ using 60 terms of the sine series.

$$f(x) = \begin{cases} -1, & 0 \le x \le \pi/2, \\ 2, & \pi/2 < x < \pi. \end{cases}$$

Then $L = \pi$ and

$$a_n(f) = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx = -6\frac{\sin\left(\frac{1}{2}\pi n\right)}{\pi n},$$

for $n > 0$ and $a_0 = 1$.

Thus

$$f(x) \sim \frac{a_0}{2} + a_1(f)\cos(x) + a_2(f)\cos(2x) + ...$$

This can also be done in Sage :

Sage

```
sage: x = var("x")
sage: f1(x) = -1
sage: f2(x) = 2
sage: f = Piecewise([[(0,pi/2),f1],[(pi/2,pi),f2]])
```

```
sage: P1 = f.plot()
sage: a10 = [f.cosine_series_coefficient(n,pi) for n in range(10)]
sage: a10
[1, -6/pi, 0, 2/pi, 0, -6/(5*pi), 0, 6/(7*pi), 0, -2/(3*pi)]
sage: a50 = [f.cosine_series_coefficient(n,pi) for n in range(50)]
sage: cs10 = a10[0]/2 + sum([a10[n]*cos(n*x) for n in range(1,len(a10))])
sage: P2 = cs10.plot(-5,5,linestyle="--")
sage: cs50 = a50[0]/2 + sum([a50[n]*cos(n*x) for n in range(1,len(a50))])
sage: P3 = cs50.plot(-5,5,linestyle=":")
sage: (P1+P2+P3).show()
```

This illustrates how the series converges to the function. The piecewise constant function $f(x)$, and some of the partial sums of its cosine series (one using 10 terms and one using 50 terms), looks like Figure 4.8.

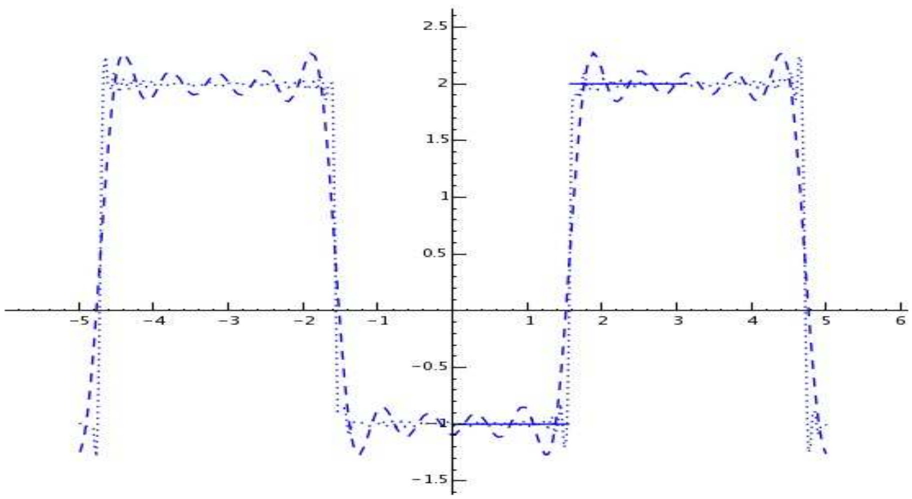


Figure 4.8: $f(x)$ and two cosine series approximations.

As you can see, taking more and more terms gives functions which better and better approximate $f(x)$.

The solution to the heat equation, therefore, is

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n(f)\cos(\frac{n\pi x}{L})\exp(-k(\frac{n\pi}{L})^2 t).$$

Using Sage , we can plot this function:

Sage

```
sage: soln50a = a50[0]/2 + sum([a50[n]*cos(n*x)*e^(-(n+1)^2*(1/100)) for n in range(1,len(a50))])
sage: soln50b = a50[0]/2 + sum([a50[n]*cos(n*x)*e^(-(n+1)^2*(1/10)) for n in range(1,len(a50))])
sage: soln50c = a50[0]/2 + sum([a50[n]*cos(n*x)*e^(-(n+1)^2*(1/2)) for n in range(1,len(a50))])
sage: P4 = soln50a.plot(0,pi)
```

```
sage: P5 = soln50b.plot(0,pi,linestyle=":")
sage: P6 = soln50c.plot(0,pi,linestyle="--")
sage: (P1+P4+P5+P6).show()
```

Taking only the first 50 terms of this series, the graph of the solution at $t = 0$, $t = 0.01$, $t = 0.1$,, $t = 0.5$, looks approximately like:

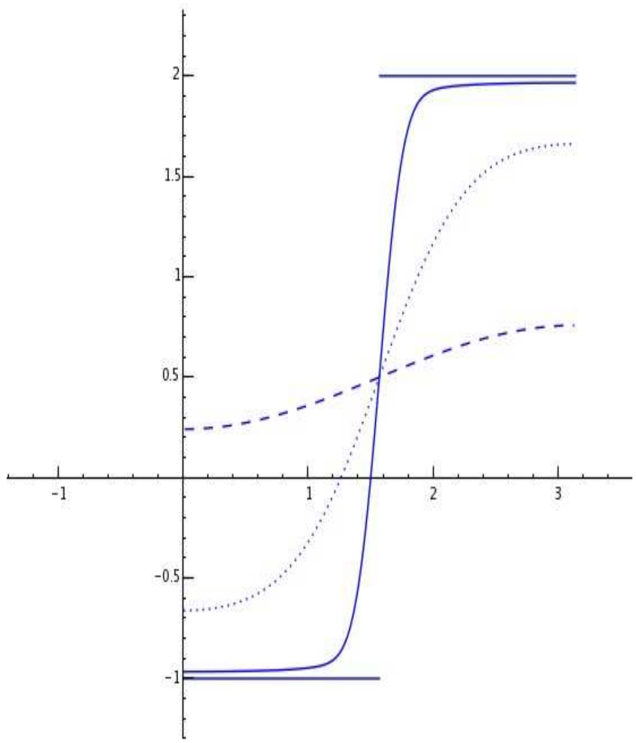


Figure 4.9: $f(x) = u(x, 0)$, $u(x, 0.01)$, $u(x, 0.1)$, $u(x, 0.5)$ using 50 terms of the cosine series.

4.3.3 Explanation

Where does this solution come from? It comes from the method of separation of variables and the superposition principle. Here is a short explanation. We shall only discuss the "zero ends" case (the "insulated ends" case is similar).

First, assume the solution to the PDE $k\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}$ has the "factored" form

$$u(x, t) = X(x)T(t),$$

for some (unknown) functions X, T. If this function solves the PDE then it must satisfy $kX''(x)T(t) = X(x)T'(t)$, or

$$\frac{X''(x)}{X(x)} = \frac{1}{k}\frac{T'(t)}{T(t)}.$$

Since x, t are independent variables, these quotients must be constant. (This of it this way: the derivative of $\frac{X''(x)}{X(x)}$ *with respect to* t is zero. Therefore, the derivative of $\frac{1}{k}\frac{T'(t)}{T(t)}$ with respect to t is zero. This implies it is a constant. In other words, there must be a constant C such that

$$\frac{T'(t)}{T(t)} = kC, \quad X''(x) - CX(x) = 0.$$

Now we have reduced the problem of solving the one PDE to two ordinary differential equations (which is good), but with the price that we have introduced a constant which we don't know, namely C (which maybe isn't so good). The first ordinary differential equation is easy to solve:

$$T(t) = A_1 e^{kCt},$$

for some constant A_1. To obtain physically meaningful solutions, we do not want the temperature of the wire to become unbounded as time increased (otherwise, the wire would simply melt eventually). Therefore, we may assume here that $C \le 0$. It is best to analyse two cases now:

Case $C = 0$: This implies $X(x) = A_2 + A_3 x$, for some constants A_2, A_3. Therefore

$$u(x,t) = A_1(A_2 + A_3 x) = \frac{a_0}{2} + b_0 x,$$

where (for reasons explained later) $A_1 A_2$ has been renamed $\frac{a_0}{2}$ and $A_1 A_3$ has been renamed b_0.

Case $C < 0$: Write (for convenience) $C = -r^2$, for some $r > 0$. The ordinary differential equation for X implies $X(x) = A_2\cos(rx) + A_3\sin(rx)$, for some constants A_2, A_3. Therefore

$$u(x,t) = A_1 e^{-kr^2 t}(A_2\cos(rx) + A_3\sin(rx)) = (a\cos(rx) + b\sin(rx))e^{-kr^2 t},$$

where $A_1 A_2$ has been renamed a and $A_1 A_3$ has been renamed b.

These are the solutions of the heat equation which can be written in factored form. By superposition, "the general solution" is a sum of these:

$$\begin{aligned} u(x,t) &= \tfrac{a_0}{2} + b_0 x + \textstyle\sum_{n=1}^{\infty}(a_n\cos(r_n x) + b_n\sin(r_n x))e^{-kr_n^2 t} \\ &= \tfrac{a_0}{2} + b_0 x + (a_1\cos(r_1 x) + b_1\sin(r_1 x))e^{-kr_1^2 t} \\ &\quad + (a_2\cos(r_2 x) + b_2\sin(r_2 x))e^{-kr_2^2 t} + \ldots, \end{aligned} \tag{4.5}$$

for some a_i, b_i, r_i. We may order the r_i's to be strictly increasing if we like.

We have not yet used the IC $u(x,0) = f(x)$ or the BCs $u(0,t) = u(L,t) = 0$. We do that next.

What do the BCs tell us? Plugging in $x = 0$ into (4.5) gives

$$0 = u(0,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-kr_n^2 t} = \frac{a_0}{2} + a_1 e^{-kr_1^2 t} + a_2 e^{-kr_2^2 t} + \dots .$$

These exponential functions are linearly independent, so $a_0 = 0$, $a_1 = 0$, $a_2 = 0$, ... This implies

$$u(x,t) = b_0 x + \sum_{n=1} b_n \sin(r_n x) e^{-kr_n^2 t} = b_0 x + b_1 \sin(r_1 x) e^{-kr_1^2 t} + b_2 \sin(r_2 x) e^{-kr_2^2 t} + \dots .$$

Plugging in $x = L$ into this gives

$$0 = u(L,t) = b_0 L + \sum_{n=1} b_n \sin(r_n L) e^{-kr_n^2 t}.$$

Again, exponential functions are linearly independent, so $b_0 = 0$, $b_n \sin(r_n L)$ for $n = 1, 2, \dots$. In other to get a non-trivial solution to the PDE, we don't want $b_n = 0$, so $\sin(r_n L) = 0$. This forces $r_n L$ to be a multiple of π, say $r_n = n\pi/L$. This gives

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{L}x) e^{-k(\frac{n\pi}{L})^2 t} = b_1 \sin(\frac{\pi}{L}x)) e^{-k(\frac{\pi}{L})^2 t} + b_2 \sin(\frac{2\pi}{L}x)) e^{-k(\frac{2\pi}{L})^2 t} + \dots, \quad (4.6)$$

for some b_i's. The special case $t = 0$ is the so-called "sine series" expansion of the initial temperature function $u(x,0)$. This was discovered by Fourier. To solve the heat eqution, it remains to solve for the "sine series coefficients" b_i.

There is one remaining condition which our solution $u(x,t)$ must satisfy.

What does the IC tell us? Plugging $t = 0$ into (4.6) gives

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{L}x) = b_1 \sin(\frac{\pi}{L}x)) + b_2 \sin(\frac{2\pi}{L}x)) + \dots .$$

In other words, if $f(x)$ is given as a sum of these sine functions, or if we can somehow express $f(x)$ as a sum of sine functions, then we can solve the heat equation. In fact there is a formula[5] for these coefficients b_n:

$$b_n = \frac{2}{L} \int_0^L f(x) \cos(\frac{n\pi}{L}x) dx.$$

It is this formula which is used in the solutions above.

Exercise: Solve the heat equation

[5] Fourier did not know this formula at the time; it was discovered later by Dirichlet.

$$\begin{cases} 2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \\ u_x(0,t) = u_x(3,t) = 0 \\ u(x,0) = x, \end{cases}$$

using Sage to plot approximations as above.

4.4 The wave equation in one dimension

The theory of the vibrating string touches on musical theory and the theory of oscillating waves, so has likely been a concern of scholars since ancient times. Nevertheless, it wasn't until the late 1700s that mathematical progress was made. Though the problem of describing mathematically a vibrating string requires no calculus, the solution does. With the advent of calculus, Jean le Rond dAlembert, Daniel Bernoulli, Leonard Euler, Joseph-Louis Lagrange were able to arrive at solutions to the one-dimensional wave equation in the eighteenth-century. Daniel Bernoulli's solution dealt with an infinite series of sines and cosines (derived from what we now call a "Fourier series", though it predates it), his contemporaries did not believe that he was correct. Bernoullis technique would be later used by Joseph Fourier when he solved the thermodynamic heat equation in 1807. It is Bernoulli's idea which we discuss here as well. Euler was wrong: Bernoulli's method was basically correct after all.

Now, d'Alembert was mentioned in the lecture on the transport equation and it is worthwhile very briefly discussing what his basic idea was. The theorem of dAlembert on the solution to the wave equation is stated roughly as follows: The partial differential equation:

$$\frac{\partial^2 w}{\partial t^2} = c^2 \cdot \frac{\partial^2 w}{\partial x^2}$$

is satisfied by any function of the form $w = w(x,t) = g(x+ct)+h(x-ct)$, where g and h are "arbitrary" functions. (This is called "the dAlembert solution".) Geometrically speaking, the idea of the proof is to observe that $\frac{\partial w}{\partial t} \pm c\frac{\partial w}{\partial x}$ is a constant times the directional derivative $D_{\vec{v_\pm}} w(x,t)$, where $\vec{v_\pm}$ is a unit vector in the direction $\langle \pm c, 1\rangle$. Therefore, you integrate

$$D_{\vec{v_-}} D_{\vec{v_+}} w(x,t) = (\text{const.})\frac{\partial^2 w}{\partial t^2} - c^2 \cdot \frac{\partial^2 w}{\partial x^2} = 0$$

twice, once in the $\vec{v_+}$ direction, once in the $\vec{v_-}$, to get the solution. Easier said than done, but still, that's the idea.

The wave equation with zero ends boundary conditions models the motion of a (perfectly elastic) guitar string of length L:

$$\begin{cases} c^2\frac{\partial^2 w(x,t)}{\partial x^2} = \frac{\partial^2 w(x,t)}{\partial t^2} \\ w(0,t) = w(L,t) = 0. \end{cases}$$

Here $w(x,t)$ denotes the displacement from rest of a point x on the string at time t. The initial displacement $f(x)$ and initial velocity $g(x)$ at specified by the equations

$$w(x,0) = f(x), \qquad w_t(x,0) = g(x).$$

Method:

- Find the sine series of $f(x)$ and $g(x)$:

$$f(x) \sim \sum_{n=1}^{\infty} b_n(f)\sin(\frac{n\pi x}{L}), \qquad g(x) \sim \sum_{n=1}^{\infty} b_n(g)\sin(\frac{n\pi x}{L}).$$

- The solution is

$$w(x,t) = \sum_{n=1}^{\infty}(b_n(f)\cos(c\frac{n\pi t}{L}) + \frac{Lb_n(g)}{cn\pi}\sin(c\frac{n\pi t}{L}))\sin(\frac{n\pi x}{L}).$$

Example 4.4.1. Let

$$f(x) = \begin{cases} -1, & 0 \le t \le \pi/2, \\ 2, & \pi/2 < t < \pi, \end{cases}$$

and let $g(x) = 0$. Then $L = \pi$, $b_n(g) = 0$, and

$$b_n(f) = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx = -2\,\frac{2\,\cos(n\pi) - 3\,\cos(1/2\,n\pi) + 1}{n}.$$

Thus

$$f(x) \sim b_1(f)\sin(x) + b_2(f)\sin(2x) + ... = \frac{2}{\pi}\sin(x) - \frac{6}{\pi}\sin(2x) + \frac{2}{3\pi}\sin(3x) + ...$$

The function $f(x)$, and some of the partial sums of its sine series, looks like

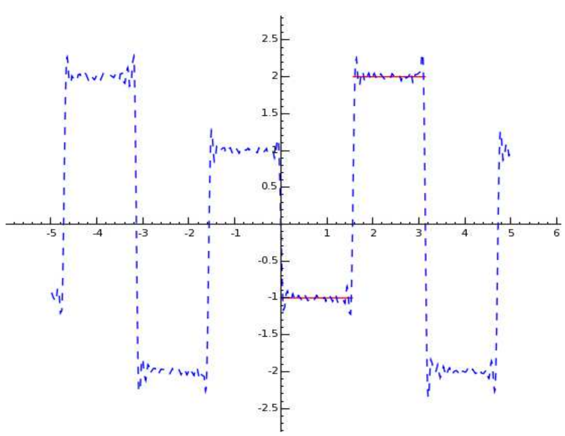


Figure 4.10: Using 50 terms of the sine series of $f(x)$.

This was computed using the following Sage commands:

Sage

```
sage: x = var("x")
sage: f1(x) = -1
sage: f2(x) = 2
sage: f = Piecewise([[(0,pi/2),f1],[(pi/2,pi),f2]])
sage: P1 = f.plot(rgbcolor=(1,0,0))
sage: b50 = [f.sine_series_coefficient(n,pi) for n in range(1,50)]
sage: ss50 = sum([b50[i-1]*sin(i*x) for i in range(1,50)])
sage: b50[0:5]
[2/pi, -6/pi, 2/3/pi, 0, 2/5/pi]
sage: P2 = ss50.plot(-5,5,linestyle="--")
sage: (P1+P2).show()
```

As you can see, taking more and more terms gives functions which better and better approximate $f(x)$.

The solution to the wave equation, therefore, is

$$w(x,t) = \sum_{n=1}^{\infty} (b_n(f)\cos(c\frac{n\pi t}{L}) + \frac{Lb_n(g)}{cn\pi}\sin(c\frac{n\pi t}{L}))\sin(\frac{n\pi x}{L}).$$

Taking only the first 50 terms of this series, the graph of the solution at $t = 0$, $t = 0.1$, $t = 1/5$, $t = 1/4$, looks approximately like:

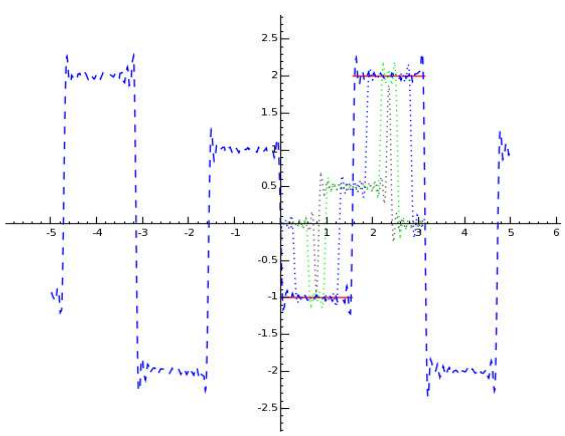


Figure 4.11: Wave equation with $c = 3$.

This was produced using the Sage commands:

Sage

```
sage: t = var("t")
sage: w50t1 = sum([b50[i-1]*sin(i*x)*cos(3*i*(1/10)) for i in range(1,50)])
sage: P3 = w50t1.plot(0,pi,linestyle=":")
sage: w50t2 = sum([b50[i-1]*sin(i*x)*cos(3*i*(1/5)) for i in range(1,50)])
```

```
sage: P4 = w50t2.plot(0,pi,linestyle=":",rgbcolor=(0,1,0))
sage: w50t3 = sum([b50[i-1]*sin(i*x)*cos(3*i*(1/4)) for i in range(1,50)])
sage: P5 = w50t3.plot(0,pi,linestyle=":",rgbcolor=(1/3,1/3,1/3))
sage: (P1+P2+P3+P4+P5).show()
```

Of course, taking terms would give a better approximation to $w(x,t)$. Taking the first 100 terms of this series (but with different times):

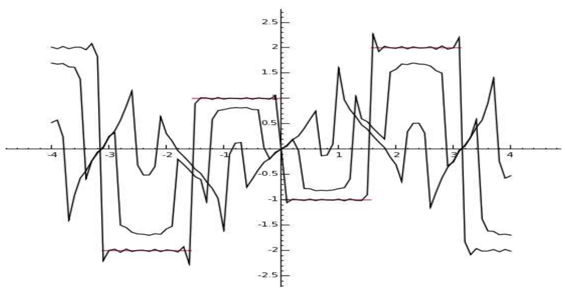


Figure 4.12: Wave equation with $c = 3$.

Exercise: Solve the wave equation

$$\begin{cases} 2\frac{\partial^2 w(x,t)}{\partial x^2} = \frac{\partial^2 w(x,t)}{\partial t^2} \\ w(0,t) = w(3,t) = 0 \\ w(x,0) = x \\ w_t(x,0) = 0, \end{cases}$$

using Sage to plot approximations as above.

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Chapter 5

Appendices

5.1 Appendix: Integral table

$$\int udv = uv - \int vdu \text{(integration by parts)}$$

$$\int x^n \; dx = x^{n+1}/n + 1 + C \quad (n \neq -1)$$

$$\int \frac{1}{x} \; dx = \ln x + C$$

$$\int \sin(x) \; dx = -\cos(x) + C$$

$$\int e^x \; dx = e^x + C$$

$$\int \frac{1}{x^2+1} \; dx = \arctan(x) + C$$

$$\int \frac{1}{ax+b} \; dx = \frac{1}{a} \ln|ax+b| + C$$

$$\int \frac{1}{(x+a)^2} \; dx = -\frac{1}{x+a} + C$$

$$\int (x+a)^n \; dx = \frac{(x+a)^{n+1}}{n+1} + c, n \neq -1$$

$$\int x(x+a)^n \; dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)} + c$$

$$\int \frac{1}{1+x^2} \; dx = \tan^{-1} x + c$$

$$\int \frac{1}{a^2+x^2} \; dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$$

$$\int \frac{x}{a^2+x^2} \; dx = \frac{1}{2} \ln|a^2+x^2| + c$$

$$\int \frac{x^2}{a^2+x^2} \; dx = x - a \tan^{-1} \frac{x}{a} + c$$

$$\int \frac{x^3}{a^2+x^2} \; dx = \frac{1}{2}x^2 - \frac{1}{2}a^2 \ln|a^2+x^2| + c$$

$$\int \frac{1}{ax^2+bx+c} \; dx = \frac{2}{\sqrt{4ac-b^2}} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}} + C$$

$$\int \frac{1}{(x+a)(x+b)} \; dx = \frac{1}{b-a} \ln \frac{a+x}{b+x}, \; a \neq b$$

$$\int \frac{x}{(x+a)^2} \; dx = \frac{a}{a+x} + \ln|a+x| + C$$

$$\int \frac{x}{ax^2+bx+c}\,dx = \frac{1}{2a}\ln|ax^2+bx+c| - \frac{b}{a\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}} + C$$

$$\int \sqrt{x-a}\,dx = \frac{2}{3}(x-a)^{3/2} + C$$

$$\int \frac{x^2}{(a^2-x^2)}\,dx = -\frac{1}{2}\,a\ln(-a+x) + \frac{1}{2}\,a\ln(a+x) - x + C$$

$$\int \frac{1}{\sqrt{x\pm a}}\,dx = 2\sqrt{x\pm a} + C$$

$$\int \frac{1}{\sqrt{a-x}}\,dx = -2\sqrt{a-x} + C$$

$$\int x\sqrt{x-a}\,dx = \frac{2}{3}a(x-a)^{3/2} + \frac{2}{5}(x-a)^{5/2} + C$$

$$\int \sqrt{ax+b}\,dx = \left(\frac{2b}{3a} + \frac{2x}{3}\right)\sqrt{ax+b} + C$$

$$\int (ax+b)^{3/2}\,dx = \frac{2}{5a}(ax+b)^{5/2} + C$$

$$\int \frac{x}{\sqrt{x\pm a}}\,dx = \frac{2}{3}(x\pm 2a)\sqrt{x\pm a} + C$$

$$\int \sqrt{\frac{x}{a-x}}\,dx = -\sqrt{x(a-x)} - a\tan^{-1}\frac{\sqrt{x(a-x)}}{x-a} + C$$

$$\int \sqrt{\frac{x}{a+x}}\,dx = \sqrt{x(a+x)} - a\ln\left[\sqrt{x} + \sqrt{x+a}\right] + C$$

$$\int x\sqrt{ax+b}\,dx = \frac{2}{15a^2}(-2b^2 + abx + 3a^2x^2)\sqrt{ax+b} + C$$

$$\int \sqrt{x(ax+b)}\,dx = \frac{1}{4a^{3/2}}\left[(2ax+b)\sqrt{ax(ax+b)} - b^2\ln\left|a\sqrt{x} + \sqrt{a(ax+b)}\right|\right] + C$$

$$\int \sqrt{x^2\pm a^2}\,dx = \frac{1}{2}x\sqrt{x^2\pm a^2} \pm \frac{1}{2}a^2\ln\left|x + \sqrt{x^2\pm a^2}\right| + C$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\tan^{-1}\frac{x}{\sqrt{a^2-x^2}} + C$$

$$\int x\sqrt{x^2\pm a^2}\,dx = \frac{1}{3}\left(x^2\pm a^2\right)^{3/2} + C$$

$$\int \frac{1}{\sqrt{x^2\pm a^2}}\,dx = \ln\left|x + \sqrt{x^2\pm a^2}\right| + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}}\ dx = \sqrt{x^2 \pm a^2} + C$$

$$\int \frac{x}{\sqrt{a^2 - x^2}}\ dx = -\sqrt{a^2 - x^2} + C$$

$$\int \frac{x^2}{\sqrt{x^2 \pm a^2}}\ dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \mp \frac{1}{2}a^2 \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$$

$$\int \sqrt{ax^2 + bx + c}\, dx = \frac{b + 2ax}{4a}\sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8a^{3/2}} \ln\left|2ax + b + 2\sqrt{a(ax^2 + bx^+ c)}\right| + C$$

$$\int \frac{1}{\sqrt{ax^2 + bx + c}}\ dx = \frac{1}{\sqrt{a}} \ln\left|2ax + b + 2\sqrt{a(ax^2 + bx + c)}\right| + C$$

$$\int \frac{x}{\sqrt{ax^2 + bx + c}}\ dx = \frac{1}{a}\sqrt{ax^2 + bx + c} + \frac{b}{2a^{3/2}} \ln\left|2ax + b + 2\sqrt{a(ax^2 + bx + c)}\right| + C$$

$$\int \ln ax\ dx = x \ln ax - x + C$$

$$\int \frac{\ln ax}{x}\ dx = \frac{1}{2}(\ln ax)^2 + C$$

$$\int \ln(ax + b)\ dx = \left(x + \frac{b}{a}\right) \ln(ax + b) - x + C, a \neq 0$$

$$\int \ln\left(a^2x^2 \pm b^2\right)\ dx = x \ln\left(a^2x^2 \pm b^2\right) + \frac{2b}{a} \tan^{-1} \frac{ax}{b} - 2x + C$$

$$\int \ln\left(a^2 - b^2x^2\right)\ dx = x \ln\left(ar - b^2x^2\right) + \frac{2a}{b} \tan^{-1} \frac{bx}{a} - 2x + C$$

$$\int \ln\left(ax^2 + bx + c\right)\ dx = \frac{1}{a}\sqrt{4ac - b^2} \tan^{-1} \frac{2ax + b}{\sqrt{4ac - b^2}} - 2x + \left(\frac{b}{2a} + x\right) \ln\left(ax^2 + bx + c\right) + C$$

$$\int x \ln(ax + b)\ dx = \frac{bx}{2a} - \frac{1}{4}x^2 + \frac{1}{2}\left(x^2 - \frac{b^2}{a^2}\right) \ln(ax + b) + C8$$

$$\int x \ln\left(a^2 - b^2x^2\right)\ dx = -\frac{1}{2}x^2 + \frac{1}{2}\left(x^2 - \frac{a^2}{b^2}\right) \ln\left(a^2 - b^2x^2\right) + C$$

$$\int e^{ax}\ dx = \frac{1}{a}e^{ax} + C$$

$$\int \sqrt{x}e^{ax}\ dx = \frac{1}{a}\sqrt{x}e^{ax} + \frac{i\sqrt{\pi}}{2a^{3/2}} \operatorname{erf}\left(i\sqrt{ax}\right) + C,\ \text{where } \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dtet$$

$$\int xe^x\ dx = (x - 1)e^x + C$$

$$\int xe^{ax}\ dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax} + C$$

$$\int x^2 e^x \, dx = \left(x^2 - 2x + 2\right) e^x + C$$

$$\int x^2 e^{ax} \, dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right) e^{ax} + C$$

$$\int x^3 e^x \, dx = \left(x^3 - 3x^2 + 6x - 6\right) e^x + C$$

$$\int x^n e^{ax} \, dx = \frac{(-1)^n}{a^{n+1}} \Gamma[1+n, -ax], \text{ where } \Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t} dt$$

$$\int e^{ax^2} \, dx = -\frac{i\sqrt{\pi}}{2\sqrt{a}} \operatorname{erf}\left(ix\sqrt{a}\right)$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax + C$$

$$\int \sin^2 ax \, dx = \frac{x}{2} - \frac{\sin 2ax}{4a} + C$$

$$\int \sin^3 ax \, dx = -\frac{3\cos ax}{4a} + \frac{\cos 3ax}{12a} + C$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax + C$$

$$\int \cos^2 ax \, dx = \frac{x}{2} + \frac{\sin 2ax}{4a} + C$$

$$\int \cos^3 ax \, dx = \frac{3\sin ax}{4a} + \frac{\sin 3ax}{12a} + C$$

$$\int \sin x \cos x \, dx = \frac{1}{2}\sin^2 x + C_1 = -\frac{1}{2}\cos^2 x + C_2 = -\frac{1}{4}\cos 2x + C_3$$

$$\int \sin^2 x \cos x \, dx = \frac{1}{3}\sin^3 x + C$$

$$\int \cos^2 ax \sin ax \, dx = -\frac{1}{3a}\cos^3 ax + C$$

$$\int \sin^2 ax \cos^2 ax \, dx = \frac{x}{8} - \frac{\sin 4ax}{32a} + C$$

$$\int \tan ax \, dx = -\frac{1}{a}\ln\cos ax + C$$

$$\int \tan^2 ax \, dx = -x + \frac{1}{a}\tan ax + C$$

$$\int \tan^3 ax \, dx = \frac{1}{a}\ln\cos ax + \frac{1}{2a}\sec^2 ax + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C = 2\tanh^{-1}\left(\tan\frac{x}{2}\right) + C$$

$$\int \sec^2 ax\ dx = \frac{1}{a}\tan ax + C$$

$$\int \sec^3 x\ dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x \tan x| + C$$

$$\int \sec x \tan x\ dx = \sec x + C$$

$$\int \sec^2 x \tan x\ dx = \frac{1}{2}\sec^2 x + C$$

$$\int \sec^n x \tan x\ dx = \frac{1}{n}\sec^n x + C, n \neq 0$$

$$\int \csc x\ dx = \ln\left|\tan\frac{x}{2}\right| + C = \ln|\csc x - \cot x| + C$$

$$\int \csc^2 ax\ dx = -\frac{1}{a}\cot ax + C$$

$$\int \csc^3 x\ dx = -\frac{1}{2}\cot x \csc x + \frac{1}{2}\ln|\csc x - \cot x| + C$$

$$\int \csc^n x \cot x\ dx = -\frac{1}{n}\csc^n x + C, n \neq 0$$

$$\int \sec x \csc x\ dx = \ln|\tan x| + C$$

$$\int x\cos x\ dx = \cos x + x\sin x + C$$

$$\int x\cos ax\ dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax + C$$

$$\int x^2\cos x\ dx = 2x\cos x + \left(x^2 - 2\right)\sin x + C$$

$$\int x^2\cos ax\ dx = \frac{2x\cos ax}{a^2} + \frac{a^2x^2 - 2}{a^3}\sin ax + C$$

$$\int x^n cosx\ dx = -\frac{1}{2}(i)^{n+1}\left[\Gamma(n+1,-ix) + (-1)^n\Gamma(n+1,ix)\right] + C$$

$$\int x^n cosax\ dx = \frac{1}{2}(ia)^{1-n}\left[(-1)^n\Gamma(n+1,-iax) - \Gamma(n+1,ixa)\right] + C$$

$$\int x\sin x\ dx = -x\cos x + \sin x + C$$

$$\int x\sin ax\ dx = -\frac{x\cos ax}{a} + \frac{\sin ax}{a^2} + C$$

$$\int x^2\sin x\ dx = \left(2 - x^2\right)\cos x + 2x\sin x + C$$

$$\int x^2 \sin ax \; dx = \frac{2 - a^2x^2}{a^3} \cos ax + \frac{2x \sin ax}{a^2} + C$$

$$\int x^n \sin x \; dx = -\frac{1}{2}(i)^n \left[\Gamma(n+1, -ix) - (-1)^n \Gamma(n+1, -ix)\right] + C$$

$$\int e^x \sin x \; dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

$$\int e^{bx} \sin ax \; dx = \frac{1}{a^2 + b^2} e^{bx} (b \sin ax - a \cos ax) + C$$

$$\int e^x \cos x \; dx = \frac{1}{2} e^x (\sin x + \cos x) + C$$

$$\int e^{bx} \cos ax \; dx = \frac{1}{a^2 + b^2} e^{bx} (a \sin ax + b \cos ax) + C$$

$$\int x e^x \sin x \; dx = \frac{1}{2} e^x (\cos x - x \cos x + x \sin x) + C$$

$$\int x e^x \cos x \; dx = \frac{1}{2} e^x (x \cos x - \sin x + x \sin x) + C$$

$$\int \cosh ax \; dx = \frac{1}{a} \sinh ax + C$$

$$\int e^{ax} \cosh bx \; dx = \begin{cases} \dfrac{e^{ax}}{a^2 - b^2} [a \cosh bx - b \sinh bx] + C & a \neq b \\ \dfrac{e^{2ax}}{4a} + \dfrac{x}{2} + C & a = b \end{cases}$$

$$\int \sinh ax \; dx = \frac{1}{a} \cosh ax + C$$

$$\int e^{ax} \sinh bx \; dx = \begin{cases} \dfrac{e^{ax}}{a^2 - b^2} [-b \cosh bx + a \sinh bx] + C & a \neq b \\ \dfrac{e^{2ax}}{4a} - \dfrac{x}{2} + C & a = b \end{cases}$$

$$\int e^{ax} \tanh bx \, dx = \begin{cases} \dfrac{1}{(a+2b)} e^{(a+2b)x} {}_2F_1\left[1 + \dfrac{a}{2b}, 1, 2 + \dfrac{a}{2b}, -e^{2bx}\right] - \dfrac{1}{a} e^{ax} {}_2F_1\left[\dfrac{a}{2b}, 1, 1E, -e^{2bx}\right] + C & a \neq b \\ \dfrac{e^{ax} - 2\tan^{-1}[e^{ax}]}{a} + C & a = b \end{cases}$$

$$\int \tanh bx \; dx = \frac{1}{a} \ln \cosh ax + C$$

$$\int \cos ax \cosh bx \; dx = \frac{1}{a^2 + b^2} \left[a \sin ax \cosh bx + b \cos ax \sinh bx\right] + C$$

$$\int \cos ax \sinh bx \; dx = \frac{1}{a^2 + b^2} \left[b \cos ax \cosh bx + a \sin ax \sinh bx\right] + C$$

$$\int \sin ax \cosh bx \; dx = \frac{1}{a^2 + b^2} \left[-a \cos ax \cosh bx + b \sin ax \sinh bx\right] + C$$

$$\int \sin ax \sinh bx \ dx = \frac{1}{a^2 + b^2} \left[b \cosh bx \sin ax - a \cos ax \sinh bx \right] + C$$

$$\int \sinh ax \cosh ax \ dx = \frac{1}{4a} \left[-2ax + \sinh 2ax \right] + C$$

$$\int \sinh ax \cosh bx \ dx = \frac{1}{b^2 - a^2} \left[b \cosh bx \sinh ax - a \cosh ax \sinh bx \right] + C$$

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