ΘΕΩΡΙΑ ΣΗΜΑΤΩΝ ΚΑΙ ΣΥΣΤΗΜΑΤΩΝ

ΛΥΣΕΙΣ 5Η ΣΕΙΡΑΣ ΑΣΚΗΣΕΩΝ

1.

a.

$$x(n) = n\left(\frac{1}{2}\right)^n u(n) + n\left(\frac{1}{2}\right)^{-n} u(-n-1) = \underbrace{n\left(\frac{1}{2}\right)^n u(n)}_{x_1(n)} + \underbrace{n \cdot 2^n u(-n-1)}_{x_2(n)}$$

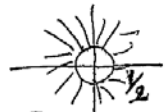
$$,\ X_1(z) = \sum_{n=-\infty}^{+\infty} n\left(\frac{1}{2}\right)^n u(n)\, z^{-n} = \sum_{n=0}^{+\infty} n\left(\frac{1}{2z}\right)^n \qquad \langle u(n) \Rightarrow n \geq 0 \rangle$$

$$= \sum_{n=0}^{+\infty} n\,(2z)^{-n} = \sum_{n=0}^{+\infty} 2^{-n} \cdot n z^{-n} \qquad \langle \text{όπου } (z^{-m})' = -m z^{-m-1} \rangle$$

$$= -\sum_{n=0}^{+\infty} 2^{-n}\left(-n z^{-n-1}\right) z = -z\sum_{n=0}^{+\infty} 2^{-n} \frac{d}{dz}\left(z^{-n}\right) = -z \frac{d}{dz} \sum_{n=0}^{\infty} (2z)^{-n}$$

• Ισχύει $\sum_{n=0}^{+\infty} \gamma^n = \frac{1}{1-\gamma}$ για $|\gamma|<1$

• $\left|\frac{1}{2z}\right| < 1 \Rightarrow \boxed{|z| > 1/2}$



ΠΣ$_1$

$$= -z \frac{d}{dz}\left(\frac{1}{1-\frac{1}{2z}}\right) = -z \frac{d}{dz}\left(\frac{z}{z-\frac{1}{2}}\right)$$

$$= -z\left(\frac{1}{z-\frac{1}{2}} - \frac{z}{(z-\frac{1}{2})^2}\right) = -z\,\frac{\cancel{z}-1/2-\cancel{z}}{(z-1/2)^2} = \frac{1}{2}\,\frac{z}{(z-1/2)^2}$$

$$= \frac{\frac{1}{2} z}{\left(z\left(1-\frac{1}{2}z^{-1}\right)\right)^2} = \frac{\frac{1}{2} z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2} \quad \text{με Π. Σ. } |z| > 1/2$$

$$X_2(z) = \sum_{n=-\infty}^{+\infty} n\, 2^n u(-n-1)\, z^{-n} = \sum_{n=-\infty}^{-1} n\, 2^n z^{-n} \qquad \left(u(-n-1) \Rightarrow n \leq -1\right)$$

$$\underset{\langle k=-n \rangle}{=} \sum_{k=1}^{+\infty} (-k)\, 2^{-k} z^{k} = -\sum_{k=1}^{\infty} k\left(\frac{z}{2}\right)^k = -\sum_{k=1}^{\infty} 2^{-k}\left(k z^{+k-1}\right) z$$

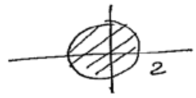
$$= -z \sum_{k=1}^{\infty} 2^{-k} \frac{d}{dz}\left(z^k\right) = -z \frac{d}{dz} \sum_{k=1}^{\infty} \left(\frac{z}{2}\right)^k$$

• $\left|\frac{z}{2}\right| < 1 \Rightarrow \boxed{|z| < 2}$

για να συγκλίνει το άθροισμα

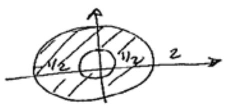
$$X_2(z) = -z \frac{d}{dz}\frac{1}{1-\frac{z}{2}} = +z\left(-\frac{1}{2}\right)\frac{1}{\left(1-\frac{z}{2}\right)^2} = \frac{-\frac{1}{2}z}{\left(\frac{z}{2}\left(2z^{-1}-1\right)\right)^2}$$

$$= \frac{-\frac{1}{2}z}{\frac{z^2}{4}\left(1-2z^{-1}\right)^2} = -\frac{2z^{-1}}{\left(1-2z^{-1}\right)^2}, \quad |z|<2$$



ΠΣ 2

άρα $X(z) = \frac{\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2} - \frac{2z^{-1}}{\left(1-2z^{-1}\right)^2}$ για $\boxed{\frac{1}{2} < |z| < 2}$



b.

$$x(n) = 4^n \cos\left(\frac{2\pi}{6}n + \frac{\pi}{4}\right) u(-n-1)$$

$$= \frac{1}{2}\, 4^n \left(e^{j\frac{\pi}{3}n} e^{j\frac{\pi}{4}} + e^{-j\frac{\pi}{3}n} e^{-j\frac{\pi}{4}}\right) u(-n-1)$$

$$= \frac{1}{2}\left(4e^{j\frac{\pi}{3}}\right)^n e^{j\frac{\pi}{4}} u(-n-1) + \frac{1}{2}\left(4e^{-j\frac{\pi}{3}}\right)^n e^{-j\frac{\pi}{4}} u(-n-1)$$

όπου για $y(n) = a^n u(-n-1)$:

$$Y(z) = \sum_{n=-\infty}^{+\infty} a^n u(-n-1) z^{-n} = \sum_{n=-\infty}^{-1} (a z^{-1})^n \underset{k=-n}{=} \sum_{k=1}^{+\infty} \left(\frac{z}{a}\right)^k$$

$\left|\frac{z}{a}\right| < 1 \Rightarrow |z| < |a|$

$$= \sum_{k=0}^{\infty}\left(\frac{z}{a}\right)^k - \underbrace{\left(\frac{z}{a}\right)^0}_{1} = -1 + \frac{1}{1-\frac{z}{a}} = \frac{\frac{z}{a}}{1-\frac{z}{a}} = -\frac{1}{1-az^{-1}}$$

άρα για:

$$x(n) = \frac{1}{2} e^{j\frac{\pi}{4}} \underbrace{\left(4e^{j\frac{\pi}{3}}\right)^n u(-n-1)}_{a^n u(-n-1)} + \frac{1}{2} e^{-j\frac{\pi}{4}} \underbrace{\left(4e^{-j\frac{\pi}{3}}\right)^n u(-n-1)}$$

έχουμε

$$X(z) = \frac{1}{2} e^{j\frac{\pi}{4}}\left(-\frac{1}{1-4e^{j\frac{\pi}{3}}z^{-1}}\right) - \frac{1}{2}e^{-j\frac{\pi}{4}} \frac{1}{1-4e^{-j\frac{\pi}{3}}z^{-1}}$$

$$= -\frac{1}{2}\left(\frac{e^{j\frac{\pi}{4}} z}{z-4e^{j\pi/3}} + \frac{e^{-j\frac{\pi}{4}} z}{z-4e^{-j\pi/3}}\right)$$

για $\left|4e^{j\frac{\pi}{3}}\right| > |z| \Rightarrow |z| < 4$

και $|z| < \left|4e^{-j\pi/3}\right| \Rightarrow |z| < 4$

$$= -\frac{1}{2}\, \frac{e^{j\frac{\pi}{4}}z^2 - 4ze^{-j\frac{\pi}{12}} + e^{-j\frac{\pi}{4}}z^2 - 4e^{j\frac{\pi}{12}}z}{(z-4e^{j\pi/3})(z-4e^{-j\pi/3})}$$

$$= -\frac{1}{2}\, \frac{\left(e^{j\frac{\pi}{4}} + e^{-j\frac{\pi}{4}}\right)z^2 - 4z\left(e^{j\pi/12} + e^{-j\pi/12}\right)}{(z-4e^{j\pi/3})(z-4e^{-j\pi/3})}$$

$$= -\frac{z\left(\cos(\pi/4)z - 4\cos(\pi/12)\right)}{(z-4e^{j\pi/3})(z-4e^{-j\pi/3})}$$

2.

a.

$$\left(\frac{1}{2}\right)^n u[n] \leftrightarrow \frac{z}{z-\frac{1}{2}} \qquad |z|>\frac{1}{2} \tag{4.58}$$

$$\left(\frac{1}{3}\right)^n u[n] \leftrightarrow \frac{z}{z-\frac{1}{3}} \qquad |z|>\frac{1}{3} \tag{4.59}$$

We see that the ROCs in Eqs. (*4.58*) and (*4.59*) overlap, and thus,

$$X(z)=\frac{z}{z-\frac{1}{2}}+\frac{z}{z-\frac{1}{3}}=\frac{2z\left(z-\frac{5}{12}\right)}{\left(s-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)} \qquad |z|>\frac{1}{2} \tag{4.60}$$

From Eq. (*4.60*) we see that $X(z)$ has two zeros at $z=0$ and $z=\frac{5}{12}$ and two poles at $z=\frac{1}{2}$ and $z=\frac{1}{3}$ and that the ROC is $|z|>\frac{1}{2}$, as sketched in Fig. 4-5(*a*).

b.

$$\left(\frac{1}{3}\right)^n u[n] \leftrightarrow \frac{z}{z-\frac{1}{3}} \qquad |z|>\frac{1}{3} \tag{4.61}$$

$$\left(\frac{1}{2}\right)^n u[-n-1] \leftrightarrow -\frac{z}{z-\frac{1}{2}} \qquad |z|<\frac{1}{2} \tag{4.62}$$

We see that the ROCs in Eqs. (*4.61*) and (*4.62*) overlap, and thus

$$X(z)=\frac{z}{z-\frac{1}{3}}-\frac{z}{z-\frac{1}{2}}=-\frac{1}{6}\frac{z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)} \qquad \frac{1}{3}<|z|<\frac{1}{2} \tag{4.63}$$

From Eq. (*4.63*) we see that $X(z)$ has one zero at $z=0$ and two poles at $z=\frac{1}{2}$ and $z=\frac{1}{3}$ and that the ROC is $\frac{1}{3}<|z|<\frac{1}{2}$, as sketched in Fig. 4-5(*b*).

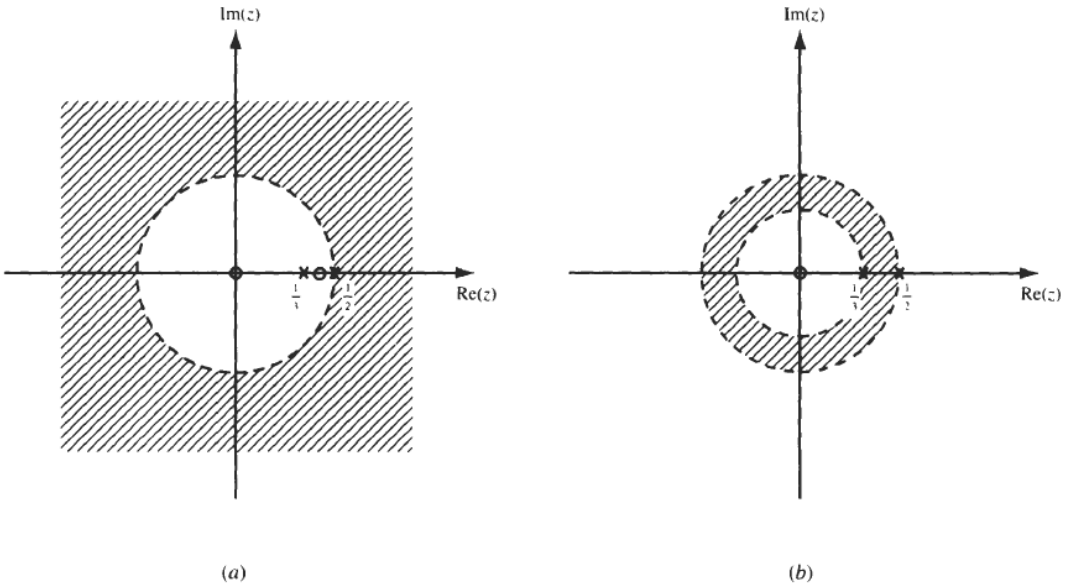


Fig. 4-5

c.

$$\left(\frac{1}{2}\right)^n u[n] \longleftrightarrow \frac{z}{z-\frac{1}{2}} \qquad |z| > \frac{1}{2} \tag{4.64}$$

$$\left(\frac{1}{3}\right)^n u[-n-1] \longleftrightarrow -\frac{z}{z-\frac{1}{3}} \qquad |z| < \frac{1}{3} \tag{4.65}$$

We see that the ROCs in Eqs. (*4.64*) and (*4.65*) do not overlap and that there is no common ROC, and thus $x[n]$ will not have $X(z)$.

3.

(*a*) Since the ROC is $|z| < \frac{1}{2}$, $x[n]$ is a left-sided sequence. Thus, we must divide to obtain a series in power of z. Carrying out the long division, we obtain

$$\begin{array}{r|l}
 & z + 3z^2 + 7z^3 + 15z^4 + \cdots \\
\hline
1 - 3z + 2z^2 & z \\
 & \underline{z - 3z^2 + 2z^3} \\
 & \quad 3z^2 - 2z^3 \\
 & \quad \underline{3z^2 - 9z^3 + 6z^4} \\
 & \qquad\quad 7z^3 - 6z^4 \\
 & \qquad\quad \underline{7z^3 - 21z^4 + 14z^5} \\
 & \qquad\qquad\quad 15z^4 \cdots
\end{array}$$

Thus,

$$X(z) = \cdots + 15z^4 + 7z^3 + 3z^2 + z$$

and so by definition (*4.3*) we obtain

$$x[n] = \{\ldots, 15, 7, 3, 1, \underset{\uparrow}{0}\}$$

(*b*) Since the ROC is $|z| > 1$, $x[n]$ is a right-sided sequence. Thus, we must divide so as to obtain a series in power of z^{-1} as follows:

$$\begin{array}{r|l}
 & \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \cdots \\
\hline
2z^2 - 3z + 1 & z \\
 & \underline{z - \frac{3}{2} - \frac{1}{2}z^{-1}} \\
 & \quad \frac{3}{2} - \frac{1}{2}z^{-1} \\
 & \quad \underline{\frac{3}{2} - \frac{9}{4}z^{-1} + \frac{3}{4}z^{-2}} \\
 & \qquad \frac{7}{4}z^{-1} - \frac{3}{4}z^{-2} \\
 & \qquad\quad \vdots
\end{array}$$

Thus,

$$X(z) = \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \cdots$$

and so by definition (*4.3*) we obtain

$$x[n] = \left\{0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots\right\}$$

4.

Using partial-fraction expansion, we have

$$\frac{X(z)}{z}=\frac{1}{(z-1)(z-2)^2}=\frac{c_1}{z-1}+\frac{\lambda_1}{z-2}+\frac{\lambda_2}{(z-2)^2} \qquad (4.83)$$

where $$c_1=\left.\frac{1}{(z-2)^2}\right|_{z=1}=1 \qquad \lambda_2=\left.\frac{1}{z-1}\right|_{z=2}=1$$

Substituting these values into Eq. (*4.83*), we have

$$\frac{1}{(z-1)(z-2)^2}=\frac{1}{z-1}+\frac{\lambda_1}{z-2}+\frac{1}{(z-2)^2}$$

Setting $z=0$ in the above expression, we have

$$-\frac{1}{4}=-1-\frac{\lambda_1}{2}+\frac{1}{4}\longrightarrow\lambda_1=-1$$

Thus,

$$X(z)=\frac{z}{z-1}-\frac{z}{z-2}+\frac{z}{(z-2)^2} \qquad |z|>2$$

Since the ROC is $|z|>2$, $x[n]$ is a right-sided sequence, we get

$$x[n]=(1-2^n+n2^{n-1})u[n]$$

5.

(*a*)

$$x[n]=u[n]\longleftrightarrow X(z)=\frac{z}{z-1} \qquad |z|>1$$

$$y[n]=2\left(\frac{1}{3}\right)^n u[n]\longleftrightarrow Y(z)=\frac{2z}{z-\frac{1}{3}} \qquad |z|>\frac{1}{3}$$

Hence, the system function $H(z)$ is

$$H(z)=\frac{Y(z)}{X(z)}=\frac{2(z-1)}{z-\frac{1}{3}} \qquad |z|>\frac{1}{3}$$

Using partial-fraction expansion, we have

$$\frac{H(z)}{z}=\frac{2(z-1)}{z\left(z-\frac{1}{3}\right)}=\frac{c_1}{z}+\frac{c_2}{z-\frac{1}{3}}$$

where $$c_1=\left.\frac{2(z-1)}{z-\frac{1}{3}}\right|_{z=0}=6 \qquad c_2=\left.\frac{2(z-1)}{z}\right|_{z=1/3}=-4$$

Thus,

$$H(z)=6-4\frac{z}{z-\frac{1}{3}} \qquad |z|>\frac{1}{3}$$

Taking the inverse z-transform of $H(z)$, we obtain

$$h[n]=6\delta[n]-4\left(\tfrac{1}{3}\right)^n u[n]$$

(*b*)

$$x[n] = \left(\frac{1}{2}\right)^n u[n] \longleftrightarrow X(z) = \frac{z}{z - \frac{1}{2}} \qquad |z| > \frac{1}{2}$$

Then,

$$Y(z) = X(z)H(z) = \frac{2z(z-1)}{(z-\frac{1}{2})(z-\frac{1}{3})} \qquad |z| > \frac{1}{2}$$

Again by partial-fraction expansion we have

$$\frac{Y(z)}{z} = \frac{2(z-1)}{(z-\frac{1}{2})(z-\frac{1}{3})} = \frac{c_1}{z-\frac{1}{2}} + \frac{c_2}{z-\frac{1}{3}}$$

where

$$c_1 = \left.\frac{2(z-1)}{z-\frac{1}{3}}\right|_{z=1/2} = -6 \qquad c_2 = \left.\frac{2(z-1)}{z-\frac{1}{2}}\right|_{z=1/3} = 8$$

Thus,

$$Y(z) = -6\frac{z}{z-\frac{1}{2}} + 8\frac{z}{z-\frac{1}{3}} \qquad |z| > \frac{1}{2}$$

Taking the inverse z-transform of $Y(z)$, we obtain

$$y[n] = \left[-6\left(\tfrac{1}{2}\right)^n + 8\left(\tfrac{1}{3}\right)^n\right]u[n]$$

6.

$$q[n] = x[n] + \frac{k}{2}q[n-1]$$

$$y[n] = q[n] + \frac{k}{3}q[n-1]$$

Taking the z-transform of the above equations, we obtain

$$Q(z) = X(z) + \frac{k}{2}z^{-1}Q(z)$$

$$Y(z) = Q(z) + \frac{k}{3}z^{-1}Q(z)$$

Rearranging, we have

$$\left(1 - \frac{k}{2}z^{-1}\right)Q(z) = X(z)$$

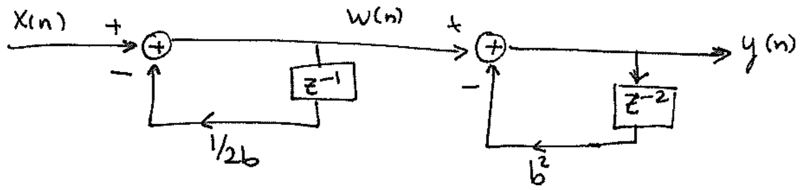
$$\left(1 + \frac{k}{3}z^{-1}\right)Q(z) = Y(z)$$

from which we obtain

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + (k/3)z^{-1}}{1 - (k/2)z^{-1}} = \frac{z + k/3}{z - k/2} \qquad |z| > \left|\frac{k}{2}\right|$$

Which shows that the system has one zero at $z = -k/3$ and one pole at $z = k/2$ and that the ROC is $|z|>|k/2|$. The system will be BIBO stable if the ROC contains the unity circle, $|z| = 1$. Hence the system is stable only if $|k|<2$.

7.



- $w(n) = x(n) - \frac{1}{2b} w(n-1) \Rightarrow W(z)\left(1 + \frac{1}{2b} z^{-1}\right) = X(z)$

$$\Rightarrow W(z) = \frac{1}{1 + \frac{1}{2b} z^{-1}} X(z)$$

- $y(n) = w(n) - b^2 y(n-2) \Rightarrow Y(z)\left(1 + b^2 z^{-2}\right) = W(z)$

$$\Rightarrow Y(z) = \frac{1}{(1 + b^2 z^{-2})} \frac{1}{(1 + \frac{1}{2b} z^{-1})} X(z)$$

$$\Rightarrow H(z) = \frac{z^3}{(z^2 + b^2)(z + 1/2b)}$$

πόλοι $\begin{cases} z_{1,2} = \pm j b \\ z_3 = \frac{1}{2b} \end{cases}$

Για να είναι αιτιατό & ευσταθές $|z| < 1$

$$\Rightarrow \begin{cases} |b| < 1 \\ \left|\frac{1}{2b}\right| < 1 \end{cases} \Rightarrow \begin{cases} |b| < 1 \\ |b| > \frac{1}{2} \end{cases} \Rightarrow \boxed{\frac{1}{2} < |b| < 1}$$