The Kernel and Bargaining Set for Convex Games[1])

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Abstract: It is shown that for convex games the bargaining set $\mathcal{M}_1^{(i)}$ (for the grand coalition) coincides with the core. Moreover, it is proved that the kernel (for the grand coalition) of convex games consists of a unique point which coincides with the nucleolus of the game.

1. Introduction

Convex games were introduced by SHAPLEY [1971], where it was shown that these are precisely the games for which the core has a certain "regular" structure (see Section 5 below). It was also shown by SHAPLEY [1971] that convex games have a unique VON NEUMANN-MORGENSTERN solution which coincides with the core, and that their SHAPLEY value is essentially the center of gravity of the extreme points of the core.

One purpose of this paper is to prove that the *kernel* (for the grand coalition) of convex games consists of a unique point (Section 7). As such, it coincides with the *nucleolus* of the game and therefore occupies a central position in the core (which is different, in general, from that of the SHAPLEY value). We also prove that the *bargaining set* $M_1^{(i)}$ (for the grand coalition) coincides with the core (Section 8). Thus, it appears that for convex games, many solution concepts either coincide with the core or occupy a central position within the core.

The proofs of these results are quite elaborate and require many lemmas drawn from various topics of game theory (Sections 2 – 6). In particular, one requires a detailed analysis of the structure of the *pre-kernel* of a game. We develop this theory, which is interesting in its own sake, in Sections 2 – 4, before specializing our attention to convex games. The pre-kernel is related to the "pseudo-kernel" used in previous investigations [MASCHLER and PELEG, 1966 and 1967], but has the advantage of a somewhat simpler definition and is invariant under strategic equivalence. If the game is 0-monotonic, i.e., strategically equivalent to a 0-normalized monotonic game, then the pre-kernel and the kernel coincide (for the grand coalition). This is the case when the game is superadditive, and hence, in particular, when the game is convex.

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2. The Pre-Kernel and its Relation to the Kernel and to the Pseudo-Kernel

In this section we shall introduce an auxiliary solution concept, called the *pre-kernel* of a game and show that if the game satisfies certain monotonicity conditions this pre-kernel coincides with the pseudo-kernel or the kernel of the game.

We shall consider a *cooperative game with side payments*, $(N; v)$, where $N = \{1,2,\dots,n\}$ is its set of *players* and v, its *characteristic function*, is an arbitrary [1]) function from the subsets of N (called *coalitions*) to the real numbers.

Given an n-tuple $x = (x_1,x_2,\dots,x_n)$ of real numbers, we define the *excess* of a coalition S with respect to x (in $(N; v)$) to be:

$$e(S,x) \equiv v(S) - x(S), \tag{2.1}$$

where $x(S)$ is a short notation for $\sum_{i\in S} x_i$ whenever $S \neq \emptyset$, and $x(\emptyset) = 0$.

An n-tuple $x = (x_1,x_2,\dots,x_n)$ of real numbers will be called a *pre-imputation* (in $(N; v)$) if it satisfies:

$$x(N) = v(N). \tag{2.2}$$

It will be called an *imputation* if it satisfies, in addition, the individual rationality condition:

$$x_i \geq v(\{i\}), \quad i = 1,2,\dots,n. \tag{2.3}$$

It will be called a *pseudo-imputation* if it satisfies (2.2) and:

$$x_i \geq 0, \quad i = 1,2,\dots,n. \tag{2.4}$$

For each n-tuple $x = (x_1,x_2,\dots,x_n)$ we define the *maximum surplus* of a player k against a player l, $k \neq l$, with respect to x, to be:

$$s_{k,l}(x) = \operatorname*{Max}_{S:k\in S,l\notin S} e(S,x). \tag{2.5}$$

Definition 2.1:

A pre-imputation x is said to belong to the *pre-kernel* of a game $\Gamma \equiv (N; v)$ (for the grand coalition) [2]), if

$$s_{k,l}(x) = s_{l,k}(x) \quad \text{for all} \quad k, l \in N, k \neq l. \tag{2.6}$$

The pre-kernel of a game Γ (for the grand coalition) will be denoted by $\mathscr{P}r\mathscr{K}(\Gamma)$ or, shortly, by $\mathscr{P}r\mathscr{K}$.

Lemma 2.2:

The per-kernel is a relative invariant under strategic equivalence [3]).

The proof is immediate.

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[1]) None of the traditional conditions are imposed on v at this point; in particular, $v(\emptyset)$ need not be 0.

[2]) The definition can be extended to cover situations in which coalition-structures other than the grand coalition are being considered.

[3]) I.e., it undergoes the transformation $x \to \alpha x + a$ when $v(S)$ is replaced by $\alpha v(S) + a(S)$ for each coalition S. Here α is a real positive constant and $a = (a_1,a_2,\dots,a_n)$ is an n-tuple of real numbers.

Definition 2.3:

A game $(N;v)$ is called *monotonic* if

$$v(S) \leq v(T) \quad \text{whenever} \quad S \subset T. \tag{2.7}$$

It is called 0-*monotonic* if it is strategically equivalent to a 0-normalized [1]) monotonic game, i.e., if

$$v(S) \leq v(T) - \sum_{i \in T-S} v(\{i\}) \quad \text{whenever} \quad S \subset T. \tag{2.8}$$

Note that the relation (2.7), unlike (2.8), is *not* invariant under strategic equivalence. In fact, every game is strategically equivalent to a monotonic game [2]).

Theorem 2.4:

If $\Gamma \equiv (N;v)$ is a 0-monotonic game and if $x \in \mathscr{P}r\mathscr{K}(\Gamma)$, then x is an imputation (see (2.3)).

Proof:

By Lemma 2.2, there is no loss of generality in assuming that Γ is already 0-normalized. The theorem is obviously true for 1-person games. Assume that Γ has at least two players, and let $x \in \mathscr{P}r\mathscr{K}(\Gamma)$. It will be convenient to denote by $\mathscr{D}(x)$ the set of all coalitions of maximum excess among the coalitions other than $\emptyset$ and N:

$$\mathscr{D}(x) = \{S : S \neq \emptyset, N \text{ and } e(S,x) \geq e(R,x) \quad \text{whenever} \quad R \neq \emptyset, N\}. \tag{2.9}$$

Suppose (2.3) is incorrect; then there exists a player k such that $x_k < 0$. We shall first show that $k \in S$ whenever $S \in \mathscr{D}(x)$. Indeed, if $k \notin S$ for some coalition S and if $S \neq N - \{k\}$ then, by (2.1) and (2.7),

$$e(S \cup \{k\}, x) = v(S \cup \{k\}) - x(S) - x_k > v(S) - x(S) = e(S,x);$$

consequently, $S \notin \mathscr{D}(x)$. Also, $N - \{k\} \notin \mathscr{D}(x)$, because, by (2.2) and (2.7):

$$e(N - \{k\}, x) = v(N - \{k\}) - x(N - \{k\}) \leq v(N) - x(N) + x_k = x_k,$$

whereas, for example $e(\{k\},x) = -x_k$ is larger.

Let R be a coalition in $\mathscr{D}(x)$ and let $l \in N - R$. Since k belongs to each coalition in $\mathscr{D}(x)$, it follows from (2.5) and (2.9) that $s_{k,l}(x) > s_{l,k}(x)$, contrary to (2.6). This contradiction shows that (2.3) is correct, thereby completing the proof.

Definition 2.5:

An imputation x (see (2.3)) is said to belong to the *kernel* of a game $\Gamma \equiv (N;v)$ (for the grand coalition) [3]), if

$$s_{k,l}(x) \leq s_{l,k}(x) \quad \text{or} \quad x_l = v(\{l\}) \quad \text{for all} \quad k, l \in N, k \neq l. \tag{2.10}$$

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[1]) A game is called 0-normalized if the value of each single-person coalition is 0.

[2]) This will be the case whenever the numbers a_i (see above) are sufficiently large.

[3]) For extension of the definition to situations in which coalition-structures other than the grand coalition are considered see, e.g., DAVIS and MASCHLER [1965].

The kernel of a game Γ (for the grand coalition) will be denoted by $\mathscr{K}(\Gamma)$ or, shortly, by $\mathscr{K}$.

Lemma 2.6:

The kernel is a relative invariant under strategic equivalence.

The proof of this well known result (see, e.g., DAVIS and MASCHLER [1965]) is immediate.

Theorem 2.7:

The kernel and the pre-kernel coincide for 0-monotonic games (and hence in particular for superadditive games).

Proof [1]*:*

By Lemmas 2.2 and 2.6, we can limit the discussion to 0-normalized games. The theorem is obviously true for a 1-person game. Let $\Gamma \equiv (N; v)$ be an n-person 0-normalized game, $n \geq 2$. By Theorem 2.4 and Definitions 2.1 and 2.5, $\mathscr{P}r\mathscr{K}(\Gamma) \subset \mathscr{K}(\Gamma)$. Let $x \in \mathscr{K}(\Gamma)$; we shall complete the proof if we show that $x \in \mathscr{P}r\mathscr{K}(\Gamma)$. Denote (see (2.9)).

$$M = \cap \{S : S \in \mathscr{D}(x)\}.$$

Clearly, $M \neq N$. If $M \neq \emptyset$, let $k \in M$ and let $l \in N - M$. Clearly, $s_{k,l}(x) > s_{l,k}(x)$; hence, by (2.10), $x_l = 0$. Let S_0 be an arbitrary coalition in $\mathscr{D}(x)$, then, by (2.1) and (2.7),

$$e(S_0, x) = v(S_0) - x(S_0) = v(S_0) - x(N) \leq v(N) - x(N) = 0.$$

If $l \in N - S_0$ then $e(\{l\}, x) = 0$ and consequently $\{l\} \in \mathscr{D}(x)$. Thus, $M = \emptyset$, since $\mathscr{D}(x)$ contains both S_0 and $\{l\}$; this contradicts the assumption $M \neq \emptyset$. It follows that, in fact, $M = \emptyset$. Suppose $x \notin \mathscr{P}r\mathscr{K}(\Gamma)$. Then players i and j exist such that

$$s_{i,j}(x) > s_{j,i}(x). \tag{2.11}$$

Consequently $x_j = 0$ (see (2.10)). There exists a coalition S_1 in $\mathscr{D}(x)$ which does not contain player i, because $M = \emptyset$. Therefore, by (2.7),

$$e(S_1 \cup \{j\}, x) = v(S_1 \cup \{j\}) - x(S_1) - x_j \geq v(S_1) - x(S_1) = e(S_1, x).$$

Thus, by (2.5) and (2.9) $s_{j,i}(x)$ cannot be smaller than $s_{i,j}(x)$, contrary to (2.11). This contradiction shows that $x \in \mathscr{P}r\mathscr{K}(\Gamma)$, thereby completing the proof of the theorem.

The following remarks are intended to orient the reader who is versed with the literature, especially MASCHLER and PELEG [1966 and 1967]. The proofs are similar to the proofs of Theorems 2.4 and 2.7 and will be omitted.

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[1]) This result follows from Theorem 2.4 and from known results stated by MASCHLER and PELEG [1967]; however, it is much more convenient to provide here an independent, shorter proof of this important result.

Remark 2.8:

The *pseudo-kernel* (for the grand coalition) of a game $\Gamma \equiv (N;v)$ is denoted by $\mathscr{P}s\mathscr{K}(\Gamma)$ and defined exactly as in Definition 2.5, except that x is assumed to be a pseudo-imputation (see (2.4)) and that $x_l = v(\{l\})$ in (2.10) is replaced by $x_l = 0$. It is an auxiliary solution concept which is *not* a relative invariant under strategic equivalence. If Γ satisfies

$$v(S) \leq v(T) \quad \text{whenever} \quad S \subset T, S \neq \emptyset, T \neq N \tag{2.12}$$

(quasi-monotonicity), and

$$v(\{i\}) + v(N) \geq v(N - \{i\}), \quad i = 1,2,\ldots,n, \tag{2.13}$$

and if $x \in \mathscr{P}r\mathscr{K}(\Gamma)$, then x is a pseudo-imputation. Moreover, under the conditions (2.12)–(2.13), $\mathscr{P}r\mathscr{K}(\Gamma) = \mathscr{P}s\mathscr{K}(\Gamma)$.

Remark 2.9:

We can interpret the pre-kernel of a game Γ as follows: Take a game Γ^* which is monotonic, satisfies $v(\emptyset) \geq 0$, and is strategically equivalent to Γ. The "inverse image" of the pseudo-kernel of Γ^* under this equivalence is the pre-kernel of Γ. Thus, loosely speaking, up to strategic equivalence, the pre-kernel is one of many pseudo-kernels a game may have.

Remark 2.10:

Since the pseudo-kernel of a game is not empty if $v(N) \geq 0$ (see Maschler and Peleg [1966 and 1967], it follows that the pre-kernel of *any* cooperative game is not empty.

3. The Structure of the Pre-Kernel

Let x be a pre-imputation in a game $\Gamma \equiv (N;v)$. We wish to find necessary and sufficient conditions that $x \in \mathscr{P}r\mathscr{K}(\Gamma)$. First, let us partition the set of all the coalitions into subsets $\mathscr{E}^1(x), \mathscr{E}^2(x), \ldots, \mathscr{E}^m(x)$ which are of highest excess, of the second highest excess, etc. Thus,

$$\mathscr{E}^1(x) \equiv \{S : e(S,x) \geq e(T,x) \text{ all } T\}, \tag{3.1}$$

$$\mathscr{E}^{i+1}(x) \equiv \left\{ S : e(S,x) \geq e(T,x) \quad \text{if and only if} \quad T \notin \bigcup_{h=1}^{i} \mathscr{E}^h(x) \right\}; \tag{3.2}$$

and $m \equiv m(x)$ is the highest index i for which $\mathscr{E}^i(x) \neq \emptyset$. Clearly, $1 \leq m \leq 2^n$.

We shall refer to the coalitions in $\mathscr{E}^i(x)$ as the *i-th stage maximum excess coalitions*. Their excess $s^i(x)$ will be called the i-th *stage maximum excess*:

$$s^i(x) = e(S,x) \quad \text{where} \quad S \in \mathscr{E}^i(x). \tag{3.3}$$

Denote:

$$i(k,l,x) \equiv \text{Min}\,\{i : \exists\, S \in \mathscr{E}^i(x), k \in S, l \notin S\}; \tag{3.4}$$

then, clearly (see (2.5)),

$$s_{k,l}(x) = s^{i(k,l,x)}(x)\,. \tag{3.5}$$

The following lemma follows from Definition 2.1 and (3.5).

Lemma 3.1:

A pre-imputation x belongs to $\mathscr{P}r\mathscr{K}(\Gamma)$ if and only if $i(k,l,x) = i(l,k,x)$ for each pair of distinct players k and l.

We can now reverse the procedure. Consider an arbitrary *ordered partition* [1]) $(\mathscr{E}^1, \mathscr{E}^2, \ldots, \mathscr{E}^m)$ of the set of all coalitions which has the property:

$$i(k,l) = i(l,k) \quad \text{for all} \quad l, k \in N, l \neq k\,, \tag{3.6}$$

where

$$i(k,l) \equiv \text{Min}\,\{i : \exists\, S \in \mathscr{E}^i, k \in S, l \notin S\}\,. \tag{3.7}$$

Every pre-imputation x satisfying

$$\mathscr{E}^i(x) = \mathscr{E}^i, \quad i = 1,2,\ldots,m\,, \tag{3.8}$$

must belong to $\mathscr{P}r\mathscr{K}(\Gamma)$.

Observe that the set of pre-imputations satisfying (3.8) for a fixed ordered partition is a (possibly empty) convex set determined by the linear inequalities:

$$\left.\begin{aligned} &x(N) = v(N) \\ &e(S,x) > e(T,x) \quad \text{whenever} \quad S \in \mathscr{E}^{\mu}, T \in \mathscr{E}^{\nu}, \mu < \nu \\ &e(S,x) = e(T,x) \quad \text{whenever} \quad S, T \in \mathscr{E}^{\mu}\,. \end{aligned}\right\} \tag{3.9}$$

Our next object is to find conditions which assure us that an ordered partition satisfies (3.6). The following definition is helpful:

Let $\mathscr{E}$ be a collection of subsets of N and let T be a non-empty subset of N. Let $\{T_1, T_2, \ldots, T_\alpha\}$ be the partition of T characterized by:

$$k, l \in T_j \Leftrightarrow (k, l \in T \text{ and } k \in A \text{ if and only if } l \in A \text{ for all } A \in \mathscr{E})\,. \tag{3.10}$$

Definition 3. 2:

The set $\{T_1, T_2, \ldots, T_\alpha\}$ defined by (3.10) will be called *the partition of T into equivalence classes induced by $\mathscr{E}$.*

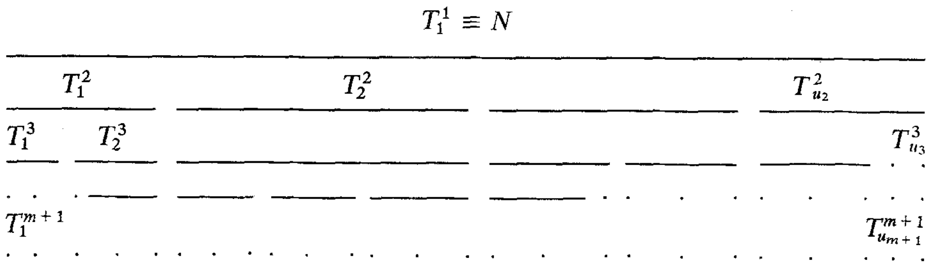
Equivalence classes in this connection mean equivalence classes determined by the relation "occur simultaneously in the coalitions of $\mathscr{E}$".

Let $\underset{\sim}{E} \equiv (\mathscr{E}^1, \mathscr{E}^2, \ldots, \mathscr{E}^m)$ be an arbitrary ordered partition of the set of coalitions. We shall now construct a sequence of successively finer partitions of N, called the *profile $P(\underset{\sim}{E})$ generated by $\underset{\sim}{E}$.*

We start by denoting $\{N\}$ as $\{T_1^1\}$. Suppose that $\{T_1^i, T_2^i, \ldots, T_{u_i}^i\}$ has been

[1]) It is important to distinguish the stages. Thus, for $N = \{1,2,3\}$, we consider $(\{\emptyset, N\}, \{\{1,2\}, \{1,3\}, \{2,3\}\}, \{\{1\}, \{2\}, \{3\}\})$ to be different from $(\{\emptyset, N\}, \{\{1\}, \{2\}, \{3\}\}, \{\{1,2\}, \{1,3\}, \{2,3\}\})$. For this reason we use the vector notation and call the partition ordered.

defined, and is a partition of N. Let $\{T_{j,1}^{i+1}, T_{j,2}^{i+1}, \dots, T_{j,\alpha_j}^{i+1}\}$ be the set of equivalence classes which are induced by $\mathscr{E}^i$ on T_j^i, $j = 1,2,\dots,u_i$. Renumber $T_{j,v}^{i+1}$ lexicographically in the lower indices to from $\{T_1^{i+1}, T_2^{i+1}, \dots, T_{u_{i+1}}^{i+1}\}$. The collection $P(E) \equiv \{T_1^1; T_1^2, \dots, T_{u_2}^2; \dots; T_1^{m+1}, \dots, T_{u_{m+1}}^{m+1}\}$ is the required profile. The term is suggested by the diagram below.



Clearly,

$$\{T_1^{m+1}, T_2^{m+1}, \dots, T_{u_{m+1}}^{m+1}\} = \{\{1\}, \{2\}, \dots, \{n\}\}, \tag{3.11}$$

but in general the equivalence classes may all become 1-person sets at an earlier stage. The next three lemmas follow directly from the definitions.

Lemma 3.3:

If $1 \le i_0 \le i_1 \le m + 1$ then

$$T_{j_0}^{i_0} \cap T_{j_1}^{i_1} \neq \emptyset \quad \textit{implies} \quad T_{j_1}^{i_1} \subset T_{j_0}^{i_0}. \tag{3.12}$$

Lemma 3.4:

If $S \in \mathscr{E}^i$ then S is a union of sets T_j^{i+1}'s.

Lemma 3.5:

If $S \in \mathscr{E}^i$ then S is a union of sets $T_j^{i_1}$'s whenever $i < i_1 \le m + 1$.

Henceforth, the profile $P(\underset{\sim}{E}(x))$ generated by the ordered partition $\underset{\sim}{E}(x) \equiv (\mathscr{E}^1(x), \mathscr{E}^2(x), \dots, \mathscr{E}^m(x))$ will be called, shortly, *the profile of* x.

Lemmas 3.3 – 3.4 indicate that the profile can be described as a "partition tree"; namely, as a tree whose vertices are the sets T_j^i, with $T_1^1 = N$ the root, such that the vertices that follow a vertex T_j^i and are adjacent to it form a partition of T_j^i.

One of the advantages of the profile is the fact that it enables one to describe condition (3.6) in a more visual fashion:

Lemma 3.6:

Let $P(\underset{\sim}{E})$ be a profile generated by an ordered partition $\underset{\sim}{E} \equiv (\mathscr{E}^1, \mathscr{E}^2, \dots, \mathscr{E}^m)$. The condition (3.6) is equivalent to the following separation condition:

If $T_k^{i+1} \subset T_j^i, T_l^{i+1} \subset T_j^i$ and $k \neq l$, then there exists a coalition S in $\mathscr{E}^i$ such that $T_k^{i+1} \subset S$ and $T_l^{i+1} \cap S = \emptyset$, $i = 1,2,3,\dots,m$.

Proof:

Name a player in T_k^{i+1} and a player in T_l^{i+1} by k and l, respectively. Then by Lemma 3.3, k, l belong to the same equivalence class $T_{v_i}^{i'}$ for each $i' \leq i$. They belong to disjoint equivalence classes T_k^{i+1} and T_l^{i+1}. By Definition 3.2,

(i) $k \in A \Leftrightarrow l \in A$ whenever $A \in \mathscr{E}^{i'}$ and $i' \leq i - 1$

and either

(ii) $\exists S \in \mathscr{E}^i$ such that $k \in S$ and $l \notin S$,

or

(iii) $\exists S \in \mathscr{E}^i$ such that $l \in S$ and $k \notin S$

(or both). Now

(i) and (ii) $\Leftrightarrow i(k,l) = i$,

(i) and (iii) $\Leftrightarrow i(l,k) = i$.

It follows that condition (3.6) is equivalent to the validity of (i), (ii), and (iii) for $i = 1, \ldots, m$ and for all $k, l \in N$, $k \neq l$.

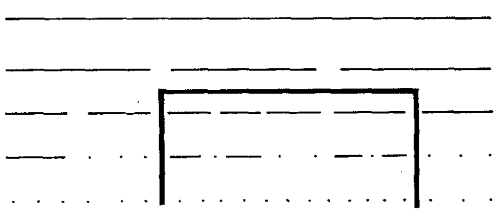
By Lemma 3.1 (see also (3.8)), and Lemma 3.6, we can now state:

Theorem 3.7:

Let x be a pre-imputation in a game Γ and let $P(\underline{E}(x))$ be the profile of x. With this notation, $x \in \mathscr{P}r\mathscr{K}(\Gamma)$ if and only if the separation condition in Lemma 3.6 is satisfied, with $\mathscr{E}^i = \mathscr{E}^i(x)$, $i = 1, 2, \ldots, m$.

4. The Stage Games

From a visual point of view, a profile may contain smaller profiles. The figure below exhibits one profile within the original one. This suggests that smaller games can be constructed from the original game, which contain fewer players. Such games can serve for induction purposes.



Theorem 3.7 indicates that the equivalence classes play a role at each stage, rather than the players. Even the maximum excess coalitions of the various stages are unions of such equivalence classes (Lemma 3.4). This suggests that it is possible under an appropriate interpretation to regard the equivalence classes themselves as players in some sense. In the present section we shall develop these heuristic ideas in a precise way.

Definition 4.1:

Let x be a pre-imputation in a game $\Gamma \equiv (N;v)$ and let $P(\underset{\sim}{E}(x)) = \{T_1^1; T_1^2, \ldots, T_{u_2}^2; \ldots; T_1^{m+1}, \ldots, T_{u_{m+1}}^{m+1}\}$ be the profile of x. Let $T^* = \{T_{j_1}^i, T_{j_2}^i, \ldots, T_{j_\alpha}^i\}$ be a fixed nonempty set of equivalence classes belonging to a fixed stage i. The *stage game generated by x and T^** is a game $(T^*;v^*)$ whose players are the members of T^* and whose characteristic function is defined by

$$\left.\begin{aligned} v^*(T^*) &= x(T_{j_1}^i) + x(T_{j_2}^i) + \cdots + x(T_{j_\alpha}^i) = x(T) \\ v^*(S^*) &= \underset{Q:Q \subset N-T}{\text{Max}} [v(S \cup Q) - x(Q)],\ S^* \subset T^*, S^* \neq T^*. \end{aligned}\right\} \tag{4.1}$$

Here, $T \equiv T_{j_1}^i \cup T_{j_2}^i \cup \ldots \cup T_{j_\alpha}^i$ and if $S^* = \{T_{v_1}^i, T_{v_2}^i, \ldots, T_{v_\beta}^i\} \subset T^*$, then $S = T_{v_1}^i \cup T_{v_2}^i \cup \ldots \cup T_{v_\beta}^i$.

Remark 4.2:

Note that $(x(T_{j_1}^i), x(T_{j_2}^i), \ldots, x(T_{j_\alpha}^i))$ is a pre-imputation in the above stage game.

Definition 4.3:

A pre-imputation x in a game Γ is said to belong to the *core* of Γ if

$$e(S,x) \leq 0 \quad \text{all} \quad S. \tag{4.2}$$

The core will be denoted by $\mathscr{C}(\Gamma)$ or, shortly, $\mathscr{C}$.

Remark 4.4:

If $x \in \mathscr{C}(\Gamma)$ then x is an imputation (see (2.3)).

Proof:

Individual rationality is nothing but (4.2) applied to single-person coalitions.

Lemma 4.5:

If $\Gamma \equiv (N;v)$ is a monotonic game (see Definition 2.3) *and if $x \in \mathscr{C}(\Gamma)$ then the stage game $(T^*;v^*)$ generated by x and T^** (see Definition 4.1) *is also a monotonic game.*

Proof:

Quasi-monotonicity (see (2.12)) follows directly from (4.1) and the monotonicity of Γ. (We even make no use of the fact that $x \in \mathscr{C}(\Gamma)$.) Let $S^* \subset T^*, S^* \neq T^*$, then, by (4.1), there exists a subset Q_0 of $N - T$ such that $v^*(S^*) = v(S \cup Q_0) - x(Q_0)$. Thus, by (4.1) and (4.2), $v^*(S^*) - v^*(T^*) = v(S \cup Q_0) - x(Q_0) - x(T) \leq v(T \cup Q_0) - x(T \cup Q_0) \leq 0$, and this concludes the proof.

We are now in a position to state the main theorem of this section:

Theorem 4.6:

If $x \in \mathscr{P}r\mathscr{K}(\Gamma)$ and if $\Gamma^* \equiv (T^*;v^*)$ is a stage game generated by x and a set $T^* = \{T_{j_1}^i, T_{j_2}^i, \ldots, T_{j_\alpha}^i\}$ of equivalence classes of the i-th stage, $1 \leq i \leq m + 1$, then the α-tuple $x^* \equiv (x(T_{j_1}^i), x(T_{j_2}^i), \ldots, x(T_{j_\alpha}^i))$ belongs to $\mathscr{P}r\mathscr{K}(\Gamma^*)$.

Proof:

We shall use stars to denote entities related to Γ^*. By Remark 4.2, x^* is a pre-imputation in Γ^*; consequently, there is nothing more to prove if $\alpha = 1$. Suppose $\alpha > 1$. We have to show that x^* satisfies the analogue of (2.6):

$$s^*_{T^i_\rho, T^i_\sigma}(x^*) = s^*_{T^i_\sigma, T^i_\rho}(x^*) \tag{4.3}$$

for all "stage players" T^i_ρ, $T^i_\sigma \in T^*$, $\rho \neq \sigma$. Here,

$$s^*_{T^i_\rho, T^i_\sigma}(x^*) \equiv \text{Max}\,\{e^*(S^*, x^*): S^* \subset T^*, T^i_\rho \in S^*, T^i_\sigma \notin S^*\} \tag{4.4}$$

and

$$e^*(S^*, x^*) \equiv v^*(S^*) - x^*(S^*) = v^*(S^*) - x(S) \tag{4.5}$$

where, as in (4.1), S is defined as $T^i_{\nu_1} \cup T^i_{\nu_2} \cup \ldots \cup T^i_{\nu_\beta}$ if $S^* = \{T^i_{\nu_1}, T^i_{\nu_2}, \ldots, T^i_{\nu_\beta}\}$ with $\{\nu_1, \nu_2, \ldots, \nu_\beta\} \subset \{j_1, j_2, \ldots, j_\alpha\}$. By (4.4), (4.5), and (4.1),

$$s^*_{T^i_\rho, T^i_\sigma}(x^*) = \text{Max}\,\{\underset{Q:Q \subset N-T}{\text{Max}}\; e(S \cup Q, x): S^* \subset T^*, T^i_\rho \in S^*, T^i_\sigma \notin S^*\}\,.$$

If k is any player in T^i_ρ and l is any player in T^i_σ, we assert that in fact

$$s^*_{T^i_\rho, T^i_\sigma}(x^*) = \text{Max}\,\{e(R, x): R \subset N, k \in R, l \notin R\} \equiv s_{k,l}(x)\,. \tag{4.6}$$

The argument for this runs as follows: *A priori*, there should be an inequality $\leq$, because the set of candidates for maximization increases. It is known however that $s_{k,l}(x) = s^{i(k,l,x)}(x)$ (see (3.5)). Since T^i_ρ and T^i_σ are distinct equivalence classes of the i-th stage, it follows that $i(k,l,x) \leq i - 1$ (see (3.4)). Let R^0 be a coalition containing k and not l such that $s_{k,l}(x) = e(R^0, x)$; then $R^0 \in \mathscr{E}^{i(k,l,x)}(x)$ (see (3.5)). Since $i(k,l,x) \leq i - 1$, it follows from Lemma 3.5 that R^0 is a union of equivalence classes of the i-th stage and, moreover, $R^0 \supset T^i_\rho$ and $R^0 \cap T^i_\sigma = \emptyset$. Thus, R^0 has the form $T^i_{\nu_1} \cup T^i_{\nu_2} \cup \ldots \cup T^i_{\nu_\beta} \cup Q$, where $\{\nu_1, \nu_2, \ldots, \nu_\beta\}$ is a subset of $\{j_1, j_2, \ldots, j_\alpha\}$ containing ρ and not σ, and $Q \subset N - T$. It is therefore a member of the smaller set of candidates, which proves (4.6). In a similar fashion we prove that $s^*_{T^i_\sigma, T^i_\rho}(x^*) = s_{l,k}(x)$. Since $x \in \mathscr{P}r\mathscr{K}(\Gamma)$, (4.3) now follows from (2.6).

Remark 4.7:

A converse theorem stating that if $x^* \in \mathscr{P}r\mathscr{K}(\Gamma^*)$ for each stage game then $x \in \mathscr{P}r\mathscr{K}(\Gamma)$ is trivially true, because the stage game $(T^*; v^*)$ where T^* is the set of all equivalence classes of the stage $m + 1$ is isomorphic to Γ under the transformation $\{k\} \to k$, $k = 1, 2, \ldots, n$ (see (3.11)).

Remark 4.8:

Theorem 4.6 generalizes results of [1]) MASCHLER and PELEG [1967]. The stage game in which T^* consists of all the equivalence classes of a given stage is known as the *intermediate game*. The stage game in which $i = m + 1$ and $\{j_1, j_2, \ldots, j_\alpha\}$ are players of a given T^i_j is known as a *reduced game*.

---

[1]) The results of MASCHLER and PELEG [1967], however, refer to the wider class of pseudo-kernels (see Remark 2.9).

5. The Stage Games Resulting from an Imputation in the Core of a Convex Game

A cooperative game $(N;v)$ is called *convex* if its characteristic function v satisfies

$$v(\emptyset) = 0\,, \tag{5.1}$$

$$v(A) + v(B) \leq v(A \cup B) + v(A \cap B) \quad \text{all} \quad A, B \subset N\,. \tag{5.2}$$

Convex games were introduced by SHAPLEY [1971], where their properties and their importance in game theory were discussed. At present, all we need to know of their properties, beyond (5.1) and (5.2), is that they have nonempty cores [1]).

The purpose of this section is to show that for an x in the core of a convex game, all the stage games are also convex. We shall also study some properties of these stage games.

Convex games are super-additive but not necessarily monotonic. However, if the characteristic function satisfies

$$v(\{i\}) \geq 0\,, \quad i = 1,2,\ldots,n\,, \tag{5.3}$$

then monotonicity follows from super-additivity. Since being a convex game is an invariant under strategic equivalence, it follows that convex games are 0-monotonic. In view of Theorem 2.7 and Remark 2.8, we can therefore state:

Theorem 5.1:

If Γ is a convex game then

$$\mathscr{K}(\Gamma) = \mathscr{P}r\mathscr{K}(\Gamma)\,. \tag{5.4}$$

If Γ is a convex game with a nonnegative characteristic function then

$$\mathscr{K}(\Gamma) = \mathscr{P}s\mathscr{K}(\Gamma)\,. \tag{5.5}$$

Note that (5.2) is equivalent to

$$e(A,x) + e(B,x) \leq e(A \cup B,x) + e(A \cap B,x) \tag{5.6}$$

for all $A,B \subset N$ and for any n-tuple x.

Theorem 5.2:

If $\Gamma \equiv (N;v)$ is a convex game and if x belongs to its core, then each stage game generated by x is convex.

Proof:

Let $(T^*;v^*)$ be a stage game generated by x and $T^* = \{T^i_{j_1}, T^i_{j_2}, \ldots, T^i_{j_\alpha}\}$. We shall show that

$$v^*(S^*) + v^*(R^*) \leq v^*(S^* \cup R^*) + v^*(S^* \cap R^*) \quad \text{all} \quad S^*, R^* \subset T^*\,. \tag{5.7}$$

---

[1]) It is proved by SHAPLEY [1971] that they can be characterized by the fact that their core is, so called, *regular* – i.e., for each x in the core, the family $\mathscr{S}_x = \{S : x(S) = v(S)\}$ is closed under union and intersection. (Compare Lemma 6.2 below.)

Relation (5.7) evidently holds if $S^* \subset R^*$ or if $R^* \subset S^*$. We can therefore assume that $S^*, R^* \neq T^*$. Let S and R be the unions of the members of S^* and R^*, respectively. By (4.1), there exists Q_1 and Q_2 in $N - T$ such that

$$\begin{aligned} v^*(S^*) + v^*(R^*) &= v(S \cup Q_1) - x(Q_1) + v(R \cup Q_2) - x(Q_2) \\ &\leq v((S \cup R) \cup (Q_1 \cup Q_2)) + v((S \cap R) \cup (Q_1 \cap Q_2)) \\ &\quad - x(Q_1 \cup Q_2) - x(Q_1 \cap Q_2) \\ &\leq \operatorname*{Max}_{Q:Q \subset N-T} [v((S \cup R) \cup Q) - x(Q)] \\ &\quad + \operatorname*{Max}_{Q:Q \subset N-T} [v((S \cap R) \cup Q) - x(Q)] \\ &= \operatorname*{Max}_{Q:Q \subset N-T} [v((S \cup R) \cup Q) - x(Q)] + v^*(S^* \cap R^*). \end{aligned}$$

If $S^* \cup R^* = T^*$, then

$$\operatorname*{Max}_{Q:Q \subset N-T} [v((S \cup R) \cup Q) - x(Q)] \leq v^*(T^*) = v^*(S^* \cup R^*),$$

because $x \in \mathscr{C}(\Gamma)$ (see Lemma 4.5). If $S^* \cup R^* \neq T^*$, then, by (4.1),

$$\operatorname*{Max}_{Q:Q \subset N-T} [v((S \cup R) \cup Q) - x(Q)] = v^*(S^* \cup R^*).$$

In any case (5.7) holds.

The following lemma furnishes important information concerning the particular Q's for which the maxima in (4.1) are achieved, when the game is convex.

Lemma 5.3:

Let $\Gamma \equiv (N;v)$ be a convex game and let x be an arbitrary n-tuple of real numbers. Let R be a coalition in $\mathscr{E}^i(x)$ and let S_1 and S_2 be subsets of R and $N - R$, respectively. Suppose Q_1 and Q_2 are subsets of $N - R$ and R, respectively, such that

$$\operatorname*{Max}_{Q:Q \subset N-R} e(S_1 \cup Q,x) = e(S_1 \cup Q_1,x), \tag{5.8}$$

and

$$\operatorname*{Max}_{Q:Q \subset R} e(S_2 \cup Q,x) = e(S_2 \cup Q_2,x). \tag{5.9}$$

Let $R \cup Q_1$ and Q_2 belong to $\mathscr{E}^{\mu_1}(x)$ and $\mathscr{E}^{\mu_2}(x)$, respectively. Under these conditions:

(i) $\mu_1 \leq i$,

(ii) $\mu_2 \leq i$,

(iii) *If* $e(S_1 \cup Q_1,x) \neq e(S_1,x)$ *then* $\mu_1 < i$,

(iv) *If* $e(S_2 \cup Q_2,x) \neq e(S_2 \cup R,x)$ *then* $\mu_2 < i$.

Proof:

By (5.6),

$$e(S_1 \cup Q_1,x) + e(R,x) \leq e(R \cup Q_1,x) + e(S_1,x). \tag{5.10}$$

By (5.8), $e(S_1 \cup Q_1,x) \geq e(S_1,x)$. Consequently,

$$e(R,x) \leq e(R \cup Q_1,x), \tag{5.11}$$

and strict inequality holds if the hypothesis of (iii) is satisfied. This proves (i) and (iii) (see (3.1), (3.2)). Similarly, by (5.6),

$$e(S_2 \cup Q_2, x) + e(R, x) \leq e(S_2 \cup R, x) + e(Q_2, x). \tag{5.12}$$

By (5.9), $e(S_2 \cup Q_2, x) \geq e(S_2 \cup R, x)$. Consequently,

$$e(R, x) \leq e(Q_2, x), \tag{5.13}$$

and strict inequality holds if the hypothesis of (iv) is satisfied. This proves (ii) and (iv).

Corollary 5.4:

Q_1 and Q_2 of Lemma 5.3 can be chosen to be unions of equivalence classes of stage $i + 1$ in the profile of x.

Proof:

Cases (i) and (ii) of Lemma 5.3 and Lemma 3.5.

Corollary 5.5:

If $R \in \mathscr{E}^1(x)$ and Γ is convex then

$$\operatorname*{Max}_{Q:Q \subset N-R} e(S \cup Q, x) = e(S, x) \quad \text{whenever} \quad S \subset R, \tag{5.14}$$

$$\operatorname*{Max}_{Q:Q \subset R} e(S \cup Q, x) = e(S \cup R, x) \quad \text{whenever} \quad S \subset N - R. \tag{5.15}$$

Proof:

Cases (iii) and (iv) of Lemma 5.3 (see (3.1)).

Lemma 5.3 can be effectively used in devising computer programs for computing the kernels of convex games. Note that it can be applied to any stage game $(T^*; v^*)$ of a stage greater than i, when the union of the members of T^* is equal to R. We shall subsequently apply Lemma 5.3 for the particular cases $i = 1,2$ and the stage game being of stage $m + 1$.

Lemma 5.6:

Let x be a pre-imputation in a game $\Gamma \equiv (N; v)$ satisfying $v(\emptyset) = 0$. Under these conditions, exactly one of the following relations holds:

(i) $\mathscr{D}(x) = \mathscr{E}^1(x)$,

(ii) $\mathscr{D}(x) \cup \{\emptyset, N\} = \mathscr{E}^1(x)$,

(iii) $\mathscr{D}(x) = \mathscr{E}^2(x)$ *and* $\mathscr{E}^1(x) = \{\emptyset, N\}$.

If $x \notin \mathscr{C}(\Gamma)$, *case* (i) *holds, and if* $x \in \mathscr{C}(\Gamma)$, *case* (ii) *or case* (iii) *holds.*

Proof:

Compare (2.9) with (3.1) and (3.2) and the definition of the core (4.2).

Lemma 5.7:

If x belongs to the core of a convex game $\Gamma \equiv (N;v)$, and it $R \in \mathscr{D}(x)$, then

$$\underset{Q:Q\subset N-R}{\text{Max}}\ e(S \cup Q, x) = \text{Max}\,[e(S,x), e(S \cup (N-R), x)] \tag{5.16}$$

whenever $S \subset R$, and

$$\underset{Q:Q\subset R}{\text{Max}}\, e(S \cup Q, x) = \text{Max}\,[e(S,x), e(S \cup R, x)] \tag{5.17}$$

whenever $S \subset N - R$.

Proof:

Corollary 5.5, if $\mathscr{D}(x) \subset \mathscr{E}^1(x)$. If this is not the case then, by Lemma 5.6, $\mathscr{E}^1(x) = \{\emptyset, N\}$ and $\mathscr{D}(x) = \mathscr{E}^2(x)$. The result now follows from Lemma 5.3, cases (iii) and (iv).

Corollary 5.8:

Let x belong to the core of a convex game and let R be a coalition in $\mathscr{D}(x)$ (see (2.9)*). Consider the stage games (T_R^*, v_R^*) and (T_{N-R}^*, v_{N-R}^*) of any stage i, such that the union of the members of T_R^* is equal to R and the union of the members of T_{N-R}^* is equal to $N - R$. Under these conditions*

$$\left.\begin{aligned} v_R^*(T_R^*) &= x(R) \\ v_R^*(S^*) &= \text{Max}\,[v(S), v(S \cup (N-R)) - x(N-R)] \end{aligned}\right\} \tag{5.18}$$

whenever $S^* \subset T_R^*$, $S^* \neq T_R^*$, and

$$\left.\begin{aligned} v_{N-R}^*(T_{N-R}^*) &= x(N-R) \\ v_{N-R}^*(S^*) &= \text{Max}\,[v(S), v(S \cup R) - x(R)] \end{aligned}\right\} \tag{5.19}$$

whenever $S^* \subset T_{N-R}^*, S^* \neq T_{N-R}^*$. Here, as before, S is defined as $T_{\mu_1}^i \cup T_{\mu_2}^i \cup \ldots \cup T_{\mu_\beta}^i$ if $S^* = \{T_{\mu_1}^i, T_{\mu_2}^i, \ldots, T_{\mu_\beta}^i\}$.

In other words, the values of v_R^* and v_{N-R}^* are reduced to two possibilities when Γ is convex, $x \in \mathscr{C}(\Gamma)$, and $R \in \mathscr{D}(x)$.

6. Separating Near-Ring Collections and Balanced Collections

Definition 6.1:

A collection $\mathscr{E}$ of subsets of a set N is called a *near-ring* [1]) if:

$$A, B \in \mathscr{E} \Rightarrow \begin{cases} A \cup B = N, & \text{or} \\ A \cap B = \emptyset, & \text{or} \\ \text{both } A \cup B \in \mathscr{E} \text{ and } A \cap B \in \mathscr{E}. \end{cases} \tag{6.1}$$

Lemma 6.2:

If Γ is a convex game and x is an arbitrary n-tuple of real numbers, then $\mathscr{D}(x)$ (see (2.9)) *is a near-ring.*

[1]) We are grateful to J. R. ISBELL for suggesting this term.

Proof:

Combine (5.6) with (2.9).

Definition 6.3:

A collection $\mathscr{E}$ of subsets of a set N is said to be *completely separating* (over N) if for each ordered pair (k, l) of distinct elements of N there exists a set in $\mathscr{E}$ containing k and not l.

Definition 6.4:

A collection $\mathscr{E}$ of subsets of a set N is called *separating* (over N) if for each ordered pair (k, l) of distinct elements of N, whenever a coalition exists in $\mathscr{E}$ that contains k and not l, another coalition exists in $\mathscr{E}$ that contains l and not k.

Let $\mathscr{E}$ be a separating collection of subsets of N. Let $T_1, T_2, \ldots, T_u$ be the equivalence classes induced by $\mathscr{E}$ on N. Let $\tilde{N}$ be a subsets of N containing exactly one member from each equivalence class. Clearly, the collection $\tilde{\mathscr{E}} \equiv \{S \cap \tilde{N}: S \in \mathscr{E}\}$ is completely separating over $\tilde{N}$.

The study of the separating and the completely separating collections has been quite useful to kernel theory (see, e.g., MASCHLER and PELEG [1966][1])). In fact, the separation condition in Lemma 3.6 simply states that the set $\{S \cap T_j^i: S \in \mathscr{E}^i\}$ is separating over the equivalence class T_j^i. A particular case of this observation is:

Lemma 6.5:

If $x \in \mathscr{P}r\mathscr{K}(\Gamma)$ *then* $\mathscr{D}(x)$ *is a separating collection.*

Proof:

Theorem 3.7 and Lemma 5.6, in view of the fact that deleting the coalitions $\emptyset$ and N from a collection of coalitions does not change its being or not being separating.

It will be convenient to associate with a subset S of N its *characteristic vector* χ^S, where

$$\chi_i^S = \begin{cases} 1 & \text{if} \quad i \in S \\ 0 & \text{if} \quad i \notin S. \end{cases} \tag{6.2}$$

Definition 6.6:

A collection $\mathscr{E} = \{S_1, S_2, \ldots, S_\alpha\}$ of subsets of a set N is called *balanced,* if positive constants $c_1, c_2, \ldots, c_\alpha$ exist, such that

$$\sum_{\nu=1}^{\alpha} c_\nu \chi^{S_\nu} = \chi^N. \tag{6.3}$$

$\mathscr{E}$ is called *minimal balanced* if it is balanced and none of its proper sub-collections is balanced. $\mathscr{E}$ is called *weakly balanced* if (6.3) is satisfied by nonnegative constants $c_1, c_2, \ldots, c_\alpha$. These constants are called *balancing coefficients* or *weights.*

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[1]) See PELEG [1968] and SHALHEVET for additional properties of separating collections.

Balanced and minimal balanced collections were introduced[1]) and studied by BONDAREVA [1963] and SHAPLEY [1967]. They are useful to the study of various solution concepts such as the core (see BONDAREVA, SCARF, SHAPLEY [1967], SHAPLEY and SHUBIK), the bargaining set (see MASCHLER), and, as we shall see here, the kernel. See PELEG [1965] for additional information concerning their structure.

Lemma 6.7:

A balanced collection is separating.

The proof is straightforward. The converse statement, however, is not true. Indeed, any set of six minimal winning coalitions in the 7-person projective game (see, e.g., VON NEUMANN and MORGENSTERN [p. 470]) is completely separating and not even weakly balanced. It turns out, however, that imposing a near-ring requirement (see Definition 6.1) is a remedy:

Theorem 6.8:

Every separating near-ring collection $\mathscr{E}$ of subsets of a set $N = \{1, 2, \ldots, n\}$, with the exception of $\mathscr{E} = \{\emptyset\}$, is weakly balanced.

Proof:

There is no loss of generality in assuming that $\mathscr{C}$ is completely separating. The theorem obviously holds for $n = 1$. Assume $n \geq 2$. Let $\mathscr{E}_i \equiv \{S : S \in \mathscr{E}, i \notin S\}$ and let $\tilde{\mathscr{E}}_i$ denote the set of elements of $\mathscr{E}_i$ which are maximal under inclusion. We shall show that $\tilde{\mathscr{E}}_i$ is a partition of $N - \{i\}$. Indeed, it follows from the complete separating property that each member of $N - \{i\}$ belongs to at least one element of $\tilde{\mathscr{E}}_i$, and by the near-ring property, the elements of $\tilde{\mathscr{E}}_i$ are disjoint. We next observe that the collection $\tilde{\mathscr{E}} \equiv \tilde{\mathscr{E}}_1 \cup \tilde{\mathscr{E}}_2 \cup \ldots \cup \tilde{\mathscr{E}}_n$ is balanced; in fact, if $c(S)$ is the number of elements i such that $S \in \tilde{\mathscr{E}}_i$, then $\{c(S)/(n-1) : S \in \tilde{\mathscr{E}}\}$ are balancing coefficients. Hence $\mathscr{E} \supset \tilde{\mathscr{E}}$ is at least weakly balanced. This completes the proof.

7. The Kernel of a Convex Game

The purpose of this section is to show that the kernel of a convex game (for the grand coalition) consists of a single point.

Lemma 7.1:

If Γ is a convex game then $\mathscr{P}r\mathscr{K}(\Gamma) \subset \mathscr{C}(\Gamma)$.

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[1]) BONDAREVA uses the term "$(q - \theta)$-covering" ["reduced $(q - \theta)$-covering"] to denote the pair consisting of the set of weights and the set of characteristic vectors of a balanced [minimal balanced] collection. Sometimes, (e.g., SCARF) "balanced" means what we here call "weakly balanced", however, note that every weakly balanced collection contains a balanced collection. It was convenient in SHAPLEY [1967] to rule out the collection $\{N\}$; this exception is not needed here.

Proof:

The theorem obviously holds if Γ is a 1-person game. Assume that Γ is a multi-person game and let $x \in \mathscr{P}r\mathscr{K}(\Gamma)$; then, since $\mathscr{D}(x)$ is a separating collection (Lemma 6.5), it follows that

$$\bigcup_{S:S\in\mathscr{D}(x)} S = N, \tag{7.1}$$

$$\bigcap_{S:S\in\mathscr{D}(x)} S = \emptyset, \tag{7.2}$$

because $\mathscr{D}(x)$ is not empty and its members are proper nonempty subsets of N.

By applying Lemma 6.2 repeatedly to unions and intersections of members of $\mathscr{D}(x)$, one concludes that either there exist two coalitions S_1 and T_1 in $\mathscr{D}(x)$ such that $S_1 \cap T_1 = \emptyset$, or there exists two coalitions S_2 and T_2 in $\mathscr{D}(x)$ such that $S_2 \cup T_2 = N$. In view of the fact that $e(N,x) = e(\emptyset,x) = 0$, it follows from (2.9) and (5.6) that $e(S,x) \leq 0$ for every coalition in Γ. Consequently, $x \in \mathscr{C}(\Gamma)$ (Definition 4.3). This concludes the proof.

Theorem 7.2:

The kernel (for the grand coalition) of a convex game consists of a single point.

Proof:

In view of Theorem 5.1 it is sufficient to prove that the pre-kernel of a convex game consists of a unique point. The theorem certainly holds for 1-person games. We shall proceed by induction, assuming that Γ is an n-person game, $n \geq 2$. Let $x, y \in \mathscr{P}r\mathscr{K}(\Gamma)$. Denote

$$s(x) = e(S,x), \quad S \in \mathscr{D}(x),$$

$$s(y) = e(R,y), \quad R \in \mathscr{D}(y).$$

Without loss of generality we may assume that

$$s(x) \leq s(y). \tag{7.3}$$

Since $\mathscr{D}(y)$ is a separating near-ring collection (Lemmas 6.2 and 6.5) which contains a nonempty subset of N, it must contain a balanced collection $\mathscr{R} = \{R_1, R_2, \ldots, R_\alpha\}$ (Theorem 6.8). If $R_j \notin \mathscr{D}(x)$ then $e(R_j,x) < s(x) \leq s(y) = e(R_j,y)$. Consequently, $x(R_j) > y(R_j)$. If $R_j \in \mathscr{D}(x)$ then we can at least conclude that $x(R_j) \geq y(R_j)$. Multiplying these inequalities by the balancing coefficients and summing over j, we obtain $x(N) \geq y(N)$, with equality occuring only if $\mathscr{R} \subset \mathscr{D}(x)$ and $s(x) = s(y)$. But equality must occur because $x(N) = v(N) = y(N)$ (see (2.2)). All we need to conclude from this observation is that there exists a coalition R in $\mathscr{D}(x) \cap \mathscr{D}(y)$ and, moreover,

$$x(R) = y(R), \qquad x(N-R) = y(N-R). \tag{7.4}$$

Now let $m(x)+1$ be the last stage of the profile of x and $m(y)+1$ the last stage of the profile of y. Consider the stage games (T_R^*, v_R^*) and (T_{N-R}^*, v_{N-R}^*)

as given in Corollary 5.8, with respect to x and $i = m(x) + 1$. Consider also the analogous stage games (T_R^{**}, v_R^{**}) and $(T_{N-R}^{**}, v_{N-R}^{**})$ with respect to y and $i = m(y) + 1$. The players in all four of these stage games are 1-element sets (see (3.11)).

Since x and y belong to $\mathscr{P}r\mathscr{K}(\Gamma)$, they *a fortiori* belong to $\mathscr{C}(\Gamma)$ (Lemma 7.1). By Theorem 5.2, all the stage games are convex; therefore their kernels and pre-kernels coincide (Theorem 5.1). Rename the players, if necessary, so that $R = \{1,2,\dots,r\}$, then, by Theorem 4.6, we conclude that

$$(x_1, x_2, \dots, x_r) \in \mathscr{P}r\mathscr{K}(T_R^*, v_R^*), \tag{7.5}$$

$$(x_{r+1}, x_{r+2}, \dots, x_n) \in \mathscr{P}r\mathscr{K}(T_{N-R}^*, v_{N-R}^*), \tag{7.6}$$

and

$$(y_1, y_2, \dots, y_r) \in \mathscr{P}r\mathscr{K}(T_R^{**}, v_R^{**}), \tag{7.7}$$

$$(y_{r+1}, y_{r+2}, \dots, y_n) \in \mathscr{P}r\mathscr{K}(T_{N-R}^{**}, v_{N-R}^{**}). \tag{7.8}$$

Now, by (7.4) and Corollary 5.8, the games $(T_R^*; v_R^*)$ and $(T_R^{**}; v_R^{**})$ are *the same game*, because they have the same set of players and the same characteristic function. Similarly, (T_{N-R}^*, v_{N-R}^*) and $(T_{N-R}^{**}, v_{N-R}^{**})$ are the same game. Identical games possess the identical kernels. Since all of them have fewer than n players, then, by the induction hypothesis, their kernels (= pre-kernels) consist of single points. Consequently, by (7.5)–(7.8), we conclude that $x = y$. This completes the proof, in view of the fact that the kernel is known not be be empty (see MASCHLER and PELEG [1966]).

Theorem 7.2 brings to an end the main part in the study of the kernel for convex games. We know exactly its shape; namely – a point. There remains, however, the problem of locating this point; i.e., stating where it lies – preferably in geometrical terms. Fortunately, general theorems are available in the literature which enable us to complete this task: SCHMEIDLER introduced the *nucleolus* of a game and proved that it is a nonempty subset of the kernel, consisting of a unique point[1]). In a subsequent paper we shall present a characterization of the location of the nucleolus for a general cooperative game. It turns out that the nucleolus lies precisely at the, so called, *lexicographic center* of the game; a point which, for games with a nonempty core, lies in the relative interior[2]) of the core and occupies there a central position. Roughly speaking, the lexicographic center is obtained by "compressing" the core, pushing inward at equal l_1-distances the hyperplanes $x(S) = v(S)$, $S \neq \emptyset, N$, but stopping the push of each

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[1]) See also KOHLBERG.

[2]) In SHAPLEY [1971] it is shown that the core of an indecomposable n-person convex game is $(n-1)$-dimensional, and that the core of a decomposable n-person convex game is the $(n-p)$-dimensional cartesian product of the cores of its minimal (i.e., indecomposable) components, which are themselves convex games.

of them just short of causing the inside to become empty. We summarize the results which are relevant to the present study in:

Corollary 7.3:

For convex games, the kernel (for the grand coalition) and the nucleolus (the lexicographic center) coincide.

8. The Bargaining Set $\mathscr{M}_1^{(i)}$ of a Convex Game

Let x be an imputation in a game $(N;v)$ (see (2.3)). An *objection* of a player k against a player l, with respect to x, is a pair $(\hat{y};C)$, where C is a coalition containing player k and not containing player l and $\hat{y}$ is a vector[1]) whose indices are the members of C, satisfying $\hat{y}(C) = v(C)$ and $\hat{y}_i > x_i$ for each i in C. A *counter objection* to this objection is a pair $(\hat{z};D)$, where D is a coalition containing player l and not containing player k and $\hat{z}$ is a vector whose indices are members of D, satisfying $\hat{z}(D) = v(D)$, $\hat{z}_i \geq \hat{y}_i$ for $i \in D \cap C$ and $\hat{z}_i \geq x_i$ for $i \in D - C$.

Definition 8.1:

An imputation x is said to belong to the *bargaining set* $\mathscr{M}_1^{(i)}(\Gamma)$ (for the grand coalition)[2]), if for any objection of one player against another with respect to x, there exists a counter objection.

Clearly, $\mathscr{M}_1^{(i)}(\Gamma) \supset \mathscr{C}(\Gamma)$, because if $x \in \mathscr{C}(\Gamma)$, no objections are possible. In this section we shall show that if Γ is convex then $\mathscr{M}_1^{(i)}(\Gamma) = \mathscr{C}(\Gamma)$. Since $\mathscr{M}_1^{(i)}(\Gamma) \supset \mathscr{K}(\Gamma)$ (see Davis and Maschler [1965]), this result furnishes another proof of Lemma 7.1.

The proof will make use of a lemma concerning convex games which is of interest in itself. First a definition: the *monotonic cover*[3]) of a game $(N;v)$ is the game $(N;\acute{v})$ defined by

$$\acute{v}(S) = \operatorname*{Max}_{R \subset S} v(R), \quad \text{all} \quad S \subset N. \tag{8.1}$$

It is clear that for any game the monotonic cover is indeed monotonic (see (2.7)), and that $\acute{v}(S) \geq v(S)$ for all $S \subset N$. The monotonic cover is of course not invariant under strategic equivalence.

Lemma 8.2:

The monotonic cover of a convex game is convex.

[1]) The circumflex reminds us that $\hat{y}$ is not an n-tuple.

[2]) The definition can be extended to cover situations in which coalition-structures other than the grand coalition are being considered (see, e.g., Davis and Maschler [1963 and 1967], Peleg [1963 and 1967]. For intuitive background and justification see Aumann and Maschler).

[3]) Called "monotonic closure" by Maschler and Peleg [1967].

Proof:

Let $(N;v)$ be convex, let $S_1,S_2 \subset N$, and let R_1,R_2 be such that $R_i \subset S_i$ and $v(R_i) = \acute{v}(S_i)$, $i = 1,2$. Then

$$\acute{v}(S_1) + \acute{v}(S_2) = v(R_1) + v(R_2) \leq v(R_1 \cup R_2) + v(R_1 \cap R_2)$$
$$\leq \acute{v}(R_1 \cup R_2) + \acute{v}(R_1 \cap R_2) \leq \acute{v}(S_1 \cup S_2) + \acute{v}(S_1 \cap S_2).$$

Since also $\acute{v}(\emptyset) = v(\emptyset) = 0$, $(N;\acute{v})$ is convex, as claimed.

Theorem 8.3:

The bargaining set $\mathcal{M}_1^{(i)}$ (for the grand coalition) of a convex game coincides with the core of the game.

Proof:

Let $\Gamma \equiv (N;v)$ be a convex game, and let x be an imputation with $x \notin \mathscr{C}(\Gamma)$. We must show that $x \notin \mathcal{M}_1^{(i)}(\Gamma)$. Let C be a maximal element of $\mathscr{D}(x)$, so that (see Lemma 5.6)

$$e(S,x) \leq e(C,x) \quad \text{for all} \quad S \subset N, \tag{8.2}$$

and

$$e(S,x) < e(C,x) \quad \text{if} \quad C \subset S \subset N, \quad S \neq C. \tag{8.3}$$

Write $e(S)$ for $e(S,x)$; then $(C;e)$ is a convex game, by (5.6). Its monotonic cover $(C;\acute{e})$ is also convex, by Lemma 8.2, and so has a nonempty core (see SHAPLEY [1971]). Take $\hat{u}$ in that core, then

$$\hat{u}(C) = \acute{e}(C) = \operatorname*{Max}_{R \subset C} e(R,x) = e(C,x) > 0 \tag{8.4}$$

(see (2.2), (8.1) and (8.2)), and, for each $R \subset C$,

$$\hat{u}(R) \geq \acute{e}(R) \geq e(R) = e(R,x) \tag{8.5}$$

(see (4.2) and (8.1)), and also, for each $i \in C$,

$$\hat{u}_i \geq \acute{e}(\{i\}) \geq \acute{e}(\emptyset) = 0 \tag{8.6}$$

(see (4.2), (8.1), and (5.1)). To construct the objection, let $k \in C$ be such that $\hat{u}_k > 0$ and let $l \in N - C$ be arbitrary. (This is possible because $\hat{u}(C) > 0$ and $C \neq \emptyset, N$.) Define

$$\hat{y}_i = x_i + \hat{u}_i + \varepsilon, \quad \text{for} \quad i \in C, i \neq k,$$

and

$$\hat{y}_k = x_k + \hat{u}_k - (c-1)\varepsilon, \tag{8.7}$$

where c is the number of elements in C and ε satisfies $\varepsilon > 0$ and $(c-1)\varepsilon < \hat{u}_k$. Then it is clear, in view of (8.4) and (8.6), that $(\hat{y};C)$ is an objection of k against l with respect to x.

Now let D be any coalition containing l but not k. By (8.5), (5.6), and (8.3) we have

$$\hat{u}(D \cap C) \geq e(D \cap C,x) \geq e(D,x) + e(C,x) - e(D \cup C,x) > e(D,x) \tag{8.8}$$

(strict inequality because $l \in (D \cup C) - C$). Hence

$$\begin{aligned} v(D) &= e(D,x) + x(D) \\ &< \hat{u}(D \cap C) + x(D) \\ &\leq \hat{y}(D \cap C) - x(D \cap C) + x(D) \\ &= \hat{y}(D \cap C) + x(D - C), \end{aligned}$$

using (8.8) and (8.7) and the fact that $k \notin D$. The strict inequality shows that the coalition D is not strong enough to support a counter objection. Hence "the objection is sustained", and $x \notin \mathscr{M}_1^{(i)}$, as was to be shown.

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