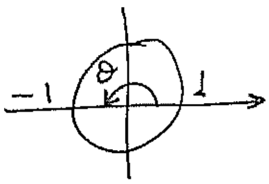
(1) a) $z_1 = e^4 e^{j\pi} = -e^4$ καί $e^{j\pi} = -1$

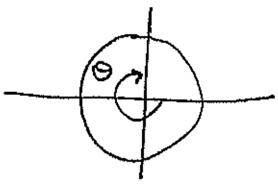


$\theta = \pi$

$\cos\theta = \cos\pi = -1$
$\sin\pi = 0$

$e^{j\pi} = \cos\pi + j\sin\pi = -1$

$z_2 = e^{-j\frac{3\pi}{2}}$



$\theta = -\frac{3\pi}{2} \Rightarrow \frac{\pi}{2}$

$\Big\}\ z_2 = e^{j\frac{\pi}{2}} = j$ καί:

$\left.\begin{array}{l} \cos\frac{\pi}{2} = 0 \\ \sin\frac{\pi}{2} = 1 \end{array}\right\} e^{j\pi/2} = \overset{0}{\cancel{\cos\frac{\pi}{2}}} + j\overset{1}{\cancel{\sin\frac{\pi}{2}}} = j$

$z_3 = (2-j)^4 = \left((2-j)^2\right)^2$

όπου $(2-j)^2 = 4 + j^2 - 4j = 3 - 4j \quad (j^2 = -1)$

άρα $\left((2-j)^2\right)^2 = (3-4j)^2 = 9 + \underbrace{16j^2}_{-16} - 24j = -7 - 24j$

$z_4 = 5j\,e^{\ln 3} = 15j$ καί $e^{\ln(x)} = x$

β)

$z_1 = \sqrt{3} + j$

$$\left.\begin{aligned} |z_1|^2 &= 3+1 = 4 \Rightarrow |z_1| = 2 \\ \phi_1 &= \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} \end{aligned}\right\} \quad z_1 = 2\,e^{j\pi/6}$$

$z_2 = e^{\pi} + j$

$$\left.\begin{aligned} |z_2|^2 &= e^{2\pi} + 1^2 = e^{2\pi} + 1 \\ \phi_2 &= \arctan\left(\frac{1}{e^{\pi}}\right) = \arctan\left(e^{-\pi}\right) \end{aligned}\right\} \quad z_2 = \left(e^{2\pi}+1\right) e^{j\tan^{-1}(e^{-\pi})}$$

② α)

$z_1 = \frac{x}{y} \qquad x = \rho_x e^{j\phi_x} \qquad y = \rho_y e^{j\phi_y}$

$$z_1 = \frac{\rho_x e^{j\phi_x}}{\rho_y e^{j\phi_y}} = \frac{\rho_x}{\rho_y} e^{j(\phi_x - \phi_y)} \Rightarrow \begin{cases} |z_1| = \rho_x / \rho_y \\ \phi_1 = \phi_x - \phi_y \end{cases}$$

$$z_2 = x \cdot y = \rho_x e^{j\phi_x} \rho_y e^{j\phi_y} = \rho_x \rho_y e^{j(\phi_x + \phi_y)}$$

$$\Rightarrow \begin{cases} |z_2| = \rho_x \cdot \rho_y \\ \phi_2 = \phi_x + \phi_y \end{cases}$$

β)

$x = a + j\beta \qquad y = \gamma + j\delta$

$$z_1 = \frac{a+j\beta}{\gamma+j\delta} = \frac{(a+j\beta)(\gamma-j\delta)}{\gamma^2+\delta^2} = \frac{a\gamma - ja\delta + j\beta\gamma + \beta\delta}{\gamma^2+\delta^2}$$

$$= \frac{a\gamma + \beta\delta}{\gamma^2+\delta^2} + j\,\frac{\beta\gamma - a\delta}{\gamma^2+\delta^2}$$

$$|z_1| = \sqrt{\left(\frac{a\gamma+\beta\delta}{\gamma^2+\delta^2}\right)^2 + \left(\frac{\beta\gamma-a\delta}{\gamma^2+\delta^2}\right)^2}$$

$$\phi_1 = \tan^{-1}\left(\frac{\beta\gamma-a\delta}{a\gamma+\beta\delta}\right)$$

$$z_2 = (a+j\beta)(\gamma+j\delta) = a\gamma + ja\delta + j\beta\gamma - \beta\delta = (a\gamma-\beta\delta) + j(a\delta+\beta\gamma)$$

$$|z_2| = \sqrt{(a\gamma-\beta\delta)^2+(a\delta+\beta\gamma)^2}$$

$$\phi_2 = \tan^{-1}\left(\frac{a\delta+\beta\gamma}{a\gamma-\beta\delta}\right)$$

β) $z_1 = \dfrac{4+j2}{\sqrt{2}-j}$

$x = 4+j2$ $\quad$ $\rho_x^2 = |x|^2 = 16+4 = 20 \Rightarrow \rho_x = 2\sqrt{5}$

$\phi_x = \tan^{-1}\left(\frac{1}{2}\right)$

$y = \sqrt{2}-j$ $\quad$ $\rho_y^2 = 2+1 = 3 \Rightarrow \rho_y = \sqrt{3}$

$\phi_y = \tan^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$ γιατί $\tan()$ είναι περιττή

δηλ. $\tan(-x) = -\tan(x)$

άρα $|z_1| = \dfrac{\rho_x}{\rho_y} = 2\sqrt{\dfrac{5}{3}}$ $\quad$ $\phi_1 = \phi_x - \phi_y = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$$z_2 = (0.5+j2)(2+j8) = x \cdot y$$

$a = 0.5 \quad \gamma = 2$

$\beta = 2 \quad \delta = 8$

Από το (a) έχουμε

$$\begin{cases} |z_2| = \sqrt{(a\gamma - \beta\delta)^2 + (a\delta + \beta\gamma)^2} \\ \phi_2 = \tan^{-1}\left(\dfrac{a\delta + \beta\gamma}{a\gamma - \beta\delta}\right) \end{cases}$$

$$\left.\begin{array}{l} a\gamma - \beta\delta = 1 - 16 = -15 \\ a\delta + \beta\gamma = 4 + 4 = 8 \end{array}\right\} \quad |z_2| = \sqrt{15^2 + 64^2} = \sqrt{289} = 17$$

$$\left.\begin{array}{l} a\delta + \beta\gamma = 4 + 4 = 8 \\ a\gamma - \beta\delta = 1 - 16 = -15 \end{array}\right\} \quad \phi_2 = \tan^{-1}\left(-\frac{8}{15}\right) = -\tan^{-1}\left(\frac{8}{15}\right)$$

(3) $z^n = r^n\left(\cos(n\theta) + j\sin(n\theta)\right)$

$$z = re^{j\theta} \Rightarrow z^n = r^n e^{jn\theta} = r^n\left(\cos(n\theta) + j\sin(n\theta)\right)$$

(4) $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

a)

$$\cos(2\theta) = \frac{e^{j2\theta} + e^{-j2\theta}}{2}$$

$$e^{j2\theta} = \left(e^{j\theta}\right)^2 = \left(\cos\theta + j\sin\theta\right)^2 = \cos^2\theta + j\,2\cos\theta\sin\theta - \sin^2\theta$$

$$\Rightarrow e^{j2\theta} = \left(\cos^2\theta - \sin^2\theta\right) + j\,2\cos\theta\sin\theta$$

$$\Rightarrow e^{-j2\theta} = \left(\cos^2\theta - \sin^2\theta\right) - j\,2\cos\theta\sin\theta$$

άρα $\cos(2\vartheta) = \frac{1}{2}\left(\cos^2\vartheta - \sin^2\vartheta + \cancel{j\,2\cos\vartheta\sin\vartheta} + \cos^2\vartheta - \sin^2\vartheta - \cancel{j\,2\cos\vartheta\sin\vartheta}\right)$

$\Rightarrow \quad \cos(2\vartheta) = \cos^2\vartheta - \sin^2\vartheta$

β) $\sin(2\vartheta) = 2\sin\vartheta\cos\vartheta$

$$\sin(2\vartheta) = \frac{e^{j2\vartheta} - e^{-j2\vartheta}}{2j} = \frac{\cancel{\cos^2\vartheta} - \cancel{\sin^2\vartheta} + j\,2\cos\vartheta\sin\vartheta - \cancel{\cos^2\vartheta} + \cancel{\sin^2\vartheta} + j\,2\cos\vartheta\sin\vartheta}{2j}$$

$\Rightarrow \quad \sin(2\vartheta) = 2\cos\vartheta\sin\vartheta$

γ) $\cos(3\vartheta) = \cos^3(\vartheta) - 3\sin^2(\vartheta)\cos(\vartheta) = 4\cos^3(\vartheta) - 3\cos\vartheta$

$$\cos(3\vartheta) = \frac{e^{j3\vartheta} + e^{-j3\vartheta}}{2}$$

$$e^{j3\vartheta} = e^{j2\vartheta}e^{j\vartheta} = \left((\cos^2\vartheta - \sin^2\vartheta) + j\,2\cos\vartheta\sin\vartheta\right)\cdot(\cos\vartheta + j\sin\vartheta)$$

$$= \cos^3\vartheta - \cos\vartheta\cdot\sin^2\vartheta + j\sin\vartheta\cos^2\vartheta - j\sin^3\vartheta$$

$$+ j\,2\cos^2\vartheta\sin\vartheta - 2\cos\vartheta\sin^2\vartheta$$

$$= \cos^3\vartheta - 3\cos\vartheta\sin^2\vartheta + j\left(3\sin\vartheta\cos^2\vartheta - \sin^3\vartheta\right)$$

$$e^{-j3\vartheta} = \cos^3\vartheta - 3\cos\vartheta\sin^2\vartheta - j\left(3\sin\vartheta\cos^2\vartheta - \sin^3\vartheta\right)$$

άρα $\cos(3\vartheta) = \cos^3\vartheta - 3\cos\vartheta\sin^2\vartheta$

$$\cos^2\theta + \sin^2\theta = 1 \Rightarrow \sin^2\theta = 1-\cos^2\theta$$

άρα

$$\cos(3\theta) = \cos^3\theta - 3\cos\theta(1-\cos^2\theta) = \cos^3\theta - 3\cos\theta + 3\cos^3\theta = 4\cos^3\theta - 3\cos\theta$$

5 a) $$\cos(j\theta) = \frac{e^{j(j\theta)}+e^{-j(j\theta)}}{2} = \frac{e^{-\theta}+e^{\theta}}{2} \in \mathbb{R}$$

β) $$i\sin(i\theta) = \cancel{i}\,\frac{e^{i(i\theta)}-e^{-i(i\theta)}}{2\cancel{i}} = \frac{e^{-\theta}-e^{\theta}}{2} \in \mathbb{R}$$