ΘΕΩΡΙΑ ΣΗΜΑΤΩΝ ΚΑΙ ΣΥΣΤΗΜΑΤΩΝ

ΛΥΣΕΙΣ 4Η ΣΕΙΡΑΣ ΑΣΚΗΣΕΩΝ

1.

a.

$$\cos\omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) = \frac{1}{2}e^{-j\omega_0 t} + \frac{1}{2}e^{j\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Thus, the complex Fourier coefficients for $\cos\omega_0 t$ are

$$c_1 = \tfrac{1}{2} \qquad c_{-1} = \tfrac{1}{2} \qquad c_k = 0,\ |k| \neq 1$$

b.

$$\sin\omega_0 t = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) = -\frac{1}{2j}e^{-j\omega_0 t} + \frac{1}{2j}e^{j\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Thus, the complex Fourier coefficients for $\sin\omega_0 t$ are

$$c_1 = \frac{1}{2j} \qquad c_{-1} = -\frac{1}{2j} \qquad c_k = 0,\ |k| \neq 1$$

c.

The fundamental angular frequency ω_0 of $x(t)$ is 2. Thus,

$$x(t) = \cos\left(2t + \frac{\pi}{4}\right) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

Now

$$x(t) = \cos\left(2t + \frac{\pi}{4}\right) = \frac{1}{2}\left(e^{j(2t+\pi/4)} + e^{-j(2t+\pi/4)}\right)$$

$$= \frac{1}{2}e^{-j\pi/4}e^{-j2t} + \frac{1}{2}e^{j\pi/4}e^{j2t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

Thus, the complex Fourier coefficients for $\cos(2t + \pi/4)$ are

$$c_1 = \frac{1}{2}e^{j\pi/4} = \frac{1}{2}\frac{1+j}{\sqrt{2}} = \frac{\sqrt{2}}{4}(1+j)$$

$$c_{-1} = \frac{1}{2}e^{-j\pi/4} = \frac{1}{2}\frac{1-j}{\sqrt{2}} = \frac{\sqrt{2}}{4}(1-j)$$

$$c_k = 0 \qquad |k| \neq 1$$

d. The fundamental period of T_0 of x(t) is π and $\omega_0 = 2\pi/T_0 = 2$

Thus,

$$x(t) = \cos 4t + \sin 6t = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

Again using Euler's formula, we have

$$x(t) = \cos 4t + \sin 6t = \frac{1}{2}(e^{j4t} + e^{-j4t}) + \frac{1}{2j}(e^{j6t} - e^{-j6t})$$

$$= -\frac{1}{2j}e^{-j6t} + \frac{1}{2}e^{-j4t} + \frac{1}{2}e^{j4t} + \frac{1}{2j}e^{j6t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

Thus, the complex Fourier coefficients for $\cos 4t + \sin 6t$ are

$$c_{-3} = -\frac{1}{2j} \qquad c_{-2} = \frac{1}{2} \qquad c_2 = \frac{1}{2} \qquad c_3 = \frac{1}{2j}$$

and all other $c_k = 0$.

e. The fundamental period of T_0 of x(t) is π and $\omega_0 = 2\pi/T_0 = 2$. Thus

$$x(t) = \sin^2 t = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

Again using Euler's formula, we get

$$x(t) = \sin^2 t = \left(\frac{e^{jt} - e^{-jt}}{2j}\right)^2 = -\frac{1}{4}(e^{j2t} - 2 + e^{-j2t})$$

$$= -\frac{1}{4}e^{-j2t} + \frac{1}{2} - \frac{1}{4}e^{j2t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

Thus, the complex Fourier coefficients for $\sin^2 t$ are

$$c_{-1} = -\tfrac{1}{4} \qquad c_0 = \tfrac{1}{2} \qquad c_1 = -\tfrac{1}{4}$$

and all other $c_k = 0$.

2.

a.

Let

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T_0}\int_0^{T_0/2} A\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{A}{-jk\omega_0 T_0} e^{-jk\omega_0 t}\Big|_0^{T_0/2} = \frac{A}{-jk\omega_0 T_0}\left(e^{-jk\omega_0 T_0/2} - 1\right)$$

$$= \frac{A}{jk2\pi}\left(1 - e^{-jk\pi}\right) = \frac{A}{jk2\pi}\left[1 - (-1)^k\right]$$

since $\omega_0 T_0 = 2\pi$ and $e^{-jk\pi} = (-1)^k$. Thus,

$$c_k = 0 \qquad k = 2m \neq 0$$

$$c_k = \frac{A}{jk\pi} \qquad k = 2m + 1$$

$$c_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\, dt = \frac{1}{T_0}\int_0^{T_0/2} A\, dt = \frac{A}{2}$$

Hence,

$$c_0 = \frac{A}{2} \qquad c_{2m} = 0 \qquad c_{2m+1} = \frac{A}{j(2m+1)\pi}$$

and we obtain

$$x(t) = \frac{A}{2} + \frac{A}{j\pi}\sum_{m=-\infty}^{\infty} \frac{1}{2m+1} e^{j(2m+1)\omega_0 t}$$

b.

$$\frac{a_0}{2} = c_0 = \frac{A}{2} \qquad a_{2m} = b_{2m} = 0,\ m \neq 0$$

$$a_{2m+1} = 2\,\mathrm{Re}[c_{2m+1}] = 0 \qquad b_{2m+1} = -2\,\mathrm{Im}[c_{2m+1}] = \frac{2A}{(2m+1)\pi}$$

Substituting these values in

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos k\omega_0 t + b_k \sin k\omega_0 t\right) \qquad \omega_0 = \frac{2\pi}{T_0}$$

We have:

$$x(t) = \frac{A}{2} + \frac{2A}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} \sin(2m+1)\omega_0 t$$

$$= \frac{A}{2} + \frac{2A}{\pi} \left(\sin \omega_0 t + \frac{1}{3} \sin 3\omega_0 t + \frac{1}{5} \sin 5\omega_0 t + \cdots \right)$$

3.

a.

Let

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T_0} \int_{-T_0/4}^{T_0/4} A e^{-jk\omega_0 t} dt$$

$$= \frac{A}{-jk\omega_0 T_0} (e^{-jk\omega_0 T_0/4} - e^{jk\omega_0 T_0/4})$$

$$= \frac{A}{-jk2\pi} (e^{-jk\pi/2} - e^{jk\pi/2}) = \frac{A}{k\pi} \sin\left(\frac{k\pi}{2}\right)$$

Thus,

$$c_k = 0 \qquad k = 2m \neq 0$$

$$c_k = (-1)^m \frac{A}{k\pi} \qquad k = 2m+1$$

$$c_0 = \frac{1}{T_0} \int_0^{T_0} x(t)\, dt = \frac{1}{T_0} \int_0^{T_0/2} A\, dt = \frac{A}{2}$$

Hence,

$$c_0 = \frac{A}{2} \qquad c_{2m} = 0,\ m \neq 0 \qquad c_{2m+1} = (-1)^m \frac{A}{(2m+1)\pi}$$

and we obtain

$$x(t) = \frac{A}{2} + \frac{A}{\pi} \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{2m+1} e^{j(2m+1)\omega_0 t}$$

b.

$$\frac{a_0}{2} = c_0 = \frac{A}{2} \qquad a_{2m} = 2\,\mathrm{Re}[c_{2m}] = 0,\ m \neq 0$$

$$a_{2m+1} = 2\,\mathrm{Re}[c_{2m+1}] = (-1)^m \frac{2A}{(2m+1)\pi} \qquad b_k = -2\,\mathrm{Im}[c_k] = 0$$

Substituting these values in

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t) \qquad \omega_0 = \frac{2\pi}{T_0}$$

We have:

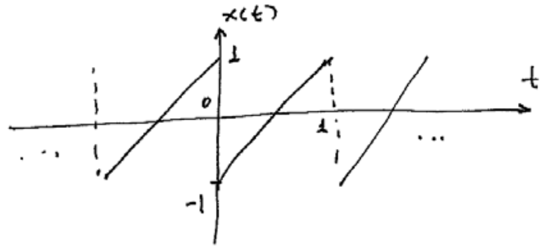
$$\begin{aligned} x(t) &= \frac{A}{2} + \frac{2A}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \cos(2m+1)\omega_0 t \\ &= \frac{A}{2} + \frac{2A}{\pi}\left(\cos \omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \cdots\right) \end{aligned}$$

Note that x(t) is even; thus x(t) contains only a dc term and cosine terms.

4.

$x(t) = 2t-1,\ 0 < t, < 1$

(a) $T = 1$



$$\left.\begin{aligned} \Omega_0 &= \frac{2\pi}{T} \\ T &= 1 \end{aligned}\right\} \Omega_0 = 2\pi$$

$$(T=1 \Rightarrow) \quad a_0 = \int_0^1 (2t-1)\,dt = \left[t^2\right]_0^1 - 1 = 0$$

$$a_k = \int_0^1 (2t-1)\, e^{-jk\Omega_0 t}\, dt = \int_0^1 (2t-1)\, e^{-j2\pi k t}\, dt$$

$$= 2\underbrace{\int_0^1 t e^{-j2\pi k t}\, dt}_{A} - \underbrace{\int_0^1 e^{-j2\pi k t}\, dt}_{B} \quad = 2A - B$$

$$A = \int_0^1 t e^{-j2\pi k t}\, dt = -\frac{1}{j2\pi k}\left[t e^{-j2\pi k t}\right]_0^1 -$$

$$+ \frac{1}{j2\pi k}\int_0^1 e^{-j2\pi k t}\, dt$$

$$= -\frac{1}{j2\pi k}\, \underbrace{\cancel{e^{-j2\pi k}}}_{1} - \frac{1}{(j2\pi k)^2}\left[e^{-j2\pi k t}\right]_0^1$$

$$= -\frac{1}{j2\pi k} - \frac{1}{(j2\pi k)^2}\underbrace{\left(e^{-j2\pi k} - 1\right)}_{\overset{\shortparallel}{0}} = \frac{j}{2\pi k}$$

$$B = \int_0^1 e^{-j2\pi k t}\,dt = -\frac{1}{j2\pi k}\left[e^{-j2\pi k t}\right]_0^1 = -\frac{1}{j2\pi k}\left(\underbrace{\cancel{e^{-j2\pi k}}}_{0}-1\right) = 0$$

άρα $\quad a_k = 2A - B = \dfrac{2j}{2\pi k} = \dfrac{j}{\pi k}$

$$a_k = \begin{cases} 0 & k=0 \\ \dfrac{j}{\pi k} & k \neq 0 \end{cases}$$

(β) $\quad$ Θ. Parseval $\quad \sum_{k=-\infty}^{+\infty} |a_k|^2 = \frac{1}{T}\int_T |x(t)|^2 dt$

• $\sum_{k=-\infty}^{+\infty} |a_k|^2 = \sum_{k=1}^{+\infty}\left|\frac{j}{\pi k}\right|^2 + \sum_{k=-\infty}^{-1}\left|\frac{j}{\pi k}\right|^2 = 2\sum_{k=1}^{+\infty}\frac{1}{\pi^2 k^2} = \frac{2}{\pi^2}\sum_{k=1}^{+\infty}\frac{1}{k^2}$

• $\frac{1}{T}\int_0^1 (2t-1)^2 dt = \int_{-1}^{1} x^2 \frac{dx}{2} = \frac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3}$

$\left\langle \begin{array}{l} \text{Θέτω } x = 2t-1 \\ \Rightarrow dx = 2dt \end{array} \right\rangle$

άρα $\quad \frac{2}{\pi^2}\sum_{k=1}^{+\infty}\frac{1}{k^2} = \frac{1}{3} \Rightarrow \sum_{k=1}^{+\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$

(γ) $\quad e_n(t) = x(t) - \sum_{k=-n}^{n} a_k e^{-jk\frac{2\pi}{T}t} \qquad (T=1)$

$\Rightarrow \quad e_n(t) = \sum_{k=-\infty}^{+\infty} a_k e^{-jk2\pi t} - \sum_{k=-n}^{n} a_k e^{-jk2\pi t}$

Αν $\quad e_n(t) = \sum_{k=-\infty}^{+\infty} c_k e^{-j2\pi k t} \quad$ θα έχουμε:

$*$ Για $|k| \leq n \Rightarrow c_k = 0$

Για $|k| > n \Rightarrow c_k = a_k$

$$\Rightarrow c_k = \begin{cases} a_k & |k| > n \\ 0 & |k| \leq n \end{cases}$$

(δ) $E_e = ?$ για $n = 1, 5, 10$:

Για $n=1 \Rightarrow E_e = \sum_k |c_k|^2 = \sum_{k=-\infty}^{-2} |c_k|^2 + \sum_{k=2}^{+\infty} |c_k|^2$

Όμως $\underbrace{E_x = \sum_{k=-\infty}^{+\infty} |a_k|^2}_{\text{ενέργεια σήματος}} = E_e + |c_1|^2 + |c_{-1}|^2 = \frac{1}{3}$

άρα $E_e = E_x - \sum_{k=-1}^{1} |c_k|^2 = \frac{1}{3} - \frac{2}{\pi^2}$

$(c_0 = 0)$ $\left(|c_1| = |c_{-1}| = |a_1| = \frac{1}{\pi}\right)$

Ομοίως για $n = 5$:

$$E_e = \frac{1}{3} - 2\sum_{k=1}^{5} |a_k|^2 = \frac{1}{3} - 2\sum_{k=1}^{5} \frac{1}{\pi^2 k^2}$$

και για $n = 10$:

$$E_e = \frac{1}{3} - 2\sum_{k=1}^{10} \frac{1}{\pi^2 k^2}$$

ε) $E_e = \sum_{k=-\infty}^{+\infty} |a_k|^2 - \sum_{k=-n}^{n} |a_k|^2$

Για $n \to \infty$:

$$\lim_{n\to\infty} E_e = \sum_{k=-\infty}^{+\infty} |a_k|^2 - \lim_{n\to\infty} \sum_{k=-n}^{n} |a_k|^2 = 0$$

5.

$$F(\Omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\Omega t} dt = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2\sigma^2}} e^{-j\Omega t} dt$$

$$= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{+\infty} e^{-\left(\frac{t^2}{2\sigma^2} + j\Omega t\right)} dt$$

όπου: $\frac{t^2}{2\sigma^2} + j\Omega t = \left(\frac{t}{\sqrt{2}\sigma}\right)^2 + 2\left(\frac{t}{\sqrt{2}\sigma}\right)\left(j\frac{\sqrt{2}\sigma\Omega}{2}\right)$

$$= \left(\frac{t}{\sqrt{2}\sigma}\right)^2 + 2\left(\frac{t}{\sqrt{2}\sigma}\right)\left(j\frac{\sigma\Omega}{\sqrt{2}}\right) + \left(j\frac{\sigma\Omega}{\sqrt{2}}\right)^2 - \left(j\frac{\sigma\Omega}{\sqrt{2}}\right)^2$$

$$= \left(\frac{t}{\sqrt{2}\sigma} + j\frac{\sigma\Omega}{\sqrt{2}}\right)^2 + \frac{\sigma^2\Omega^2}{2}$$

άρα $F(\Omega)=\frac{1}{2\pi\sigma^2}\int_{-\infty}^{+\infty} e^{-\left(\frac{t}{\sqrt{2}\sigma}+j\frac{\sigma\Omega}{\sqrt{2}}\right)^2} e^{-\frac{\sigma^2\Omega^2}{2}}\, dt$

Για $x=\frac{t}{\sqrt{2}\sigma}+j\frac{\sigma\Omega}{\sqrt{2}} \Rightarrow dt=\sqrt{2}\sigma\, dx$ και:

$$F(\Omega)=\frac{1}{2\pi\sigma^2}\int_{-\infty}^{+\infty} e^{-x^2} e^{-\sigma^2\Omega^2/2}\sqrt{2}\sigma\, dx \Rightarrow$$

$$F(\Omega)=\frac{1}{\sqrt{2}\pi\sigma}\underbrace{\left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)}_{=\sqrt{\pi}} e^{-\sigma^2\Omega^2/2}$$

άρα $F(\Omega)=\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\sigma^2\Omega^2/2}$

6.

a. Για x[n]<->x[n-n_0], το 2ο μέλος => x[n-n_0-1]+x[n-n_0+1]
   Για n<->n-n_0, y[n-n_0]= x[n-n_0-1]+x[n-n_0+1]
   Άρα είναι χρονικά αμετάβλητα.

b. Για x[n]<->x[n-n_0], το 2ο μέλος => $a^n x[n-n_0-1]+x[n-n_0+1]$
   Για n<->n-n_0, $y[n-n_0]=a^{n-n_0}x[n-n_0-1]+x[n-n_0+1]$
   Άρα δεν είναι χρονικά αμετάβλητα.

7.

$$\nabla(\alpha\cdot x_1[n]+\beta\cdot x_2[n])=\alpha\cdot x_1[n]+\beta\cdot x_2[n]-\alpha\cdot x_1[n-1]-\beta\cdot x_2[n-1]$$
$$=\alpha\cdot(x_1[n]-x_1[n-1])+\beta\cdot(x_2[n]-x_2[n-1])=\alpha\cdot y_1[n]+\beta\cdot y_2[n]$$

Άρα είναι γραμμικό.

$y[n]=\nabla(x[n])=x[n]-x[n-1]$

Για x[n]<->x[n-n_0], το 2ο μέλος => x[n-n_0]+x[n-n_0-1]

Για n<->n-n_0, y[n-n_0]= x[n-n_0]+x[n-n_0-1]

Άρα είναι χρονικά αμετάβλητα.