Coding theory and cryptography with Sage

a free and open-source mathematics package

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Sage homepage: http:/www.sagemath.org/

Sage

What is

What is Sage?

First, let us take a tour of the

`http://www.sagemath.org`

website ...

What is Sage?

As we saw, Sage includes are: `Maxima`, `pynac` (a Python-icized GiNaC), and `SymPy` (for calculus and other symbolic computation), `Singular` and `GAP` (for algebra), `R` (for statistics), `Pari` (for number theory), `SciPy` (for numerical computation), `libcrypt` for cryptography, and over 60 more.

Sage is based on the mainstream programming language Python.

Sage is headed by the mathematician William Stein, who is at the University of Washington, in Seattle.

What is in Sage?

Other packages available in Sage:

Basic Arithmetic	`GMP`, `NTL`, `flint`
Command Line	`IPython`
Graphical Interface	Sage Notebook
Graphics	`jmol`, `Matplotlib`, ...
Graph theory	`NetworkX`
Interpreted programming language	Python
Networking	`Twisted`
Applied Math.	`SciPy`, `GSL`, `GLPK`, etc.
Source control system	`Mercurial`
Symbolic computation, calculus	`SymPy`, `pynac`

To be a component of Sage, the software must be: free, open source, robust, high quality, and portable.

Some history: Sage 0.1 to Sage 4.5

- Nov 2004: William Stein developed Manin, a precursor to Sage.
- Feb 2005: Sage *0.1*. This included `Pari`.
- Oct 2005, Sage *0.8*: `GAP` and `Singular` included as standard.
- Feb 2006: Sage **Days 1** workshop, UCSD – Sage 1.0
- May-July, 2006 (Sage 1.3.*) GUI Notebook developed by William Stein, Alex Clemsha and Tom Boothby.
- Sage Days Workshops at UCLA, UW, Cambridge, Bristol, Austin, France, San Diego, Seattle, MSRI, Barcelona, ...
- Sage won first prize in the Trophees du Libre (November 2007)
- Sage Days 23.5 – Kaiserslautern, Germany on “Singular and Sage integration,” ends July 9, 2010.

See `http://wiki.sagemath.org/` for more details.

Sage now has a *huge* range of functionality.

The Sage Command Line

When you start Sage you will get a small Sage banner and then the Sage command-line prompt `sage:`.

If you are happy to work at the command line, here is an example of what a short Sage session could look like:

Sage

```
sage: 2^3
8
sage: t = var("t")
sage: integrate(t*sin(t^2),t)
-cos(t^2)/2
sage: plot[TAB]
plot                    plot_slope_field     plotkin_bound_asymp
plot3d                  plot_vector_field    plotkin_upper_bound
```

Tab-completion helps you select the command you want with less effort.

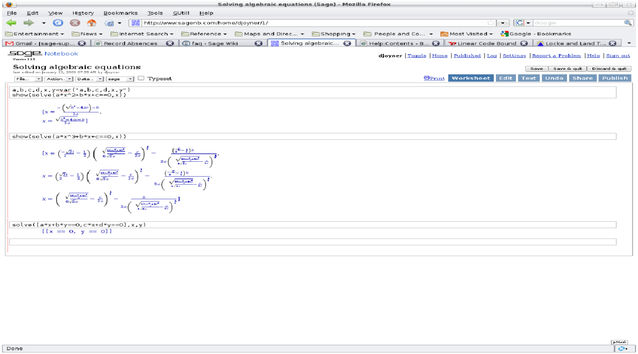
The Sage Notebook

The Sage Notebook can be tried out for free by anyone with an internet connection and a good browser at **http://www.sagenb.org**.

- Connect to Sage running locally *or elsewhere* (via internet).
- Create embedded graphics (in 2- and 3-d).
- Typeset mathematical expressions using LaTeX.
- Add and delete input, re-executing entire block of commands at once.
- Start and interrupt multiple calculations at once.
- The notebook also works with `Maxima`, Python, `R`, `Singular`, LaTeX, html, etc.!

The Sage Notebook

The following screenshot illustrates a Notebook worksheet.



The Sage Notebook

Here are the commands used to create the output in the Notebook session in the above screenshot:

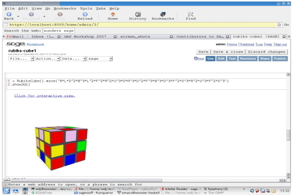
Sage Notebook

```
a,b,c,d,x,y=var('a,b,c,d,x,y')
show(solve(a*x^2+b*x+c==0,x))
show(solve(a*x^3+b*x+c==0,x))
solve(a*x+b*y==0,c*x+d*y==0,x,y)
```

Worksheets can be **saved** (as text or as an sws file in Sage worksheet format), **downloaded** and emailed (for use by someone else), **shared** (with colleagues or students), or **published** (if created on a public Sage server).

Sage notebook screenshot (an uploaded *.sws file)

- If you enjoy playing with the Rubik's cube, there are several programs for solving the Rubik's cube in Sage:



You can rotate the Rubik's cube interactively with your mouse.

Open source philosophy

Sage is Free!

- Sage is free software. You can check the algorithms yourself in the source code.
- You can legally serve all its functionality over the web (unlike Magma, Maple, Mathematica, and Matlab).
- Everything in Sage is 100% GPL-compatible (except jsmath, which is Apache licensed and runs in browser).
- A lot of work has went into "clarifying" licenses on existing math software (... the `Singular`/oMalloc story).
- Sometimes we reimplement major algorithms from the ground up because of license problems (... the Nauty/NICE story).
- You can change absolutely anything in Sage or any of its dependencies and definitely rebuild or publicly redistribute the result.

Why is open source relevant for mathematics? From a recent interview published in the AMS Notices:

I think we need a symbolic standard to make computer manipulations easier to document and verify. And with all due respect to the free market, perhaps we should not be dependent on commercial software here. An open source project could, perhaps, find better answers to the obvious problems such as availability, bugs, backward compatibility, platform independence, standard libraries, etc. One can learn from the success of TeX and more specialized software like Macaulay2. I do hope that funding agencies are looking into this.

Andrei Okounkov, *2006 Fields Medalist*

Open Source in Government and Industry

Open source software is part of the integrated network fabric which connects and enables our command and control system to work effectively, as people's lives depend on it.

Open source software is all about "playing nice with others." It is all about "citizenship." We need more software collaboration in the DoD. My challenge to you: Become a citizen of the OSS community.

Brig. Gen. N. G. Justice, *U. S. Army*

Elliptic curves

Elliptic curves

- All standard algorithms
- p-adic L-functions, complex L-functions
- Heegner points
  Euler system and Iwasawa-theoretic bounds on Shafarevich-Tate groups
- Group structure over finite fields
- Fast point counting modulo p
- Plotting pictures of elliptic curves

Number theory

Extensive collection of number theory functions. However, for factoring of large integers, only select algorithms are implemented.

Example

```
sage: zeta(0.5+14.0*I)
0.0222411426099936 - 0.103258123266450*I
sage: zeta(0.5+14.1*I)
0.00469840018348919 - 0.0270582823742510*I
sage: zeta(0.5+14.2*I)
-0.00681621815859797 + 0.0515969909777821*I
sage: zeta(0.5+14.3*I)
-0.0119878243107407 + 0.132231368469266*I
```

Modular forms

Probably Sage is the best software for this area of computational mathematics.

Example

```
sage: m = ModularForms(Gamma0(389),6)
sage: m.eisenstein_submodule()
Eisenstein subspace of dimension 2 of Modular Forms
 space of dimension 163 for Congruence Subgroup
 Gamma0(389) of weight 6 over Rational Field
```

Rings

- Weyl character ring and group rings,
- Algebraic rings: All of the standard rings, such as $\mathbb{Z}$, $\mathbb{Q}$, finite fields $GF(p^k)$, and polynomial, power series and Laurant series rings over any other ring in Sage. Threes models of p-adic numbers.

  The algebraic closure of Q and its maximal totally real subfield are also implemented, using intervals.
- Numerical: Real and complex numbers of any fixed precision. Rings that model $\mathbb{R}$ and $\mathbb{C}$ with intervals (interval arithmetic).
- Symbolic rings (for calculus, etc).

Number fields

- Absolute, relative, arbitrary towers (built on Pari but offers much more flexibility)
- Class groups, units, norm equations, maximal orders, reduction mod primes

Commutative Algebra

- Clean, structured, object-oriented multivariate polynomial rings, coordinate rings of varieties, and ideals
- Uses `Singular` as backend when possible for arithmetic speed and certain algorithms
- Groebner Basis computations

Algebraic geometry

Algebraic geometry

- Varieties and Schemes
- Genus 2 curves and their Jacobians (including fast p-adic point counting algorithms of Kedlaya and Harvey)
- Implicit plotting of curves and surfaces

Linear algebra

Linear algebra

- Sparse and dense linear algebra over many rings
- Highly optimized in many cases
- In somes cases, possibly the fastest money can buy

Algebraic topology

- The Steenrod algebra
- Simplical complexes and their homology

Graph theory

- Sage may overall be the best graph theory software money can buy...

(Thanks to Robert Miller, Nathann Cohen, Emily Kirkman, ...)

Sage and graph theory

Sage

```
sage: graph_dict = {0: [1,4,5], 1: [2,6], 2: [3,7], 3: [4,2],
                                 4: [0,1], 5: [7, 6], 6: [2], 7: [2]}
sage: G = Graph(graph_dict)
sage: G.show(graph_border=True)
```

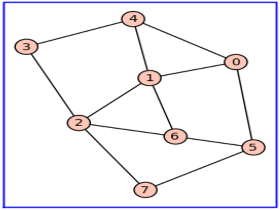


Figure: **A graph created using Sage.**

Combinatorics

Sage has excellent functionality in algebraic combinatorics

- Nicolas Thiery: Mupad-combinat $\mapsto$ Sage-combinat
- Symmetric functions, partitions, Lie algebras and root systems, enumeration, crystals, species, etc.

Group theory

Group theory

- Sage includes GAP
- Weyl groups and Coxeter groups,
- Sage includes some “native” permutation group functions
- Sage includes “native” abelian group functions
- Sage includes a matrix group class, abelian group class and a permutation group class
- Sage has some native group cohomology functions

Sage lacks a free group class (for example).

Applied math

- Sage includes sympy
- Sage will include GLPK
- Sage includes scipy, numpy, and GSL
- Sage includes R
  Sage can solve some ODEs using maxima or sympy.

Statistics

- Sage includes R
- Sage includes scipy.stats
- Sage includes a finance module

Python

Sage is based on Python

Python is a powerful modern interpreted general programming language, which happens to be very well-suited for scientific programming.

- "Python is fast enough for our site and allows us to **produce maintainable features in record times**, with a minimum of developers," said Cuong Do, Software Architect, `YouTube.com`.

Sage is based on Python

- "Google has made no secret of the fact they use Python a lot for a number of internal projects. Even knowing that, once **I was an employee, I was amazed at how much** Python **code there actually is in the Google source code system**.", said Guido van Rossum, Google, creator of Python.
- "Python plays a key role in our production pipeline. Without it a project the size of **Star Wars: Episode II** would have been very difficult to pull off. From crowd rendering to batch processing to compositing, Python binds all things together," said Tommy Burnette, Senior Technical Director, Industrial Light & Magic.

Python is...

- Easy for you to define your own data types and methods on it. symbolic expressions, graphics types, vector spaces, special functions, whatever.
- Very clean language that results in easy to *read* code.
- Easy to learn:
  - Free: Dive into Python `http://www.diveintopython.org/`
  - Free: Python Tutorial `http://docs.python.org/tut/`
- A *huge* number of libraries: statistics, networking, databases, bioinformatic, physics, video games, 3d graphics, ...

Python is...

- Easy to use any C/C++ libraries from Python.
- Excellent support for string manipulation and file manipulation.
- `Cython` – a Python compiler (`http://www.cython.org`).

xkcd: import antigravity

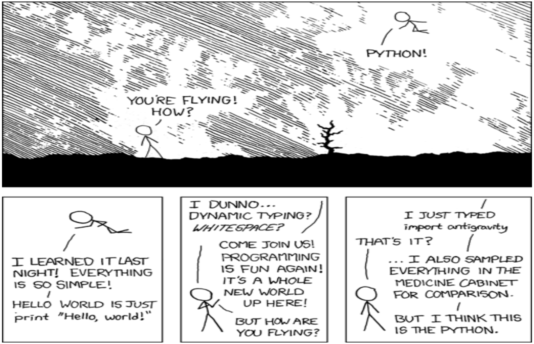


Figure: Python. xkcd.com license:
http://creativecommons.org/licenses/by-nc/2.5/

Python is...

- The Python programming language has a specific syntax (form) and semantics (meaning) which enables it to express computations and data manipulations which can be performed by a computer.
- Python's implementation was started in 1989 by Guido van Rossum, while at CWI .
- Python is an "interpreted' language, i.e., Python programs are not directly executed by the host CPU but rather executed by a program known as an "interpreter."
- The source code of a Python program is translated or (partially) compiled to a "bytecode" form of a Python "process virtual machine" language.

Python is dynamically typed

Because Python is dynamically typed, Python can figure out the type from the command at run-time.

Python

```
>>> a = 2012
>>> type(a)
<type 'int'>
>>> b = 2.011
>>> type(b)
<type 'float'>
```

Python is object-oriented

Python is an object-oriented language. Objects are data structures consisting of datafields and methods. Here is an example of a method, `sort`, which applies to the object `L` of type `list`.

Python

```
>>> L = [2,1,4,3]
>>> type(L)
<type 'list'>
>>> L.sort()
>>> L
[1, 2, 3, 4]
```

Some Python data types

Python data types are described in
`http://docs.python.org/library/datatypes.html`.

Type	*Description*	*Syntax example*
`str`	An immutable sequence of Unicode characters	`"string", """\python is great""", '2012'`
`list`	Mutable, can contain mixed types	`[1.0, 'list', True]`
`tuple`	Immutable, can contain mixed types	`(-1.0, 'tuple', False)`
`dict`	A mutable group of key and value pairs	`{'key1': 1.0, 'key2': False}`
`int`	immutable fixed precision	`42`
`float`	immutable floating point	`2.71828`
`bool`	An immutable Boolean value	`True, False`

An example of a Python dictionary

You can create a dictionary "from scratch," adding entries "manually" and using `pop` to remove items. Otherwise, a dictionary is like a list.

Sage

```
sage: d = {}
sage: d["1"] = 2
sage: d[2010] = "year"
sage: d
{'1': 2, 2010: 'year'}
sage: type(d)
<type 'dict'>
sage: d.pop(2010)
'year'
sage: d
{'1': 2}
```

Python keywords

Keyword	*meaning*
`and`	boolean operator
`as`	used with `import` and `with`
`assert`	used for debugging
`break`	used in a `for/while` loop
`class`	creates a class
`continue`	used in `for/while` loops
`def`	defines a function or method
`del`	deletes a reference to a object instance
`elif`	used in `if ... then` statements
`else`	used in `if ... then` statements
`except`	used in `if ... then` statements
`exec`	executes a system command
`finally`	used in `if ... then` statements
`for`	used in a `for` loop
`from`	used in a `for` loop
`global`	this is a (constant) data type
`if`	used in `if ... then` statements

Python keywords

Keyword	*meaning*
`import`	loads a file of data or Python commands
`in`	boolean operator on a set
`is`	boolean operator
`lambda`	defines a simple “one-liner” function
`not`	boolean operator
`or`	boolean operator
`pass`	allows and `if-then-elif` statement to skip a case
`print`	prints the value of the argument
`raise`	used for error messages
`return`	output of a function
`try`	allows you to test for an error
`while`	used in a `while` loop
`with`	used in `try` statements
`yield`	used for iterators and generators

(Type `import keyword; keyword.kwlist` for this list within Python.)

The Zen of Python, I

Beautiful is better than ugly.
Explicit is better than implicit.
Simple is better than complex.
Complex is better than complicated.
Flat is better than nested.
Sparse is better than dense.
Readability counts.
Special cases aren't special enough to break the rules.
Although practicality beats purity.
Errors should never pass silently.
Unless explicitly silenced.

Type `import this` to see the rest!

`for` loops

A `for` loop:

Python

```
>>> for n in range(10,14):
...         if not(n%4 == 2):
...             print n
...
11
12
13
>>> [n for n in range(10,20) if not(n%4==2)]  #  list comprehension
[11, 12, 13, 15, 16, 17, 19]
```

Note the indentation after the ":".

Python function template

Here is a template of a properly documented Python function.

Python

```
def my_function(my_input1, my_input2 = my_default_value2):
    """
    Your docstring (see next slide).
    """
    command1    # comment 1
    command2    # comment 2
    return output
```

Documenting appropriately for Sage submissions is required.

Python function template

Here is a docstring of a properly documented Python function. (Add an `AUTHOR(s)` field if appropriate).

Python

```
"""
Description.

 INPUT:
   my_input1 - the type of the 1st input
   my_input2 - the type of the 2nd input
OUTPUT:
   the type of the output

EXAMPLES:
    >>> my_function(arg1,arg2)
    <the output>

REFERENCES:
    [1] <A Wikipedia article describing the algorithm used>, <url>
    [2] <A book on algorithms describing the algorithm used>,
    <page numbers>
"""
```

Hello World!

The example below gives an interactive example requiring user input.

Python

```
>>> def hello():
...          name = raw_input('What is your name?\n')
...          print "Hello World! My name is %s"%name
...
>>> hello()
What is your name?                        ###  This is output
David                                     ###  This is input
Hello World! My name is David             ###  This is output
>>>
```

xgcd

Python

```
def extended_gcd(a, b):
    """
    Implements Euclid's extended greatest common divisor
    algorithm (returns (x,y) s.t. a*x+b*y=gcd(a,b)).

    EXAMPLES:
        >>> extended_gcd(12,15)
        (-1, 1)
    """
    if a%b == 0:
        return (0, 1)
    else:
        (x, y) = extended_gcd(b, a%b)
        return (y, x-y*int(a/b))
```

xkcd and Python

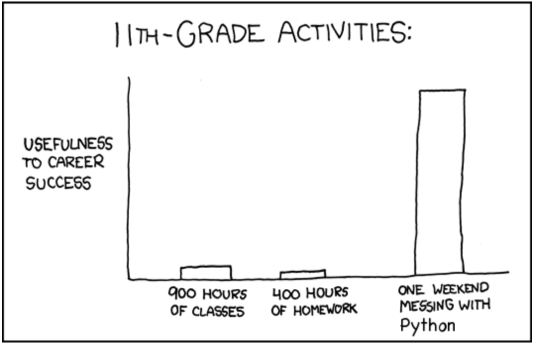


Figure: **11th grade.** `xkcd.com` license: `http://creativecommons.org/licenses/by-nc/2.5/`

lambda functions

The command `lambda` allows you to create a one-line function which does not have any local variables except those used to define the function.

Python

```
>>> f = lambda x,y: x+y
>>> f(1,2)
3
```

Your own functions using Sage classes

The function below is in Python but uses Sage classes.

Sage

```
def Hexacode():
    """
    This function returns the [6,3,4] hexacode over GF(4).
    It is an extremal (Hermitian) self-dual Type IV code.

    EXAMPLES:
        sage: C = Hexacode()
        sage: C.minimum_distance()
        4
    """
    F = GF(4,"z")
    z = F.gen()
    MS = MatrixSpace(F, 3, 6)
    G = MS([[1, 0, 0, 1, z, z ], [0, 1, 0, z, 1, z ], [0, 0, 1, z, z, 1 ]])
    return LinearCode(G)
```

Collatz conjecture

The **Collatz conjecture** (or the $3n + 1$ **conjecture**, or as the **Syracuse problem**): Start with any integer n greater than 1. If n is even, we halve it ($n/2$), else we "triple it plus one" ($3n + 1$). According to the conjecture, for all positive numbers this process eventually converges to 1.
For example,

$$10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1.$$

Exercise: Write a Python function to test this conjecture.

Collatz conjecture

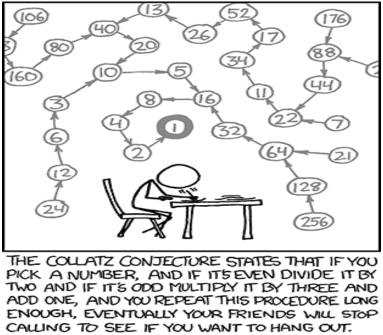


Figure: **The Collatz Conjecture.** `xkcd` license:
`http://creativecommons.org/licenses/by-nc/2.5/`

Example: Compute x^{13}.
Use the "binary decomposition": $13 = 1 + 2^2 + 2^3$. First compute x^1 (0 steps), then x^4 (2 steps, namely $x^2 = x \cdot x$ and $x^4 = x^2 \cdot x^2$), and finally x^8 (1 more step, namely $x^8 = x^4 \cdot x^4$). Now (3 more steps)

$$x^{13} = x \cdot x \cdot x^4 \cdot x^8.$$

In general, **we can compute x^n in about $O(\log n)$ steps**.

Repeated squaring algorithm

Python

```
def power(x,n):
    """
    INPUT:
        x - a number
        n - an integer > 0
    OUTPUT:
        x^n

    EXAMPLES:
        >>> power(3,13)
        1594323
        >>> 3**(13)
        1594323
    """
    if n == 1:
        return x
    if n%2 == 0:
       return power(x, int(n/2))**2
    if n%2 == 1:
       return x*power(x, int((n-1)/2))**2
```

Fibonacci numbers

Leonardo of Pisa, known as Fibonacci, who mentioned the $\{f_n\}_{n=0}^{\infty}$ in a book he wrote in the 1200's. The recursion equation

$$f_n = f_{n-1} + f_{n-2}, \quad n > 1, \quad f_1 = 1, \ f_0 = 0,$$

defines the sequence of **Fibonacci numbers**.

Fibonacci numbers

Computes the f_n very slowly (note: the input n requires $O(\log n)$ bits).

Python

```
def my_fibonacci(n):
    """
    This is really really slow.
    """
    if n==0:
        return 0
    elif n==1:
        return 1
    else:
        return my_fibonacci(n-1)+my_fibonacci(n-2)
```

In fact, the “complexity" of this algorithm to compute f_n is about equal to f_n. This is $O(\phi^n)$, where $\phi = \frac{1+\sqrt{5}+1}{2}$. (Think about the associated binary tree ...)

Fibonacci numbers

The following is left as an exercise.

Lemma

For each $n > 0$, we have $F^n = \begin{pmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{pmatrix}$, where $F = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

Thanks to “repeated squaring,” the “complexity" of this algorithm to compute f_n is about equal to $O(\log n)$

A simple Python class for a prime finite fields, 1

Python

```
"""
Prime finite fields in Python.

"""

class FF:
    """
    Implements "prime" finite fields.

    EXAMPLES:
        sage: F = FF(5)
        sage: print F
        Finite field with 5 elements
        sage: F
        FF(5)

    """
    def __init__(self, p):
        self.characteristic = p
```

A simple Python class for a prime finite fields, 2

Continued (note the indentation):

Python

```
def __repr__(self):
    """
    Called to compute the "official" string representation of an object.
    If at all possible, this should look like a valid Python expression
    that could be used to recreate an object with the same value.

    EXAMPLES:
        sage: F = FF(5)
        sage: F
        FF(5)

    """
    return "FF(%s)"%self.characteristic
```

A simple Python class for a prime finite fields, 3

Continued (note the indentation):

Python

```
def __str__(self):
    """
    Called to compute the "informal" string description of an object.

    EXAMPLES:
        sage: F = FF(5)
        sage: print F
        Finite field with 5 elements

    """
    return "Finite field with %s elements"%self.characteristic
```

A simple Python class for a prime finite fields, 4

Python

```
    def char(self):
        """
        Returns the characteristic of the finite field.

        EXAMPLES:
            sage: FF(5).char()
            5
        """
        return self.characteristic

    def __eq__(self, other):
        """
        Returns True of self = other and False otherwise.

        EXAMPLES:
            sage: FF(5) == FF(7)
            False
        """
        p = self.char()
        q = other.char()
        return p == q
```

A simple Python class for a prime finite fields, 5

Continued (note the indentation):

Python

```
    def __call__(self, a):
        """
        Reduces $a \pmod p$, returning an element of the FF (``coercion'').

        EXAMPLES:
            sage: F = FF(5)
            sage: F(12)
            2
        """
        p = self.characteristic
        return FFElement(p, a)
```

A simple Python class for a prime finite fields, 6

Python

```
    def __contains__(self, a):
        """
        Tests if a is in the FF.

        EXAMPLES:
            sage: F = FF(5)
            sage: 2 in F
            True
            sage: 6 in F
            False

        """
        p = self.characteristic
        if a>=0 and a<p:
            return True
        else:
            return False
```

A simple Python class for a prime finite fields, 7

A new class:

Python

```
class FFElement:
    """
    A class for elements of a FF.
    """
    def __init__(self, p, a):
        self.characteristic = p
        self.element = a%p
        self.base_field = FF(p)
```

Continued (note the indentation):

Python

```
def __repr__(self):
    """
    Called to compute the "official" string representation of an object.
    If at all possible, this should look like a valid Python expression
    that could be used to recreate an object with the same value.

    EXAMPLES:
        sage: F = FF(5)
        sage: a = F(3)
        sage: a
        FFElement(5.3)

    """
    return "FFElement(%s, %s)"%(self.characteristic, self.element)
```

A simple Python class for a prime finite fields, 9

Continued (note the indentation):

Python

```
    def __str__(self):
        """
        Called to compute the "informal" string description of an object.

        EXAMPLES:
            sage: F = FF(5)
            sage: a = F(3)
            sage: print a
            Finite field element 3 in Finite field with 5 elements

        """
        return "Finite field element %s in %s"%(self.element, self.base_field)
```

A simple Python class for a prime finite fields, 10

Continued (note the indentation):

Python

```
def __add__(self, other):
    """
    Implements +.

    EXAMPLES:
        sage: F = FF(7)
        sage: a = F(102); b = F(-2)
        sage: a; b; print a; print b; a+b
        FFElement(7, 4)
        FFElement(7, 5)
        Finite field element 4 in Finite field with 7 elements
        Finite field element 5 in Finite field with 7 elements
        2
    """
    p = self.characteristic
    return (self.element+other.element)%p
```

A simple Python class for a prime finite fields, 11

Continued (note the indentation):

Python

```
    def __sub__(self, other):
        """
        Implements -.

        EXAMPLES:
            sage: F = FF(7)
            sage: a = F(102); b = F(-2)
            sage: a; b; print a; print b; a-b
            FFElement(7, 4)
            FFElement(7, 5)
            Finite field element 4 in Finite field with 7 elements
            Finite field element 5 in Finite field with 7 elements
            6
        """
        p = self.characteristic
        return (self.element-other.element)%p
```

A simple Python class for a prime finite fields, 12

Continued (note the indentation):

Python

```
def __mul__(self, other):
    """
    Implements multiplication *.

    EXAMPLES:
        sage: F = FF(7)
        sage: a = F(102); b = F(-2)
        sage: a; b; print a; print b; a*b
        FFElement(7, 4)
        FFElement(7, 5)
        Finite field element 4 in Finite field with 7 elements
        Finite field element 5 in Finite field with 7 elements
        6
    """
    p = self.characteristic
    return (self.element*other.element)%p
```

A simple Python class for a prime finite fields, 13

Continued (note the indentation):

Python

```
    def __div__(self, other):
        """
        Implements /. (Assumes other is not = 0.)

        EXAMPLES:
            sage: F = FF(7)
            sage: a = F(102); b = F(-2)
            sage: a; b; print a; print b; a/b
            FFElement(7, 4)
            FFElement(7, 5)
            Finite field element 4 in Finite field with 7 elements
            Finite field element 5 in Finite field with 7 elements
            5
        """
        p = self.characteristic
        a = self.element
        b = other.element
        return (a*b.__pow__(-1))%p
```

A simple Python class for a prime finite fields, 14a

Continued (note the indentation):

Python

```
    def __pow__(self, n):
        """
        Implements ^ or **.

        EXAMPLES:
            sage: F = FF(7)
            sage: a = F(102); b = F(-2)
            sage: a; b; a**(-1); b^2
            FFElement(7, 4)
            FFElement(7, 5)
            2
            4
        """
        p = self.characteristic
        a = self.element
        n = int(n)
```

A simple Python class for a prime finite fields, 14b

Continued (note the indentation):

Python

```
        if a%p == 0 and not(n<0):
            return 0
        if p == 2 and n == -1:
            return a%p
        if n == 0:
            return 1
        if n == 1:
            return a%p
        if n>1:
            if n%2 == 0:
                return ((a.__pow__(int(n/2)))**2)%p      # repeated squaring
            if n%2 == 1:
                return (a*(a.__pow__(int(n/2)))**2)%p    # repeated squaring
        if n == -1:
            return (a.__pow__(p-2))%p
        if n<-1:
            return ((a.__pow__(-1))**(-n))%p
        return 0 # should never happen
```

A simple Python class for a prime finite fields, 15

Python

```
def inverse(self):
    """
    Implements the inverse.

    EXAMPLES:
        sage: F = FF(7)
        sage: a = F(102); b = F(-2)
        sage: a.inverse(); b.inverse()
        2
        3
    """
    p = self.characteristic
    a = self.element
    if a%p == 0:
        raise ValueError, "Element must be non-zero."
    if p == 2:
        return a%p
    return (a.__pow__(p-2))%p
```

A Python class for finite fields

The Python class `FF` for finite fields $GF(p)$, p prime, is given in above. Modify this class as follows.

Exercise: Make your own class that implements the class `FFVectorSpace` and `FFVectors`.

- The vector space class must be able to take a prime p (for the characteristic) and an integer n (for the dimension) as arguments.
- The vectors class must be able to take a prime p, an integer n and a list of length n of integers (for the coordinates of the vector) as arguments.
- Implement $=$, vector addition, subtraction and scalar multiplication.
- Document your code with standard Python docstrings.

Coding theory in Sage

Basic notation and terms

A **code** is a linear block code over a finite field $\mathbb{F} = GF(q)$, i.e., a subspace of $\mathbb{F}^n$ with a fixed basis. In the exact sequence

$$0 \to \mathbb{F}^k \xrightarrow{G} \mathbb{F}^n \xrightarrow{H} \mathbb{F}^{n-k} \to 0, \qquad (1)$$

- G represents a generating matrix,
- H represents a check matrix,
- $C = Image(G) = Kernel(H)$ is the code.

General constructions

Sage contains `GAP` but not Guava (which can be loaded as an optional package via `sage -i`).

General constructions	`LinearCode`, `LinearCodeFromCheckMatrix` `LinearCodeFromVectorSpace`, `RandomLinearCode`

LinearCode

Sage

```
sage: MS = MatrixSpace(GF(2),4,7)
sage: G  = MS([[1,1,1,0,0,0,0], [1,0,0,1,1,0,0],
               [0,1,0,1,0,1,0], [1,1,0,1,0,0,1]])
sage: C  = LinearCode(G); C
Linear code of length 7, dimension 4 over Finite Field of size 2
sage: C.base_ring()
Finite Field of size 2
sage: C.length(); C.dimension(); C.minimum_distance()
7
4
3
sage: C.weight_distribution()
[1, 0, 0, 7, 7, 0, 0, 1]
```

LinearCodeFromCheckMatrix

Sage

```
sage: MS = MatrixSpace(GF(2),4,7)
sage: G  = MS([[1,1,1,0,0,0,0], [1,0,0,1,1,0,0], [0,1,0,1,0,1,0], [1,1,0,1,0,0,1]])
sage: C = LinearCodeFromCheckMatrix(G); C
Linear code of length 7, dimension 3 over Finite Field of size 2
sage: C.length(); C.dimension(); C.minimum_distance()
7
3
4
sage: C.weight_distribution()
[1, 0, 0, 0, 7, 0, 0, 0]
```

LinearCodeFromVectorSpace

Sage

```
sage: V = GF(2)^7
sage: S = V.subspace([[1,1,1,0,0,0,0], [1,0,0,1,1,0,0], [0,1,0,1,0,1,0]])
sage: S.dimension()
3
sage: C = LinearCodeFromVectorSpace(S); C
Linear code of length 7, dimension 3 over Finite Field of size 2
sage: C.length(); C.dimension(); C.minimum_distance()
7
3
3
```

Hamming weight, etc.

Hamming metric is the function $d : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{R}$,

$$d(\mathbf{v}, \mathbf{w}) = |\{i \mid v_i \neq w_i\}| = d(\mathbf{v} - \mathbf{w}, \mathbf{0}).$$

- the **weight** is $wt(\mathbf{c}) = d(\mathbf{c}, \mathbf{0})$
- **minimum distance of** C is $d(C) = \min_{\mathbf{c} \neq \mathbf{0}} wt(\mathbf{c})$.
- **weight distribution** (or spectrum) of C is $\mathrm{spec}(C) = (A_0, A_1, ..., A_n)$, where

$$A_i = |\{\mathbf{c} \in C \mid wt(\mathbf{c}) = i\}|.$$

Coding theory functions

coding theory functions	spectrum, minimum_distance characteristic_function, binomial_moment gen_mat, check_mat, support, decode standard_form,

Coding theory functions

coding theory functions	divisor, genus random_element, redundancy_matrix weight_enumerator, chinen_polynomial zeta_polynomial, zeta_function

Corresponding GAP functions.

Coding theory and Sage

David Joyner

What is Sage?
What is in Sage?
The CLI
The GUI
Python
What is Python?
for loops
XGCD, lambda, Sage examples
Repeated squaring algorithm
Fibonacci numbers
Classes
Coding theory functionality in Sage
General constructions
Coding theory functions
Coding theory bounds
Coding theory not implemented in Sage
Cryptography
Classical cryptography
Algebraic cryptosystems
LFSRs
Blum-Goldwasser
Miscellaneous topics
Guava
Duursma zeta functions
Self-dual codes

Some associated GAP functions

- `AClosestVectorCombinationsMatFFEVecFFECoords` (for $d(C)$)
- `DistancesDistributionMatFFEVecFFE` (for $\mathrm{spec}(C)$)
- `WeightVecFFE`, `DistanceVecFFE` (for $wt(v)$, $d(v,w)$)
- `ConwayPolynomial` (uses database of polynomials used to construct $GF(q)$)
- `RandomPrimitivePolynomial`

Examples: gen_mat, check_mat, support

Sage

```
sage: C = HammingCode(3,GF(2))
sage: C.gen_mat()
[1 0 0 1 0 1 0]
[0 1 0 1 0 1 1]
[0 0 1 1 0 0 1]
[0 0 0 0 1 1 1]
sage: C.check_mat()
[1 0 0 1 1 0 1]
[0 1 0 1 0 1 1]
[0 0 1 1 1 1 0]
sage: C.support()
[0, 3, 4, 7]
```

Examples: `characteristic_polynomial`, `support`

Sage

```
sage: C = HammingCode(3,GF(2))
sage: Cd = C.dual_code()
sage: Cd.support()
[0, 4]
sage: C.support()
[0, 3, 4, 7]
sage: C.characteristic_polynomial()
-2*x + 8
sage: Cd.characteristic_polynomial()
-4/21*x^3 + 8/3*x^2 - 244/21*x + 16
```

Binomial moment

The i-th binomial moment of the $[n, k, d]_q$-code C is

$$B_i(C) = \sum_{S, |S|=i} \frac{q^{k_S} - 1}{q - 1}$$

where k_S is the dimension of the shortened code C_{J-S}, where $J = [1, 2, ..., n]$.

Examples: binomial_moment

Sage

```
sage: C = HammingCode(3,GF(2))
sage: C.binomial_moment?         # this gives you the docstring
Type:           instancemethod
String Form:  <bound method LinearCode.binomial_moment of Linear code of length 7,
  dimension 4 over Finite Field of size 2>
File: .../sage-4.4.rc0/local/lib/python2.6/site-packages/sage/coding/linear_code.py
Definition:     C.binomial_moment(self, i)
Docstring:
       Returns the i-th binomial moment of the [n,k,d]_q-code C:

          B_i(C) = sum_{S, |S|=i} frac{q^{k_S}-1}{q-1}

       where k_S is the dimension of the shortened code C_{J-S},
       J=[1,2,...,n]. (The normalized binomial moment is b_i(C) =
       binom(n,d+i)^{-1}B_{d+i}(C).) In other words, C_{J-S} is
       isomorphic to the subcode of C of codewords supported on S.

<snip>
```

Examples: binomial_moment

Sage

```
sage: C = HammingCode(3,GF(2))
sage: C.binomial_moment??  # this gives you the source code listing
  <snip>
        n = self.length()
        k = self.dimension()
        d = self.minimum_distance()
        F = self.base_ring()
        q = F.order()
  <snip>
sage: [(i,C.binomial_moment(i)) for i in range(8)]
[(0, 0), (1, 0), (2, 0), (3, 0), (4, 35), (5, 63), (6, 49), (7, 15)]
```

Examples: standard_form

Sage

```
sage: C = HammingCode(3,GF(2)); C.gen_mat()
[1 0 0 1 0 1 0]
[0 1 0 1 0 1 1]
[0 0 1 1 0 0 1]
[0 0 0 0 1 1 1]
sage: Cs, p = C.standard_form()
sage: Cs
Linear code of length 7, dimension 4 over Finite Field of size 2
sage: p; p in SymmetricGroup(7)
(4,5)
True
sage: Cs.gen_mat()
[1 0 0 0 1 1 0]
[0 1 0 0 1 1 1]
[0 0 1 0 1 0 1]
[0 0 0 1 0 1 1]
```

Examples: decode

Sage

```
sage: C = HammingCode(3,GF(2))
sage: C.decode??
 <snip>
File:  ...sage-4.4.rc0/local/lib/python2.6/site-packages/sage/coding/linear_code.py
Definition:     C.decode(self, right, method='syndrome')
Source:
    def decode(self, right, method="syndrome"):
        r"""
        Decodes the received vector ``right`` to an element `c` in this code.
        Optional methods are "guava", "nearest neighbor" or "syndrome". The
        ``method="guava"`` wraps GUAVA's ``Decodeword``.  Hamming codes have a
        special decoding algorithm; otherwise, ``"syndrome"`` decoding is
        used.
 <snip>
        from decoder import decode
        if method=="syndrome" or method=="nearest neighbor":
            return decode(self,right)
 <snip>
```

(The bit about Hamming codes is not true for Sage.)

Examples: decode

Sage

```
sage: decode??
Object `decode` not found.
sage: from sage.coding.decoder import decode
sage: decode??
<snip>
File: ... sage-4.4.rc0/local/lib/python2.6/site-packages/sage/coding/decoder.py
Definition:     decode(C, v, method='syndrome')
Source:
def decode(C, v, method="syndrome"):
    """
    The vector v represents a received word, so should
    be in the same ambient space V as C. Returns an
    element in C which is closest to v in the Hamming
    metric.

    Methods implemented include "nearest neighbor" (essentially
    a brute force search) and "syndrome".
<snip>
```

Examples: decode

decode is slow and only a few algoritms have been implemented.

Sage

```
sage: C = HammingCode(3,GF(2)); V = GF(2)^7
sage: v = V([1,1,0,1,1,0,1])
sage: v in V; v in C
True
False
sage: c = C.decode(v); c; c in C
(1, 0, 0, 1, 1, 0, 1)
True
```

This used syndrome decoding.

Definitions

Weight enumerator polynomial -

$$A_C(x,y) = \sum_{i=0}^{n} A_i x^{n-i} y^i = x^n + A_d x^{n-d} y^d + \cdots + A_n y^n,$$

where

$$A_i = |\{c \in C \mid \mathrm{wt}(c) = i\}| = \#\ \text{of codewds wt } i.$$

Weight enumerators

Examples:

- $W_5(x, y) = x^8 + 14x^4y^4 + y^8$ is the weight enumerator of the Type II $[8, 4, 4]$ code C constructed by extending the binary $[7, 4, 3]$ Hamming code by a check bit. This is the smallest Type II code.
- $W_6(x, y) = x^{24} + 759x^{16}y^8 + 2576x^{12}y^{12} + 759x^8y^{16} + y^{24}$ is the weight enumerator of the extended the binary Golay code with parameters $[24, 12, 8]$.

Example: weight_enumerator, ...

Sage can verify the fact from the previous slide.

Sage

```
sage: C = HammingCode(3,GF(2))
sage: Cx = C.extended_code()
sage: Cx.weight_enumerator()
x^8 + 14*x^4*y^4 + y^8
sage: C = ExtendedBinaryGolayCode()
sage: C.weight_enumerator()
x^24 + 759*x^16*y^8 + 2576*x^12*y^12 + 759*x^8*y^16 + y^24
```

More on these later.

Examples: genus, weight_enumerator, ...

The Duursma zeta function is implemented.

Sage

```
sage: C = HammingCode(3,GF(2))
sage: C.genus() # n+1-k-d
1
sage: C.weight_enumerator()
x^7 + 7*x^4*y^3 + 7*x^3*y^4 + y^7
sage: C.zeta_function()
(2/5*T^2 + 2/5*T + 1/5)/(2*T^2 - 3*T + 1)
sage: C.zeta_polynomial()
2/5*T^2 + 2/5*T + 1/5
```

More on these later.

Coding constructions

code constructions	dual_code, extended_code, direct_sum, punctured, shortened, permuted_code, galois_closure

Guava has a lot more constructions, but does not have galois_closure.

Examples: extended_code

extended_code simply adds a check-bit at the end.

Sage

```
sage: C = HammingCode(3,GF(2))
sage: Cx = C.extended_code()
sage: Cx.is_self_orthogonal()
True
sage: Cx.is_self_dual()
True
sage: Cx.divisor()
4
sage: Cx.spectrum()
[1, 0, 0, 0, 14, 0, 0, 0, 1]
```

More on self-dual codes later.

Examples: direct_sum

An example of Komichi (master's thesis, unpublished).

Sage

```
sage: C = HammingCode(3,GF(2))
sage: C1 = HammingCode(3,GF(2))
sage: C2 = C1.extended_code()
sage: C3 = (C2.direct_sum(C2)).direct_sum(C2)
sage: R.<T> = PolynomialRing(CC, "T")
sage: f = C3.zeta_polynomial(); f = R(f); rts = f.roots()
sage: [abs(z[0]*sqrt(2.0)) for z in rts]
[0.733550688875582, 1.36323230986647, 1.00000000000000,
 1.00000000000000, 1.00000000000000, 1.00000000000000,
 1.00000000000000, 1.00000000000000, 1.00000000000000,
 1.00000000000000, 1.00000000000000, 1.00000000000000,
 1.00000000000000, 1.00000000000000, 1.00000000000000,
 1.00000000000000, 1.00000000000000, 1.00000000000000]
```

Komichi's example, continued.

Sage

```
sage: P1 = list_plot([(z[0].real(),z[0].imag()) for z in f.roots()])
sage: pts = lambda t: [cos(t)/sqrt(2),sin(t)/sqrt(2)]
sage: t = var("t")
sage: P2 = parametric_plot(pts(t),(0,2*pi),linestyle="--",rgbcolor=(1,0,0))
sage: show(P1+P2)
```

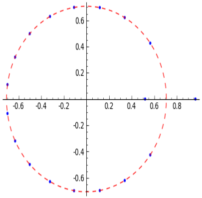


Figure: Zeros of the Duursma zeta function of Komichi's code.

Examples: galois_closure

`galois_closure` of a code C defined over $GF(p^k)$ returns the smallest code defined over $GF(p^k)$ closed under the Galois action of $\mathrm{Gal}(GF(p^k)/GF(p))$.

Sage

```
sage: C = HammingCode(3,GF(4,'a'))
sage: Cc = C.galois_closure(GF(2))
sage: C; Cc
Linear code of length 21, dimension 18 over Finite Field in a of size 2^2
Linear code of length 21, dimension 20 over Finite Field in a of size 2^2
sage: C.is_subcode(Cc)
True
sage: Cc.is_galois_closed()
True
```

Automorphism group of a code

What is an automorphism of a code?

Let S_n denote the symmetric group on n letters. The **(permutation) automorphism group** of a code C of length n is simply the group

$$\mathrm{Aut}(C) = \{\sigma \in S_n \mid (c_1, ..., c_n) \in C \implies (c_{\sigma(1)}, ..., c_{\sigma(n)}) \in C\}.$$

There are no known methods for computing these groups which are polynomial time in the length n of C.

Automorphism group of a code

If

(a) $C_1, C_2 \subset \mathbb{F}^n$ are codes, and

(b) $\exists \sigma \in S_n$ for which $(c_1, ..., c_n) \in C_1 \iff (c_{\sigma(1)}, ..., c_{\sigma(n)}) \in C_2$,

then $C_1 \cong C_2$ (i.e., C_1 and C_2 are **permutation equivalent**).

Examples: permuted_code

Sage

```
sage: C = HammingCode(3,GF(2))
sage: g = SymmetricGroup(7).random_element(); g
(1,2)(3,7,4)
sage: Cg = C.permuted_code(g)
sage: Cg.is_permutation_equivalent(C)
True
sage: G = C.automorphism_group_binary_code(); G
Permutation Group with generators [(3,4)(5,6), (3,5)(4,6), (2,3)(5,7), (1,2)(5,6)]
sage: g = G("(2,3)(5,7)")
sage: Cg = C.permuted_code(g)
sage: C == Cg
True
```

Examples: punctured, shortened

The code C^L obtained from C by **puncturing** at the positions in L is the code of length $n - |L|$ consisting of codewords of C which have their i-th coordinate deleted if $i \in L$ and left alone if $i \notin L$.

Sage

```
sage: C = HammingCode(3,GF(2))
sage: C.punctured([1,2])
Linear code of length 5, dimension 4 over Finite Field of size 2
sage: C.shortened([1,2])
Linear code of length 5, dimension 2 over Finite Field of size 2
```

The subcode $C(L)$ is all codewords $c \in C$ which satisfy $c_i = 0$ for all $i \in L$. The punctured code $C(L)^L$ is called the **shortened code** on L and is denoted C_L.

Coding constructions

coding theory functions (boolean)	is_self_dual, == is_self_orthogonal, is_subcode, is_permutation_automorphism, is_permutation_equivalent, is_galois_closed

Examples of most of these have been seen already.

Examples: permuted_code

Sage

```
sage: C = HammingCode(3,GF(2))
sage: g = SymmetricGroup(7).random_element(); g
(1,6,4,7,3)(2,5)
sage: C.is_permutation_automorphism(g)
False
sage: Cg = C.permuted_code(g)
sage: Cg.is_permutation_equivalent(C)
True
```

All this is expected behavior.

Coding constructions

coding theory functions (group theoretical)	`module_composition_factors`, `automorphism_group_binary_code`

`module_composition_factors` prints the `GAP` record of the `Meataxe` composition factors module in `Meataxe` notation.

Examples: module_composition_factors

Sage

```
sage: C = HammingCode(3,GF(2))
sage: Cx = C.extended_code()
sage: G = Cx.automorphism_group_binary_code()
sage: G.order()
1344
sage: Cx.module_composition_factors(G)
[ rec(
      field := GF(2),
      isMTXModule := true,
      dimension := 1,
      generators := [ [ [ Z(2)^0 ] ], [ [ Z(2)^0 ] ], [ [ Z(2)^0 ] ],
          [ [ Z(2)^0 ] ], [ [ Z(2)^0 ] ], [ [ Z(2)^0 ] ] ],
<snip>
      IsIrreducible := true ) ]
```

(A lot is omitted for space reasons.)

Coding theory functions

coding theory functions (combinatorial)	assmus_mattson_designs

Block designs

- A **block design**: a pair (X, B), where X is a non-empty finite set of $v > 0$ elements called **points**, and B is a non-empty finite multiset of size b whose elements are called **blocks**, such that each block is a non-empty finite multiset of k points.
- If every subset of points of size t is contained in exactly λ blocks the block design is called a $t-(v, k, \lambda)$ **design**.
- When $\lambda = 1$ then the block design is called a $S(t, k, v)$ **Steiner system**.

The Assmus-Mattson Theorem

Assmus and Mattson Theorem: Let $A_0, A_1, \ldots, A_n$ be the weights of the codewords in a binary linear $[n, k, d]$ code C, and let $A_0^*, A_1^*, \ldots, A_n^*$ be the weights of the codewords in its dual $[n, n-k, d^*]$ code C^*. Fix a t, $0 < t < d$, and let $s = |\{i \mid A_i^* not = 0, 0 < i \leq n - t\}|$. Assume $s \leq d - t$.

- If $A_i \neq 0$ and $d \leq i \leq n$ then $C_i = \{c \in C \mid wt(c) = i\}$ holds a simple t-design.
- If $A_i^* \neq 0$ and $d^* \leq i \leq n - t$ then $C_i^* = \{c \in C^* \mid wt(c) = i\}$ holds a simple t-design.

Examples: assmus_mattson_designs

Sage

```
sage: C = HammingCode(3,GF(2))
sage: Cx = C.extended_code()
sage: Cx.assmus_mattson_designs(3)
['weights from C: ', [4, 8],
 'designs from C: ', [[3, (8, 4, 1)], [3, (8, 8, 1)]],
 'weights from C*: ', [4],
 'designs from C*: ', [[3, (8, 4, 1)]]]
```

Special constructions

Special constructions	`BinaryGolayCode`, `ExtendedBinaryGolayCode`, `TernaryGolayCode`, `ExtendedTernaryGolayCode`, `CyclicCode`, `BCHCode`, `CyclicCodeFromCheckPolynomial`, `DuadicCodeEvenPair`, `DuadicCodeOddPair`, `HammingCode`,

Special constructions

Special constructions (cont.)	`QuadraticResidueCodeEvenPair`, `QuadraticResidueCodeOddPair`, `QuadraticResidueCode`, `ExtendedQuadraticResidueCode`, `ReedSolomonCode`, `self_dual_codes_binary`, `ToricCode`, `WalshCode`

Examples: `ReedSolomonCode`

`ReedSolomonCode` - Also called a "generalized Reed-Solomon code".

The "narrow" RS codes codes are also cyclic codes; they are part of GUAVA but have not been ported over to natice Python/Sage (yet).

Examples: `ReedSolomonCode`

- Let $\mathbb{F} = GF(q)$,
- let n and k be such that $1 \leq k \leq n \leq q$,
- pick n distinct elements of $\mathbb{F}$, $\{x_1, x_2, ..., x_n\}$.

Define the GRS code by

$$C = \{(f(x_1), f(x_2), ..., f(x_n)) \mid f \in \mathbb{F}[x], \deg(f) < k\} .$$

This is an $[n, k, n-k+1]$ code.

Examples: ReedSolomonCode

Sage

```
sage: C = ReedSolomonCode(6,4,GF(7)); C
Linear code of length 6, dimension 4 over Finite Field of size 7
sage: C.minimum_distance()
3
sage: F.<a> = GF(3^2,"a")
sage: pts = [0,1,a,a^2,2*a,2*a+1]
sage: len(Set(pts)) == 6 # to make sure there are no duplicates
True
sage: C = ReedSolomonCode(6,4,F,pts); C
Linear code of length 6, dimension 4 over Finite Field in a of size 3^2
sage: C.minimum_distance()
3
```

Examples: `ExtendedTernaryGolayCode`

The permutation automorphism group of the extended ternary Golay code is the Mathieu group M_{11}.

Sage

```
sage: C = ExtendedTernaryGolayCode(); C; C.minimum_distance()
Linear code of length 12, dimension 6 over Finite Field of size 3
6
sage: G = C.permutation_automorphism_group(); G
Permutation Group with generators [(5,7)(6,11)(8,9)(10,12), (4,6)(5,10)(7,8)(9,12),
      (3,4)(6,8)(9,11)(10,12), (2,3)(5,7)(8,10)(9,12), (1,2)(5,12)(6,11)(7,10)]
sage: G.order(); G.is_simple()
7920
True
sage: M11 = MathieuGroup(11); G.is_isomorphic(M11)
True
```

(The full “monomial” automorpism group is larger, but Sage lacks the functionality to compute that at this point.)

Examples: ToricCode

ToricCodes can be bad or very good.

Sage

```
sage: C = ToricCode([[-2,-2],[-1,-2],[-1,-1],[-1,0],[0,-1],
  [0,0],[0,1],[1,-1],[1,0]],GF(5))
sage: C
Linear code of length 16, dimension 9 over Finite Field of size 5
sage: C.minimum_distance()
6
```

(Ask Diego Ruano if you have more questions about this family of codes.)

Examples: self_dual_codes_binary

Sage has a small database of self_dual_codes_binarys.

Sage

```
sage: C = self_dual_codes_binary(10)
sage: C.keys()
['10']
sage: C['10'].keys()
['1', '0']
sage: C['10']['0']
{'Comment': 'No Type II of this length.', 'Type': 'I',
 'code': Linear code of length 10, dimension 5 over Finite Field of size 2,
 'order autgp': 3840, 'spectrum': [1, 0, 5, 0, 10, 0, 10, 0, 5, 0, 1]}
sage: C = self_dual_codes_binary(10)
sage: C = C['10']['0']['code']
sage: C
Linear code of length 10, dimension 5 over Finite Field of size 2
sage: C.divisor()
2
```

Code bounds

code bounds	`best_known_linear_code_www`, `bounds_minimum_distance` `codesize_upper_bound(n,d,q)`, `dimension_upper_bound(n,d,q)` `gilbert_lower_bound(n,q,d)`, `plotkin_upper_bound(n,q,d)` `griesmer_upper_bound(n,q,d)`, `elias_upper_bound(n,q,d)`

Code bounds

code bounds	`hamming_upper_bound(n,q,d)`, `singleton_upper_bound(n,q,d)` `gv_info_rate(n,delta,q)`, `gv_bound_asymp(delta,q)` `plotkin_bound_asymp(delta,q)`, `elias_bound_asymp(delta,q)` `hamming_bound_asymp(delta,q)`, `singleton_bound_asymp(delta,q)` `mrrw1_bound_asymp(delta,q)`

Examples: best_known_linear_code_www

`best_known_linear_code_www` (interface with codetables.de since A. Brouwer's online tables have been disabled).

Explains the construction of the best known linear code over GF(q) with length n and dimension k, courtesy of the www page `http://www.codetables.de/`.

INPUT:

- `n` – integer, the length of the code
- `k` – integer, the dimension of the code
- `F` – finite field, whose field order must be in [2, 3, 4, 5, 7, 8, 9]
- `verbose` – bool (default=False), print verbose message

Examples: best_known_linear_code_www

Sage

```
sage: L = best_known_linear_code_www(72, 36, GF(2)) # requires internet
sage: print L
Construction of a linear code [72,36,15] over GF(2):
[1]:  [73, 36, 16] Cyclic Linear Code over GF(2)
      CyclicCode of length 73 with generating polynomial x^37 + x^36
      + x^34 + x^33 + x^32 + x^27 + x^25 + x^24 + x^22 + x^21 + x^19
      + x^18 + x^15 + x^11 + x^10 + x^8 + x^7 + x^5 + x^3 + 1
[2]:  [72, 36, 15] Linear Code over GF(2)
           Puncturing of [1] at 1
last modified: 2002-03-20
```

Asymptotic bounds

Theorem

(Manin) There exists a continuous decreasing function

$$\alpha_q : [0,1] \to [0,1],$$

such that

- α_q *is strictly decreasing on* $[0, \frac{q-1}{q}]$,
- $\alpha_q(0) = 1$,
- *if* $\frac{q-1}{q} \leq x \leq 1$ *then* $\alpha_q(x) = 0$,
- $\Sigma_q = \{(\delta, R) \in [0,1]^2 \mid 0 \leq R \leq \alpha_q(\delta)\}$.

Asymptotic bounds

Not a single value of $\alpha_q(x)$ is known for $0 < x < \frac{q-1}{q}$! It is not known whether or not the maximum value of the bound, $R = \alpha_q(\delta)$ is attained by a sequence of linear codes. It is not known whether or not $\alpha_q(x)$ is differentiable for $0 < x < \frac{q-1}{q}$, nor is it known if $\alpha_q(x)$ is convex on $0 < x < \frac{q-1}{q}$.

Asymptotic bounds

Theorem

(Gilbert-Varshamov) We have

$$\alpha_q(x) \geq 1 - x\log_q(q-1) - x\log_q(x) - (1-x)\log_q(1-x).$$

In other words, for each fixed $\epsilon > 0$, there exists an (n,k,d)-code C (which may depend on ϵ) with

$$R(C)+\delta(C) \geq 1-\delta(C)\log_q(\frac{q-1}{q})-\delta(C)\log_q(\delta(C))-(1-\delta(C))\log_q(1$$

The curve $(\delta, 1-\delta\log_q(\frac{q-1}{q})-\delta\log_q(\delta)-(1-\delta)\log_q(1-\delta)))$ is called the **Gilbert-Varshamov curve**.

Examples: A plot with gv_bound_asymp

Sage has excellent plotting functionality.

Sage

```
sage: f = lambda x: gv_bound_asymp(x,2)
sage: P1 = plot(f,0,1/2)
sage: P2 = list_plot([(3/7,4/7)])
sage: P3 = text('$Hamm(7,4,3)$', (0.4,0.62), rgbcolor=(0,1,0))
sage: P4 = text('$*$', (4/8,4/8), rgbcolor=(1,0,0))
sage: P5 = text('$Hamm^+(8,4,4)$', (0.45,0.4), rgbcolor=(0,1,0))
sage: show(P1+P2+P3+P4+P5)
```

Examples: A plot with `gv_bound_asymp`

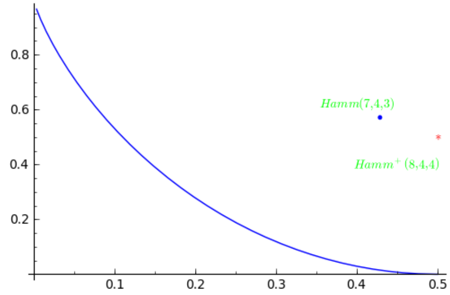


Figure: Gilbert-Varshamov curve plotted with the $[7, 4, 3]_2$ and extended $[8, 4, 4]_2$ Hamming codes.

Examples: A plot with `mrrw1_bound_asymp`

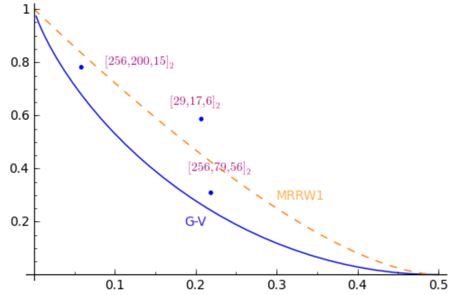


Figure: Gilbert-Varshamov curve and MRRW1 curve plotted with some "good" codes.

Asymptotic bounds

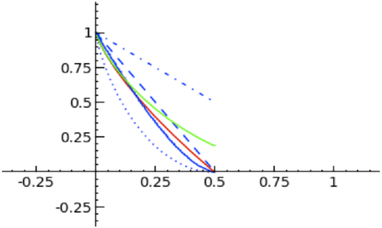


Figure: Plot of the Gilbert-Varshamov (dotted), Elias (red), Plotkin (dashed), Singleton (dash-dotted), Hamming (green), and MRRW (blue) curves using Sage.

Coding theory not yet in Sage

Minimum distance for non-binary codes

- The only fast implementation of `minimum_distance` is in the binary case (and due to Robert Miller).
- Guava has a fast implementation of `MinimumDistance` in the ternary case.
- Sage needs a fast implementation of `minimum_distance` is in the non-binary case.

Automorphism groups for non-binary codes

- The only fast implementation of automorphism groups is in the binary case (and also due to Robert Miller).
- Sage needs a fast implementation of automorphism groups is in the non-binary case.

AG codes

- AG codes are implemented in `Singular`, but not yet completely implemented in Sage.
- There is a module `ag_code` in Sage's `coding` directory but it does not work at present and is not imported.
- This needs to be fixed! (See also trac ticket # 8997.)

Decoding

- Sage has no special decoding algorithms. (Not even for Hamming codes!)
- Guava has some but still is very limited.
- Sage needs a lot of work in this area!

Gray codes

- Sage has nothing on Gray codes
- A lot of Python modules exists that could be submitted.

  `http://boxen.math.washington.edu/home/wdj/research/coding-theory/graycode.sage`
- Lack of developers in this area is the main problem.

More on this later.

Cycle and cocycle codes

- Sage has nothing on graph-theoretic cycle or cocycle codes.
- Python modules do exist that could be submitted.

  `http://boxen.math.washington.edu/home/wdj/research/coding-theory/cycle-space.sage`
- Lack of developers in this area is the main problem.

More on this later.

LDPC codes

- Sage has nothing on LDPC codes.
- I think there is C code which possibly could be “wrapped”?
- Guava has very limited functionality.

Guava homepage: `http://sage.math.washington.edu/home/wdj/guava/`

Recent contributors: David Joyner (USNA), Cen Tjhal (Univ Plymouth), Robert Miller (Univ Wash.), Tom Boothby (Univ Wash.).
Joe Fields (S. Conn. St. Univ.) is lead maintainer

Figure: Robert Miller

Figure: Cen Tjhal ("CJ")

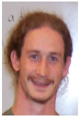


Figure: Tom Boothby

Figure: Joe

Guava port

Guava is not part of Sage, though it can be loaded easily. Finish porting or "wrapping" everything in Guava to Sage.

Sage

```
sage: install_package("gap_packages")
sage: gap.eval('LoadPackage("guava")')
'true'
sage: C = gap("HammingCode(3,GF(2))")
sage: C.MinimumDistance()
3
```

Codes over finite rings

- Sage has nothing on ring codes.
- There is Cython code written (mostly) by Cesar A. Garcia-Vazquez.
- Cesar's code can go in with some extra effort (see trac #6452).

Sage

```
sage: M = Matrix(IntegerModRing(12), [[0, 1, 6, -1],[1, 6, 1, 2],[6, 1, 1, 0]])
sage: C = RingCode(M) ; C
 (4, 1728, 2)-code over the Ring of integers modulo 12
sage: c = C.minimum_weight_codeword(); c
 (0, 1, 0, 5)
sage: c in C
True
```

Circuit and cocircuit codes from matroids

- Sage has nothing on matroids, much less circuit or cocircuit codes.
- There is some Python code which could be submitted.
- Lack of developers in this area is the main problem.

More on this later.

Gray codes

Here's an example after `attach`'ing the module `graycode.sage`.

Sage

```
sage: graycode_GF(2,GF(2))
  [[0, 0], [1, 0], [1, 1], [0, 1]]
sage: graycode_GF(2,GF(3))
  [[0, 0], [1, 0], [2, 0], [2, 1], [1, 1], [0, 1], [0, 2], [1, 2], [2, 2]]
sage: graycode_GF(2,GF(4,"a"))
  [[0, 0], [a, 0], [a + 1, 0], [1, 0], [1, a], [a + 1, a],
   [a, a], [0, a], [0, a + 1], [a, a + 1], [a + 1, a + 1],
   [1, a + 1], [1, 1], [a + 1, 1], [a, 1], [0, 1]]
```

Cycle and cocycle codes

It is easy to load and run your own Sage modules. You can even access your own docstrings as usual.

Sage

```
sage: attach "/Users/wdj/sagefiles/cycle-space.sage"
sage: cycle_code?
<snip>
Definition:     cycle_code(G)
Docstring:
       Returns a "circuit code", called here a cycle code, as described by
       Hakimi-Bredeson, IEE Trans Info Thry 14(1968).

       INPUT:
          G - a simple connected graph with n edges.

       OUTPUT:
          a binary code of length n
<snip>
```

Cycle and cocycle codes

Sage

```
sage: G = graphs.HeawoodGraph()
sage: G.girth()
6
sage: C = cycle_code(G); C; C.minimum_distance()
Linear code of length 21, dimension 8 over Finite Field of size 2
6
```

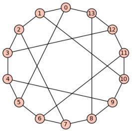


Figure: Heawood graph of girth 6

Cryptography in Sage

A **cryptosystem** is an injection

$$E : KS \to \mathrm{Hom}_{\mathrm{Set}}(MS, CS),$$

where

- KS is the key space,
- MS is the plaintext (or message) space , and
- CS is the ciphertext space.

Classical ciphers

Sage's modules on "classical" ciphers was created by David Kohel and Minh van Nyugen.

- Hill, substitution, transposition, shift cipher, affine cipher and Vigenere cryptosystems are implemented.

Affine cryptosystem

Let $A = \{a_0, a_1, a_2, \ldots, a_{n-1}\}$ be an alphabet. Define an injection $f : A \longrightarrow \mathbb{Z}/n\mathbb{Z}$ given by $f(a_i) = i$.
Set $MS = CS = \mathbb{Z}/n\mathbb{Z} \cong A$
key space: $KS = \{(a, b) \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \mid \gcd(a, n) = 1\}$. Let $(a, b) \in KS$.
Encryption: For $p \in MS$, define $c \in CS$ by $c \equiv ap + b \pmod{n}$
Decryption: For $c \in CS$, define $p \in MS$ by $p \equiv a^{-1}(c - b) \pmod{n}$ where a^{-1} is the inverse of a modulo n.

Affine cryptosystem

Sage

```
sage: A = AffineCryptosystem(AlphabeticStrings())
sage: P = A.encoding("`Hello` to everyone in Spain!!"); P
HELLOTOEVERYONEINSPAIN
sage: a, b = (3, 7)
sage: C = A.enciphering(a, b, P); C
CTOOXMXTSTGBXUTFUJAHFU
sage: L = A.brute_force(C)
sage: sorted(L.items())[30:35]
[((3, 4), IFMMPUPFWFSZPOFJOTQBJO), ((3, 5), ZWDDGLGWNWJQGFWAFKHSAF),
((3, 6), QNUUXCXNENAHXWNRWBYJRW), ((3, 7), HELLOTOEVERYONEINSPAIN),
((3, 8), YVCCFKFVMVIPFEVZEJGRZE)]
sage: L = A.brute_force(C, ranking="chisquare")
sage: L[0]
((3, 7), HELLOTOEVERYONEINSPAIN)
```

Shift cryptosystem

Sage

```
sage: S = ShiftCryptosystem(AlphabeticStrings())
sage: P = S.encoding("Shift from Mathematica to Sage!."); P
SHIFTFROMMATHEMATICATOSAGE
sage: K = 3
sage: C = S.enciphering(K, P); C
VKLIWIURPPDWKHPDWLFDWRVDJH
sage: S.enciphering(26-K, C)
SHIFTFROMMATHEMATICATOSAGE
sage: S.deciphering(K, C)
SHIFTFROMMATHEMATICATOSAGE
```

Algebraic cryptosystems

Sage's modules on algebraic cryptosystems was created by Martin Albrecht and Minh van Nyugen.

- mini-DES,

  based on: E. Schaefer. A simplified data encryption algorithm. Cryptologia(1996)77-84.
- mini-AES,

  based on: R. C.-W. Phan. Mini advanced encryption standard, Cryptologia (2002) 283-306.
- Small Scale Variants of the AES Polynomial System Generator
- Multivariate Polynomial Systems

See also:
Martin Albrecht. Algebraic Attacks against the Courtois Toy Cipher in Cryptologia (2008) 220-276.

RSA

RSA is a deterministic public key encryption algorithm which relies on

- the extended Euclidean algorithm, and
- Euler's theorem in the special case of a modulus which is a product of two primes.

RSA: Key generation

PKC generalities:

- Two keys - a public key and a private key.
- public key - known to everyone, used for encryption.
- private key - Known only to the receiver, ciphertext can only be decrypted using the private key.
- The security of the RSA cryptosystem relies on that belief that it is computationally infeasible to compute the private key from the public key.

RSA: Key generation

Suppose Alice wants to send a message to Bob using RSA.

She says, “Bob, I need to tell you something.”

Bob says, “Hang on a second while I generate the keys.”

RSA: Key generation

Bob then

- chooses two distinct prime numbers p and q (only Bob knows these),
- computes $n = pq$ (n is used for both the public and private keys),
- computes $\phi(pq) = (p-1)(q-1)$ (ϕ = Euler's function),
- chooses an integer e such that $1 < e < \phi(pq)$ and $gcd(e, \phi(pq))$ (e is the **public key exponent**),
- determines d which satisfies $de \equiv 1 \pmod{\phi(pq)}$ (d is the **private key exponent**).

The **public key** consists of (n, e).
The **private key** consists of (n, d) .

RSA

Alice wants to send a message to Bob.
Bob selects $p = 1009$ and $q = 1013$, so $n = pq = 1022117$. Bob computes $\phi(n) = 1020096$. If he selects $e = 123451$, then he can compute $d = 300019$.
Alice wants to send Bob the message $m = 46577$. She encrypts it using $46577^{123451} \pmod{1022117}$, which is the ciphertext $c = 622474$.

RSA

Sage

```
sage: p = next_prime(1000); q = next_prime(1010); n = p*q; n
1022117
sage: k = euler_phi(n); e = 123451
sage: k; xgcd(k, e)
1020096
(1, -36308, 300019)
sage: x = xgcd(k, e)[1]; y = xgcd(k, e)[2]
sage: d = y%k
sage: y*e%k; d*e%k
1
1
sage: m = randint(100, k); m
46577
sage: c = power_mod(m,e,n) # faster than m^e%n
622474
sage: power_mod(c,d,n) # so m was correctly decrypted
46577
```

Discrete logs

The **discrete logarithm problem** is the following: Let G be a multiplicative abelian group and let $a, b \in G$. Find $x \in \mathbb{Z}$ such that

$$b^x = a,$$

if it exists.

Discrete logs

Sage

```
sage: p = next_prime(10^30)
sage: F = GF(p)
sage: b = F(2); b.multiplicative_order()
500000000000000000000000000028
sage: b = F(3); b.multiplicative_order()
416666666666666666666666666669
sage: b = F(5); b.multiplicative_order()
1000000000000000000000000000056
sage: a = F.random_element(); a
837776537981704766224734890062
sage: time a.log(b)
CPU times: user 0.04 s, sys: 0.01 s, total: 0.06 s
Wall time: 0.22 s
972953394163188347701109599170
```

Diffie-Hellman

Alice and Bob want to share a secret key.

- Alice and Bob agree on a finite cyclic group G and a generating element $g \in G$. (g is assumed to be known by all attackers.) Assume G has order n.
- Alice picks a random a, $1 < a < n$, and sends g^a to Bob.
- Bob picks a random b, $1 < b < n$, and sends g^b to Alice.
- Alice computes $(g^b)^a$.
- Bob computes $(g^a)^b$.
- Both Alice and Bob posses a **shared secret key**, g^{ab}.

Discrete logs

Sage

```
sage: G = IntegerModRing(101)
sage: g = G.random_element(); g; g.multiplicative_order()
3
100
sage: a = randint(1,50); b = randint(1,50)
sage: a; b
35
36
sage: ga = g^a; gb = g^b
sage: ga^b; ga^b == gb^a
36
True
```

Elgamal

The Elgamal cryptosystem and the Elgamal digital signature system have been implemented as Sage modules, but not yet submitted to Sage.

`http://boxen.math.washington.edu/home/wdj/teaching/python-and-coding-theory/sm450_python-notes4.pdf`

LFSRs

Let q be a prime power, $\ell > 1$ be an integer, and let $c_1, \ldots, c_\ell$ are given elements of $GF(q)$.
A **linear feedback shift register sequence** (LFSR) modulo p of *length* ℓ is a sequence $s_0, s_1, s_2, \ldots \in GF(q)$ such that

- $s_0, s_1, \ldots, s_{\ell-1}$ are given, and
- $s_n + c_1 s_{n-1} + c_2 s_{n-2} + \ldots + c_\ell s_{n-\ell} = 0\ ,\ n \geq \ell.$

LFSRs

Terminology:

- **key** - the list of coefficients $[c_1, c_2, \ldots, c_\ell]$
- **fill** - the list of initial values $s_0, s_1, \ldots, s_{\ell-1}$.
- **connection polynomial** - $c(x) = 1 + c_1 x + \ldots c_\ell x^\ell$.

LFSRs

Sage

```
sage: F = GF(2); l = F(1); o = F(0)
sage: fill = [o,l]; key = [1,l]; n = 20
sage: c = lfsr_sequence(key, fill, n); c
[0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1]
sage: f = lfsr_connection_polynomial(c); f
x^2 + x + 1
sage: f.is_primitive()
True
```

Notice that this Fibonacci sequence mod 2 seems to be periodic with period 3 $(= q^{\deg(c(x))} - 1)$.

LFSRs

Sage

```
sage: F = GF(3); l = F(1); o = F(0)
sage: fill = [o,l]; key = [1,l]; n = 20
sage: c = lfsr_sequence(key, fill, n); c
[0, 1, 1, 2, 0, 2, 2, 1, 0, 1, 1, 2, 0, 2, 2, 1, 0, 1, 1, 2]
sage: f = lfsr_connection_polynomial(c); f
2*x^2 + 2*x + 1
sage: f.is_primitive()
True
```

Notice that this Fibonacci sequence mod 3 seems to be periodic with period 8 $(= q^{\deg(c(x))} - 1)$.

LFSRs

Theorem

Let $S = \{s_i\}$ be a LFSR over $GF(p)$. The period of S is at most $p^k - 1$. It's period is exactly $P = p^k - 1$ if and only if the characteristic polynomial of

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & & \dots & & \\ 0 & 0 & \dots & 0 & 1 \\ -c_\ell & -c_{\ell-1} & \dots & & -c_1 \end{pmatrix},$$

is irreducible and primitive over $GF(p)$.

BBS streamcipher

Definition

Let p, q be two distinct prime numbers such that $p \equiv 3 \pmod 4$ and $q \equiv 3 \pmod 4$. Let $n = pq$ and let $0 < r < n$ be a random number. We define x_0, the first "seed" of the Blum-Blum-Shub pseudorandom number generator as

$$x_0 = r^2 \pmod n.$$

Each proceeding seed can be defined as

$$x_{i+1} = {x_i}^2 \pmod n.$$

The streamcipher, $b = b_1 b_2 \dots b_t$, is created by setting $b_i = x_i \mod 2$.

BBS streamcipher

Sage

```
sage: from sage.crypto.stream import blum_blum_shub
sage: p0 = next_prime(1015); q0 = next_prime(1100)
sage: blum_blum_shub(length=50, seed=999, p=p0, q=q0)
11111000110010101001001100100001010100101001011010
sage: from sage.crypto.util import carmichael_lambda as carmichael
sage: carmichael(carmichael(p0*q0))
32004
```

The last output tells us the maximum possible value of period of the BBS sequence.

Blum-Goldwasser

- Alice wants to send a message m to Bob.
- Bob generates two distinct prime numbers p and q such that $p \equiv 3 \pmod 4$, $q \equiv 3 \pmod 4$.
- Bob computes $n = pq$.
- Using the extended Euclidean algorithm, Bob computes a, b such that $ap + bq = 1$.

The **public key** is n. The **private key** is (p, q, a, b).

Blum-Goldwasser

Let x_0 be a random QR $\pmod{n}$.

- Plaintext: $m = m_1 m_2 \ldots m_t$ - a binary string of length t.
- Let $b = b_1 b_2 \ldots b_t$ be the BBS streamcipher of length t associated to x_0, n.
- Ciphertext: $c = b \oplus m$, where $\oplus$ indicates the XOR operation.

Alice sends the ciphertext c along with a number $y = x_0^{2^{t+1}} \pmod{n}$.

BBS streamcipher

Sage

```
sage: from sage.crypto.public_key.blum_goldwasser import BlumGoldwasser
sage: bg = BlumGoldwasser(); bg
The Blum-Goldwasser public-key encryption scheme.
sage: p = 499; q = 547
sage: pubkey = bg.public_key(p, q); pubkey
272953
sage: prikey = bg.private_key(p, q); prikey
(499, 547, -57, 52)
sage: p*q; p*prikey[2]+q*prikey[3]
272953
1
sage: M = "10011100000100001100"
sage: C = bg.encrypt(M, pubkey, seed=159201); C
([[0, 0, 1, 0], [0, 0, 0, 0], [1, 1, 0, 0], [1, 1, 1, 0], [0, 1, 0, 0]], 139680)
sage: M0 = bg.decrypt(C, prikey); M0
[[1, 0, 0, 1], [1, 1, 0, 0], [0, 0, 0, 1], [0, 0, 0, 0], [1, 1, 0, 0]]
sage: M = "".join(map(lambda x: str(x), flatten(M))); M
'10011100000100001100'
```

NTRU

NTRU has been **partially** implemented as a Sage module, but not yet submitted to Sage.
`http://boxen.math.washington.edu/home/wdj/teaching/python-and-coding-theory/sm450_python-notes4.pdf`
This would be a welcomed addition!

Miscellaneous topics

Guava, Duursma zeta functions, self-dual codes, cool examples.

A brief tour of Guava

homepage:

`http://sage.math.washington.edu/home/wdj/guava/`

Basic Guava functions

- `MinimumDistance`
- `MinimumDistanceLeon` (does not call Leon's C code)
- `MinimumDistanceRandom`
- `CoveringRadius`
- `WeightDistribution` (for *spec*(C), should call Leon?)
- `DistancesDistribution` (the distribution of the distances of elements of C to a vector w)

Leon's code.

Leon's C code for computing automorphism groups of matrices and designs and linear codes is now GPL'd. Good news:

- it's GPL'd, optimized C code,
- Joe Fields is working on Guava

Drawbacks:

- it has memory leaks and "home-brewed" finite fields (should use Conway polynomials),
- Guava only interfaces a small part of what it does.

Robert Miller and Tom Boothby have tried to fix up Leon's code.

Leon's code.

Guava functions interfacing with Leon's code:

- `IsEquivalent`,
- `CodeIsomorphism`,
- `AutomorphismGroup`,
- `ConstantWeightSubcode`,
- `PermutationDecode` - see below.

Guava's non-linear codes

"Unrestricted" codes:

- `ElementsCode`, `RandomCode`
- `HadamardCode` (assumes Guava has associated Hadamard matrix in it database to construct `HadamardMat( ... ))`
- `ConferenceCode`
- `MOLSCode` (from mutually orthogonal Latin squares)
- `NordstromRobinsonCode`
- `GreedyCode`, `LexiCode`

General linear code constructions.

From the check/generator matrix or tables:

- `GeneratorMatCode`
- `CheckMatCodeMutable`, `CheckMatCode`
- `RandomLinearCode`
- `OptimalityCode`, `BestKnownLinearCode`

The last command uses tables developed by Cen Tjhal. (Much larger “best known” codes tables are needed.)

Common linear code constructions.

- HammingCode, ReedMullerCode,
- SrivastavaCode, GeneralizedSrivastavaCode
- FerreroDesignCode (uses SONATA)
- (classical) GoppaCode

Figure: Richard Hamming (1915-1998)

Special covering codes.

The **covering radius** of a linear code C is the smallest number r with the property that each element $\mathbf{v} \in \mathbb{F}^n$ there must be a codeword $\mathbf{c} \in C$ with $d(\mathbf{c}, \mathbf{c}) \leq r$.

- `GabidulinCode`
- `EnlargedGabidulinCode`
- `DavydovCode`
- `TombakCode`
- `EnlargedTombakCode`

Much larger covering codes tables are needed.

Golay codes.

- `BinaryGolayCode`
- `ExtendedBinaryGolayCode`
- `TernaryGolayCode`
- `ExtendedTernaryGolayCode`

Figure: Marcel Golay (1902-1989)

Cyclic codes.

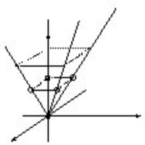
From the check/generator poly, etc:

- `GeneratorPolCode`, `CheckPolCode`
- `RootsCode`, `FireCode`
- `ReedSolomonCode`
- `BCHCode`, `AlternantCode`
- `QRCode`, `QQRCodeNC`
- `CyclicCodes`, `NrCyclicCodes`

Figure: Irving Reed, Gustave Solomon

Evaluation codes

- `EvaluationCode`
- `GeneralizedReedSolomonCode`
- `GeneralizedReedMullerCode`
- `ToricCode`
- `GoppaCodeClassical`
- `EvaluationBivariateCode`, `EvaluationBivariateCodeNC`
- `OnePointAGCode`



ToricCode example

This code was once best known:

Example

```
gap> C := ToricCode([ [0,0],[1,1],[1,2],[1,3],[1,4],\
 [2,1],[2,2],[2,3],[3,1],[3,2],[4,1]],GF(8));
a linear [49,11,1..39]25..38  toric code over GF(8)
```

min. dist. = 28.

- Diego Ruano and many others have also searched for other "new and good" toric-like codes, finding many more.
- Choosing the polytope carefully, the code can be constructed to have a large automorphism group.

Decoding methods

`Decode(C,r)` uses syndrome decoding or nearest-neighbor except for:

- Hamming codes (the usual trick),
- GRS codes - see below,
- cyclic codes (error-trapping - sometimes), and
- BCH codes (Sugiyama decoding).

Decoding methods

The default algorithm used for generalized Reed-Solomon codes is the interpolation algorithm. Gao's decoding method for GRS codes is also available as an option.

generalized Reed-Solomon codes

Decoding codes obtained from evaluating polynomials at lots of points "should be easy".

Rough idea: codewords are values of polynomial and $\#$ values known is $> \deg$(polynomials), so the vector overdetermines the polynomial. If the number of errors is "small" then the polynomial can still be reconstructed....

generalized Reed-Solomon codes

Syntax: `Decodeword( C, r )`, where C is a GRS code. This does “interpolation decoding”.

`GeneralizedReedSolomonDecoderGao` is a version which uses an algorithm of Gao.

`GeneralizedReedSolomonListDecoder( C, r, tau )` implements Sudan’s list-decoding algorithm for “low rate” GRS codes. It returns the list of all codewords in C which are a distance of at most τ from r.

Permutation decoding

Permutation decoding

Here is the basic idea.

- C is a code, $v \in \mathbb{F}^n$ is a received vector, $G = Aut(C)$ is the perm. automorphism group.
- Assume C is in standard form , with check matrix H.

Permutation decoding

The algorithm runs through the elements g of $G = Aut(C)$, checking if $\mathrm{wt}(H(g \cdot v)) < (d-1)/2$. If it is then the vector $g \cdot v$ is used to decode v: $c = g^{-1} \cdot Gm$ is the decoded word, where m is the information digits part of $g \cdot v$.

If no such g exists then "fail" is returned.

- This generalizes "error-trapping" for decoding cyclic codes,

Guava functions: `PermutationDecodeNC( C, v, G )`, `PermutationDecode( C, v )`

Sage and Guava

In Sage, **bad news** :

- most `GUAVA` functions are not wrapped or ported,
- most Leon functions are not wrapped (nor have they been rewritten in Cython)

Lots of work to be done.

Duursma zeta functions

In Sage, computing Duursma zeta functions of codes is implemented.

Figure: Tom Hoeholdt talking to Iwan Duursma at the IMA coding theory conference, May 2007.

Duursma zeta functions

C is an $[n, k, d]_q$ code
$C^\perp$ is an $[n, k^\perp, d^\perp]_q$ code
Motivated by local CFT, Iwan Duursma introduced the zeta function $Z = Z_C$ associated to C:

$$Z(T) = \frac{P(T)}{(1-T)(1-qT)}, \tag{2}$$

where $P(T)$ is a polynomial of degree $n + 2 - d - d^\perp$, called the zeta polynomial.

The *genus* of an $[n, k, d]_q$-code C is defined by

$$\begin{aligned}\gamma(C) &= n + 1 - k - d \\ &= \text{“distance code is from being MDS''}.\end{aligned}$$

For AG codes, it often is equal to the genus of the associated curve

Note that if C is a self-dual code then its genus satisfies

$$\gamma = n/2 + 1 - d.$$

Definitions

Weight enumerator polynomial -

$$A_C(x,y) = \sum_{i=0}^{n} A_i x^{n-i} y^i = x^n + A_d x^{n-d} y^d + \cdots + A_n y^n,$$

where

$$A_i = |\{c \in C \mid \mathrm{wt}(c) = i\}| = \#\text{ of codewds wt } i.$$

$A_C(x,y) = A_{C^\perp}(x,y)$ iff C is *formally self-dual code*

There exist a SD MDS code $[10,5,6]_{41}$ (due to J.-L. Kim, Y. Lee).

Definition of the zeta polynomial

A polynomial $P(T)$ for which

$$\frac{(xT+(1-T)y)^n}{(1-T)(1-qT)}P(T) = \cdots + \frac{A_C(x,y)-x^n}{q-1}T^{n-d} + \dots .$$

is called a *Duursma zeta polynomial of C*. (The Duusma zeta polynomial $P = P_C$ exists and is unique.)

The functional equation holds:

$$P^{\perp}(T) = P(\frac{1}{qT})q^g T^{g+g^{\perp}}, \qquad (3)$$

where $g = n/2 + 1 - d$ and $g^{\perp} = n/2 + 1 - d^{\perp}$.

The Riemann hypothesis is the statement that all zeros of $P(T)$ lie on the circle $|T| = 1/\sqrt{q}$.

Mallows-Sloane bounds

Let C be a fsd b-divisible $[n,k,d]_q$-code.
We say C is *Type I* if $q=b=2$, and n is even.
We say C is *Type II* if $q=2$, $b=4$, and $8|n$.
We say C is *Type III* if $q=b=3$, and $4|n$.
If $q=4$, $b=2$, and n is even then C is said to be *Type IV*.

Lemma (*Mallows-Sloane bounds*) If C is SD then

$$d \leq \begin{cases} 2[n/8]+2, & \text{if } C \text{ is Type I}, \\ 4[n/24]+4, & \text{if } C \text{ is Type II}, \\ 3[n/12]+3, & \text{if } C \text{ is Type III}, \\ 2[n/6]+2, & \text{if } C \text{ is Type IV}. \end{cases}$$

Virtual weight enumerators

Virtual weight enumerator - a homogeneous polynomial $F(x,y) = x^n + \sum_{i=1}^n f_i x^{n-i} y^i$ of degree n with complex coefficients.

If $F(x,y) = x^n + \sum_{i=d}^n f_i x^{n-i} y^i$ with $f_d \neq 0$ then we say that the *length* of F is n and the *minimum distance* of F is d.

Virtual weight enumerators

Formally self-dual weight enumerator - Such an F of even degree invariant under $\sigma = \frac{1}{\sqrt{q}} \begin{pmatrix} 1 & 1 \\ q-1 & -1 \end{pmatrix}$

Genus of a FSDWE: $\gamma(F) = n/2 + 1 - d$.

A virtual weight enumerator F is formally identified with an object we call a *virtual code* C subject only to the following condition: we formally extend the definition of $C \longmapsto A_C$ to all virtual codes by $A_C = F$.

Extremal FSDWEs

Theorem: If F is a FSDWE with length n and minimum distance d then

$$d \leq \begin{cases} 2[n/8]+2, & \text{if } C \text{ is Type I,} \\ 4[n/24]+4, & \text{if } C \text{ is Type II,} \\ 3[n/12]+3, & \text{if } C \text{ is Type III,} \\ 2[n/6]+2, & \text{if } C \text{ is Type IV.} \end{cases}$$

A FSDWE F (ie, a virtual SD code) is called extremal if the bound in the theorem holds with equality.

The statement of the RH for codes

A code is called *optimal* if its minimum distance is maximal among all linear codes of that length and dimension.

It is known that any two extremal codes (if they exist) have the same weight enumerator polynomial.

Duusma's conjecture:

The RH holds for $Z(T)$ for all extremal virtual codes.

Sage examples

Example

```
sage: C = HammingCode(3,GF(2))
sage: C.zeta_function()
 (1/5 + 2/5*T + 2/5*T^2)/(1 - 3*T + 2*T^2)
sage: C = ExtendedTernaryGolayCode()
sage: C.zeta_function()
(1/7 + 3/7*T + 3/7*T^2)/(1 - 4*T + 3*T^2)
```

These satisfy the RH.

An optimal FSD code

Consider the $[26, 13, 6]_{13}$ code with weight distribution

$$[1, 0, 0, 0, 0, 0, 39, 0, 455, 0, 1196, 0, 2405,$$
$$0, 2405, 0, 1196, 0, 455, 0, 39, 0, 0, 0, 0, 0, 1].$$

This is an optimal formally self-dual code C.

C has zeta polynomial

$$\begin{aligned} P(T) &= \frac{3}{17710} + \frac{6}{8855}T + \frac{611}{336490}T^2 + \frac{9}{2185}T^3 + \frac{3441}{408595}T^4 + \\ &+ \frac{6448}{408595}T^5 + \frac{44499}{1634380}T^6 + \frac{22539}{520030}T^7 + \frac{66303}{1040060}T^8 + \\ &+ \frac{22539}{260015}T^9 + \frac{44499}{408595}T^{10} + \frac{51584}{408595}T^{11} + \frac{55056}{408595}T^{12} + \\ &+ \frac{288}{2185}T^{13} + \frac{19552}{168245}T^{14} + \frac{768}{8855}T^{15} + \frac{384}{8855}T^{16}. \end{aligned}$$

Using Sage, it can be checked that only 8 of the 12 zeros of this function have absolute value $\sqrt{2}$.

The statement of the RH for codes

- Duursma has “explicitly” computed all zeta functions of extremal virtual SD codes.
- Duursma verified the RH for Type IV codes.
- For all low values of the parameters, computations using Sage have shown that the RH holds.

Sage has good functionality for working with

Self-dual codes

Self-dual codes database

Sage includes a database of all self-dual binary codes of length ≤ 20 (and some of length 22). The main function is `self_dual_codes_binary`, which is a list of Python dictionaries.

Format of each entry: dictionary with keys `order autgp`, `spectrum`, `code`, `Comment`, `Type`, where

- `code` - a self-dual code C of length n, dimension $n/2$, over $GF(2)$,
- `order autgp` - order of the permutation autom. group of C,
- `Type` - the type of C (which can be "I" or "II", in the binary case),
- `spectrum` - the spectrum $[A_0, A_1, \ldots, A_n]$,
- `Comment` - possibly an empty string.

Self-dual codess database

Sage

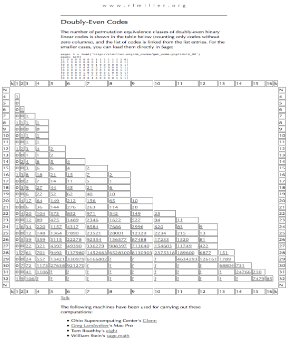
```
sage: C = self_dual_codes_binary(10)["10"]
sage: C["0"]["code"] == C["0"]["code"].dual_code()
True
sage: C["1"]["code"] == C["1"]["code"].dual_code()
True
sage: len(C.keys()) # number of inequiv sd codes of length 10
2
sage: C = self_dual_codes_binary(12)["12"]
sage: C["0"]["code"] == C["0"]["code"].dual_code()
True
sage: C["1"]["code"] == C["1"]["code"].dual_code()
True
sage: C["2"]["code"] == C["2"]["code"].dual_code()
True
```

These commands check that some of the database entries (of length 10 and of length 12), are indeed self dual.

Self-orthogonal codes

For classification of doubly even self-orthogonal codes using Sage, see `http://www.rlmiller.org/de_codes/`.
The number of permutation equivalence classes of all doubly even $[n, k]$–codes is shown in the table at `http://www.rlmiller.org/de_codes/`, and the list of codes so far discovered is linked from the list entries.

Self-orthogonal codes



w w w . r l m i l l e r . o r g

Doubly-Even Codes

The number of permutation equivalence classes of doubly-even binary linear codes is shown in the table below (counting only codes without zero columns), and the list of codes is linked from the list entries. For the smaller cases, you can load them directly in Sage:

```
sage: L = load('http://rlmiller.org/de_codes/get_code.php?id=16_08')
sage: L[0]
[1 0 0 0 0 0 0 0 1 1 1 1 1 1 1 0]
[0 1 0 0 0 0 0 0 1 1 1 1 1 1 0 1]
[0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 1]
[0 0 0 1 0 0 0 0 0 1 0 0 0 0 1 1]
[0 0 0 0 1 0 0 0 0 0 1 0 0 0 1 1]
[0 0 0 0 0 1 0 0 0 0 0 1 0 0 1 1]
[0 0 0 0 0 0 1 0 0 0 0 0 1 0 1 1]
[0 0 0 0 0 0 0 1 0 0 0 0 0 1 1 1]
```

N \ k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	N
4	1																4
5	0																5
6	0	1															6
7	0	0	1														7
8	1	1	1	1													8
9	0	0	0	0													9
10	0	1	1	1													10
11	0	0	1	1													11
12	1	2	3	4	2												12
13	0	0	1	1	2												13
14	0	2	4	6	5	4											14
15	0	0	3	6	6	4	2										15
16	1	3	8	18	21	15	7	2									16
17	0	0	2	7	14	11	5	1									17
18	0	3	9	27	44	45	21	6									18
19	0	0	6	22	52	62	40	10									19
20	1	4	17	64	149	212	156	65	10								20
21	0	0	6	36	144	276	263	114	28								21
22	0	4	20	104	373	852	971	542	149	25							22
23	0	0	12	89	475	1489	2346	1622	527	94	11						23
24	1	6	34	220	1157	4317	8584	7686	2996	620	83	9					24
25	0	0	12	148	1364	7890	23521	28001	12329	2234	215	13					25
26	0	5	39	359	3115	22278	92354	156577	87488	17233	1520	81					26
27	0	0	22	321	4397	49390	336279	908397	713640	154603	11749	422					27
28	1	7	61	701	9492	137980	1452663	6528300	8130903	2375318	189600	6877	151				28
29	0	0	24	557	13421	330979	6166802	?	?	?	4634293	126161	1789				29
30	0	7	72	1173	27638	921270	?	?	?	?	?	?	68804	731			30
31	0	0	41	1106	?	?	?	?	?	?	?	?	?	24766	210		31
32	1	9	106	?	?	?	?	?	?	?	?	?	?	?	7479	85	32
N \ k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	N

Talk

The following machines have been used for carrying out these computations:

- Ohio Supercomputing Center's Glenn
- Greg Landweber's Mac Pro
- Tom Boothby's eight
- William Stein's sage.math

Figure: http://www.rlmiller.org/de_codes/.

Self-orthogonal codes

Each link on that webpage points to a Sage object file, which when loaded. For example

```
sage:  L = load('24_12_de_codes.sobj')
```

is a list of matrices in standard form.

Cool example (on self-dual codes).

Recall:

- $W_5(x, y) = x^8 + 14x^4y^4 + y^8$ is the weight enumerator of the Type II $[8, 4, 4]$ code C constructed by extending the binary $[7, 4, 3]$ Hamming code by a check bit. This is the smallest Type II code.
- $W_6(x, y) = x^{24} + 759x^{16}y^8 + 2576x^{12}y^{12} + 759x^8y^{16} + y^{24}$ is the weight enumerator of the extended binary Golay code with parameters $[24, 12, 8]$.

Cool example (on self-dual codes).

Theorem

Assume C is a formally self-dual divisible code of Type II. Then $A_C(x, y)$ is invariant under the group

$$G_{II} = \langle \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \rangle$$

of order 192. Moreover, $\mathbb{C}[x, y]^{G_{II}} = \mathbb{C}[W_5, W_6]$.

Cool example (on self-dual codes).

Consider the group G generated by

$$g_1 = \begin{pmatrix} 1/\sqrt{q} & 1/\sqrt{q} \\ (q-1)/\sqrt{q} & -1/\sqrt{q} \end{pmatrix}, g_2 = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, g_3 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix},$$

with $q = 2$. This group leaves invariant the weight enumerator of any self-dual doubly even binary code, e.g., `ExtendedBinaryGolayCode`.

Cool example (on self-dual codes).

Sage code below cals GAP to construct the matrix group.

Example

```
sage: F = CyclotomicField(8)
sage: z = F.gen()
sage: a = z+1/z
sage: b = z^2
sage: MS = MatrixSpace(F,2,2)
sage: g1 = MS([[1/a,1/a],[1/a,-1/a]])
sage: g2 = MS([[1,0],[0,b]])
sage: g3 = MS([[b,0],[0,1]])
sage: G = MatrixGroup([g1,g2,g3])
sage: G.order()
192
```

Cool example (on self-dual codes).

Sage code below calls `Singular` for computing the invariants of G. We see that the invariants are indeed as predicted.

Example

```
sage: G.invariant_generators()
[x1^8 + 14*x1^4*x2^4 + x2^8,
 x1^24 + 10626/1025*x1^20*x2^4 + 735471/1025*x1^16*x2^8\
 + 2704156/1025*x1^12*x2^12 + 735471/1025*x1^8*x2^16\
 + 10626/1025*x1^4*x2^20 + x2^24]
```

Cool example (on self-dual codes).

The above result implies that any such weight enumerator must be a polynomial in

$$x^8 + 14x^4y^4 + y^8$$

and

$$1025x^{24} + 10626x^{20}y^4 + 735471x^{16}y^8 + 2704156x^{12}y^{12} + 735471x^8y^{16} + 10626x^4y^{20} + 1025y^{24}.$$

(Consistent with the previously mentioned result.)

The end.

Sage is a community. **Please** join us!

Have fun with Sage!

The End.