

COST ALLOCATION

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1. Introduction

Organizations of all kinds allocate common costs. Manufacturing companies allocate overhead expenses among various products and divisions. Telephone companies allocate the cost of switching facilities and lines among different types of calls. Universities allocate computing costs among different departments. Cost allocation is also practiced by public agencies. Aviation authorities set landing fees for aircraft based on their size. Highway departments determine road taxes for different classes of vehicles according to the amount of wear and tear they cause to the roadways. Regulatory commissions set rates for electricity, water, and other utilities based on the costs of providing these services. Cost allocation is even found in voluntary forms of organization. When two doctors share an office, for example, they need to divide the cost of office space, medical equipment, and secretarial help. If several municipalities use a common water supply system, they must reach an agreement on how to share the costs of building and operating it. When the members of NATO cooperate on common defense, they need to determine how to share the burden. The common feature in all of these examples is that prices are not determined externally by market forces, but are set internally by mutual agreement or administrative decision.

While cost allocation is an interesting accounting problem, however, it is not clear that it has much to do with game theory. What does the division of defense costs in NATO or overhead costs in General Motors have in common with dividing the spoils of a game? The answer is that cost allocation *is* a kind of game in which costs (and benefits) are shared among different parts of an organization. The organization wants an allocation mechanism that is efficient, equitable, and provides appropriate incentives to its various parts. Cooperative game theory provides the tools for analyzing these issues. Moreover, cooperative game theory and cost allocation are closely intertwined in practice. Some of the central ideas in cooperative game theory, such as the core, were prefigured in the early theoretical literature on cost allocation. Others, such as the Shapley value, have long been used implicitly by some organizations. Like Moliere's M. Jourdain, who was delighted to hear that he had been speaking prose all his life, there are people who use game theory all the time without ever suspecting it.

This chapter provides an overview of the game theoretic literature on cost allocation. The aim of the chapter is two-fold. First, it provides a concrete motivation for some of the central solution concepts in cooperative game theory. Axioms and conditions that are usually presented in an abstract setting often seem more compelling when interpreted in the cost allocation framework. Second, cost allocation is a practical problem in which the salience of the solution depends on contextual and institutional details. Thus the second objective of the chapter is to illustrate various ways of *modelling* a cost allocation situation.

In order to keep both the theoretical and practical issues constantly in view, we have organized the chapter around a series of examples (many based on real data) that motivate general definitions and theorems. Proofs are provided when they are relatively brief. The reader who wants to delve deeper may consult the bibliography at the end.

2. An illustrative example

Consider the following simple example. Two nearby towns are considering whether to build a joint water distribution system. Town A could build a facility for itself at a cost of \$11 million, while town B could build a facility at a cost of \$7 million. If they cooperate, however, then they can build a facility serving both communities at a cost of \$15 million. (See Figure 1.) Clearly it makes sense to cooperate since they can jointly save \$3 million. Cooperation will only occur, however, if they can agree on how to divide the charges.

One solution that springs to mind is to share the costs equally – \$7.5 million for each. The argument for equal division is that each town has equal power to enter into a contract, so each should shoulder an equal burden. This argument is plausible if the towns are of about the same size, but otherwise it is suspect. Suppose, for example, that town A has 36 000 residents and town B has 12 000 residents. Equal division between the towns would imply that each resident of A pays only *one-third* as much as each resident of B, even though they are served by the same system. This hardly seems fair, and one can imagine that town B will not agree to it. A more plausible solution would be to divide the costs equally among the *persons* rather

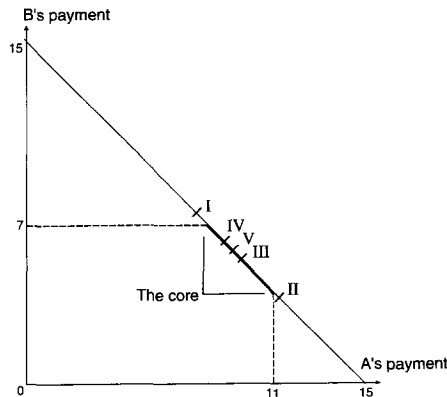


Figure 1. The core of the cost-sharing game.

Table 1
Five cost allocations for two towns

	Town A	Town B
I. Equal division of costs between towns	7.50	7.50
II. Equal division of costs among persons	11.25	3.75
III. Equal division of savings between towns	9.50	5.50
IV. Equal division of savings among persons	8.75	6.25
V. Savings (and costs) proportional to opportunity costs	9.17	5.83

than the towns. This results in a charge of \$312.50 per capita, and altogether the citizens of town A pay \$11.25 million while the citizens of town B pay \$3.75 million (see Table 1).

Unfortunately, neither of these proposals takes into account the opportunity costs of the parties. B is not likely to agree to equal division, because \$7.5 million exceeds the cost of building its own system. Similarly, A is not likely to agree to equal division per capita, since \$11.25 exceeds the cost of building its own system. Thus the equity issue is complicated by the need to give the parties an incentive to cooperate. Without such incentives, cooperation will probably not occur and the outcome will be inefficient. Thus we see that the three major themes of cost allocation – efficiency, equity, and incentives – are closely intertwined.

Let us consider the incentives issue first. The simplest way to ensure that the parties have an incentive to cooperate is to focus on the amounts that they *save* rather than on the amounts that they *pay*. Three solutions now suggest themselves. One is to divide the \$3 million in savings equally among the towns. In this case town A would pay $\$11 - 1.5 = \9.5 million and town B would pay $\$7 - 1.5 = \5.5 million. A second, and perhaps more plausible, solution is to divide the savings equally among the residents. Thus everyone would save \$62.50, and the total cost assessments would be \$8.75 million for town A and \$6.25 million for town B. Yet a third solution would be to allocate the savings in proportion to each town's opportunity cost. This yields a payment of \$9.17 million for A and \$5.83 million for B. (Note that this is the same thing as allocating total cost in proportion to each town's opportunity cost.)

All three of these allocations give the parties an incentive to cooperate, because each realizes positive savings. Indeed, any solution in which A pays at most \$11 million and B pays at most \$7 million creates no *disincentive* to cooperation. The set of all such solutions is known as the *core* of the cost-sharing game, a concept that will be defined more generally in Section 4 below. In the present case the core is the line segment shown in Figure 1.

This example illustrates several points. First, there is no completely obvious answer to the cost allocation problem even in apparently simple cases. Second, the problem cannot be avoided: costs must be allocated for the organization to be viable. Third, there is no external market mechanism that does the job. One

might try to mimic the market by setting price equal to marginal cost, but this does not work either. In the preceding example, the marginal cost of including town A is $\$15 - 7 = \8 million (the difference between the cost of the project with A and the cost of the project without A), while the marginal cost of including town B is $\$15 - 11 = \4 million. Thus the sum of the marginal costs does not equal total cost – indeed it does not even *cover* total cost. Hence we must find some other means to justify a solution. This is where ideas of equity come to the fore: they are the instruments that the participants use to reach a joint decision. Equity principles, in other words, are not merely normative or philosophical concepts. Like other kinds of norms, they play a crucial economic role by coordinating players' expectations, without which joint gains cannot be realized.

3. The cooperative game model

Let us now formulate the problem in more general terms. Let $N = \{1, 2, \dots, n\}$ be a set of *projects, products, or services* that can be provided jointly or severally by some organization. Let $c(i)$ be the cost of providing i by itself, and for each subset $S \subseteq N$, let $c(S)$ be the cost of providing the items in S jointly. By convention, $c(\emptyset) = 0$. The function c is called a *discrete cost function* or sometimes a *cost-sharing game*. An *allocation* is a vector (x_1, \dots, x_n) such that $\sum x_i = c(N)$, where x_i is the amount charged to project i . A *cost allocation method* is a function $\phi(c)$ that associates a unique allocation to every cost-sharing game.

In some contexts it is natural to interpret $c(S)$ as the *least costly* way of carrying out the projects in S . Suppose, for example, that each project involves providing a given amount of computing capability for each department in a university. Given a subset of departments S , $c(S)$ is the cost of the most economical system that provides the required level of computing for all the members of S . This might mean that each department is served by a separate system, or that certain groups of departments are served by a common system while others are served separately, and so forth. In other words, the cost function describes the *cost* of the most economical way of combining activities, it does not describe the physical *structure* of the system. If the cost function is interpreted in this way, then for any partition of a subset of projects into two disjoint subsets S' and S'' , we have

$$c(S' \cup S'') \leq c(S') + c(S''). \quad (1)$$

This property is known as *subadditivity*.

A second natural property of a cost function is that costs increase the more projects there are, that is, $c(S) \leq c(S')$ for all $S \subseteq S'$. Such a cost function is *monotonic*. Neither monotonicity nor subadditivity will be assumed in subsequent results unless we specifically say so.

The reason for carrying out projects separately rather than jointly is that it generates cost savings. Hence it often makes sense to focus attention on the

cost-savings directly. For each subset S of projects the potential *cost saving* is

$$v(S) = \sum_{i \in S} c(i) - c(S). \tag{2}$$

The function v is called the *cost-savings game*.

If c is subadditive, then v is nonnegative and monotone increasing in S . Indeed, for every S and $i \notin S$, subadditivity implies that $c(S + i) \leq c(S) + c(i)$, from which it follows that $v(S) \leq v(S + i)$. (We write “ $S + i$ ” instead of “ $S \cup \{i\}$ ”.) Since $v(\emptyset) = 0$, it follows that v is nonnegative and monotonic. It also follows that $v(N)$ is the largest among all $v(S)$, so from a purely formal point of view, N is the efficient set of projects to undertake. This assumption will be implicit throughout the remainder of the chapter. We do not, however, need to assume subadditivity and monotonicity of the cost function in order to prove many of the theorems quoted below, and we shall not assume them unless explicitly noted.

4. The Tennessee Valley Authority

The Tennessee Valley Authority was a major regional development project created by an act of Congress in the 1930s to stimulate economic activity in the mid-southern United States. The goal was to construct a series of dams and reservoirs along the Tennessee River to generate hydroelectric power, control flooding, and improve navigational and recreational uses of the waterway. Economists charged with analyzing the costs and benefits of this project observed that there is no completely obvious way to attribute costs to these purposes, because the system is designed to satisfy all of them simultaneously. The concepts that they devised to deal with this problem foreshadow modern ideas in game theory, and one of the formulas they suggested has since become (after minor modifications) the standard method for allocating the cost of multi-purpose reservoirs.

Table 2
Cost function for navigation (1), flood control (2), and power (3), in thousands of dollars

Subsets S	Cost $c(S)$
\emptyset	0
{1}	163 520
{2}	140 826
{3}	250 096
{1, 2}	301 607
{1, 3}	378 821
{2, 3}	367 370
{1, 2, 3}	412 584

Table 2 shows the cost function for the TVA case as analyzed by Ransmeier (1942, p. 329). There are three purposes: navigation (1), flood control (2), and power (3).

Ransmeier (1942, p. 220) suggested the following criteria for a cost allocation formula:

The method should have a reasonable logical basis... It should not result in charging any objective with a greater investment than would suffice for its development at an alternate single-purpose site. Finally, it should not charge any two or more objectives with a greater investment than would suffice for alternate dual or multiple purpose development.

In terms of the joint cost function $c(S)$ these requirements state that, if x_i is the charge to purpose i , then the following inequality should hold for every subset S of purposes (including singletons),

$$x(S) \leq c(S), \quad (3)$$

where $x(S) = \sum_{i \in S} x_i$. Condition (3) is known as the *stand-alone cost test*. Its rationale is evident: if cooperation among the parties is voluntary, then self-interest dictates that no participant – or group of participants – be charged more than their stand-alone (opportunity) cost. Otherwise they would have no incentive to agree to the proposed allocation.

A related principle known as the “incremental cost test” states that no project should be charged *less* than the marginal cost of including it. In Table 2, for example, the cost of including project 1 at the margin is

$$c(1, 2, 3) - c(2, 3) = 45\,214.$$

In general, the *incremental* or *marginal cost* of a subset S is defined to be $c(N) - c(N - S)$, and the *incremental cost test* requires that the allocation $x \in R^N$ satisfy

$$x(S) \geq c(N) - c(N - S) \quad \text{for all } S \subseteq N. \quad (4)$$

Whereas (3) provides incentives for voluntary cooperation, (4) arises from considerations of equity. For, if (4) were violated for some S , then it could be said that the coalition $N - S$ is *subsidizing* S . In other words, even if there is no need to give the parties an incentive to cooperate, there is still an argument for a core allocation on equity grounds¹.

It is easily seen that conditions (3) and (4) are equivalent given that costs are allocated exactly, namely,

$$x(N) = c(N). \quad (5)$$

¹This idea has been extensively discussed in the literature on public utility pricing. See for example Faulhaber (1975), Sharkey (1982a, b, 1985), and Zajac (1993).

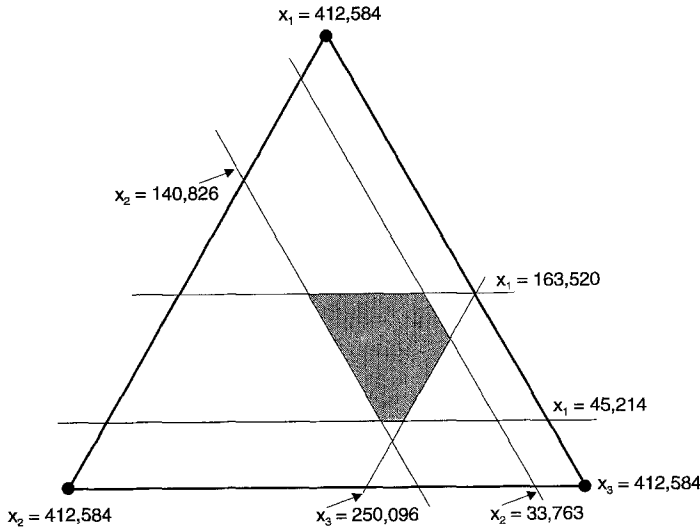


Figure 2. The core of the TVA cost-sharing game.

The *core* of the cost-sharing game c , written $\text{Core}(c)$, is the set of all allocations $x \in R^N$ such that (3) and (5) [equivalently (4) and (5)] hold for all $S \subseteq N$.

The core of the TVA cost game is illustrated in Figure 2. The top vertex x_1 represents the situation where all costs are allocated to purpose 1 (navigation); the right-hand vertex allocates all costs to purpose 3 (power), and the left-hand vertex allocates all costs to purpose 2 (flood control). Each point in the triangle represents a division of the \$412 584 among the three purposes.

In this example the core is fairly large, because there are strongly increasing returns to scale. This is due to the fact that the marginal cost of building a higher dam (to serve more purposes) decreases with the height of the dam.

The core is a closed, compact, convex subset of R^N . It may, however, be empty. Consider the cost function

$$\begin{aligned} c(1) &= c(2) = c(3) = 6 \\ c(1, 2) &= c(1, 3) = c(2, 3) = 7 \\ c(1, 2, 3) &= 11. \end{aligned}$$

If x is in the core, then

$$x_1 + x_2 \leq 7, x_1 + x_3 \leq 7, x_2 + x_3 \leq 7.$$

However, the sum of these inequalities yields $2(x_1 + x_2 + x_3) \leq 21$, which contradicts the break-even requirement $x_1 + x_2 + x_3 = 11$. Hence the core is empty. Furthermore this is true even though the cost function is both subadditive and monotonic. A sufficient condition that the core be nonempty is that the cost function exhibit increasing returns to scale, as we shall show in Section 9.

Table 3
The ACA method applied to the TVA data

	Purpose			Total
	1	2	3	
1. Alternate cost $c(i)$	163 520	140 826	250 096	554 442
2. Separable cost $c(N) - c(N - i)$	147 630	136 179	213 393	497 202
3. Alternate cost avoided (1-2)	15 890	4 647	36 704	57 240
4. Allocation of nonsep. cost	4 941	1 445	11 412	17 798
5. Allocation (2 + 4)	152 571	137 624	224 805	515 000

Although the Tennessee Valley Authority did not adopt a formal method for allocating costs, they took as a basis an approach known as the “alternative justifiable expenditure method” and then rounded off the results according to “judgment.”² This has become, after further refinements, the principal textbook method used by civil engineers to allocate the costs of multi-purpose reservoirs, and is known as the “separable costs remaining benefits method” [James and Lee (1971)]. We shall now describe a simple version of this method.

Given a cost-sharing game c , define the separable cost of a purpose $i \in N$ to be its marginal cost

$$s_i = c(N) - c(N - i).$$

The *alternate cost* for i is the stand-alone cost $c(i)$. The difference between the alternate cost and the separable cost is the *alternate cost avoided*

$$r_i = c(i) - s_i.$$

The *alternate cost avoided method* (ACA) assigns costs according to the formula

$$x_i = s_i + [r_i/r(N)][c(N) - s(N)].^3 \tag{6}$$

In other words, each project pays its separable cost and the “nonseparable cost” $c(N) - s(N)$ is allocated in proportion to the numbers r_i . The implicit assumption here is that all $r_i \geq 0$, which is the case if c is subadditive. Table 3 illustrates the calculation for the TVA cost data.

The ACA method can be given a more succinct and intuitive formulation in terms of the cost-savings game. For each project $i \in N$ define i 's *marginal savings*

²As Ransmeier remarks, “there is little to recommend the pure judgement method for allocation. In many regards it resembles what Professor Lewis has called the ‘trance method’ of utility valuation.” (1942, p.342).

³The more sophisticated *separable costs remaining benefits method* (SCRB) incorporates benefits as follows. Let $b(i)$ be the benefit from undertaking project i by itself. Then the *maximum justifiable expenditure* for i is $\min \{b(i), c(i)\}$ and r_i in formula (6) is defined to be $r_i = \min \{b(i), c(i)\} - s_i$. [James and Lee (1971)].

to be

$$v^i(N) = v(N) - v(N - i). \quad (7)$$

Given that i is charged x_i , the resulting savings are $y_i = c(i) - x_i$. A simple manipulation of (6) shows that the ACA imputes savings according to the formula

$$y_i = \left[v^i(N) / \sum_{j \in N} [v^j(N)] \right] v(N). \quad (8)$$

In other words, the ACA allocates cost savings in proportion to each project's marginal contribution to savings [Straffin and Heaney (1981)]. This solution was proposed independently in the game theory literature as a means of minimizing players' "propensity to disrupt" the solution [Gately (1974), Littlechild and Vaidya (1976), Charnes et al. (1979)].

There is no reason to think that the ACA method yields a solution in the core, and indeed it does not in general (see Section 6 below for an example). However, when there are at most three projects, and the cost function is subadditive, then it is in the core provided that the core is nonempty. Indeed we can show more. Define the *semicore* of a cost function to be the set of all allocations x such that,

$$\text{for every } i, \quad x_i \leq c(i), \quad x(N - i) \leq c(N - i), \quad \text{and } x(N) = c(N). \quad (9)$$

In the case of two or three projects the *semicore* is clearly the same as the core.

Theorem 1. *If c is subadditive, then the alternate cost avoided method is in the *semicore* whenever the latter is nonempty.*

Proof. It is easiest to work in terms of the cost-savings game v . By assumption c is subadditive, so v is nonnegative and monotone. By assumption, c has a nonempty *semicore*, so there is an allocation of savings y such that

$$v^i(N) \geq y_i \geq v(i) = 0 \quad \text{for every } i \in N.$$

Hence

$$y(N) = v(N) \leq \sum_{j \in N} v^j(N). \quad (10)$$

The ACA allocation is defined by

$$y_i^* = v^i(N) \left[v(N) / \sum_{j \in N} v^j(N) \right].$$

From this and (10) we conclude that

$$0 \leq y_i^* \leq v^i(N). \quad (11)$$

Letting $x_i^* = c(i) - y_i^*$ it follows that x^* is in the *semicore* of c . \square

5. Equitable core solutions

Core allocations provide incentives for cooperation. They are also *fair* in the sense that no subgroup subsidizes any other. If core allocations exist, however, there are usually an uncountable number of them. Which is most equitable? Here we shall suggest one answer to this question.

Let us begin by noticing that in the case of two projects the natural solution is to choose the *midpoint* of the core. This solution treats the two projects equally in the sense that they save equal amounts relative to their opportunity costs. There is another way of justifying this answer however. Consider a cost allocation situation in which some costs can be attributed directly to particular projects. In the TVA case, for example, the cost of the generators is directly attributable to hydropower generation, the cost of constructing levees is directly attributable to controlling flooding, and the cost of building locks is directly attributable to navigation. Sometimes the distinction between direct and joint costs is not so clear, however. Deepening the river channel probably benefits navigation most, but it has a favorable impact on flood control and hydropower generation as well.

From a formal point of view, we say that a cost function c *decomposes into direct costs* $d = (d_1, d_2, \dots, d_n)$ and *joint costs* c^* if c can be written in the form

$$c(S) = d(S) + c^*(S) \quad \text{for every } S \subseteq N. \tag{12}$$

A cost allocation method ϕ is *invariant in direct costs* if whenever c satisfies (12), then

$$\phi(c) = d + \phi(c^*), \tag{13}$$

that is, $\phi_i(c) = d_i + \phi_i(c^*)$ for every i .

A cost allocation method is *symmetric* if it is invariant under any renaming of the projects. In other words, given any cost function c on N , and any permutation π of N , if we define the cost function πc such that $\pi c(\pi S) = c(S)$, then

$$\phi_{\pi(i)}(\pi c) = \phi_i(c) \quad \text{for every } i \in N. \tag{14}$$

Theorem 2. *If ϕ is symmetric and invariant in direct costs, then for any two-project cost function c defined on $N = \{1, 2\}$,*

$$\phi_i(c) = c(i) - s/2 \quad \text{where } s = c(1) + c(2) - c(1, 2). \tag{15}$$

The proof of this result is straightforward and is left to the reader.

The method defined by (15) is called the *standard two-project solution*. When total savings s are nonnegative (which is true if c is subadditive), the standard solution is simply the midpoint of the core. Thus Theorem 2 provides an axiomatic justification of a solution that is intuitively appealing on equity grounds.

When there are more than two projects, it seems natural to apply the same idea and divide the cost-savings equally among the various projects. In the TVA case, this yields the cost allocation (116 234, 93 540, 202 810) which is in the core. But

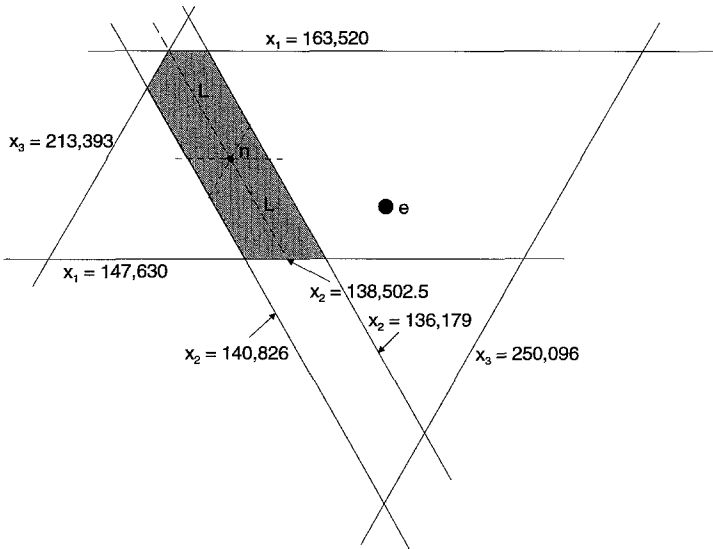


Figure 3. The core of the modified TVA cost game.

there are many situations where the equal savings allocation is *not* in the core. Consider the following variation of the TVA data: total costs are \$515 000 000 and all other costs remain the same as before. The core of this modified TVA game is shown in Figure 3, and the equal division of savings (e) is not in it. Hence we must find some other principle for determining an equitable allocation in the core.

Consider the following approach: instead of insisting that all projects be treated equally, let us treat all projects – and all subgroups of projects – as *nearly equally as possible*. By this we mean the following. Let c be a cost function and let x be an allocation of total cost $c(N)$. Each subset of projects $S \subseteq N$ realizes cost savings equal to $e(S, x) = c(S) - x(S)$. We shall say that the set S is strictly *better off* than the set T if $e(S, x) > e(T, x)$. A natural criterion of equity is to *maximize the position of the least well-off subset*, that is, to find an allocation x that maximizes $\min_S e(S, x)$. This is known as the *maximin criterion*.⁴ A maximin allocation is found by solving the linear programming problem

$$\begin{aligned}
 &\max \varepsilon \\
 &\text{subject to } e(S, x) \geq \varepsilon \text{ for all } S, \phi \subset S \subset N \\
 &\text{and } x(N) = c(N).
 \end{aligned} \tag{16}$$

The set of all solutions to x (16) is called the *least core* of c .

If there is more than one allocation in the least core, we may whittle it down further by extending the maximin criterion as follows. Order the numbers $e(S, x)$,

⁴This idea is central to John Rawls's *A Theory of Justice* (1971).

$\phi \subset S \subset N$, from lowest to highest, and denote this vector of dimension $(2^n - 2)$ by $\theta(x)$. The *prenucleolus* is the allocation x that maximizes $\theta(x)$ lexicographically, that is, if y is any other cost allocation and k is the first index such that $\theta_k(x) \neq \theta_k(y)$, then $\theta_k(x) > \theta_k(y)$ [Schmeidler (1969)].⁵

The prenucleolus occupies a central position in the core in the sense that the minimal distance from any boundary is as large as possible. In Figure 3 it is the point labelled “ n ”, with coordinates $x_1 = \$155\,367.2$, $x_2 = \$138\,502.5$, $x_3 = \$221\,130.25$. (Note that this is *not* the center of gravity.)

We claim that the prenucleolus is a natural extension of the standard two-project solution. Let us begin by observing that, when there are just two projects, the prenucleolus agrees with the standard two-project solution. Indeed, in this case there are just two proper subsets $\{1\}$ and $\{2\}$, and the allocation that maximizes the smaller of $c(1) - x_1$ and $c(2) - x_2$ is clearly the one such that $c(1) - x_1 = c(2) - x_2$.

When there are more than two projects, the prenucleolus generalizes the standard two-project solution in a more subtle way. Imagine that each project is represented by an agent, and that they have reached a preliminary agreement on how to split the costs. It is natural for each subgroup of agents to ask whether they fairly divide the cost allocated to them as a subgroup. To illustrate this idea, consider Figure 3 and suppose that the agents are considering the division $n = (155\,367.2, 138\,502.5, 221\,130.25)$. Let us restrict attention to agents 1 and 3. If they view agent 2’s allocation as being fixed at $\$138\,502.50$, then they have $\$376\,497.50$ to divide between them. The range of possible divisions that lie within the core is represented by the dotted line segment labelled L . In effect, L is the core of a smaller or “reduced” game on the two-player set $\{1, 3\}$ that results when 2’s allocation is held fixed. Now observe that (n_1, n_3) is the midpoint of this segment. In other words, it is the standard two-project solution of the reduced game. Moreover the figure shows that the prenucleolus n bisects *each* of the three line segments through n in which the charge to one agent is held fixed. This observation holds in general: *the prenucleolus is the standard two-project solution when restricted to each pair of agents*. This motivates the following definition. Let c be any cost function defined on the set of projects N , and let x be any allocation of $c(N)$. For each proper subset T of N , define the *reduced cost function* $c_{T,x}$ as follows:

$$\begin{aligned} c_{T,x}(S) &= \min_{S' \subseteq N-T} \{c(S \cup S') - x(S')\} \quad \text{if } \phi \subset S \subset T, \\ c_{T,x}(T) &= x(T), \\ c_{T,x}(\phi) &= 0. \end{aligned} \tag{17}$$

A cost allocation method ϕ is *consistent* if for every N , every cost function c on

⁵If x' and x'' are distinct allocations that maximize $\theta(\cdot)$ lexicographically, then it is relatively easy to show that $\theta(x'/2 + x''/2)$ is strictly larger lexicographically than both $\theta(x')$ and $\theta(x'')$. Hence θ has a unique maximum. The lexicographic criterion has been proposed as a general principle of justice by Sen (1970).

N , and every proper subset T of N ,

$$\phi(c) = x \Rightarrow \phi(c_{T,x}) = x_T. \quad (18)$$

If (18) holds for every subset T of cardinality two, then ϕ is *pairwise consistent*. Note that this definition applies to all cost games c , whether or not they have a nonempty core.⁶

Consistency is an extremely general principle of fair division which says that if an allocation is fair, then every subgroup of claimants should agree that they share the amount allotted to them fairly. This idea has been applied to a wide variety of allocation problems, including the apportionment of representation [Balinski and Young (1982)], bankruptcy rules [Aumann and Maschler (1985)], surplus sharing rules [Moulin (1985)], bargaining problems [Harsanyi (1959), Lensberg (1985, 1987, 1988)], taxation [Young (1988)], and economic exchange [Thomson (1988)]. For reviews of this literature see Thomson (1990) and Young (1994).

To state the major result of this section we need one more condition. A cost allocation rule is *homogeneous* if for every cost function c and every positive scale factor λ , $\phi(\lambda c) = \lambda \phi(c)$.

Theorem 3 [Sobolev (1975)]. *The prenucleolus is the unique cost allocation method that is symmetric, invariant in direct costs, homogeneous, and consistent.*

We remark that it will not suffice here to assume pairwise consistency instead of consistency. Indeed, it can be shown that, for any cost function c , the set of all allocations that are pairwise consistent with the standard two-project solution constitutes the *prekernel* of c [Peleg (1986)]. The prekernel contains the prenucleolus but possibly other points as well. Hence pairwise consistency with the standard solution does not identify a unique cost allocation.

6. A Swedish municipal cost-sharing problem

In this section we analyze an actual example that illustrates some of the practical problems that arise when we apply the theory developed above. The Skåne region of southern Sweden consists of eighteen municipalities, the most populous of which is the city of Malmö (see Figure 4). In the 1940s several of them, including Malmö, banded together to form a regional water supply utility known as the Sydsvatten (South Water) Company. As water demands have grown, the Company has been under increasing pressure to increase long-run supply and incorporate outlying municipalities into the system. In the late 1970s a group from the International Institute for Applied Systems Analysis, including this author, were invited to

⁶For an alternative definition of the reduced game see Hart and Mas-Colell (1989).

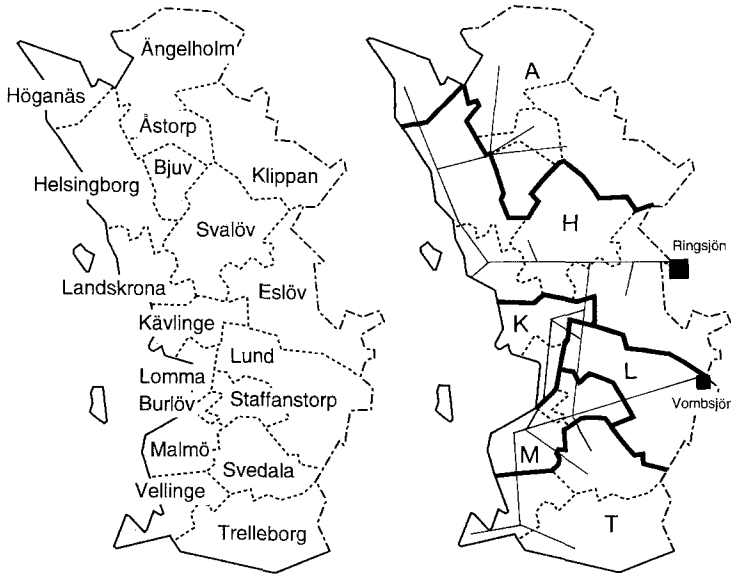


Figure 4. The region of Skåne, Sweden and its partition into groups of municipalities.

analyze how the cost of expanding the system should be allocated among the various townships, We took the system as it existed in 1970 and asked how much it would cost to expand it in order to serve the water demands projected by 1980.

In theory we should have estimated the system expansion cost for each of the $2^{18} = 262\,144$ possible subgroups, but this was clearly infeasible. To simplify the problem, we noted that the municipalities fall into natural groups based on past associations, geographical proximity, and existing water transmission systems. This led us to group the eighteen municipalities into the six units shown in Figure 4. We treated these groups as single actors in developing the cost function. Of course, once a cost allocation among the six groups is determined, a second-stage allocation must be carried out within each subgroup. This raises some interesting modelling issues that will be discussed in Section 10 below. Here we shall concentrate on the problem of allocating costs among the six units in the aggregated cost-sharing game.

One of the first problems that arises in defining the cost function is how to distinguish between direct costs and joint costs. Within each municipality, for example, a local distribution network is required no matter where the water comes from, so one might suppose that this is a direct cost. However, in some cases the water delivered by the regional supply network must first be pumped up to a reservoir before being distributed within the municipality. The cost of these pumping facilities depends on the pressure at which the water is delivered by the regional system. Hence the cost of the local distribution facilities is *not* completely

independent of the method by which the water is supplied. This illustrates why the borderline between direct and joint costs is somewhat fuzzy, and why it is important to use a method that does not depend on where the line is drawn, i.e., to use a method that is invariant in direct costs.

For each subset of the six units, we estimated the cost of expanding the system to serve the members of this subset using standard engineering formulas, and the result is shown in Table 4. [See Young et al. (1982) for details.]

Note the following qualitative features of the cost function. Even though L is close to the two major sources of supply (lakes Ringsjön and Vombsjön), it has a high stand-alone cost because it does not have rights to withdraw from these sources. Hence we should expect L's charge to be fairly high. By contrast, H and M have relatively low stand-alone costs that can be reduced even further by including other municipalities in the joint scheme. However, the system owned by H (Ringsjön) has a higher incremental capacity than the one owned by M (Vombsjön). Hence the incremental cost of including other municipalities in a coalition with M is higher than the incremental cost of including them in a coalition with H. In effect, H has more to offer its partners than M does, and this should be reflected in the cost allocation.

Table 4

Costs of alternative supply systems, in millions of Swedish crowns. Coalitions are separated by commas if there are no economies of scale from integrating them into a single system, that is, we write S, S' if $c(S) + c(S') = c(S \cup S')$

Group	Total cost	Group	Total cost	Group	Total cost
A	21.95	AHK	40.74	AHKL	48.95
H	17.08	AHL	43.22	AHKM	60.25
K	10.91	AH, M	55.50	AHK, T	62.72
L	15.88	AH, T	56.67	AHL, M	64.03
M	20.81	A, K, L	48.74	AHL, T	65.20
T	21.98	A, KM	53.40	AH, MT	74.10
		A, K, T	54.84	A, K, LM	63.96
AH	34.69	A, LM	53.05	A, K, L, T	70.72
A, K	32.86	A, L, T	59.81	A, K, MT	72.27
A, L	37.83	A, MT	51.36	A, LMT	73.41
A, M	42.76	HKL	27.26	HKL, M	48.07
A, T	43.93	HKM	42.55	HKL, T	49.24
HK	22.96	HK, T	44.94	HKMT	59.35
HL	25.00	HL, M	45.81	HLMT	64.41
H, M	37.89	HL, T	46.98	KLMT	56.61
H, T	39.06	H, MT	56.49	AHKL, T	70.93
K, L	26.79	K, LM	42.01	AHKLM	69.76
KM	31.45	K, L, T	48.77	AKHMT	77.42
K, T	32.89	K, MT	50.32	AHLMT	83.00
LM	31.10	LMT	51.46	AKLMT	73.97
L, T	37.86			HKLMT	66.46
MT	39.41			AHKLMT	83.82

Table 5
 Cost allocation of 83.82 million Swedish crowns by four methods

	A	H	K	L	M	T
Stand-alone cost	21.95	17.08	10.91	15.88	20.81	21.98
Prop. to pop.	10.13	21.00	3.19	8.22	34.22	7.07
Prop. to demand	13.07	16.01	7.30	6.87	28.48	12.08
ACA	19.54	13.28	5.62	10.90	16.66	17.82
Prenucleolus	20.35	12.06	5.00	8.61	18.32	19.49

Table 5 shows cost allocations by four different methods. The first two are “naive” solutions that allocate costs in proportion to population and water demand respectively. The third is the standard engineering approach described in Section 4 (the ACA method), and the last is the prenucleolus. Note that both of the proportional methods charge some participant more than its stand-alone cost. Allocation by demand penalizes M, while allocation by population penalizes both H and M. These two units have large populations but they have ready access to the major sources of supply, hence their stand-alone costs are low. A and T, by contrast, are not very populous but are remote from the sources and have high stand-alone costs. Hence they are favored by the proportional methods. Indeed, neither of these methods charges A and T even the *marginal* cost of including them.

The ACA method is apparently more reasonable because it does not charge any unit more than its stand-alone cost. Nevertheless it fails to be in the core, which is nonempty. H, K, and L can provide water for themselves at a cost of 27.26 million Swedish crowns, but the ACA method charges them a total of 29.80 million Swedish crowns. In effect they are subsidizing the other participants. The prenucleolus, by contrast, is in the core and is therefore subsidy-free.

7. Monotonicity

Up to this point we have implicitly assumed that all cost information is in hand, and the agents need only reach agreement on the final allocation. In practice, however, the parties may need to make an agreement before the actual costs are known. They may be able to *estimate* the total cost to be divided (and hence their prospective shares), but in reality they are committing themselves to a *rule* for allocating cost rather than to a single cost allocation. This has significant implications for the type of rule that they are likely to agree to. In particular, if total cost is higher than anticipated, it would be unreasonable for anyone’s charge to go down. If cost is lower than anticipated, it would be unreasonable for anyone’s charge to go up. Formally, an allocation rule ϕ is *monotonic in the aggregate* if for any set of projects N , and any two cost functions c and c' on N

$$\begin{aligned}
 c'(N) \geq c(N) \quad \text{and} \quad c'(S) = c(S) \quad \text{for all } S \subset N \\
 \text{implies } \phi_i(c') \geq \phi_i(c) \quad \text{for all } i \in N.
 \end{aligned}
 \tag{19}$$

Table 6
Allocation of a cost overrun of 5 million Swedish crowns by two methods

	A	H	K	L	M	T
Preruleolus	0.41	1.19	-0.49	1.19	0.84	0.84
ACA	1.88	0.91	-0.16	0.07	0.65	0.65

This concept was first formulated for cooperative games by Megiddo (1974). It is obvious that any method based on a proportional criterion is monotonic in the aggregate, but such methods fail to be in the core. The alternate cost avoided method is neither in the core nor is it monotonic in the aggregate. For example, if the total cost of the Swedish system increases by 5 million crowns to 87.82, the ACA method charges K less than before (see Table 6). The preruleolus is in the core (when the core is nonempty) but it is also not monotonic in the aggregate, as Table 6 shows.

The question naturally arises whether *any* core method is monotonic in the aggregate. The answer is affirmative. Consider the following variation of the preruleolus. Given a cost function c and an allocation x , define the *per capita savings* of the proper subset S to be $d(x, S) = (c(S) - x(S))/|S|$. Order the $2^n - 2$ numbers $d(x, S)$ from lowest to highest and let the resulting vector be $\gamma(x)$. The *per capita preruleolus* is the unique allocation that lexicographically maximizes $\gamma(x)$ [Grotte (1970)].⁷ It may be shown that the per capita preruleolus is monotonic in the aggregate and in the core whenever the core is nonempty. Moreover, it allocates any increase in total cost in a natural way: the increase is split equally among the participants [Young et al. (1982)]. In these two respects the per capita preruleolus performs better than the preruleolus, although it is less satisfactory in that it fails to be consistent.

There is a natural generalization of monotonicity, however, that both of these methods fail to satisfy. We say that the cost allocation method ϕ is *coalitionally monotonic* if an increase in the cost of any particular coalition implies, *ceteris paribus*, no decrease in the allocation to any member of that coalition. That is, for every set of projects N , every two cost functions c, c' on N , and every $T \subseteq N$,

$$c'(T) \geq c(T) \quad \text{and} \quad c'(S) = c(S) \quad \text{for all } S \neq T \\ \text{implies } \phi_i(c') \geq \phi_i(c) \quad \text{for all } i \in T. \quad (20)$$

It is readily verified that (20) is equivalent to the following definition: ϕ is *coalitionally monotonic* if for every N , every two cost functions c' and c on N , and every $i \in N$,

$$\text{if } c'(S) \geq c(S) \text{ for all } S \text{ containing } i \text{ and } c'(S) = c(S) \text{ for all } S \text{ not containing } i, \\ \text{then } \phi_i(c') \geq \phi_i(c).$$

⁷Grotte (1970) uses the term "normalized nucleolus" instead of "per capita nucleolus".

The following “impossibility” theorem shows that coalitional monotonicity is incompatible with staying in the core.

Theorem 4 [Young (1985a)]. *For $|N| \geq 5$ there exists no core allocation method that is coalitionally monotonic.*

Proof. Consider the cost function c defined on $N = \{1, 2, 3, 4, 5\}$ as follows:

$$\begin{aligned} c(S_1) = c(3, 5) &= 3, & c(S_2) = c(1, 2, 3) &= 3, \\ c(S_3) = c(1, 3, 4) &= 9, & c(S_4) = c(2, 4, 5) &= 9, \\ c(S_5) = c(1, 2, 4, 5) &= 9, & c(S_6) = c(1, 2, 3, 4, 5) &= 11. \end{aligned}$$

For $S \neq S_1, \dots, S_5, S_6, \phi$, define

$$c(S) = \min_k \{c(S_k) : S \subseteq S_k\}.$$

If x is in the core of c , then

$$x(S_k) \leq c(S_k) \quad \text{for } 1 \leq k \leq 5. \tag{21}$$

Adding the five inequalities defined by (21) we deduce that $3x(N) \leq 33$, whence $x(N) \leq 11$. But $x(N) = 11$ because x is an allocation. Hence, all inequalities in (21) must be equalities. These have the unique solution $x = (0, 1, 2, 7, 1)$, which constitutes the core of c .

Now consider the game c' , which is identical to c except that $c'(S_5) = c'(S_6) = 12$. A similar argument shows that the unique core element of this game is $x' = (3, 0, 0, 6, 3)$. Thus the allocation to both 2 and 4 decreases even though the cost of some of the sets containing 2 and 4 monotonically increases. This shows that no core allocation procedure is monotonic for $|N| = 5$, and by extension for $|N| \geq 5$. \square

8. Decomposition into cost elements

We now turn to a class of situations that calls for a different approach. Consider four homeowners who want to connect their houses to a trunk power line (see Figure 5). The cost of each segment of the line is proportional to its length, and a segment costs the same amount whether it serves some or all of the houses. Thus the cost of segment OA is the same whether it carries power to house A alone or to A plus all of the houses more distant than A, and so forth.

If the homeowners do not cooperate they can always build parallel lines along the routes shown, but this would clearly be wasteful. The efficient strategy is to construct exactly four segments OA, AB, BC, and BD and to share them. But what is a reasonable way to divide the cost?

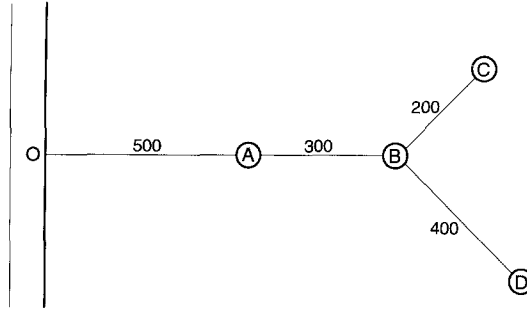


Figure 5. Cost of connecting four houses to an existing trunk power line.

The answer is transparent. Since everyone uses the segment OA, its cost should be divided equally among all four homeowners. Similarly, the cost of segment AB would be divided equally among B, C, D, the cost of BC should be borne exclusively by C and the cost of BD exclusively by D. The resulting cost allocation is shown in Table 7.

Let us now generalize this idea. Suppose that a project consists of m distinct components or *cost elements*. Let $C_\alpha \geq 0$ be the cost of component $\alpha, \alpha = 1, 2, \dots, m$. Denote the set of potential beneficiaries by $N = \{1, 2, \dots, n\}$. For each cost element α , let $N_\alpha \subseteq N$ be the set of parties who use α . Thus the stand-alone cost of each subset $S \subseteq N$ is

$$c(S) = \sum_{N_\alpha \cap S \neq \emptyset} C_\alpha. \tag{22}$$

A cost function that satisfies (22) *decomposes* into nonnegative cost elements. *The decomposition principle states that when a cost function decomposes, the solution is to divide each cost element equally among those who use it and sum the results.*

It is worth noting that the decomposition principle yields an allocation that is in the core. Indeed, $c(S)$ is the sum of the cost elements used by members of S , but the charge for any given element is divided equally among all users, some of

Table 7
Decomposition of electrical line costs

Cost elements	Homes				Segment cost
	A	B	C	D	
OA	125	125	125	125	500
AB		100	100	100	300
BC			200		200
BD				400	400
Charge	125	225	425	625	1400

Table 8
Aircraft landings, runway costs, and charges at Birmingham airport, 1968–69

Aircraft type	No. landings	Total cost*	Shapley value
Fokker Friendship 27	42	65 899	4.86
Viscount 800	9555	76 725	5.66
Hawker Siddeley Trident	288	95 200	10.30
Britannia	303	97 200	10.85
Caravelle VIR	151	97 436	10.92
BAC 111 (500)	1315	98 142	11.13
Vanguard 953	505	102 496	13.40
Comet 4B	1128	104 849	15.07
Britannia 300	151	113 322	44.80
Corvair Corronado	112	115 440	60.61
Boeing 707	22	117 676	162.24

*Total cost of serving this type of plane and all smaller planes.

which may not be in S . Hence the members of S are collectively not charged *more* than $c(S)$.

As a second application of the decomposition principle consider the problem of setting landing fees for different types of planes using an airport [Littlechild and Thompson (1977)]. Assume that the landing fees must cover the cost of building and maintaining the runways, and that runways must be longer (and therefore more expensive) the larger the planes are. To be specific, let there be m different types of aircraft that use the airport. Order them according to the length of runway that they need: type 1 needs a short runway, type 2 needs a somewhat longer runway, and so forth. Schematically we can think of the runway as being divided into m sections. The first section is used by all planes, the second is used by all but the smallest planes, the third by all but the smallest two types of planes, and so forth.

Let the annualized cost of section α be $c_\alpha, \alpha = 1, 2, \dots, m$. Let n_α be the number of landings by planes of type α in a given year, let N_α be the set of all such landings, and let $N = \cup N_\alpha$. Then the cost function takes the form

$$c(S) = \sum_{N_\alpha \cap S \neq \phi} c_\alpha,$$

so it is decomposable. Table 8 shows cost and landing data for Birmingham airport in 1968/69, as reported by Littlechild and Thompson (1977), and the charges using the decomposition principle.

9. The Shapley value

The decomposition principle involves three distinct ideas. The first is that everyone who uses a given cost element should be charged equally for it. The second is that

those who do not use a cost element should not be charged for it. The third is that the results of different cost allocations can be added together. We shall now show how these ideas can be extended to cost functions that do *not* necessarily decompose into nonnegative cost elements.

Fix a set of projects N and let ϕ be a cost allocation rule defined for every cost function c on N . The notion that everyone who uses a cost element should be charged equally for it is captured by symmetry (see p. 1203). The idea that someone should not be charged for a cost element he does not use generalizes as follows. Say that project i is a *dummy* if $c(S + i) = c(S)$ for every subset S not containing i . It is natural to require that the charge to a dummy is equal to zero.

Finally, suppose that costs can be broken down into different categories, say operating cost and capital cost. In other words, suppose that there exist cost functions c' and c'' such that

$$c(S) = c'(S) + c''(S) \quad \text{for every } S \subseteq N.$$

The rule ϕ' is *additive* if

$$\phi(c) = \phi(c') + \phi(c'').$$

Theorem 5. [Shapley (1953a, b)]. *For each fixed N there exists a unique cost allocation rule ϕ defined for all cost functions c on N that is symmetric, charges dummies nothing, and is additive, namely the Shapley value*

$$\phi_i(c) = \sum_{S \subseteq N-i} \frac{|S|!(|N-S|-1)!}{|N|!} [c(S+i) - c(S)].$$

When the cost function decomposes into cost elements, it may be checked that the Shapley value gives the same answer as the decomposition principle. In the more general case the Shapley value may be calculated as follows. Think of the projects as being added one at a time in some arbitrary order $R = i_1, i_2, \dots, i_n$. The *cost contribution of project $i = i_k$* relative to the order R is

$$\gamma_i(R) = c(i_1, i_2, \dots, i_k) - c(i_1, i_2, \dots, i_{k-1}).$$

It is straightforward to check that the Shapley value for i is just the average of $\gamma_i(R)$ over all $n!$ orderings R .

When the cost function decomposes into distinct cost elements, the Shapley value is in the core, as we have already noted. Even when the game does not decompose, the Shapley value may be in the core provided that the core is large enough. In the TVA game, for example, the Shapley value is (117 829, 100 756.5, 193 998.5), which is comfortably inside the core. There are perfectly plausible examples, however, where the cost function has a nonempty core and the Shapley value fails to be in it. If total cost for the TVA were 515 000, for example (see

Figure 3) the Shapley value would be

$$(151\ 967\ 2/3, \ 134\ 895\ 1/6, \ 228\ 137\ 1/6).$$

This is not in the core because the total charges for projects 1 and 3 come to $380\ 104\ 5/6$, which exceeds the stand-alone cost $c(1,3) = 378\ 821$.

There is, however, a natural condition under which the Shapley value is in the core – namely, if the marginal cost of including any given project decreases the more projects there are. In other words, the Shapley value is in the core provide there are increasing (or at least not decreasing) returns to scale. To make this idea precise, consider a cost function c on N . For each $i \in N$ and $S \subseteq N$, i 's *marginal cost contribution relative to S* is

$$c^i(S) = \begin{cases} c(S) - c(S - i), & \text{if } i \in S, \\ c(S + i) - c(S), & \text{if } i \notin S. \end{cases} \quad (23)$$

The function $c^i(S)$ is called the *derivative* of c with respect to i . The cost function is *concave* if $c^i(S)$ is a nonincreasing function of S for every i , that is, if $c^i(S) \geq c^i(S')$ whenever $S \subset S' \subseteq N$.⁸

Theorem 6 [Shapley (1971)]. *The core of every concave cost function is nonempty and contains the Shapley value.*

10. Weighted Shapley values

Of all the properties that characterize the Shapley value, symmetry seems to be the most innocuous. Yet from a modelling point of view this assumption is perhaps the trickiest, because it calls for a judgment about what should be treated equally. Consider again the problem of allocating water supply costs among two towns A and B as discussed in Section 2. The Shapley value assigns the cost savings (\$3 million) equally between them. Yet if the towns have very different populations, this solution might be quite inappropriate. This example illustrates why the symmetry axiom is not plausible when the partners or projects differ in some respect *other than cost* that we feel has a bearing on the allocation.

Let us define the *cost objects* to be the things we think deserve equal treatment provided that they contribute equally to cost. They are the “elementary particles” of the system. In the municipal cost-sharing case, for example, the objects might be the *towns* or the *persons* or (conceivably) the *gallons of water* used. To apply

⁸An equivalent condition is that c be submodular, that is, for any $S, S' \subseteq N$, $c(S \cup S') + c(S \cap S') \leq c(S) + c(S')$ for all $S, S' \subseteq N$.

the Shapley value we would then compute the cost function for all subsets of the cost objects.

If the objects are very numerous, however, this approach is impractical. Simplifying assumptions must be made. We could assume, for instance, that serving any part of a town costs the same amount as serving the whole town. Thus, if persons are the cost objects, and N^* is the set of all 48 000 persons, we would define the cost function c^* on N^* as follows:

$$c^*(S) = 11 \text{ if } S \subseteq A, \quad c^*(S) = 7 \text{ if } S \subseteq B, \quad c^*(S) = 15 \text{ otherwise.}$$

This game has the feature that it is composed of distinct “blocks” or “families.” More generally, given a cost function c^* defined on a set N^* , a *family* of c^* is a nonempty subset $S \subseteq N^*$ such that

$$\text{for every } T \subseteq N, \quad S \cap T \neq \phi \text{ implies } c^*(T \cup S) = c^*(T). \tag{24}$$

In other words, S is a family if we incur the full cost of serving S whenever we have to serve at least one member of S .

Consider any partition of N^* into families S_1, S_2, \dots, S_m , some or all of which may be singletons. Let w_j be the number of persons in family j , $1 \leq j \leq m$. Define the *aggregated cost function* c on the index set $M = \{1, 2, \dots, m\}$ as follows:

$$c(T) = c^*\left(\bigcup_{j \in T} S_j\right), \quad T \subseteq M.$$

Consider all $m!$ orderings of the m families. For any such ordering $R = (i(1), i(2), \dots, i(m))$ define the *probability of R with respect to w* to be

$$P_w(R) = \prod_{k=1}^m \left(w_{i(k)} \left/ \sum_{j=k}^m w_{i(j)} \right. \right).$$

One way of constructing such a probability distribution over orderings is as follows: Add one family at a time, where the probability that a family is chosen equals its weight divided by the weight of all partnerships remaining to be chosen.

The Shapley value of c^* may now be computed in two steps. First we compute the expected marginal contribution to cost of each family j over all orderings. In other words, for each ordering R , let $\gamma_j(R)$ be the difference between the cost of family j together with all its predecessors in R , and the cost of all j 's predecessors in R (excluding j). Define

$$(\phi_w)_j(c) = \sum_R P_w(R) \gamma_j(R). \tag{25}$$

The function ϕ_w is called the *weighted Shapley value* of c with weights w [see Loehman and Whinston (1976), Shapley (1981), Kalai and Samet (1987)]. It can be shown that the Shapley value of individual project i in the *original* cost function c^* is just the weighted Shapley value of the family to which i belongs, divided

by the number of members in that family, that is,

$$\text{for all } i \in S_j, \quad \phi_i(c^*) = (\phi_w)_j(c)/w_j.$$

The question now arises whether we can justify the weighted Shapley value from first principles without the symmetry axiom. Here is one such axiomatization. [For alternative approaches see Loehman and Whinston (1976), Shapley (1981).] Fix the set N , and consider a cost allocation method ϕ defined for all cost functions c on N . The method ϕ is *positive* if whenever c is monotonic [$S \subseteq S'$ implies $c(S) \leq c(S')$] and contains no dummy players, then $\phi_i(c) > 0$ for all i .

For every $S \subseteq N$, let u_S be the cost function such that $u_S(T) = 1$ whenever $T \cap S \neq \emptyset$ and $u_S(T) = 0$ whenever $T \cap S = \emptyset$. The method ϕ is *family consistent* if for any family S in c , $\phi_i(c) = \phi_i(\gamma u_S)$, where $\gamma = \sum_{i \in S} \phi_i(c)$. Family consistency means that the total charge to any family is divided among its members just as it would be if this were the unique maximal family and all other players were dummies.

Theorem 7 [Kalai and Samet (1987)]. *A cost allocation method ϕ on N is additive, positive, family consistent, and charges dummies nothing if and only if there is a vector of weights $w \in \mathbb{R}_{++}$ such that $\phi = \phi_w$.*

For a review of the literature on weighted Shapley values see Kalai and Samet (1988).

11. Cost allocation in the firm

We shift our attention now to firms that want to allocate common costs among various product lines or divisions. The reason for allocating such costs is to make division managers sensitive to the burden that they are placing on shared facilities. Ideally, therefore, the allocation should provide an incentive for managers to operate more efficiently, thereby reducing the incremental burden they place on these facilities.

This problem can be modelled using cooperative game theory as follows. [This approach was pioneered by Shubik (1962)]. Let N denote a set of n divisions or product lines in a firm, each of which is represented by a division manager. For each $S \subseteq N$, let $c(S)$ represent the overhead cost that the divisions in S would incur if they were in business by themselves, that is, if the firm were stripped of the other divisions. Suppose, for example, that the firm consists of two divisions that share warehouse space for their products. We assume that the volume of business is fixed, and the issue is how to divide the cost of the warehouse among the divisions given the volume of business that they do. (The case where volume varies in response to price will be taken up in the next section.) The cost function might

be as follows:

$$c(\emptyset) = 0, \quad c(1) = 40, \quad c(2) = 60, \quad c(1, 2) = 75.$$

A customary solution under these circumstances is to divide total cost in proportion to some rough measure of each division's usage of the facility, such as stand-alone costs. In this case the allocation would be $x_1 = 30$, $x_2 = 45$.

Suppose now that division manager 1 institutes a policy that reduces his demands on warehouse space, so that the cost of all coalitions containing 1 decreases. We would certainly want this change to be reflected in a reduced charge for division 1, for otherwise there would be no incentive for the division manager to undertake the cost-cutting moves. This condition translates into the requirement that the cost allocation rule be *coalitionally monotonic* in the sense of (20). It is easy to see that allocating in proportion to stand-alone cost has this property (assuming that stand-alone costs are positive), as do a number of other natural methods, such as dividing cost savings equally.

We know from Theorem 4, however, that coalitional monotonicity is inconsistent with staying in the core when $|N| \geq 5$. Thus in more complicated examples we are faced with a choice: we can adopt an allocation rule that gives managers appropriate incentives to cut costs, or that gives them an incentive to cooperate, but not both. One could argue that in the present context the incentives created by monotonicity are more compelling than the incentives embodied in the core. The reason is that a firm is not a voluntary association of divisions that must be given an incentive to remain together (which is the argument for a core allocation), rather, the problem for the firm is to send the right signals to divisions to get them to act efficiently.

When we probe a bit deeper, however, we see that coalitional monotonicity is not sufficient to create the cost-cutting incentives that the firm would like. It encourages cost-cutting moves by some divisions *provided* that there are no offsetting cost increases by other divisions. Yet this is exactly what might happen in practice. Suppose, for example, that division 1 reduces the demands that it places on warehouse space by 1 unit, and simultaneously division 2 introduces wasteful policies that increase demands on space by 11 units. The new cost function is as follows:

$$c(\emptyset) = 0, \quad c(1) = 39, \quad c(2) = 71, \quad c(1, 2) = 85.$$

The principle of coalitional monotonicity does not apply here because the efficiency losses due to division 2 offset the efficiency gains by division 1. Nevertheless it seems clear that division 1 should be charged less than before and that division 2 should be charged more than before. The reason is that the *marginal* cost contribution of division 1 has decreased, whereas the marginal cost contribution of 2 has increased. Indeed, the partial derivative $c^1(S)$ has decreased by one unit for every coalition S , while $c^2(S)$ has increased by 11 units for every coalition. Allocation in proportion to stand-alone cost, however, yields the new charges $x_1 = 30.1$, $x_2 = 54.9$. In other words, division 1 is penalized for backsliding by division 2. This seems unreasonable.

To generalize these observations, fix a firm with a set of divisions N . We shall say that a cost allocation rule ϕ is *strongly monotonic* (does not create perverse incentives) if for every two cost functions c and C on N , and for every $i \in N$,

$$c^i(S) \leq C^i(S) \quad \text{for all } S \subseteq N \quad \text{implies } \phi_i(c) \leq \phi_i(C).$$

The reader may check that strong monotonicity implies coalitional monotonicity, but not vice versa.

Theorem 8 [Young (1985a)]. *For every set N the Shapley value is the unique cost allocation method that is symmetric and strongly monotonic.*

12. Cost allocation with variable output

We turn now to the problem of allocating joint costs when output can vary. Let there be n goods that are jointly produced by a firm, and let $C(q)$ be the joint cost of producing the bundle $q = (q_1, \dots, q_n)$, where $q_i \geq 0$ is the quantity of good i . We shall assume that $C(0) = 0$ and that C has continuous first partial derivatives on the domain R_+^n (one-sided on the boundary). (The latter condition can be relaxed slightly as will be indicated below.)

Given a target level of production $q^* > 0$, the goal is to find unit prices $p = (p_1, \dots, p_n)$ such that costs are exactly covered:

$$\sum_{i=1}^n p_i q_i^* = C(q^*). \quad (26)$$

This condition is known as the *break-even* or *zero-profit* constraint. Normally, C is defined to include the cost of capital, so (26) means that there is no profit after stockholders are allowed a normal return on their investment.

13. Ramsey prices

The traditional approach to allocating costs in this setting is to consider how demands adjust in response to the cost of the goods. In other words, we interpret the allocated cost of product i as a published price, and consumers demand any amount they wish at that price (there is no rationing). An example would be a publicly regulated utility that sets prices to cover total cost, and supplies whatever the market demands at these prices.

To simplify matters, we shall assume that demands for the products are independent. Let $q_i = Q_i(p_i)$ be the amount demanded of product i when prices are set at p_i . The inverse demand function Q_i is assumed to be strictly monotone decreasing in p_i and continuously differentiable. In a competitive setting the firm would produce quantities that clear the market, and price would equal marginal cost. As we have already noted, however, this will not work in a regulated setting

because marginal cost pricing will not normally cause the firm to break even. Indeed, many public utilities have the property that marginal costs *decrease* as quantity increases, in which case marginal cost pricing does not even *cover* total cost.

The solution proposed by Ramsey (1927) is to determine a price–quantity pair (p, q) that *maximizes consumer surplus subject to the break-even constraint*, that is, which maximizes

$$S(q) = \sum_{i=1}^n \int_0^{q_i} Q_i^{-1}(t) dt - C(q), \quad \text{subject to} \quad \sum_{i=1}^n p_i q_i - C(q) = 0. \quad (27)$$

This is a standard exercise in constrained optimization. Form the Lagrangian

$$L(q) = S(q) + \lambda \left(\sum_{i=1}^n q_i Q_i^{-1}(q_i) - C(q) \right).$$

Let $p_i = Q_i^{-1}(q_i)$ and $c_i(q) = \partial C(q) / \partial q_i$. A necessary condition that the pair (p, q) be optimal is that there exist a real number λ such that for all i

$$p_i - c_i(q) + \lambda \left(p_i + q_i \frac{\partial p_i}{\partial q_i} - c_i(q) \right) = 0. \quad (28)$$

Let $\mu_i = -(p_i/q_i)(\partial q_i / \partial p_i)$ be the demand elasticity for i at q . Then (28) is equivalent to

$$(p_i - c_i)/p_i = \lambda/(1 + \lambda)\mu_i. \quad (29)$$

This is the *Ramsey formula* and prices satisfying it are known as *Ramsey prices*.⁹ Their essential property is that *the percentage difference between price and marginal cost for each good is inversely proportional to the elasticity of demand for that good*. As W. Arthur Lewis (1949, p. 21) put it:

The principle is ... that those who cannot escape must make the largest contribution to indivisible cost, and those to whom the commodity does not matter much may escape. The man who has to cross Dupuit's bridge to see his dying father is mulcted thoroughly; the man who wishes only to see the scenery on the other side gets off lightly.

14. Aumann–Shapley prices

A drawback of Ramsey pricing is that it is highly sensitive to demand elasticities, which in practice may not be known with much accuracy. In this section we

⁹Ramsey first proposed this approach as a way of setting optimal tax rates on consumer goods (Ramsey, 1927). Later it was applied to public utility pricing by Manne (1952), Baumol and Bradford (1970), Boiteux (1971) and others.

examine an alternative approach that does not rely heavily on an analysis of demands. We posit instead that the quantities to be produced q^* are given exogenously (perhaps as the result of a back-of-the-envelope demand analysis), and the object is to allocate the cost $C(q^*)$ fairly among the n products. In this set-up a *cost allocation method* is a function $f(C, q^*) = p$ where p is a nonnegative vector of prices satisfying $\sum p_i q_i^* = C(q^*)$.

Many of the concepts introduced for discrete cost functions carry over to this case. Suppose, for example, that C is the sum of two cost functions, say capital cost C' and operating cost C'' . The cost allocation method f is *additive* if

$$C(q) = C'(q) + C''(q) \quad \text{for all } q \leq q^* \text{ implies } f(C, q^*) = f(C', q^*) + f(C'', q^*).$$

Let us now consider the analog of symmetry. We want to say that when two products look alike from the standpoint of costs, their prices should be equal. To see why this issue is not quite as straightforward as it first appears, consider a refinery that makes gasoline for the U.S. market (q_1) and gasoline for the British market (q_2). The quantity q_1 is expressed in U.S. gallons and the quantity q_2 in Imperial gallons; otherwise they are the same. Thus the cost function takes the form $C(q_1, q_2) = C'(q)$ where $q = 0.833q_1 + q_2$ is the total quantity in Imperial gallons. Note that the cost function C is not symmetric in q_1 and q_2 ; nevertheless 1 and 2 are essentially the same products. In this situation it is natural to require that the prices satisfy $p_1 = 0.833p_2$.

More generally, we say that f is *weakly aggregation invariant* [Billera and Heath (1982)] if, for every C, C' and q^* ,

$$C(q_1, q_2, \dots, q_n) = C'(\sum \lambda_i q_i) \quad \text{for all } q \leq q^* \\ \text{implies } f_i(C, q^*) = \lambda_i f(C', \sum \lambda_i q_i^*).$$

The example of U.S. and Imperial gallons may seem a bit contrived. Consider, however, the following situation. A refinery blends m grades of petroleum distillate to make n grades of gasoline for sale at the pump. Assume that the cost of blending is negligible. Let $C(y_1, y_2, \dots, y_m)$ be the joint cost of producing the m refinery grades in amounts y_1, y_2, \dots, y_m . Suppose that one unit of blend j uses a_{ij} units of grade i , $1 \leq i \leq m$, $1 \leq j \leq n$. Let $x = (x_1, x_2, \dots, x_n)$ be the amounts produced of the various blends. The joint cost of producing x is $C(Ax)$.

The cost allocation method f is *aggregation invariant* [Young (1985b)] if, for every $m \times n$ nonnegative matrix A , and every target level of production $x^* > 0$ such that $Ax^* > 0$,

$$f(C(Ax^*), x^*) = f(C, Ax^*) A.^{10}$$

¹⁰Several weaker forms of this axiom have been proposed in the literature. Suppose, for example, that A is a square diagonal matrix, that is, $a_{ii} > 0$ for all i and $a_{ij} = 0$ for all $i \neq j$. Thus each product is simply rescaled by a positive factor. Then the axiom says that the prices should be scaled accordingly. This is the *rescaling axiom* [Mirman and Tauman (1982a)]. Another variation is the following. Suppose that the n products can be divided into m disjoint, nonempty subgroups S_1, S_2, \dots, S_m such that the products within any subgroup are equivalent for cost purposes. By this we mean that the cost of

A third natural condition on the allocation rule is that, if costs are increasing (or at least nondecreasing) in every product, then all prices should be nonnegative. Formally, we say that the cost allocation method f is *nonnegative* if, for every cost function C and target $q^* > 0$,

$$C(q) \leq C(q') \text{ for all } 0 \leq q < q' \leq q^* \text{ implies } f(C, q^*) \geq 0. \tag{30}$$

Theorem 9 [Billera and Heath (1982), Mirman and Tauman (1982a)].¹¹ *There exists a unique cost allocation method f that is additive, weakly aggregation invariant, and nonnegative, namely*

$$\forall i, p_i = f_i(C, q^*) = \int_0^1 (\partial C(tq^*)/\partial q_i) dt. \tag{31}$$

In other words, the price of each product is its marginal cost *averaged* over all vectors $tq^*: 0 \leq t \leq 1$ that define the ray from 0 to q^* . These are known as *Aumann–Shapley (AS) prices* and are based on the Aumann–Shapley value for nonatomic games [Aumann and Shapley (1974)].

We remark that marginal cost pricing has all of these properties except that it fails to satisfy the break-even requirement. (In the special case where the cost function exhibits constant returns to scale, of course, the two methods are identical.) Samet and Tauman (1982) characterize marginal cost pricing axiomatically by dropping the break-even requirement and strengthening nonnegativity to require that $f(C, q^*) \geq 0$ whenever C is nondecreasing in a neighborhood of q^* .

Billera et al. (1978) describe how the AS method was used to price telephone calls at Cornell University. The university, like other large organizations, can buy telephone service in bulk from the telephone company at reduced rates. Two types of contracts are offered for each class of service. The first type of contract requires that the university buy a large amount of calling time at a fixed price, and any amount of time exceeding this quota is charged at a small incremental cost. The second type of contract calls for the university to buy a relatively small amount of time at a lower fixed price, and the incremental cost for calls over the quota is higher. There are seven classes of service according to the destination of the call: five classes of WATS lines, overseas (FX) lines, and ordinary direct distance dialing (DDD). The university buys contracts for each class of service based on its estimate of expected demand. Calls are then broken down into types according to three

producing any quantities x_1, \dots, x_n can be written in the form $C(\sum_{i \in S_1} x_i, \sum_{i \in S_2} x_i, \dots, \sum_{i \in S_m} x_i)$, for some cost function C defined on m variables. In this case the A -matrix consists of zeros and ones, and the sum of the rows is $(1, 1, \dots, 1)$. This version of aggregation invariance was proposed by Mirman and Neyman (1983), who dubbed it “consistency”. (Note that this use of the term consistency is not the same as in Section 5). This condition is very natural, and says that splitting a product into several equivalent products does not effectively change their prices. When the A -matrix is a *single* row vector of ones, the condition is called *weak consistency* [Mirman and Tauman (1982a)].

¹¹Mirman and Tauman (1982a) prove this result under the assumptions of rescaling and weak consistency instead of weak aggregation invariance.

criteria: the time of day the call is placed, the day on which it is placed (business or nonbusiness), and the type of line along which it is routed (five types of WATS, FX, or DDD). Time of day is determined by the hour in which the call begins: midnight to 1 A.M., 1–2 A.M., and so forth. Thus there are $n = 24 \times 7 \times 2 = 338$ types of “products.” Quantities are measured in minutes. For each combination of products demanded $q = q_1, q_2, \dots, q_n$, the least cost $C(q)$ of meeting these demands can be computed using an optimization routine. The cost is then allocated using Aumann–Shapley prices, that is, for each demand vector q^* the unit price is computed according to formula (31), which determines the rate for each type of call.¹²

15. Adjustment of supply and demand

When a regulated firm uses Aumann–Shapley prices, it is natural to ask whether there exists a level of production q^* such that supply equals demand, i.e., such that markets clear. The answer is affirmative under fairly innocuous assumptions on the demand and cost functions. Assume that: (i) there are m consumers $j = 1, 2, \dots, m$ and that each has a utility $u_j(q)$ for bundles $q = (q_1, q_2, \dots, q_n)$ that is continuous, quasi-concave, and monotonically increasing in q ; (ii) each consumer j has an initial money budget equal to $b_j > 0$. The cost function is assumed to satisfy the following conditions: (iii) there are no fixed costs [$C(0) = 0$]; (iv) $C(q)$ is continuous and nondecreasing; (v) $\partial C / \partial q_i$ is continuously differentiable except for at most a finite number of points on the ray $\{tq: 0 \leq t \leq q^*\}$; moreover, $\partial C / \partial q_i$ is continuous for all q such that $q_i = 0$ except perhaps when $q = 0$.

Theorem 10 [Mirman and Tauman (1982a, b)]. *Under assumptions (i)–(v) there exists a level of output q^* such that Aumann–Shapley prices clear the market, that is, there exists a distribution of q^* among the m consumers q^1, q^2, \dots, q^m such that, at the AS prices p , q^j maximizes $u_j(q)$ among all bundles $q \geq 0$ such that $p \cdot q \leq b_j$.*

For related results see Mirman and Tauman (1981, 1982a, b) Boes and Tillmann (1983), Dierker et al. (1985), and Boehm (1985). An excellent survey of this literature is given by Tauman (1988).

16. Monotonicity of Aumann–Shapley prices

Aumann–Shapley prices have an important property that is analogous to strong monotonicity in discrete cost-sharing games. Consider a decentralized firm in which

¹²The cost function computed in this manner is not differentiable, i.e., there may be abrupt changes in slope for neighboring configurations of demands. It may be proved, however, that the characterization of Aumann–Shapley prices in Theorem 9 also holds on the larger domain of cost functions for which the partial derivatives exist almost everywhere on the diagonal and are integrable.

each of the n product lines is supervised by a division manager. Corporate headquarters wants a cost accounting scheme that encourages managers to innovate and reduce costs. Suppose that the cost function in period 1 is C and the cost function in period 2 is C' . We can say that i 's contribution to costs decreases if $\partial C'(q)/\partial q_i \leq \partial C(q)/\partial q_i$ for all $q \leq q^*$. A cost allocation method f is *strongly monotonic* if whenever i 's contribution to cost decreases, then i 's unit price does not increase, that is, if $f_i(C', q^*) \leq f_i(C, q^*)$.

Theorem 11 [Young (1985b)]. *For every set N there is a unique cost allocation method that is aggregation invariant and strongly monotonic, namely, Aumann–Shapley pricing.*¹³

17. Equity and competitive entry

Cost allocation not only provides internal signals that guide the firm's operations, it may also be a response to external competitive pressures. As before, we consider a firm that produces n products and whose cost of production is given by $C(q)$, where $C(0) = 0$ and $C(q)$ is continuous and nondecreasing in q . The firm is said to be a *natural monopoly* if the cost function is subadditive:

$$\text{for all } q, q' \geq 0, \quad C(q) + C(q') \geq C(q + q').$$

(This is the analog of condition (1) for discrete cost functions.) Typical examples are firms that rely on a fixed distribution network (subways, natural gas pipelines, electric power grids, telephone lines), and for this reason their prices are often regulated by the state. If such a firm is subjected to potential competition from firms that can enter its market, however, then it may be motivated to regulate its own prices in order to deter entry. Under certain conditions this leads to a price structure that can be justified on grounds of equity as well. This is the subject of *contestable market theory*.¹⁴

The *core* of the cost function C for a given level of production q^* is the set of all price vectors $p = p_1, p_2, \dots, p_n$ such that

$$\sum p_i q_i^* = C(q^*) \quad \text{and} \quad p \cdot q \leq C(q) \quad \text{whenever } 0 \leq q \leq q^*.$$

Let $q = Q(p)$ be the inverse demand function for the firm's products. A price vector p is *anonymously equitable* if p is in the core of C given $q^* = Q(p)$.

Consider a firm that is currently charging prices p^* and is subject to competitive entry. For the prices to be sustainable, revenue must cover cost given the demand at these prices, that is,

$$p^* \cdot q^* \geq C(q^*) \quad \text{when } q^* = Q(p^*).$$

¹³ Monderer and Neyman (1988) show that the result holds if we replace aggregation invariance by the weaker conditions of rescaling and consistency (see footnote 10).

¹⁴ See Baumol et al. (1977), Panzar and Willig (1977), Sharkey and Telser (1978), Baumol et al. (1982).

If p^* is not in the core, then there exists some bundle $q \leq q^*$ such that $p^* \cdot q > C(q)$. This means, however, that another firm can profitably enter the market. The new firm can undercut the old firm's prices and capture the portion q of the old firm's market; moreover it can choose these prices so that $p \cdot q \geq C(q)$. (We assume the new firm has the same production technology, and hence the same cost function, as the old firm.) Thus to deter entry the old firm must choose prices p^* that are in the core.

In fact, entry deterrence requires more than being in the core. To see why, consider a subset of products S , and let $Q_S(p_S, p_{N-S}^*)$ be the inverse demand function for S when the entering firm charges prices p_S and the original firm charges prices p_{N-S}^* for the other products. The entering firm can undercut p^* on some subset S and make a profit unless it is the case that

$$\text{for all } S \subseteq N, \quad p_S \leq p_S^* \text{ and } q_S \leq Q_S(p_S, p_{N-S}^*) \text{ implies } p_S q_S \leq C(q_S, 0_{N-S}). \quad (32)$$

A vector of prices p^* that satisfies (32) is said to be *sustainable*. Sustainable prices have the property that no entrant can anticipate positive profits by entering the market and undercutting these prices. This means, in particular, that sustainable prices yield zero profits, that is, costs are covered exactly.

We now examine conditions under which AS prices are sustainable. The cost function C exhibits *cost complementarity* if it is twice differentiable and all second-order partial derivatives are nonincreasing functions of q :

$$\frac{\partial^2 C(q)}{\partial q_i \partial q_j} \leq 0 \quad \text{for all } i, j.$$

The inverse demand function $Q(p)$ satisfies *weak gross substitutability* if, for every i , Q_i is differentiable, and $\partial Q_i / \partial p_j \geq 0$ for every distinct i and j . Q is *inelastic below* p^* if

$$\frac{\partial Q_i(p) / Q_i(p)}{\partial p_i / p_i} \geq -1 \quad \text{for every } i \text{ and all } p \leq p^*.$$

Theorem 12 [Mirman, Tauman, and Zang (1985a)]. *If C satisfies cost complementarity and $Q(p)$ is upper semicontinuous, then there exists an AS vector p^* that is in the core of C given $q^* = Q(p^*)$. Moreover, if Q satisfies weak gross substitutability and is inelastic below p^* then p^* is sustainable.*

18. Incentives

One of the reasons why firms allocate joint costs is to provide their divisions with incentives to operate more efficiently. This problem can be modelled in a variety of ways. On the one hand we may think of the cost allocation mechanism as an incentive to change the cost function itself (i.e., to innovate). This issue was discussed

in Section 16, where we showed that it leads to AS prices. In this section we take a somewhat different view of the problem. Let us think of the firm as being composed of n divisions that use inputs provided by the center. Suppose, for simplicity, that each division i uses one input, and that the cost of jointly producing these inputs is $C(q_1, q_2, \dots, q_n)$. Each division has a technology for converting q_i into a marketable product, but this technology is unknown to the center and cannot even be observed *ex post*. Let $r_i(q_i)$ be the *maximum revenue* that division i can generate using the input q_i and its most efficient technology. Assume that the revenue generated by each division is independent of the revenue generated by the other divisions. The *firm's* objective is to maximize net profits

$$\max_q F(q) = \sum r_i(q_i) - C(q). \quad (33)$$

Since the true value of r_i is known only to division i , the firm cannot solve the profit maximization problem posed by (33). Instead, it would like to design a cost allocation scheme that will give each division the incentive to “do the right thing”. Specifically we imagine the following sequence of events. First each division sends a message $m_i(q_i)$ to the center about what its revenue function is. The message may or may not be true. Based on the vector of messages $m = (m_1, m_2, \dots, m_n)$, the center determines the quantities of inputs $q_i(m)$ to provide and allocates the costs according to some scheme $t = g(m, C)$, where t_i is the total amount that the division is assessed for using the input q_i (i.e., t_i/q_i is its unit “transfer” price). The combination of choices $(q(m), g(m, C))$ is called a *cost allocation mechanism*. Note that the function g depends on the messages as well as on the cost function C , so it is more general than the cost allocation methods discussed in earlier sections. Note also that the cost assessment does not depend on the divisional revenues, which are assumed to be unobservable by the center.

We impose the following requirements on the cost allocation mechanism. First, the assessments $t = g(m, C)$ should exactly equal the center's costs:

$$\sum g_i(m, C) = C(q(m)). \quad (34)$$

Second, the center chooses the quantities that would maximize profit assuming the reported revenue functions are accurate:

$$q(m) = \operatorname{argmax} \sum m_i(q_i) - C(q). \quad (35)$$

Each division has an incentive to reveal its true revenue function provided that reporting some other message would never yield a higher profit, that is, if reporting the true revenue function is a dominant strategy. In this case the cost allocation mechanism $(q(m), g(m, C))$ is *incentive-compatible*:

$$\text{for every } i \text{ and every } m, r_i(q_i(r_i, m_{-i})) - g_i((r_i, m_{-i}), C) \geq r_i(q_i(m)) - g_i(m, C). \quad (36)$$

Theorem 13 [Green and Laffont (1977), Hurwicz (1981), Walker (1978)]. *There exists no cost allocation mechanism $(q(m), g(m, C))$ that, for all cost functions C and all revenue functions r_i , allocates costs exactly (34), is efficient (35), and incentive-compatible (36).*

One can obtain more positive results by weakening the conditions of the theorem. For example, we can devise mechanisms that are efficient and incentive-compatible, though they may not allocate costs exactly. A particularly simple example is the following. For each vector of messages m let

$$q(m_{-i}) = \operatorname{argmax}_q \sum_{j \neq i} m_j(q_j) - C(q). \tag{37}$$

Thus $q(m_{-i})$ is the production plan the center would adopt if i 's message (and revenue) is ignored. Let

$$P_i(m) = \sum_{j \neq i} m_j(q_j(m)) - C(q(m)),$$

where $q(m)$ is defined as in (35), and let

$$P_i(m_{-i}) = \sum_{j \neq i} m_j(q_j(m_{-i})) - C(q(m_{-i})).$$

$P_i(m)$ is the profit from adopting the optimal production plan based on *all* messages but not taking into account i 's reported revenue, while $P_i(m_{-i})$ is the profit if we ignore both i 's message and its revenue. Define the following cost allocation mechanism: $q(m)$ maximizes $\sum m_i(q_i) - C(q)$ and

$$g_i(m, C) = P_i(m_{-i}) - P_i(m). \tag{38}$$

This is known as the *Groves mechanism*.

Theorem 14 [Groves (1973, 1985)]. *The Groves mechanism is incentive-compatible and efficient.*

It may be shown, moreover, that any mechanism that is incentive-compatible and efficient is equivalent to a cost allocation mechanism such that $q(m)$ maximizes $\sum m_i(q_i) - C(q)$ and $g_i(m, C) = A_i(m_{-i}) - P_i(m)$, where A_i is any function of the messages that does not depend on i 's message [Green and Laffont (1977)].

Under more specialized assumptions on the cost and revenue functions we can obtain more positive results. Consider the following situation. Each division is required to meet some exogenously given demand or target q_i^0 that is unknown to the center. The division can buy some or all of the required input from the center, say q_i , and make up the deficit $q_i^0 - q_i$ by some other (perhaps more expensive) means. (Alternatively we may think of the division as incurring a penalty for not meeting the target.)

Let $(q_i^0 - q_i)_+$ denote the larger of $q_i^0 - q_i$ and 0. Assume that the cost of covering the shortfall is linear:

$$c_i(q_i) = a_i(q_i^0 - q_i)_+ \quad \text{where } a_i > 0.$$

The division receives a fixed revenue r_i^0 from selling the target amount q_i^0 . Assume that the center knows each division's unit cost a_i but not the values of the targets. Consider the following mechanism. Each division i reports a target $m_i \geq 0$ to the center. The center then chooses the efficient amount of inputs to supply assuming that the numbers m_i are true. In other words, the center chooses $q(m)$ to minimize total revealed cost:

$$q(m) = \operatorname{argmin} [a_i(m_i - q_i)_+ + C(q)]. \quad (39)$$

Let the center assign a nonnegative weight λ_i to each division, where $\sum \lambda_i = 1$, and define the cost allocation scheme by

$$g_i(m, C) = a_i q_i(m) - \lambda_i [\sum a_j q_j(m) - C(q(m))]. \quad (40)$$

In other words, each division is charged the amount that it saves by receiving $q_i(m)$ from the center, minus the fraction λ_i of the joint savings. Notice that if $q_i(m) = 0$, then division i 's charge is zero or negative. This case arises when i 's unit cost is lower than the center's marginal cost of producing q_i .

The cost allocation scheme g is *individually rational* if $g_i(m, C) \leq a_i q_i^0$ for all i , that is, if no division is charged more than the cost of providing the good on its own.

Theorem 15 [Schmeidler and Tauman (1994)]. *The mechanism described by (39) and (40) is efficient, incentive-compatible, individually rational, and allocates costs exactly.*

Generalizations of this result to nonlinear divisional cost functions are discussed by Schmeidler and Tauman (1994). It is also possible to implement cost allocations via mechanisms that rely on other notions of equilibrium, e.g., Nash equilibrium, strong equilibrium or dominance-solvable equilibrium. For examples of this literature see Young (1980, 1985c, Ch. 1), Jackson and Moulin (1992), and Moulin and Schenker (1992).

19. Conclusion

In this chapter we have examined how cooperative game theory can be used to justify various methods for allocating common costs. As we have repeatedly emphasized, cost allocation is not merely an exercise in mathematics, but a practical problem that calls for translating institutional constraints and objectives into mathematical language.

Among the practical issues that need to be considered are the following. What are the relevant units on which costs are assessed – persons, aircraft landings, towns, divisions in a firm, quantities of product consumed? This is a nontrivial issue because it amounts to a decision about what is to be treated equally for purposes of the cost allocation. A second practical issue concerns the amount of available information. In allocating the cost of water service between two towns, for example, it is unrealistic to compute the cost of serving all possible subsets of individuals. Instead we would probably compute the cost of serving the two towns together and apart, and then allocate the cost savings by some method that is weighted by population. Another type of limited information concerns levels of demand. In theory, we might want to estimate the cost of different quantities of water supply, as well as the demands for service as a function of price. Ramsey pricing requires such an analysis. Yet in most cases this approach is infeasible. Moreover, such an approach ignores a key institutional constraint, namely the need to allocate costs so that both towns have an incentive to accept. (Ramsey prices need not be in the core of the cost-sharing game.)

This brings us to the third modelling problem, which is to identify the purpose of the cost allocation exercise. Broadly speaking there are three objectives: the allocation decision should be *efficient*, it should be *equitable*, and it should create appropriate *incentives* for various parts of the organization. These objectives are closely intertwined. Moreover, their interpretation depends on the institutional context. In allocating water supply costs among municipalities, for example, efficiency calls for meeting fixed demands at least cost. An efficient solution will not be voluntarily chosen, however, unless the cost allocation provides an incentive for all subgroups to participate. This implies that the allocation lie in the core. This condition is not sufficient for cooperation, however, because the parties still need to coordinate on a *particular solution* in the core. In this they are guided by principles of equity. In other words, equity promotes efficient solutions because it helps the participants realize the potential gains from cooperation.

Cost allocation in the firm raises a somewhat different set of issues. Efficiency is still a central concern, of course, but creating voluntary cooperation among the various units or divisions is not, because they are already bound together in a single organization. Incentives are still important for two reasons, however. First, the cost allocation mechanism sends price signals within the firm that affects the decisions of its divisions, and therefore the efficiency of the outcome. Second, it creates external signals to potential competitors who may be poised to enter the market. For prices to be sustainable (i.e., to deter entry) they need to lie in the core; indeed that must satisfy a somewhat stronger condition than being in the core. Thus incentive considerations prompted by external market forces are closely related to incentives that arise from the need for cooperation.

If the firm has full information on both costs and demands, and is constrained to break even, then the efficient (second-best) solution is given by Ramsey pricing. This solution may or may not be sustainable. If demand data is not known but

the cost function is, there is a good case for using Aumann–Shapley pricing, since this is essentially the only method that rewards innovations that reduce marginal costs. Under certain conditions Aumann–Shapley prices are also sustainable. If key aspects of the cost structure are known only to the divisions, however, then it is impossible to design a general cost allocation mechanism that implements an efficient outcome in dominant strategies and fully allocates costs. More positive results are attainable for particular classes of cost functions, and for mechanisms that rely on weaker forms of equilibrium.

We conclude from this discussion is that there is no single, all-purpose solution to the cost allocation problem. Which method suits best depends on context, organizational goals, and the amount of information available. We also conclude, however, that cost allocation is a significant real-world problem that helps motivate the central concepts in cooperative game theory, and to which the theory brings important and unexpected insights.

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