ΘΕΩΡΙΑ ΣΗΜΑΤΩΝ ΚΑΙ ΣΥΣΤΗΜΑΤΩΝ

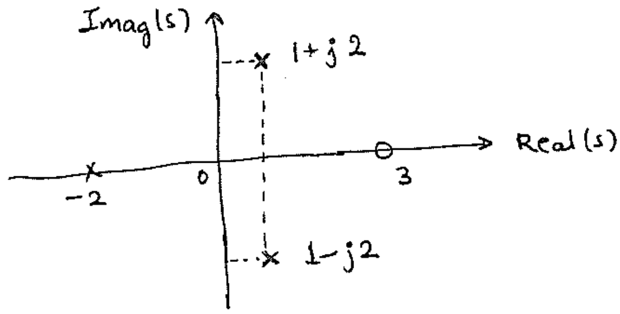
ΛΥΣΕΙΣ 6Η ΣΕΙΡΑΣ ΑΣΚΗΣΕΩΝ

1.

α) Μηδενικό : $s=3$

Πόλοι : $s=-2$

$$\Delta = 4 - 4\cdot 5 = -16 \Rightarrow s_{1,2} = \frac{2 \pm j}{2} = 1 \pm j2$$



β) $$H(s) = \frac{Y(s)}{X(s)} = \frac{5(s-3)}{(s+2)(s^2-2s+5)} = \frac{5(s-3)}{s^3-\cancel{2s^2}+5s+\cancel{2s^2}-4s+10} = \frac{5(s-3)}{s^3+s+10}$$

$$s^3 Y(s) + s\,Y(s) + 10\,Y(s) = 5s\,X(s) - 15\,X(s)$$

$$\frac{d^3y(t)}{dt^3} + \frac{dy(t)}{dt} + 10\,y(t) = 5\frac{dx(t)}{dt} - 15\,x(t)$$

γ) Αν είναι αιτιατό, θα έχει Π. Σ. $Re(s) > 1$

$$H(s) = \frac{5(s+3)}{(s+2)(s^2-2s+5)} = \frac{A}{s+2} + \frac{Bs+C}{s^2-2s+5}$$

$$A = \frac{-25}{13} \qquad B = +\frac{25}{13} \qquad C = -\frac{35}{13}$$

$$H(s) = \frac{1}{13}\left(-25\frac{1}{s+2} + 25\frac{s}{s^2-2s+5} - 35\frac{1}{s^2-2s+5}\right)$$

όπου $s^2-2s+5 = s^2-2s+1+4 = (s-1)^2 + 2^2$ άρα:

$$H(s) = \frac{1}{13}\left(-25\,\frac{1}{s+2} + 25\,\frac{s-1}{(s-1)^2+2^2} + 25\,\frac{1}{(s-1)^2+2^2} - 35\,\frac{1}{(s-1)^2+2^2}\right)$$

$$= \frac{1}{13}\left(-25\,\frac{1}{s+2} + 25\,\frac{s-1}{(s-1)^2+2^2} - 10\,\frac{1}{(s-1)^2+2^2}\right)$$

Το σύστημα είναι αιτιατό $\Rightarrow$ ΠΣ $Re(s) > 1$

$$\left.\begin{array}{l} \dfrac{1}{s+2} \Leftrightarrow e^{-2t}u(t) \\[2ex] \dfrac{s-1}{(s-1)^2+2^2} \longleftrightarrow e^{t}\cos(2t)u(t) \\[2ex] \dfrac{2}{(s-1)^2+2^2} \longleftrightarrow e^{t}\sin(2t)u(t) \end{array}\right\} \quad \begin{array}{l} h(t) = -\frac{25}{13}e^{-2t}u(t) + \frac{25}{13}e^{t}\cos(2t)\,u(t) \\ \qquad -\frac{5}{13}e^{t}\sin(2t)\,u(t) \end{array}$$

δ) Για να είναι ευσταθές, η ΠΣ πρέπει να περιέχει το $j\omega$ άξονα

$$\Rightarrow \quad \text{ΠΣ} \quad -2 < Re(s) < 1$$

$$u(t)\cos(\omega_0 t) \longleftrightarrow \frac{s}{s^2+\omega_0^2} \qquad \text{άρα} \qquad \cos(\omega_0 t)u(-t) \longleftrightarrow \frac{-s}{s^2+\omega_0^2}$$

ΠΣ $Re(s) > 0$ ΠΣ $Re(s) < 0$

γιατί

$$x(-t) \leftrightarrow X(-s)$$

Τότε $$\frac{-(s-1)}{(s-1)^2+\omega_0^2} \longleftrightarrow e^{t}\cos(\omega_0 t)\,u(-t), \quad Re(s) < 1$$

$$\sin(\omega_0 t)u(t) \longleftrightarrow \frac{\omega_0}{s^2+\omega_0^2} \qquad \text{άρα} \qquad -\sin(\omega_0 t)u(-t) \longleftrightarrow \frac{\omega_0}{s^2+\omega_0^2}$$

ΠΣ $Re(s) > 0$ ΠΣ $Re(s) < 0$

Τότε $$\frac{\omega_0^2}{(s-1)^2+\omega_0^2} \longleftrightarrow -e^{t}\sin(\omega_0 t)\,u(-t), \quad Re(s) < 1$$

$$h(t) = -\frac{25}{13}e^{-2t}u(t) - \frac{25}{13}e^{t}\cos(2t)\,u(-t) + \frac{5}{13}e^{t}\sin(2t)\,u(-t)$$

2.

$$y'(t) \leftrightarrow sY_I(s) - y(0^-) = sY_I(s) - 2$$
$$y''(t) \leftrightarrow s^2Y_I(s) - sy(0^-) - y'(0^-) = s^2Y_I(s) - 2s - 1$$

And

$$x(t) \leftrightarrow X_I(s) = \frac{1}{s+1}$$

Taking the unilateral Laplace transform we obtain

$$\left[s^2Y_I(s) - 2s - 1\right] + 5\left[sY_I(s) - 2\right] + 6Y_I(s) = \frac{1}{s+1}$$

or
$$(s^2 + 5s + 6)Y_I(s) = \frac{1}{s+1} + 2s + 11 = \frac{2s^2 + 13s + 12}{s+1}$$

Thus,

$$Y_I(s) = \frac{2s^2 + 13s + 12}{(s+1)(s^2 + 5s + 6)} = \frac{2s^2 + 13s + 12}{(s+1)(s+2)(s+3)}$$

Using partial-fraction expansions, we obtain

$$Y_I(s) = \frac{1}{2}\frac{1}{s+1} + 6\frac{1}{s+2} - \frac{9}{2}\frac{1}{s+3}$$

Taking the inverse Laplace transform of $Y_I(s)$, we have

$$y(t) = \left(\tfrac{1}{2}e^{-t} + 6e^{-2t} - \tfrac{9}{2}e^{-3t}\right)u(t)$$

Notice that $y(0^+) = 2 = y(0)$ and $y'(0^+) = 1 = y'(0)$; and we can write $y(t)$ as

$$y(t) = \tfrac{1}{2}e^{-t} + 6e^{-2t} - \tfrac{9}{2}e^{-3t} \qquad t \geq 0$$

3.

(*a*) From the given initial conditions, we have

$$v_{C_1}(0^-) = 1 \text{ V} \qquad \text{and} \qquad v_{C_2}(0^-) = 2 \text{ V}$$

we construct a transform circuit as shown in Fig. 3-17(*b*).

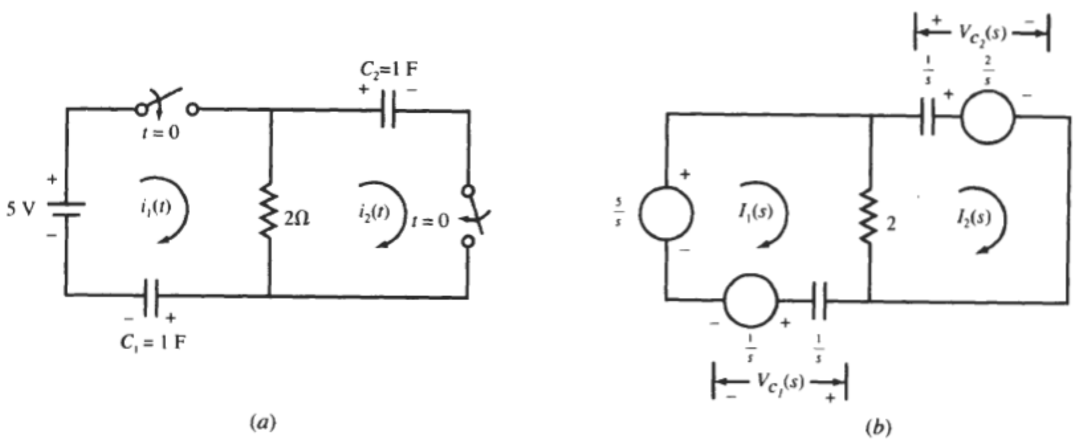


Fig. 3-17

Fig. 3-17(b) the loop equations can be written directly as

$$\left(2+\frac{1}{s}\right)I_1(s) - 2I_2(s) = \frac{4}{s}$$

$$-2I_1(s) + \left(2+\frac{1}{s}\right)I_2(s) = -\frac{2}{s}$$

Solving for $I_1(s)$ and $I_2(s)$ yields

$$I_1(s) = \frac{s+1}{s+\frac{1}{4}} = \frac{s+\frac{1}{4}+\frac{3}{4}}{s+\frac{1}{4}} = 1 + \frac{3}{4}\frac{1}{s+\frac{1}{4}}$$

$$I_2(s) = \frac{s-\frac{1}{2}}{s+\frac{1}{4}} = \frac{s+\frac{1}{4}-\frac{3}{4}}{s+\frac{1}{4}} = 1 - \frac{3}{4}\frac{1}{s+\frac{1}{4}}$$

Taking the inverse Laplace transforms of $I_1(s)$ and $I_2(s)$, we get

$$i_1(t) = \delta(t) + \tfrac{3}{4}e^{-t/4}u(t)$$

$$i_2(t) = \delta(t) - \tfrac{3}{4}e^{-t/4}u(t)$$

(b) From Fig. 3-17(b) we have

$$V_{C_1}(s) = \frac{1}{s}I_1(s) + \frac{1}{s}$$

$$V_{C_2}(s) = \frac{1}{s}I_2(s) + \frac{2}{s}$$

Substituting $I_1(s)$ and $I_2(s)$ obtained in part (a) into the above expressions, we get

$$V_{C_1}(s) = \frac{1}{s}\frac{s+1}{s+\frac{1}{4}} + \frac{1}{s}$$

$$V_{C_2}(s) = \frac{1}{s}\frac{s-\frac{1}{2}}{s+\frac{1}{4}} + \frac{2}{s}$$

Then, using the initial value theorem we have

$$v_{C_1}(0^+) = \lim_{s\to\infty} sV_{C_1}(s) = \lim_{s\to\infty}\frac{s+1}{s+\frac{1}{4}} + 1 = 1 + 1 = 2\text{ V}$$

$$v_{C_2}(0^+) = \lim_{s\to\infty} sV_{C_2}(s) = \lim_{s\to\infty}\frac{s-\frac{1}{2}}{s+\frac{1}{4}} + 2 = 1 + 2 = 3\text{ V}$$

Note that $v_{C_1}(0^+) \neq v_{C_1}(0^-)$ and $v_{C_2}(0^+) \neq v_{C_2}(0^-)$. This is due to the existence of a capacitor loop in the circuit resulting in a sudden change in voltage across the capacitors. This step change in voltages will result in impulses in $i_1(t)$ and $i_2(t)$. Circuits having a capacitor loop or an inductor star connection are known as *degenerative circuits*.

4.

$$x(n) = \cos(n\omega_0) = \frac{1}{2} e^{jn\omega_0} + \frac{1}{2} e^{-jn\omega_0}$$

$$X(k) = \sum_{n=0}^{N-1} \frac{1}{2} e^{jn\omega_0} e^{-j2\pi kn/N} + \frac{1}{2} e^{-jn\omega_0} e^{-j2\pi kn/N}$$

$$= \frac{1}{2} \sum_{n=0}^{N-1} e^{jn(\omega_0 - 2\pi k/N)} + \frac{1}{2} \sum_{n=0}^{N-1} e^{-jn(\omega_0 + 2\pi k/N)}$$

όμως $\displaystyle\sum_{n=0}^{N-1} e^{jn\left(\omega_0 - \frac{2\pi k}{N}\right)} = \sum_{n=0}^{N-1} e^{jn\frac{2\pi}{N}(k_0 - k)}$ $\boxed{\text{για } \omega_0 = \frac{2\pi k_0}{N}}$ (A)

$$X(k) = \frac{1}{2} \sum_{n=0}^{N-1} e^{-j\frac{2\pi n}{N}(k-k_0)} + \frac{1}{2} \sum_{n=0}^{N-1} e^{-j\frac{2\pi n}{N}(k+k_0)}$$

$\underline{k \neq k_0}$: $\displaystyle\sum_{n=0}^{N-1} e^{-j\frac{2\pi n}{N}(k-k_0)} = \frac{e^{-j\frac{2\pi n}{N}(k-k_0)N} - 1}{e^{-j\frac{2\pi n}{N}} - 1} = \frac{1 - e^{-j2\pi n(k-k_0)}}{1 - e^{-j\frac{2\pi n}{N}}} = 0$

$\underline{k = k_0}$: $\displaystyle\sum_{n=0}^{N-1} 1 = N$

Ομοίως για $k + k_0 = N \Rightarrow k = N - k_0$: $\displaystyle\sum_{n=0}^{N-1} e^{-j\frac{2\pi n}{N}(k+k_0)} = 0$, αλλιώς N

$$X(k) = \frac{N}{2} \delta(k - k_0) + \frac{N}{2} \delta(k - N + k_0)$$

(B) $\boxed{\text{Για } \omega_0 \neq 2\pi k_0/N}$

$$X(k) = \frac{1}{2} \frac{1 - e^{j\left(\omega_0 - \frac{2\pi k}{N}\right)N}}{1 - e^{j\left(\omega_0 - \frac{2\pi k}{N}\right)}} + \frac{1}{2} \frac{1 - e^{-j\left(\omega_0 + \frac{2\pi k}{N}\right)N}}{1 - e^{-j\left(\omega_0 + \frac{2\pi k}{N}\right)}}$$

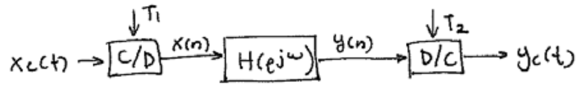
$$= \frac{1}{2} \frac{e^{j\frac{N}{2}\left(\omega_0 - \frac{2\pi k}{N}\right)}}{e^{-j\frac{1}{2}\left(\omega_0 - \frac{2\pi k}{N}\right)}} \frac{\sin\left(\frac{N\omega_0}{2} - \pi k\right)}{\sin\left(\frac{\omega_0}{2} - \frac{\pi k}{N}\right)} + \frac{1}{2} \frac{e^{-j\frac{N}{2}\left(\omega_0 + \frac{2\pi k}{N}\right)}\left(e^{j\frac{N}{2}\left(\omega_0 + \frac{2\pi k}{N}\right)} - e^{-j\frac{N}{2}\left(\omega_0 + \frac{2\pi k}{N}\right)}\right)}{e^{-j\frac{1}{2}\left(\omega_0 + \frac{2\pi k}{N}\right)}\left(e^{j\frac{1}{2}\left(\omega_0 + \frac{2\pi k}{N}\right)} - e^{-j\frac{1}{2}\left(\omega_0 + \frac{2\pi k}{N}\right)}\right)}$$

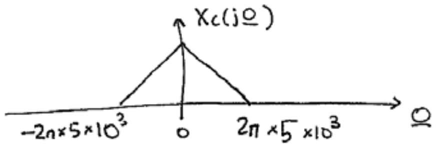
$$= \frac{1}{2} e^{j\left(\frac{N+1}{2}\right)\left(\omega_0 - \frac{2\pi k}{N}\right)} \frac{\sin\left(\pi k - \frac{N\omega_0}{2}\right)}{\sin\left(\frac{\pi k}{N} + \frac{\omega_0}{2}\right)} + \frac{1}{2} e^{-j\frac{(N-1)}{2}\left(\omega_0 + \frac{2\pi k}{N}\right)} \frac{\sin\left(\pi k + \frac{N\omega_0}{2}\right)}{\sin\left(\frac{\pi k}{N} + \frac{\omega_0}{2}\right)}$$

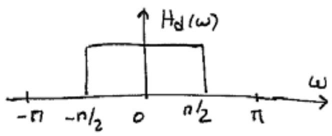
5.

$T_1 = T_2 = 10^{-4}$

X_d = DTFT
(d για discrete)







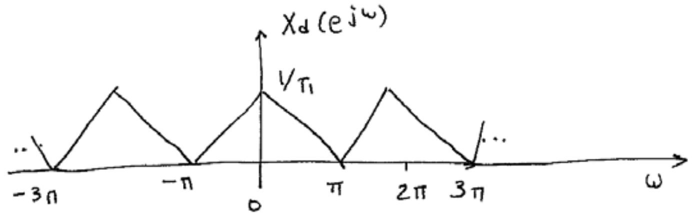
$f_{max} = 5 \times 10^3 \Rightarrow f_s \geq 10^4 \Rightarrow T_s \leq 10^{-4}$

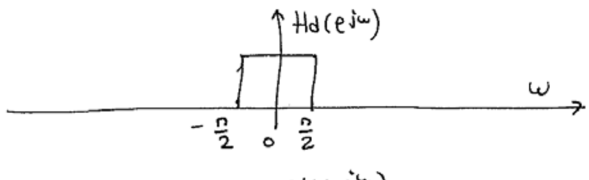
Εδώ έχουμε $T_1 = T_2 = 10^{-4}$ οπότε δεν περιμένουμε να δούμε aliasing.

Ισχύουν
$$\begin{cases} X(e^{j\omega}) = \dfrac{1}{T} \displaystyle\sum_{n=-\infty}^{+\infty} X_c\left(\dfrac{\omega + 2\pi n}{T}\right) & \omega = \Omega T \quad (*) \\ X(e^{j\Omega T}) = \dfrac{1}{T} \displaystyle\sum_{n} X_c(\Omega + 2\pi n) & \end{cases}$$

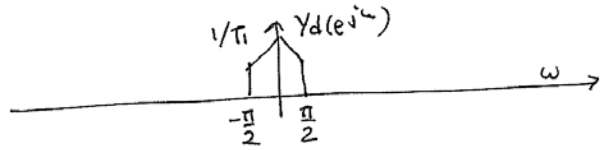
$\omega = \Omega T$ άρα με $\begin{cases} \Omega_1 = 2\pi \times 5 \times 10^3 \\ T_1 = 10^{-4} \end{cases} \Rightarrow \omega_1 = \pi$

Το φάσμα επαναλαμβάνεται κάθε $2\pi n$ (από την (*)) άρα:

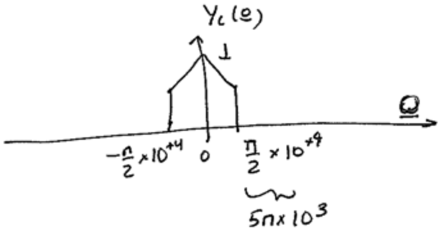




$\Rightarrow$



Με το Ο/C έχουμε $Y_c(\Omega) = \begin{cases} T\, Y_d(e^{j\Omega T}) & |\Omega| \le \frac{\pi}{T} \\ 0 & \text{αλλού} \end{cases}$ $\left(\begin{matrix} |\omega| \le \pi \\ \Omega = \frac{\omega}{T} \end{matrix} \right)$

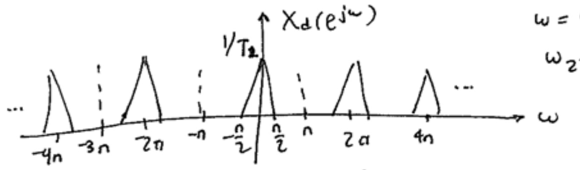


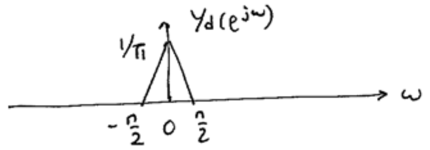
β) $T_1 = T_2 = 0.5\times10^{-4}$

Έχουμε $T_s \le 10^{-4}$ οπότε δεν περιμένουμε να δούμε αναδίπλωση.

$\omega = \Omega\cdot T \Rightarrow$

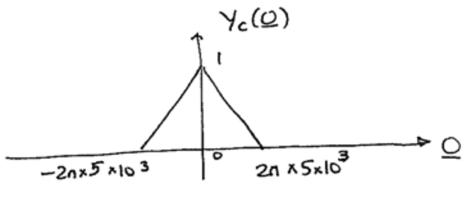
$\omega_2 = 2\pi \times 5\times10^3 \times \frac{1}{2}\times10^{-4} = \frac{\pi}{2}$





$Y_c(\Omega) = \begin{cases} T\, Y_d(e^{j\Omega T}) & |\Omega| \le \frac{\pi}{T} \\ 0 & \text{αλλού} \end{cases}$

$\omega = \frac{\pi}{2}$ γίνεται $\Omega = \frac{\omega}{T} = \frac{\pi}{2} \times \frac{1}{\frac{1}{2}10^{-4}} = \pi\times10^4 = 2\pi\times10^3\times5$



6.

$$X[N-k] = \sum_{n=0}^{N-1} x[n] W_N^{(N-k)n} = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)(N-k)n}$$

Now $$e^{-j(2\pi/N)(N-k)n} = e^{-j2\pi n}\, e^{j(2\pi/N)kn} = e^{j(2\pi/N)kn}$$

Hence, if $x[n]$ is real, then $x^*[n] = x[n]$ and

$$X[N-k] = \sum_{n=0}^{N-1} x[n]\, e^{j(2\pi/N)kn} = \left[\sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \right]^* = X^*[k]$$

7.

$$x[n] = \frac{1}{N}\left[\sum_{n=0}^{N-1} X[k]\, e^{j(2\pi/N)kn}\right] = \frac{1}{N}\left[\sum_{n=0}^{N-1} X^*[k]\, e^{-j(2\pi/N)nk}\right]^*$$

Noting that the term in brackets in the last term is the DFT of $X^*[k]$, we get

$$x[n] = \mathrm{IDFT}\{X[k]\} = \frac{1}{N}[\mathrm{DFT}\{X^*[k]\}]^*$$

which shows that the same algorithm used to evaluate the DFT can be used to evaluate the IDFT.