Nash Equilibria in Parallel Downloading with Multiple Clients*

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Abstract

Recently, the scheme of parallel downloading has been proposed as a novel approach to expedite the reception of a large file from the Internet. Experiments with a single client have shown that the client can improve its performance significantly by using the scheme. Simulations and experiments with multiple clients using the scheme have been conducted in [8, 9] to investigate the impact that this technique might have on the network if it is widely adopted. Contrast to the methodology used in [8, 9], we formulate parallel downloading as a non-cooperative game. Within this framework, we present a characterization of the traffic configuration at Nash equilibrium in a general network, and analyze its properties in a specific network. We also establish the dynamic convergence to equilibrium from an initial non-equilibrium state for a specific network. Finally, we investigate the efficiency of Nash equilibrium from the point of view of the clients and the system respectively, i.e., downloading latencies perceived by individual clients and total latencies over all connections. We find that although the traffic configuration at Nash equilibrium is optimal from the point of view of the clients, it may be bad from the point of view of the system.

1. Introduction

Recently, the scheme of parallel downloading has been proposed as a novel approach to expedite the reception of a large file from the Internet. With this scheme, a single client establishes concurrent connections to multiple servers with replicated content. Recent solutions for content deliver use the digital fountain encoding approach, which further facilitates the employment of the scheme since it can seamlessly accommodate connection migration and parallel transfer [3, 4]. Experiments with a single client have shown that the client can improve its performance significantly [16] by using the scheme. With the wide deployment of content distribution networks (CDNs) and peer-to-peer (P2P) networks, parallel downloading is expected to become more and more popular. To understand the impact that this technique might have on the network if it is widely adopted, simulations and experiments with multiple clients using the scheme have been conducted in [8, 9].

In large-scale communication networks, it is usually impossible to globally management the entire networks. Therefore, a reasonable assumption is that network users are rational and strategic participants who wish to minimize their own individual cost (such as latency) rather than acting selflessly to achieve some system-wide social optimum. This motivates the analysis of parallel downloading using models from non-cooperative game theory, in which an appropriate concept for the solution is Nash equilibrium. Based on the state of the network, clients change their behavior to minimize their own downloading latencies, and the change in behavior of one client is likely to cause changes in other clients' behavior. Under these assumptions, the process should reach the Nash equilibrium, in which unilateral deviation does not help any client to improve its performance.

Within this framework, we consider in this paper the parallel downloading problem, in which each client has a file download task and has to determine how to split the task between the servers with the complete copies of the file. If a client is slow in obtaining "the last few packets" from some server, the client still cannot reconstruct the original content even if all other packets have arrived earlier. Therefore, it is natural that every rational client would aim to minimize its own maximal downloading latency among all servers to which the download task is assigned. Under such objective functions, we present a characterization of the traffic configuration at Nash equilibrium in a general network, and analyze its properties in a specific network.

In general, the dynamic system is not initially in equilib-

* This work is supported by a grant from the Ministry of Science and Technology (grant ♯2001CCA03000), National Natural Science Fund (grant ♯60273045) and Shanghai Science and Technology Development Fund (grant ♯025115032)

rium, nor is there any coordination among clients to reach equilibrium. What they do is just to use best response actions to minimize their own downloading latencies based on the state of the network. On the other hand, even if every client starts at equilibrium, clients may join or leave the system. It is therefore important to know whether the best response actions can converge to equilibrium. We follow the Elementary Stepwise System (ESS) introduced in [1, 13] and establish the dynamic convergence to equilibrium from an initial non-equilibrium state for a specific network.

It is well known that Nash equilibria do not always optimize the overall performance of the system. In order to quantify the efficiency of Nash equilibrium, we study the coordination ratio, initially investigated in [11], which is the ratio between the worst possible Nash equilibrium and the overall optimum. We consider two types of social optimum from the point of view of the clients and the system respectively, i.e., downloading latencies perceived by individual clients and total latencies over all connections. We find that although the traffic configuration at Nash equilibrium is optimal from the point of view of the clients, it may be bad from the point of view of the system.

The rest of this paper is organized as follows. Section 2 reviews some related work. Section 3 introduces formally the problem of parallel downloading game. We present the characterization and properties of Nash equilibrium in section 4. We establish convergence to Nash equilibrium and investigate its efficiency in section 5 and section 6 respectively. Finally, section 7 concludes our paper.

2. Related work

The idea of parallel access to multiple servers with replicated content has been proposed in the recent literature ([3, 15, 16]). Rodriguez et al. [16] introduce a dynamic parallel-access scheme to access multiple mirror servers, and find that their scheme achieves significant downloading speedup from the perspective of a single client. As parallel downloading is becoming more and more popular, it is imperative to understand the impact if multiple clients use it. More recently, Gkantsidis et al. [8] investigate experimentally the performance of parallel downloading when performed by multiple clients. Simulative and analytical results in [9] have shown that while the parallel downloading scheme may achieve a shorter downloading time, its impact on the network and server is significant. But analytical models established in [9] are under the simple assumption that each client is requesting $1/K$ of the file from each of the K servers. Contrast to models established in [9], we formulate parallel downloading as a non-cooperative game, which is essentially a multi-user routing game.

The questions of existence, uniqueness, stability, and efficiency of Nash equilibria for non-cooperative routing have been studied over various networking settings. Orda et al. [13] consider the competitive routing problem, where there are finitely many users, each of whom controls a non-negligible amount of flow. They investigate existence, uniqueness and properties of Nash equilibria, and analyze stability briefly. Based on the model in [13], Altman et al. [1] study stability under several dynamic policy adjustment schemes. Rather than just showing a convergence in the limit, Even-Dar et al. [7] are concerned with the time it takes for the system to converge to equilibrium and provide quantitative bounds in a load balancing scenario. To quantify the efficiency of the Nash equilibrium, Koutsoupias and Papadimitriou [11] initially studied the coordination ratio for a network topology of parallel links with linear cost functions, and the results in [11] have been later largely extended in [5, 6, 12]. In the traffic routing model of Wardrop[21] where there are infinitely many users, each controlling a negligible fraction of the overall traffic and pursuing a minimum latency path, Roughgarden et al. [17, 18, 19] study the cost of selfish routing and determine the worst-case inefficiency of equilibria.

Similar to the competitive routing problem in [13], we consider the problem of parallel downloading game, where there are n clients, each of whom has a non-negligible amount of traffic with the aim at minimizing the maximal latency over all connections to which it has assigned positive traffic.

3. Model

We consider a general network $G = (V, E)$ with node set V, link set E. Let $T(T \subseteq V)$ be the set of n clients and $S(S \subseteq V)$ be the set of m servers. A fixed path (connection) from each server $s_i \in S(1 \leq i \leq m)$ to each client $t_j \in T(1 \leq j \leq n)$, denoted by P_j^i, is given (Since a client typically issues several requests to the same server during the download of a file, TCP persistent connections are used between the client and every server to minimize the overhead of opening multiple TCP connections [14, 16].). We denote the set of connections to t_j by $P_j = \bigcup_i P_j^i$ and denote the set of all connections by $P = \bigcup_j P_j$. Each client t_j has an amount of traffic w_j (i.e., a download task with length w_j), and is able to determine at any time how its traffic is assigned among each server s_i. That is, client t_j determines what fraction of w_j should be transferred through each connection P_j^i. We denote by f_j^i the fractional traffic assigned to server s_i by client t_j. The traffic configuration $\mathbf{f}_j$ of client t_j is the vector $\mathbf{f}_j$=$(f_j^1, f_j^2, \cdots, f_j^m)$ and the system traffic configuration **f** is the vector of all clients traffic configurations, **f**=($\mathbf{f}_1$, $\mathbf{f}_2$, $\cdots$, $\mathbf{f}_n$). We define indicator variables $I_j^i \in \{0, 1\}$ for each connection P_j^i, such that $I_j^i = 1$ if and only if $f_j^i > 0$. A system traffic configuration **f** is said to be feasible if it satisfies the following two constraints:

- (R1) Non-negativity constraints: $f_j^i \geq 0,\ \ \forall i, j$;
- (R2) Demand constraints: $\sum_{i=1}^{m} f_j^i = w_j,\ \ \forall j$.

We denote by $\mathbf{F}$ the set of all feasible $\mathbf{f}$'s and $\mathbf{F}_j$ the set of all configuration $\mathbf{f}_j$'s of client t_j that lead to feasible configuration $\mathbf{f}$'s. For a fixed traffic configuration $\mathbf{f}$, the load f_e on each link $e \in E$ is the total traffic being routed over it and the load f_i on each server $s_i \in S$ is the total traffic assigned to it. Then, $f_e = \sum_{P_j^i : e \in P_j^i, P_j^i \in P} f_j^i$ and $f_i = \sum_{j=1}^{n} f_j^i$.

Finally, Each link $e \in E$ and server $s_i \in S$ are given a load-dependent latency function that we denote by $l_e(\cdot)$ and $l_i(\cdot)$ respectively. We assume that the latency functions $l_e(\cdot)$ and $l_i(\cdot)$ are nonnegative, continuous, and nondecreasing. The latency over a connection P_j^i with respect to a configuration $\mathbf{f}$, denoted by $L_j^i(\mathbf{f})$, is defined as the sum of the latencies of the links on P_j^i and the latency on server s_i. That is, $L_j^i(\mathbf{f}) = l_i(f_i) + \sum_{e \in P_j^i} l_e(f_e)$. The cost function of a client $t_j \in T$, denoted by $J_j(\mathbf{f})$, is defined as the maximal latency over all connections to which it has assigned positive traffic. That is, $J_j(\mathbf{f}) = \max_i L_j^i(\mathbf{f}) I_j^i$.

The aim of each client is to minimize its cost. Since the cost functions depend on the traffic assignment of all clients, the optimal decision of each client depends on the decisions made by other clients. We formulate this problem as a non-cooperative game since each client is rational and selfish. The concept of the solution is that of Nash equilibrium, i.e., we seek a feasible traffic configuration $\mathbf{f}$ such that no client finds it beneficial to change its traffic on any connection. A formal definition is as follows.

Definition 1 A feasible system traffic configuration $\mathbf{f}$=($\mathbf{f}_1$, $\mathbf{f}_2$, $\cdots$, $\mathbf{f}_n$) is at Nash equilibrium if for all $t_j \in T$, the following condition holds:

$$\begin{aligned} J_j(\mathbf{f}) &= J_j(\mathbf{f}_1, \cdots, \mathbf{f}_{j-1}, \mathbf{f}_j, \mathbf{f}_{j+1}, \cdots, \mathbf{f}_n) \\ &= \min_{\tilde{\mathbf{f}}_j \in \mathbf{F}_j} J_j(\mathbf{f}_1, \cdots, \mathbf{f}_{j-1}, \tilde{\mathbf{f}}_j, \mathbf{f}_{j+1}, \cdots, \mathbf{f}_n). \end{aligned} \qquad (1)$$

4. Characterizations and properties of Nash equilibrium

In this section, we first present a characterization of the traffic configuration at Nash equilibrium in a general network, and then establish existence and essentially uniqueness of Nash equilibrium using the characterization. Finally, we analyze its properties in a specific network.

4.1. Characterizing traffic configurations at Nash equilibrium

We present a characterization of traffic configurations at Nash equilibrium in the following theorem.

Theorem 1 *A feasible system traffic configuration* $\boldsymbol{f}$ *will be at Nash equilibrium if and only if for every* $t_j \in T$ *and* $P_j^1, P_j^2 \in P_j$ *with* $f_j^1 > 0,\ \ L_j^1(\boldsymbol{f}) \leq L_j^2(\boldsymbol{f})$.

Proof: We first prove the necessary condition by contradiction argument. Assume that there exists a client $t_j \in T$ and $P_j^1, P_j^2 \in P_j$ with $f_j^1 > 0$ and $L_j^1(\mathbf{f}) > L_j^2(\mathbf{f})$.

Note that $J_j(\mathbf{f}) = \max_i L_j^i(\mathbf{f}) I_j^i$. We partition the set S into three subsets: S_A, S_B, and S_C where $S_A = \{s_i \in S | L_j^i(\mathbf{f}) < J_j(\mathbf{f})\}$, $S_B = \{s_i \in S | L_j^i(\mathbf{f}) I_j^i = J_j(\mathbf{f})\}$, and $S_C = \{s_i \in S | L_j^i(\mathbf{f}) \geq J_j(\mathbf{f}) \wedge I_j^i = 0\}$. We can easily see that $s_2 \in S_A$ and the set S_B is not empty since $f_j^1 > 0$ and $L_j^1(\mathbf{f}) > L_j^2(\mathbf{f})$. But the set S_C may be empty.

We now construct a new traffic configuration $\tilde{\mathbf{f}}$ as follows:

- For all $t_{j'} \in T$ and $t_{j'} \neq t_j$, $\ \tilde{\mathbf{f}}_{j'} = \mathbf{f}_{j'}$;
- For t_j , we construct the configuration $\tilde{\mathbf{f}}_j$ as:
  - for all $s_a \in S_A$, $\ \tilde{f}_j^a = f_j^a + \delta_a\ \ (\delta_a > 0)$;
  - for all $s_b \in S_B$, $\ \tilde{f}_j^b = f_j^b - \delta_b\ \ (\delta_b > 0)$;
  - for all $s_c \in S_C$, $\ \tilde{f}_j^c = f_j^c$.

We choose $\delta_a' s$ and $\delta_b' s$ such that (2), (3), (4) are satisfied.

$$\begin{aligned} \sum_{s_a \in S_A} \delta_a &= \sum_{s_b \in S_B} \delta_b && (2) \\ L_j^a(\tilde{\mathbf{f}}) &< J_j(\mathbf{f}) \quad \forall s_a \in S_A && (3) \\ L_j^b(\tilde{\mathbf{f}}) &< J_j(\mathbf{f}) \quad \forall s_b \in S_B && (4) \end{aligned}$$

It is clear that $\tilde{\mathbf{f}}$ is a feasible traffic configuration and that $J_j(\tilde{\mathbf{f}}) = \max_i L_j^i(\tilde{\mathbf{f}}) I_j^i < J_j(\mathbf{f})$, which contradicts the fact that the client t_j has no better traffic configuration to strictly decrease its maximal downloading latency when $\mathbf{f}$ is at Nash equilibrium.

We now prove the other direction. From the condition, we get that for every $t_j \in T$ and for all $P_j^{i'}, P_j^{i''} \in P_j$ with $f_j^{i'} > 0, f_j^{i''} > 0,\ \ L_j^{i'}(\mathbf{f}) = L_j^{i''}(\mathbf{f})$. Therefore, no client will change its traffic configuration, which means that $\mathbf{f}$ is at Nash equilibrium. ■

From Theorem 1, we know that if $\mathbf{f}$ is the traffic configuration at Nash equilibrium, the client $t_j \in T$ experiences the same latency, denoted by $L_j(\mathbf{f})$, over all connections P_j^i with $f_j^i > 0$.

Note that the parallel downloading game we consider is essentially a nonatomic congestion game and the characterization of Nash equilibrium here is right the definition of that in [20]. Therefore, we can establish the existence and essentially uniqueness of Nash equilibrium from this characterization.

Theorem 2 *([2, 18])There exists a traffic configuration at Nash equilibrium in a network G with nonnegative, continuous, and nondecreasing latency functions. Moreover, if* $\boldsymbol{f}$

and $\tilde{f}$ are traffic configurations at Nash equilibrium, then $l_e(f_e) = l_e(\tilde{f}_e)$ for all $e \in E$ and $l_i(f_i) = l_i(\tilde{f}_i)$ for all $s_i \in S$.

4.2. Properties of Nash equilibrium

In this subsection, we analyze properties of Nash equilibrium in a specific network $G1 = (V, E)$ where each path P_j^i connecting a server $s_i \in S(1 \leq i \leq m)$ with a client $t_j \in T(1 \leq j \leq n)$ consists of only one (disjoint) link. Moreover, the latency function $l_e(\cdot)$ for each link $e \in E$ is the same and strictly increasing while latency functions for servers can be different.

Theorem 3 *A traffic configuration $\boldsymbol{f}$ at the Nash equilibrium in $G1$ has the following two properties:*

(1) If $f_a^{i_0} > f_b^{i_0}$ for a certain server $s_{i_0} \in S$ and clients $t_a, t_b \in T$, then $f_a^i \geq f_b^i$ holds for all $s_i \in S$. Moreover, if $f_b^{i_0} > 0$ then $f_a^i > f_b^i$.

(2) If $f_a^{i_1} > f_a^{i_2}$ for a certain client $t_a \in T$ and servers $s_{i_1}, s_{i_2} \in S$, then $f_b^{i_1} \geq f_b^{i_2}$ holds for all $t_b \in T$. Moreover, if $f_b^{i_2} > 0$ then $f_b^{i_1} > f_b^{i_2}$.

Proof: (1) Considering $\forall s_i \in S$.

For the case $f_b^i = 0$, it is clear that $f_a^i \geq f_b^i$.

For the case $f_b^i > 0$, we have $L_b^i(\mathbf{f}) \leq L_b^{i_0}(\mathbf{f})$ according to Theorem 1. That is

$$l_i(f_i) + l_e(f_b^i) \leq l_{i_0}(f_{i_0}) + l_e(f_b^{i_0}). \tag{5}$$

Since $f_a^{i_0} > f_b^{i_0}$, it is clear that $f_a^{i_0} > 0$. Thus, we get $L_a^{i_0}(\mathbf{f}) \leq L_a^i(\mathbf{f})$. That is

$$l_{i_0}(f_{i_0}) + l_e(f_a^{i_0}) \leq l_i(f_i) + l_e(f_a^i). \tag{6}$$

Because the latency function $l_e(\cdot)$ is strictly increasing, we have

$$l_{i_0}(f_{i_0}) + l_e(f_b^{i_0}) < l_{i_0}(f_{i_0}) + l_e(f_a^{i_0}). \tag{7}$$

From (5), (6), and (7), we get

$$l_i(f_i) + l_e(f_b^i) < l_i(f_i) + l_e(f_a^i). \tag{8}$$

Therefore $l_e(f_b^i) < l_e(f_a^i)$ which implies that $f_b^i < f_a^i$.

(2) Considering $\forall t_b \in T$.

For the case $f_b^{i_2} = 0$, it is clear that $f_b^{i_1} \geq f_b^{i_2}$.

For the case $f_b^{i_2} > 0$, we have $L_b^{i_2}(\mathbf{f}) \leq L_b^{i_1}(\mathbf{f})$, i.e.,

$$l_{i_2}(f_{i_2}) + l_e(f_b^{i_2}) \leq l_{i_1}(f_{i_1}) + l_e(f_b^{i_1}). \tag{9}$$

Also, since $f_a^{i_1} > 0$, we have $L_a^{i_1}(\mathbf{f}) \leq L_a^{i_2}(\mathbf{f})$, i.e.,

$$l_{i_1}(f_{i_1}) + l_e(f_a^{i_1}) \leq l_{i_2}(f_{i_2}) + l_e(f_a^{i_2}). \tag{10}$$

Moreover, since $f_a^{i_1} > f_a^{i_2}$, we have $l_e(f_a^{i_1}) > l_e(f_a^{i_2})$. Then, from (10) we get

$$l_{i_1}(f_{i_1}) < l_{i_2}(f_{i_2}). \tag{11}$$

Finally, from (9) and (11), we have $l_e(f_b^{i_2}) < l_e(f_b^{i_1})$ which implies that $f_b^{i_2} < f_b^{i_1}$. ■

Theorem 4 *[13] A traffic configuration $\boldsymbol{f}$ is at the Nash equilibrium in $G1$. For any clients $t_a, t_b \in T, w_a \geq w_b$ implies that $f_a^i \geq f_b^i$ for all $s_i \in S$ and if $w_a > w_b$ then $f_a^i = f_b^i$ only for $f_a^i = f_b^i = 0$.*

Corollary 1 *[13] A traffic configuration $\boldsymbol{f}$ is at the Nash equilibrium in $G1$. For all clients $t_a, t_b \in T, w_a = w_b$, holds $f_a^i = f_b^i$ for all $s_i \in S$.*

5. Convergence to Nash equilibrium

In this section we consider the stability of Nash equilibrium using the Elementary Stepwise System (ESS) introduced in [1, 13], in which clients update their actions one after the other, in the order $1, 2, \cdots, n, 1, 2, \cdots, n$, etc, and where at each update a client uses the best response action against the actions of the other clients.

Relabel the clients so that client t_1 is the first to update, client t_2 the second and so on. The system starts with a initial feasible configuration denoted by $\mathbf{f}(0) = (\mathbf{f}_1(0), \mathbf{f}_2(0), \cdots, \mathbf{f}_n(0))$. The resulting traffic configuration after the first step is denoted by $\mathbf{f}(1) = (\mathbf{f}_1(1), \mathbf{f}_2(0), \cdots, \mathbf{f}_n(0))$ where $\mathbf{f}_1(1)$ is optimal traffic configuration against $\mathbf{f}_2(0), \cdots, \mathbf{f}_n(0)$. At the i^{th} step client t_i updates its traffic configuration and the result is $\mathbf{f}(i) = (\mathbf{f}_1(1), \cdots, \mathbf{f}_i(1), \mathbf{f}_{i+1}(0), \cdots, \mathbf{f}_n(0))$. At the $(kn + j)^{th}$ step client t_j updates and the resulting traffic configuration is $\mathbf{f}(kn + j) = (\mathbf{f}_1(k+1), \cdots, \mathbf{f}_j(k+1), \mathbf{f}_{j+1}(k), \cdots, \mathbf{f}_n(k))$.

In the following, we investigate the stability of Nash equilibrium in a specific network.

We consider a specific network $G2$ which is a simpler version of $G1$ with two servers s_1, s_2 and n clients $t_1, t_2, \cdots, t_n$, each client having the same traffic demand w. The latency functions for the two servers s_1, s_2 are given by $l_1(x)$ and $l_2(x)$ respectively, where $l_1(x) = a_1 x$ and $l_2(x) = a_2 x (a_1, a_2 > 0)$. Each link e has the same latency function given by $l_e(x) = bx (b > 0)$.

We first compute the Nash equilibrium in $G2$.

Theorem 5 *There exists a unique traffic configuration $\boldsymbol{f}(E)$ at Nash equilibrium in $G2$. Moreover, at the Nash equilibrium, each client $t_j (1 \leq j \leq n)$ has the same traffic configuration given by*

$$\begin{aligned} \boldsymbol{f}_j(E) &= (f_j^1(E), f_j^2(E)) \\ &= \left(\frac{(a_2 n + b)w}{(a_1 + a_2)n + 2b}, \frac{(a_1 n + b)w}{(a_1 + a_2)n + 2b} \right). \end{aligned}$$

Proof: The existence and uniqueness of the Nash equilibrium in $G2$ follows directly from Theorem 2.

Assume that $\mathbf{f}(E)$ is at the Nash equilibrium in $G2$.

For each client t_j such that $0 < f_j^1 < w$, from Theorem 1, we have

$$a_1 \sum_{i=1}^{n} f_i^1(E) + b f_j^1(E) = a_2 \sum_{i=1}^{n} f_i^2(E) + b f_j^2(E). \quad (12)$$

Since all clients have the same traffic demand w, we know that they have the same traffic configuration at the Nash equilibrium $\mathbf{f}(E)$ from Corollary 1, that is,

$$f_1^1(E) = f_2^1(E) = \cdots = f_n^1(E). \quad (13)$$

According to the Demand constraints $R2$, we have

$$f_j^1(E) + f_j^2(E) = w \quad \forall j = 1, 2, \cdots, n. \quad (14)$$

From (12), (13), and (14), we have

$$\begin{aligned} \mathbf{f}_j(E) &= (f_j^1(E), f_j^2(E)) \\ &= \left(\frac{(a_2 n + b) w}{(a_1 + a_2) n + 2b}, \frac{(a_1 n + b) w}{(a_1 + a_2) n + 2b} \right). \end{aligned}$$

■

Now we prove that the ESS do converge to the Nash equilibrium $\mathbf{f}(E)$ in $G2$ from an arbitrary initial feasible configuration $\mathbf{f}(0)$.

Considering $(kn + j)^{th}$ step, client t_j uses the best response action to update its traffic configuration against $\mathbf{f}_1(k+1), \cdots, \mathbf{f}_{j-1}(k+1), \mathbf{f}_{j+1}(k), \cdots, \mathbf{f}_n(k)$. That is, client t_j configures its traffic $\mathbf{f}_j(k+1)$ such that

$$\begin{aligned} & a_1 \sum_{i=1}^{j} f_i^1(k+1) + a_1 \sum_{i=j+1}^{n} f_i^1(k) + b f_j^1(k+1) \\ = \; & a_2 \sum_{i=1}^{j} (w - f_i^1(k+1)) + a_2 \sum_{i=j+1}^{n} (w - f_i^1(k)) + \\ & b(w - f_j^1(k+1)). \end{aligned} \quad (15)$$

From (15), we have

$$\begin{aligned} f_j^1(k+1) \;=\; & \frac{(a_2 n + b) w}{a_1 + a_2 + 2b} - \\ & \frac{a_1 + a_2}{a_1 + a_2 + 2b} \sum_{i=1}^{j-1} f_i^1(k+1) - \\ & \frac{a_1 + a_2}{a_1 + a_2 + 2b} \sum_{i=j+1}^{n} f_i^1(k). \end{aligned} \quad (16)$$

Define

$$\begin{aligned} \Delta f_j^1(k) &\equiv f_j^1(k) - f_j^1(E) \\ &= f_j^1(k) - \frac{(a_2 n + b) w}{(a_1 + a_2) n + 2b}. \end{aligned} \quad (17)$$

From (16) and (17), we obtain

$$\begin{aligned} f_j^1(k+1) \;=\; & \frac{(a_2 n + b) w}{(a_1 + a_2) n + 2b} - \\ & \frac{a_1 + a_2}{a_1 + a_2 + 2b} \sum_{i=1}^{j-1} \Delta f_i^1(k+1) - \\ & \frac{a_1 + a_2}{a_1 + a_2 + 2b} \sum_{i=j+1}^{n} \Delta f_i^1(k). \end{aligned} \quad (18)$$

Therefore,

$$\begin{aligned} \Delta f_j^1(k+1) \;&=\; f_j^1(k+1) - f_j^1(E) \\ &=\; -\frac{a_1 + a_2}{a_1 + a_2 + 2b} \sum_{i=1}^{j-1} \Delta f_i^1(k+1) - \\ &\quad\; \frac{a_1 + a_2}{a_1 + a_2 + 2b} \sum_{i=j+1}^{n} \Delta f_i^1(k). \end{aligned} \quad (19)$$

Define vector

$$\Delta f^1(k) = \left(\Delta f_1^1(k), \Delta f_2^1(k), \cdots, \Delta f_n^1(k) \right).$$

Let $\rho = \frac{a_1 + a_2 + 2b}{a_1 + a_2}$ $(\rho > 1)$. Then,

$$\begin{aligned} \Delta f^1(k+1) = & -\frac{1}{\rho} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \Delta f^1(k+1) \\ & -\frac{1}{\rho} \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Delta f^1(k) \\ = & -\frac{1}{\rho} \times A \times \Delta f^1(k+1) - \\ & \frac{1}{\rho} \times B \times \Delta f^1(k). \end{aligned} \quad (20)$$

Let I be the identity matrix of order n. Then,

$$\Delta f^1(k+1) = -\left(\rho \times I + A\right)^{-1} B \Delta f^1(k). \quad (21)$$

Theorem 6 *The Elementary Stepwise System converges to the Nash equilibrium in $G2$.*

Proof: The sufficient condition for the ESS to converge to the Nash equilibrium in $G2$ is that $\lim_{k \to \infty} \Delta f^1(k+1) = \vec{0}$ ([1]). That is, $\lim_{k \to \infty} \left(-\left(\rho I + A\right)^{-1} B \right)^k = \vec{0}$. Therefore, all left is to show that all eigenvalues of $-\left(\rho I + A\right)^{-1} B$ is in the interior of the unit circle. Let λ be the eigenvalue of $-\left(\rho I + A\right)^{-1} B$, then we have

$$\left| (\rho I + A)^{-1} B + \lambda I \right| = 0. \quad (22)$$

We get $|B + \rho\lambda I + \lambda A| = 0$, i.e.,

$$\begin{vmatrix} \rho\lambda & 1 & 1 & \cdots & 1 & 1 \\ \lambda & \rho\lambda & 1 & \cdots & 1 & 1 \\ \lambda & \lambda & \rho\lambda & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda & \lambda & \lambda & \cdots & \rho\lambda & 1 \\ \lambda & \lambda & \lambda & \cdots & \lambda & \rho\lambda \end{vmatrix} = 0. \tag{23}$$

It is clear that $\lambda = 1$ is not a solution of (23). So we assume in the following that $\lambda \neq 1$.

Simplify (23), we get

$$\frac{\lambda^n(\rho-1)^n - \lambda(\rho\lambda-1)^n}{1-\lambda} = 0. \tag{24}$$

Now we prove that all zeroes of (24) are in the interior of the unit circle.

Let $g_1(\lambda) = \lambda^n(\rho-1)^n$ and $g_2(\lambda) = -\lambda(\rho\lambda-1)^n$. We consider a closed contour C consisting of an arc $C1$ and an arc $C2$ shown in Figure 1.

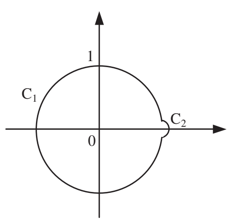


Figure 1. A closed contour C consisting of an arc $C1$ and an arc $C2$.

It is clear that the functions $g_1(\lambda)$ and $g_2(\lambda)$ are analytic on and inside C.

We consider the following two cases.

Case 1: $\lambda \in C_1$.

Let $\lambda = \cos\theta + i\sin\theta \;\; (0 < \theta < 2\pi)$. Then,

$$|g_1(\lambda)| = |\lambda^n(\rho-1)^n| = (\rho-1)^n. \tag{25}$$

$$\begin{aligned} |g_2(\lambda)| &= |-\lambda(\rho\lambda-1)^n| \\ &= \left(\sqrt{(\rho\cos\theta-1)^2 + (\rho\sin\theta)^2}\right)^n \\ &> (\rho-1)^n. \end{aligned} \tag{26}$$

Case 2: $\lambda \in C_2$.

Let $\lambda = \gamma\cos\beta + i\gamma\sin\beta \;\; (\gamma > 1,\;\; \beta \geq 0)$. Then,

$$|g_1(\lambda)| = |\lambda^n(\rho-1)^n| = (\rho\gamma-\gamma)^n. \tag{27}$$

$$\begin{aligned} |g_2(\lambda)| &= |-\lambda(\rho\lambda-1)^n| \\ &= \gamma\left(\sqrt{(\rho\gamma\cos\beta-1)^2 + (\rho\gamma\sin\beta)^2}\right)^n \\ &> \left(\sqrt{(\rho\gamma\cos\beta-1)^2 + (\rho\gamma\sin\beta)^2}\right)^n \\ &\geq (\rho\gamma-1)^n \\ &> (\rho\gamma-\gamma)^n. \end{aligned} \tag{28}$$

Therefore, we have $|g_2(\lambda)| > |g_1(\lambda)|$ for all $\lambda \in C$.

Now we apply Rouche's Theorem to the contour C and get that the functions $g_2(\lambda)$ and $g_1(\lambda)+g_2(\lambda)$ have the same number of zeroes inside C. Since $g_2(\lambda)$ has $n+1$ zeroes inside the unit circle, we get that all zeroes of (24) are in the interior of the unit circle. ■

6. Efficiency of Nash equilibrium

In this section, we study the coordination ratio, which is the ratio between the worst possible Nash equilibrium and the social optimum. We consider two types of social optimum from the point of view of the clients and the system respectively, i.e., downloading latencies perceived by individual clients and total latencies over all connections. Formally, we define the following two types of social cost functions of a traffic configuration **f** in G:

Type A: the maximal latency over all connections, that is,

$$C_A(\mathbf{f}) = \max_{1\leq i\leq m, 1\leq j\leq n} I_j^i L_j^i(\mathbf{f}).$$

Type B: the total latencies over all connections, that is,

$$C_B(\mathbf{f}) = \sum_{j=1}^{n}\sum_{i=1}^{m} I_j^i L_j^i(\mathbf{f}).$$

We assume that an optimal configuration minimizes the social cost. That is, for the cost function of Type A, we define

$$OPT_A = \min_{\mathbf{f}^*\in\mathbf{F}} C_A(\mathbf{f}^*) = \min_{\mathbf{f}^*\in\mathbf{F}}\left(\max_{1\leq i\leq m, 1\leq j\leq n} I_j^i L_j^i(\mathbf{f}^*)\right).$$

For the cost function of Type B, we define

$$OPT_B = \min_{\mathbf{f}^*\in\mathbf{F}} C_B(\mathbf{f}^*) = \min_{\mathbf{f}^*\in\mathbf{F}} \sum_{j=1}^{n}\sum_{i=1}^{m} I_j^i L_j^i(\mathbf{f}^*).$$

Let $\mathbf{f}(E)$ be the traffic configuration at the Nash equilibrium in G. We denote the coordination ratio against the cost functions of Type A and Type B by R_A and R_B respectively (Note that the Nash equilibrium is not influenced by how our social cost functions are defined), i.e.,

$$R_A = \frac{C_A(\mathbf{f}(E))}{OPT_A}$$

and

$$R_B = \frac{C_B(\mathbf{f}(E))}{OPT_B}.$$

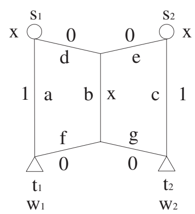


Figure 2. Example the performance loss of the traffic configuration at Nash equilibrium from the point of view of the system.

Theorem 7 *In a network G with nonnegative, continuous, and nondecreasing latency functions, $R_A = 1$.*

Proof: According to the definition of the Nash equilibrium, each client $t_j \in T$ minimizes its maximal latency over all connections to which it has assigned positive traffic. As a result, the maximal latency over all connections is minimized. Therefore, if $\mathbf{f}(E)$ is the traffic configuration at the Nash equilibrium in G, then $C_A(\mathbf{f}(E)) = OPT_A$, i.e., $R_A = 1$. ■

The above Theorem shows that the traffic configuration at Nash equilibrium is an optimal one from the point of view of the clients. But it may not be an optimal one from the point of view of the system. Figure 2 shows an example. In Figure 2, the two given connections for client t_1 are $s_1 - a - t_1$ and $s_2 - e - b - f - t_1$, and those for client t_2 are $s_2 - c - t_2$ and $s_1 - d - b - g - t_2$. The latency functions for the two servers s_1, s_2, and the seven links a, b, c, d, e, f and g are described as follows: $l_1(x) = x, l_2(x) = x, l_a(x) = 1, l_b(x) = x, l_c(x) = 1, l_d(x) = 0, l_e(x) = 0, l_f(x) = 0$, and $l_g(x) = 0$. The traffic demands w_1 and w_2 of the two clients t_1 and t_2 are both 1. It is easy to see that if the traffic configuration $\mathbf{f}$ is at Nash equilibrium, i.e., $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) = ((1/2, 1/2), (1/2, 1/2))$, both clients t_1 and t_2 experience the minimal latency of 2. However, at another traffic configuration $\tilde{\mathbf{f}}$ where $\tilde{\mathbf{f}} = (\tilde{\mathbf{f}}_1, \tilde{\mathbf{f}}_2) = ((1, 0), (0, 1))$, both clients also experience the minimal latency of 2, without using the parallel downloading scheme. Note that the configuration $\tilde{\mathbf{f}}$ is not at Nash equilibrium since each client can change its traffic configuration to make its maximal latency decreased. The main difference between the two configurations lies in that each client establishes more connections at the configuration $\mathbf{f}$ than at $\tilde{\mathbf{f}}$. This suggests that, in the worst case, the scheme of parallel downloading does not necessarily do better than that of downloading from a single server from the point of view of the clients, but it may do worse from the point of view of the system since it consumes more resources (such as connections). The following Theorem 8 further quantifies the performance loss from the point of view of the system.

Lemma 1 *In a network G with nonnegative, continuous, and nondecreasing latency functions, if $\tilde{\boldsymbol{f}}$ is an arbitrary feasible traffic configuration and $\boldsymbol{f}(E)$ is that at Nash equilibrium, then*

$$\sum_{j=1}^{n} L_j(\boldsymbol{f}(E)) \leq \sum_{j=1}^{n} \sum_{i=1}^{m} I_j^i L_j^i(\tilde{\boldsymbol{f}}).$$

Proof: It follows directly from the definition of Nash equilibrium where each client $t_j \in T$ minimizes its maximal latency over all connections to which it has assigned positive traffic. ■

Theorem 8 *In a network G with nonnegative, continuous, and nondecreasing latency functions, $R_B \leq m$ where m is the number of servers. Moreover, the bound is tight.*

Proof: Assume that $\mathbf{f}(E)$ is the traffic configuration at Nash equilibrium and $\mathbf{f}^*$ is the optimal traffic configuration about the cost function of Type B. Then,

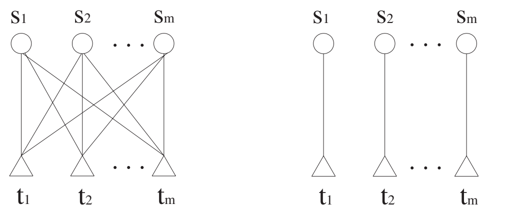
$$\begin{aligned} C_B(\mathbf{f}(E)) &= \sum_{j=1}^{n} \sum_{i=1}^{m} I_j^i L_j^i(\mathbf{f}(E)) \\ &= \sum_{j=1}^{n} \left(L_j(\mathbf{f}(E)) \sum_{i=1}^{m} I_j^i \right) \\ &\leq m \sum_{j=1}^{n} L_j(\mathbf{f}(E)) \\ &\leq m \sum_{j=1}^{n} \sum_{i=1}^{m} I_j^i L_j^i(\mathbf{f}^*) \\ &= m C_B(\mathbf{f}^*). \end{aligned}$$

Now we construct an instance to show that the bound is tight. We consider a specific network $G3$ where there are m clients and m servers, all clients having the same traffic demand of 1 and all servers having the same latency function (say $l(\cdot)$). Moreover, there is only one link between each sever and each client, and the latencies over all links are equal and constant (say 1). At one traffic configuration at Nash equilibrium, each client assigns $1/m$ traffic to each server and experiences latency of $l(1) + 1$, as is shown in Figure 3(a). However, if each client performs download from a single server, i.e., assigns the whole traffic to a single server as in Figure 3(b), it also experiences latency of $l(1) + 1$. For this case, $R_B = m$. ■

It should be noted that the scheme of downloading from a single server can do as good as that of parallel downloading only in the best case where complicated server selection algorithms are usually needed.

7. Conclusion

Content delivery in the current Internet has employed the scheme of parallel downloading. With the wide deployment



(a) Each client assigns $1/m$ traffic to each server.

(b) Each client assigns the whole traffic to a single server.

Figure 3. An instance with a tight upper bound of R_B.

of content distribution networks (CDNs) and peer-to-peer (P2P) networks, parallel downloading is expected to become more and more popular. To investigate in depth what would happen if this technique is widely adopted, we formulate parallel downloading as a non-cooperative game. To the best of our knowledge this is the first paper that analyzes the parallel downloading problem from a noncooperative game theoretical perspective.

In this paper, we have established existence and essentially uniqueness of Nash equilibrium, and analyzed its properties in a specific network. We have also analyzed the stability of Nash equilibrium using the Elementary Stepwise System (ESS) for a specific network. Finally, we have investigated the efficiency of Nash equilibrium from the point of view of the clients and the system respectively. We find that although the traffic configuration at Nash equilibrium is optimal from the point of view of the clients, it may be bad from the point of view of the system. However, to achieve optimal traffic configuration from the point of view of the system complicated server selection algorithms are usually needed. Therefore, if systems have sufficient resources to accommodate more connections, parallel downloading is still a promising technique as clients can always experience minimal downloading latency regardless of what the networking settings are.

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