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Introduction

… what do you learn …

This learning material is to teach you basics of *mathematical models of negotiation analysis*. We cover some ideas from:

- Multiple criteria decision making in which decision maker's desires are modelled mathematically and he/she is aided to find out the best alternative
- Game theory that analyses different outcomes of the negotiations
- Axiomatic bargaining that pays attention on fair outcomes of the negotiations
- Negotiation analysis that provides aid to reach agreements in negotiations

… everyday negotiations …

Negotiations take place in everyday life, different levels, e.g.,

- Parents and children divide household duties
- Labour and management negotiate employment contracts
- Producer and its client negotiate delivery, amount and price contracts
- In the management of natural resources stakeholders with different interests negotiate a common policy

... the elements of the negotiations ...

Each negotiation is different, but there are some common basic elements:

1. The negotiating parties
2. The issues that are included in the negotiation
3. Best agreement to a negotiated agreement (BATNA). What happens if no agreement is reached?

... the parties ...

The identification of the parties is not necessarily a trivial task because in many problems there are many stakeholders involved in. For instance in environmental decision problems, there can be thousands stakeholders but in real negotiations they need to be splintered into groups that have only a single representative in the negotiation.

... the issues ...

The issues that are agreed upon in the negotiations are another important element in negotiations. For instance, if two persons are dividing a cake the problem consists only of one issue. However, in that division problem money can be included resulting to a two issue problem. In real life problems, there can be hundreds of issues causing complex problems.

BATNA has a remarkable effect in the negotiation settlement. It is the result for the parties that they could achieve without negotiations. Basically, BATNA is an insurance that acts as a reference against which any outcome should be compared to. It protects the parties from accepting unfavorable agreements and from rejecting favorable ones.

Multiple criteria decision making

The basic concepts of multiple criteria decision making (MCDM) are presented by using an energy production example. It shows how a decision-maker faces a problem in which he has two objectives. Those objectives are conflicting: he is unable to perform well in both of them simultaneously, which makes the problem complex. Therefore, he is forced to make compromises and analyse his preferences carefully to be able to make the best possible choice.

Here, we first define the multiple criteria decision problem. Then we describe the decision-maker's preferences by both single and multiple criteria value functions and finally we apply the value functions in multiple criteria decision making and let the decision-maker to make optimal choice.

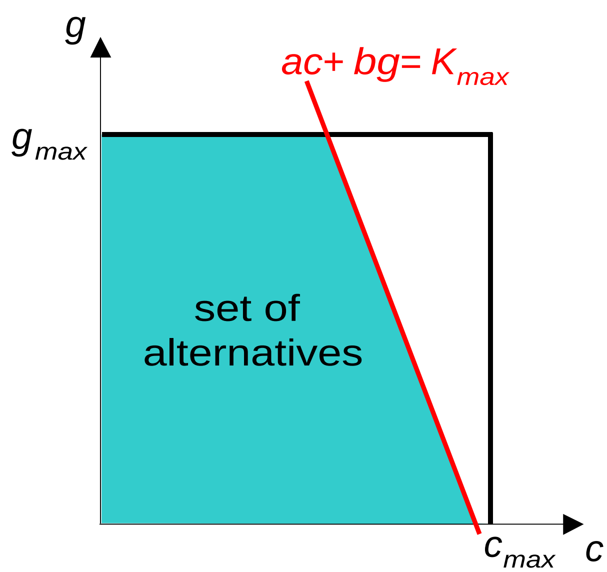
The multiple criteria decision problem

... the problem of the decision-maker ...

Harold, director and owner of a power company, has two power plants that are fired by coal and gas. He sits in his office making a production plan for the following month and he thinks the possible decisions he can take; namely he is deciding how many tons of coal and gas he uses for energy production in the following month. Those amounts are called the *decision variables*. In this example we denote the decision variables by c and g respectively.

... the set of alternatives ...

The capacities of the power plants are limited and hence there are *constraints* $0 \leq c \leq c_{max}$ and $0 \leq g \leq g_{max}$ where c_{max} and g_{max} denote the maximum amount of coal and gas that can be fired in a month. Additionally, Harold must pay the fuel at the beginning of the month and he has only limited capital K_{max} available. The prices of coal and gas are *a* and *b* euros per ton and hence we obtain another constraint $ac + bg \leq K_{max}$. By taking these constraints into account we can determine the set of *alternatives*, see figure below.



... what does the decision-maker want? ...

Harold knows his alternatives, but his decision is still unclear because he has not yet analysed either his own preferences.

Harold produces power to earn money, but on the other hand he enjoys clean environment as well. Therefore, he concludes that he has at least two *objectives*:

1. He wants to maximise his profit.

2. He wants to minimise the pollution.

By definition, the *objectives* are statements that delineate the desires of a decision-maker.

Harold's objectives are obviously conflicting. The more he produces electricity the more he earns money but the more he pollutes. This makes the problem complex.

... how to measure the objectives? ...

Harold must be able measure the objectives to describe the degree to which they are achieved:

1. He measures maximise profit by profit in euros per month, denoted by f_p, and

2. minimise pollution by tons of emitted sulphur dioxide pollutants per month, denoted by f_e.

These measures are depending on the decision variables and hence they are functions of the decision variables and actually

$f_p = f_p(c,g)$ and $f_e = f_e(c,g)$.

By definition the measures of the objectives are called *criteria*.

Sometimes the criteria are referred to as *attributes*.

... multiple criteria optimisation problem ...

Harold's problem can be presented as a *multiple criteria optimisation problem*:

$$\max f_p(c, g)$$
$$\min f_e(c, g)$$

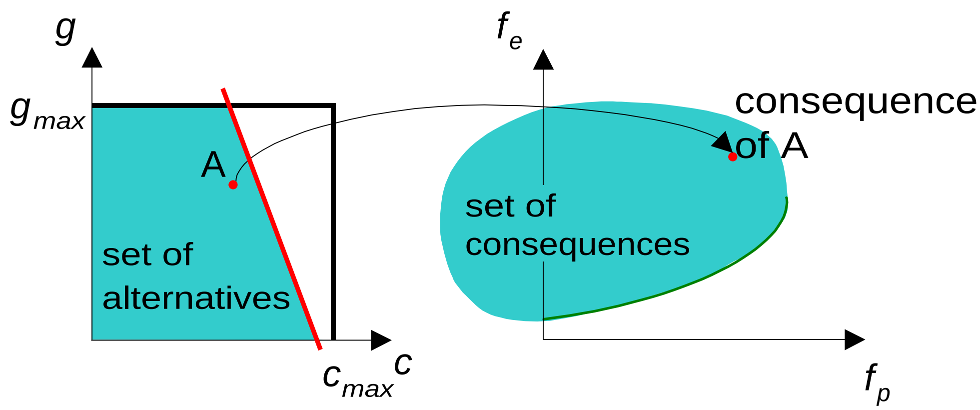
such that

$$ac + bg \le K_{\max}$$
$$0 \le c \le c_{\max}$$
$$0 \le g \le g_{\max}$$

The *optimal solution* to the problem is unambiguous, because there are two conflicting criteria.

... the set of consequences ...

Harold can describe his problem also in the set of *consequences*. He draws the set of consequences by evaluating each alternative (c,g) by a criterion pair $[f_p(c,g), f_e(c,g)]$ which is called the consequence of the alternative. The set of consequences can be developed by plotting each consequence in the (f_p,f_e)-plane.



By definition, the values of the criterion associated with an alternative are called the *consequence* of the alternative.

… what is an efficient alternative? …

In Harold's case, an alternative is efficient if its profit cannot be increased without increasing the emissions, or the emissions cannot be reduced without decreasing the profit. Evidently, Harold should make his choice among the efficient alternatives and his proper choice depends on his willingness to protect the nature. The efficient alternatives are called *Pareto optimal*. In the figure above, the green darkened frontier of the set of consequences presents the Pareto optimal consequences.

By definition, an alternative is *Pareto optimal*, if for any other alternative at least one criteria is worse.

Value function for single criteria

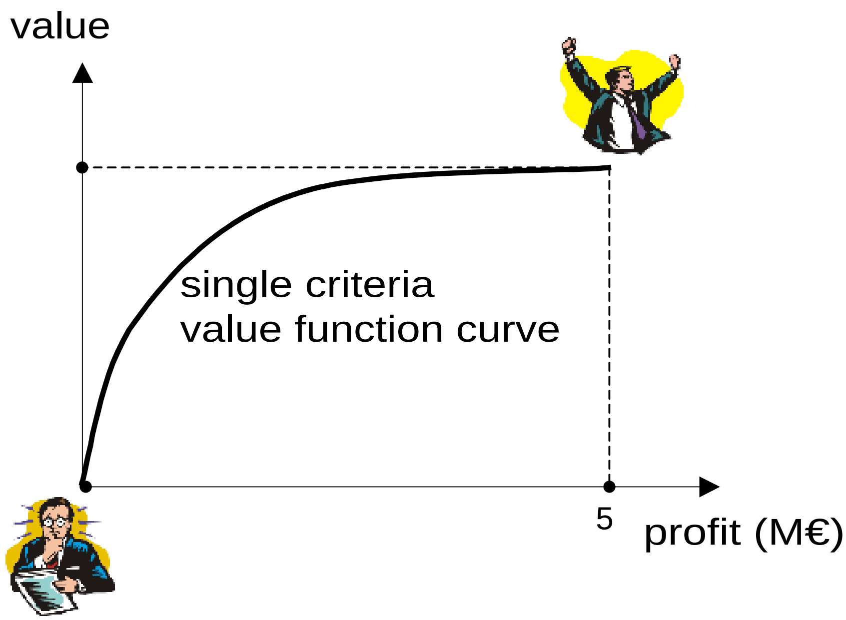
... what is the value of money? ...

First, Harold considers what he thinks about money alone. He has calculated that he can achieve a profit ranging from 0 to 5 million euros and he concludes that

1. if he earns much, say 4 M€, having thousand euros more is nearly insignificant for him
2. but if he earns only a little, say 10 000€, the significance of having thousand euros more is really high.

Therefore, the value of having thousand euros more does depend on his profit, which can be described by a *single criteria value function curve* in the figure below.

By definition, the *single criteria value function* measures the desirability of each performance level of criteria.



You can see that, the lower the profit is the greater is the slope of the value function curve. Thus, the more Harold earns the lower the value of having one euro more is for him. A curve obeying this property is called *concave* and, in classical economics, this feature is referred to as *decreasing marginal utility*.

… how to develop a single criteria value function? …

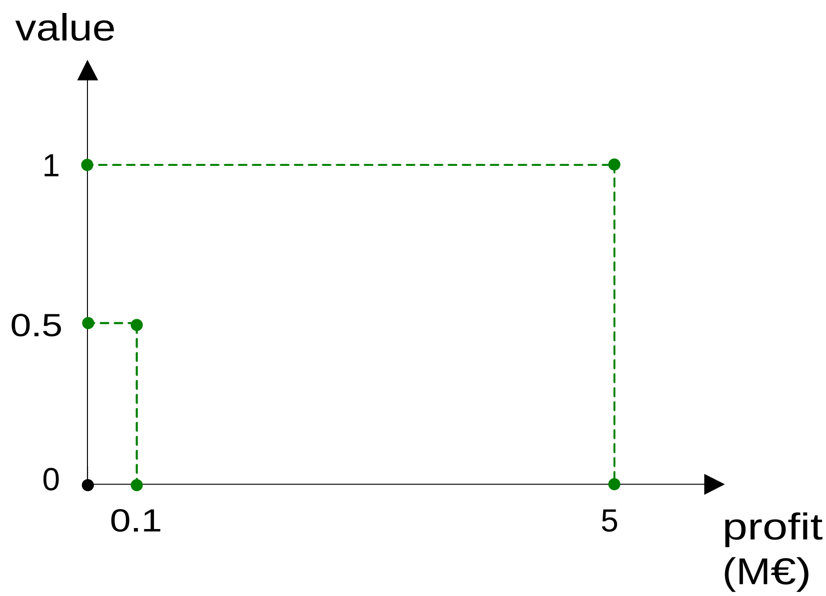
There are a great variety of different possible methods to construct the single criteria value function. Here we present one, namely the *bisection method*.

In the bisection method Harold arbitrarily assigns the value of zero for the worst profit and one for the best. Then he searches for a bisection profit that is half as desirable as the greatest profit available compared to the worst possibility.

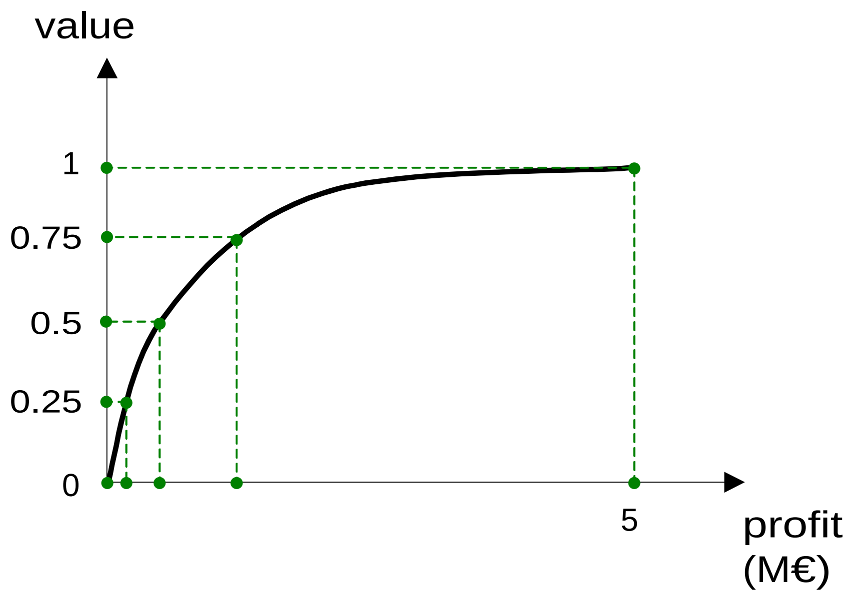
First he asks himself: ”Is the pleasure of having 2.5M€ instead of 0€ equal to pleasure of having 5M€ instead of 2.5M€?”. He answers in his mind: ”No, it is more desirable to increase earnings from the poor 0 € to 2.5M€ than to increase earnings from 2.5M€ to 5M€!”. After some probing he realises that he considers the following options equally desirable:

1. to increase earnings from 0€ to 0.1M€
2. to increase earnings from 0.1€ to 5M€.

This implies that the value of having 0.1M€ is halfway between the values 0 and 1 and, therefore, the value of 0.1M€ is equal to 0.5, see the figure below.



Harold can repeat the bisection for the profit intervals [0, 0.1] and [0.1, 5] separately and thus construct more points on his value function curve.



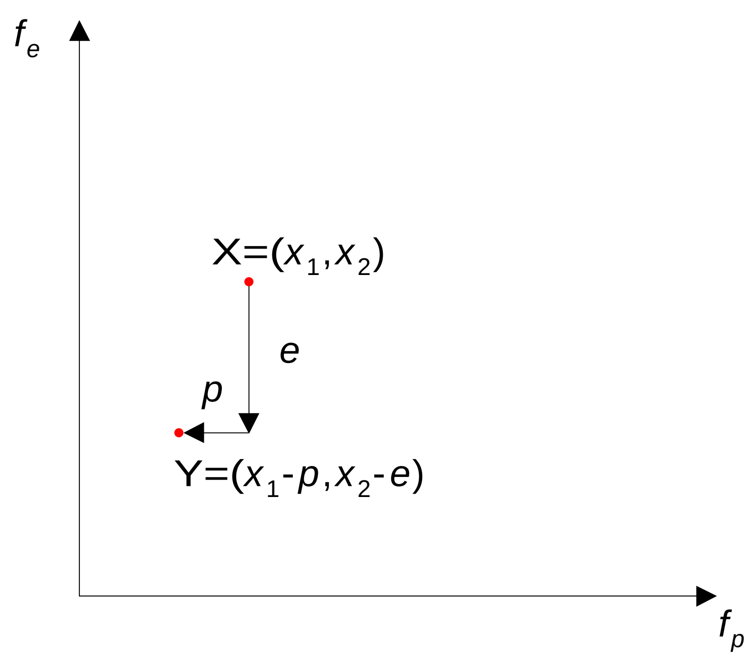
After constructing as many points as necessary Harold can approximate his value function by drawing a curve through the constructed points.

Note that, if Harold makes his decision based on this single criteria value function only, he selects the alternative that maximises his profit. Obviously, Harold is not happy with it because his other criteria, the emissions, are not taken into account at all. Hence, he should construct a two criteria value function and apply it in the decision making.

Multiple criteria value function

... making tradeoffs ...

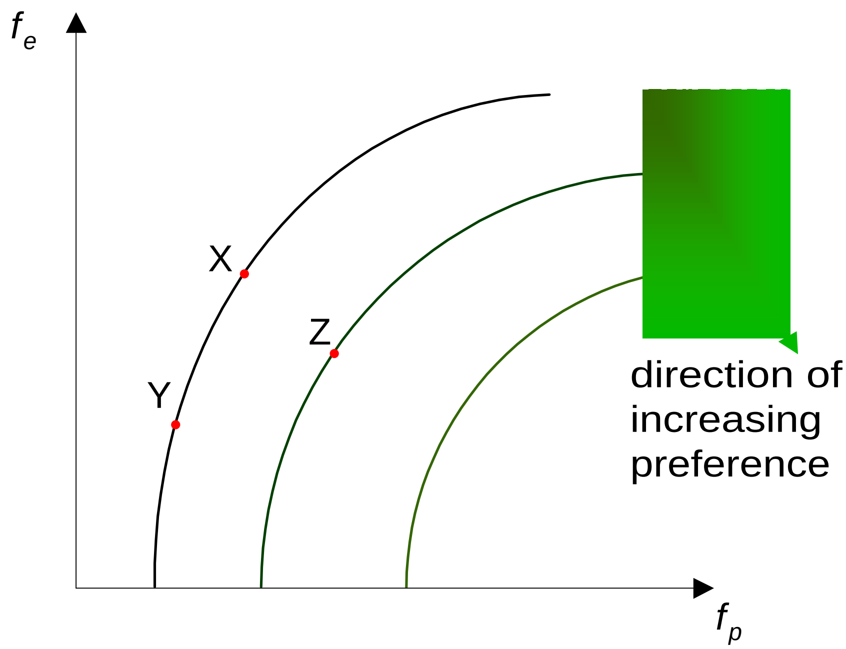
Consider consequence X=(x_1,x_2) in the figure below. Harold has two objectives and he may ask himself: ”How much am I willing to pay for decreasing the emissions by a certain amount *e*?” He answers his question: ”I do not want to pay more than *p* euros for that.”



Hence Harold considers consequences X=(x_1,x_2) and Y=(x_1-p,x_2-e) equally desirable, or in other words, he is indifferent between X and Y.

... indifference curves ...

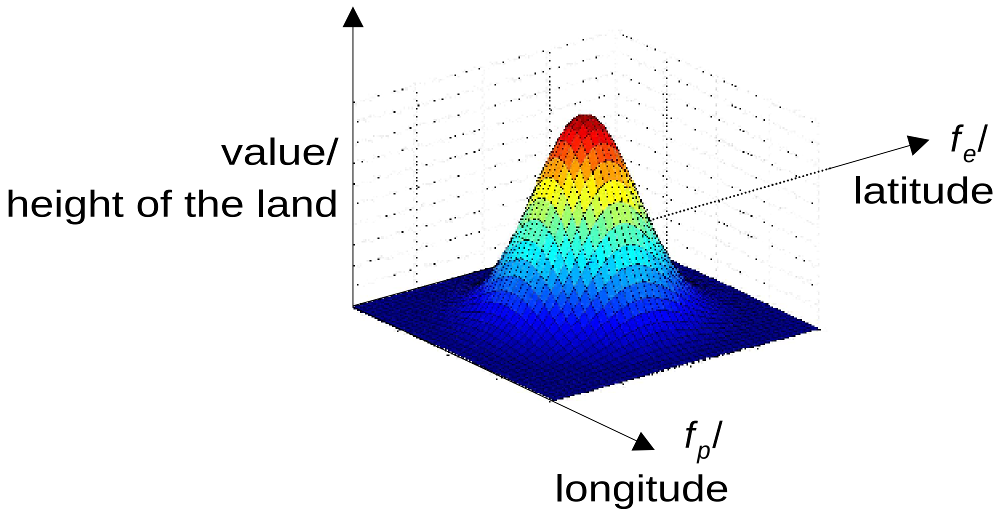
By varying the change *e* Harold can generate more and more consequences that he considers equally preferable to X. Those consequences are shown by the black *indifference curve* in the figure below.

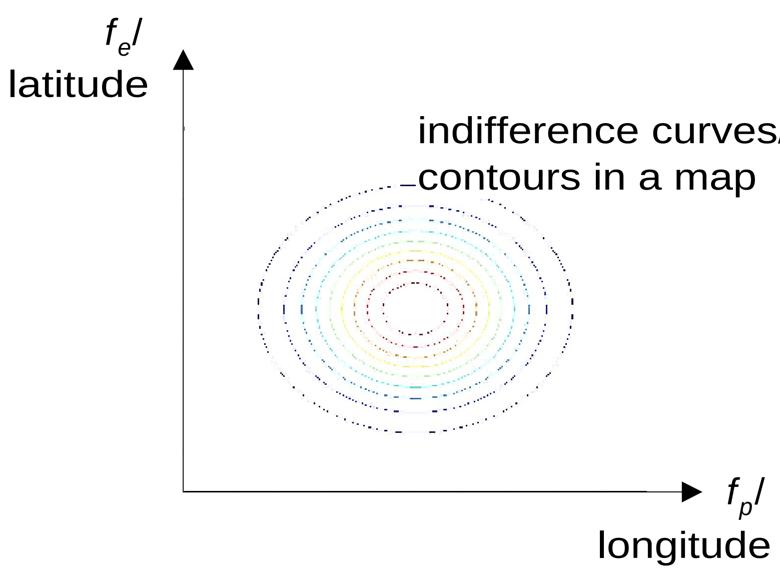


There are many indifference curves, some of which are included in the figure. For instance, you can see that Harold is indifferent between X and Y but he prefers Z to X and Z to Y, because he obviously prefers the inner contours.

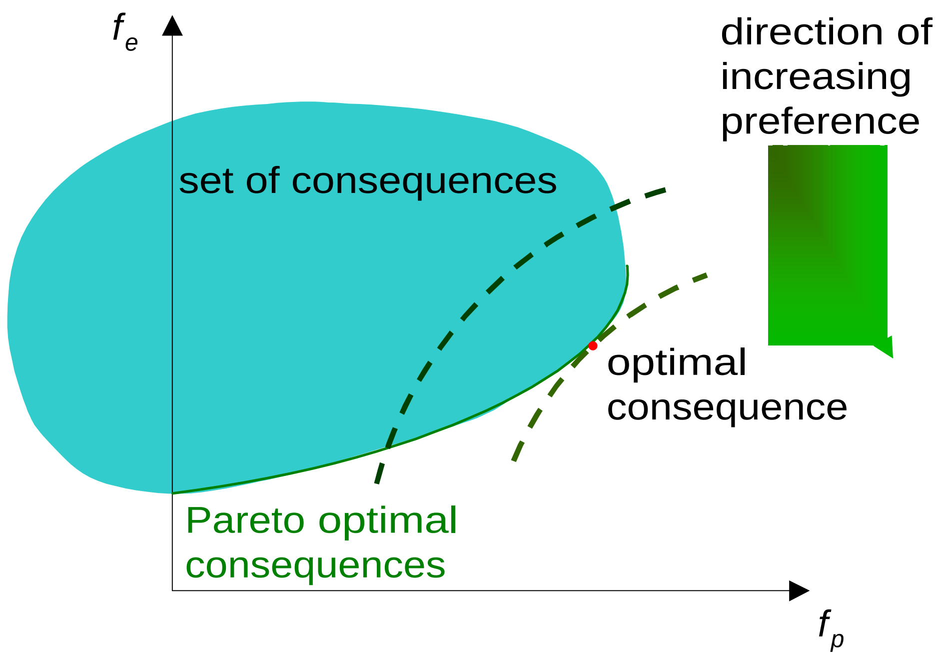
... indifference curves and contours ...

The indifference curves are analogous to the altitude *contours* in a countryside map provided that the decision-maker prefers climbing to the top of a hill. The indifference curves represent the preferences of a decision-maker, see two figures below.





Now after developing the contours Harold is able to choose the optimal consequence. He draws all the possible consequences together with the indifference curves in the same figure:

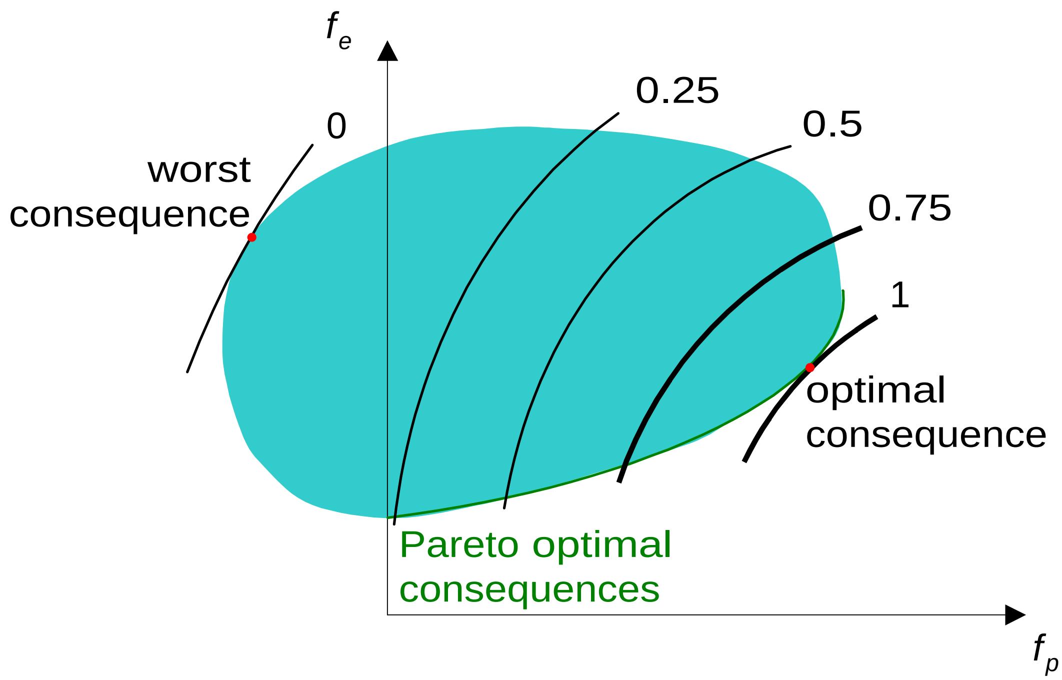


The optimal consequence is found geometrically by searching the point in which an indifference curve and the set of consequences touch each other because the inner contours stand for more preferable consequences.

... describing the strength of the preferences ...

The presented indifference curves express only the preferential ordering of the different possible consequences. However, a real number can be assigned to each indifference curve to describe the strength of the preferences.

Arbitrarily, Harold sets the value of one for the best possible consequence and zero for the worst. Now he can assign values for all the other indifference curves according to his preferences, for instance, similarly as in the bisection method.



In the figure above, you can see, for instance, that the consequences on the indifference curve labelled by 0.5 are two times more preferable than the consequences on the curve labelled by 0.25.

Math.: **... multiple criteria value function ...**

Let $g(f_p,f_e)$ denote a function that maps the consequences to the real numbers. If that function satisfies the following two conditions it is called *ordinal* multiple criteria value function:

1. if the decision-maker is indifferent between consequences (p_1,e_1) and (p_2,e_2) then $g(p_1,e_1)=g(p_2,e_2)$
2. if the decision-maker prefers (p_1,e_1) to (p_2,e_2) then $g(p_1,e_1)>g(p_2,e_2)$.

Additionally, the value function can include the information on the strength of the preferences as in the preceding chapter. In that case, the value function is called *cardinal*.

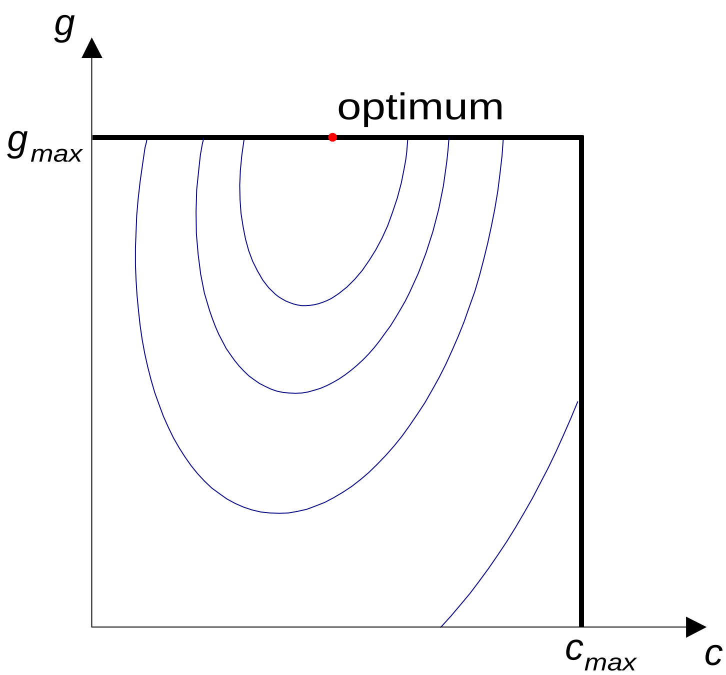
Note that the criterion f_p and f_e are functions of decision variables c and g and hence Harold's value function can be presented as $g(f_p(c,g), f_e(c,g))=v(c,g)$. Hence, we have a value function v that maps the alternatives to real numbers. Now the optimal alternative can be found by maximising $v(c,g)$ subject to c and g.

By definition,

- the *ordinal multiple criteria value function* expresses the preferential ordering of each consequence and alternative.
- the *cardinal multiple criteria value function* measures the desirability of each consequence and alternative.

Finding the optimal alternative

Harold can present his preferences in the set of alternatives as in the consequence set and draw his contours in the set of alternatives. Note that here we have excluded the budget constraint.



It is easy for Harold to point out the optimal alternative in the figure, but how to search it if the contours are not explicitly known.

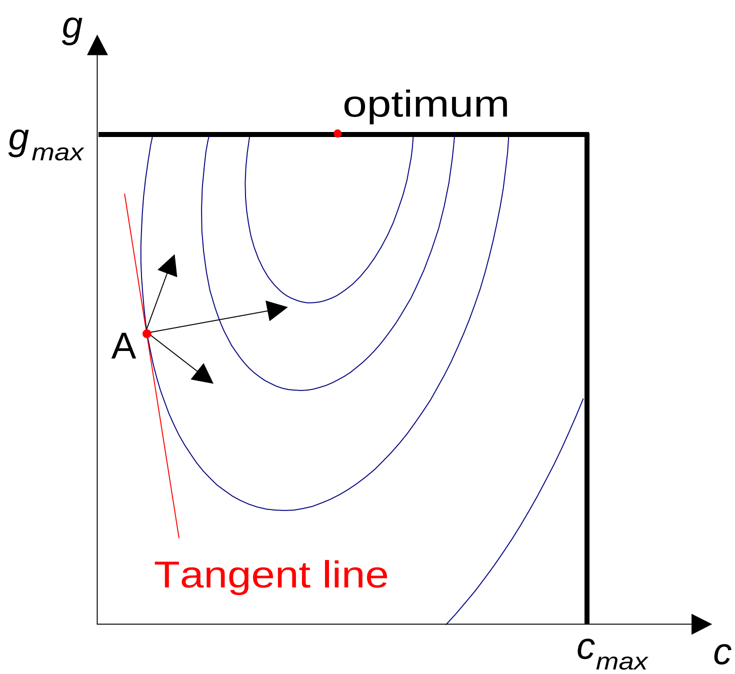
... searching better alternatives? ...

Harold considers alternative A and he wants to improve it. He selects first an *improving direction*, on which he can find better alternatives, and then moves a suitable step along that direction.

> By definition, a direction is *improving* if by moving a sufficiently small step along that direction a better alternative is found.

... the set of improving directions ...

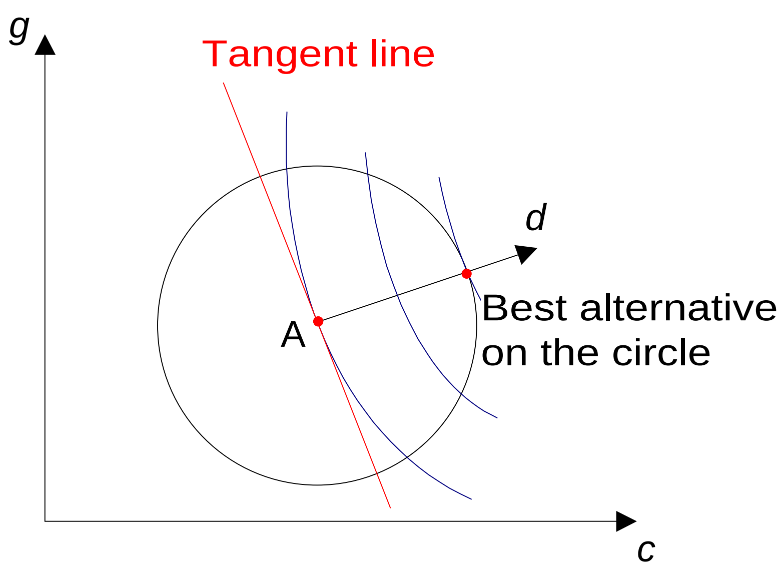
Harold can find the set of improving directions by drawing draws a line that touches a contour at A. That line is called *tangent line* and it determines the set of improving directions. He can see that the directions starting from A and pointing to the direction of inner contours are improving directions. This is because the inner contours stand for more preferable alternatives.



... how to find an improving direction? ...

If Harold does not know explicitly his contours, he can search for an improving direction, for instance, by

1. drawing a small circle around A,
2. selecting the best alternative B on that circle
3. and drawing a direction from A and going through B.



The direction *d*, in figure above, is improving for Harold because by moving along that direction he reaches better alternatives.

If the circle is decreased infinitely, then *d* becomes perpendicular to the tangent line.

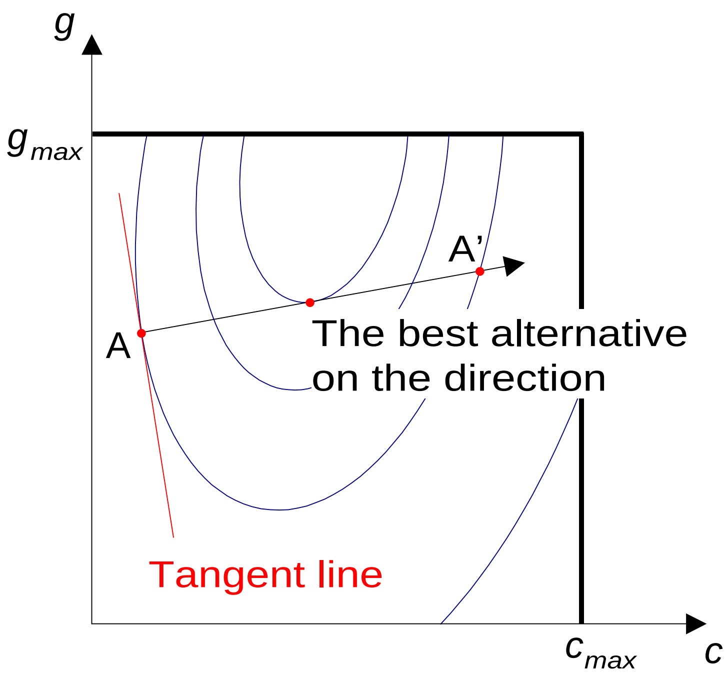
A direction that is perpendicular to the tangent line of a contour of the value function at A is called the *gradient* of the value function at A.

The gradient is also called *steepest ascent direction* of the value function.

… searching better alternatives …

Next, Harold takes a step along the improving direction he selected to find improving alternative.

If Harold knew contours of his value function, he finds the improving alternatives on the line segment AA’, see the figure below. At the best alternative a contour touches the direction.



If the contours are not known, Harold can select a better alternative by taking a step along the improving direction. He takes a step, finds an alternative and asks himself: “Do I prefer that alternative to A”. If yes, then he has found a better alternative. Otherwise, he should try selecting a shorter step length.

Further reading

- Keeney, R. and H. Raiffa. (1976). *Decisions with Multiple Objectives: Preferences and Value Tradeoffs*. John Wiley & Sons.
- Keeney, R. (1992). *Value Focused Thinking: A Path to Creative Decision Making*. Harvard University Press.
- Von Winterfeldt and D., Ward, E. (1986). *Decision Analysis and Behavioral Research*. Cambridge University Press.
- Bernoulli, D. (1738). "Specimen theoriae novae de mensura sortis". *Commentari Academica Scientiarum Imperalis Petropolitanae*, 175-192; English translation: Sommer, L., 1954, *Econometrica*, 22, 23-36.
- von Neumann, J. and Morgenstern O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press.
- Lung, P.-L. (1987). *Multiple Criteria Decision Making*. Pluton. A book for mathematical analysis of multiple criteria decision making.

- Bazaraa, M., Sherali, H. and Shetty C. (1993). *Nonlinear Programming*. Theory and Algorithms, Second Edition, John Wiley&Sons. A book on optimisation theory and methods.

Game theory

Here you learn the basic concepts in game theory. They are presented by two classical examples. The first one, called the prisoners' dilemma game, shows that applying the main solution concept, the Nash equilibrium solution (Nash 1951), gives both players worse outcome than they could achieve if they would have the possibility to co-operate. This problem is verbally presented Raiffa (1957) who says that the original game is contributed by A. W. Tucker.

The other problem is called the problem of the commons, which is known at least since Hume (1739). In this game, a common property is being exhausted because the Nash equilibrium produces worse outcome than the players could achieve by co-operating as in the first game. Such a behaviour may constitute a tragedy to the society; it appears to be a striking feature especially of the human societies.

The prisoners’ dilemma

... the old story about two friends...

Two suspects, Harold and William, are taken into custody and separated. The policeman accuses them of a crime but lacks sufficient evidence to convict them, unless at least one of them confesses. He explains the consequences following the two *actions* they could take: namely confessing the crime or not confessing it.

... what are the consequences? ...

The policeman says:

”If neither of you confess, then both will be convicted of a minor offence and sentenced to **1** month in jail.

If you both confess, then I will sentence you to jail for **6** months.

Finally, if only one of you confesses then he will be treated leniently and will be freed; while his confession is used as a witness against the other, who will be sentenced to **9** months in jail; 6 for the crime and 3 for obstructing the justice."

... the consequences written in a simple form ...

The *bi-matrix* form below presents the four possible *outcomes*, or *payoff* pairs, depending on the actions chosen by Harold and William. They can be presented as pairs of prison years. The first number in each cell is Harold's payoff and the latter is that for William:

	William does not confess	William confesses
Harold does not confess	-1, -1	-9, 0
Harold confesses	0, -9	-6, -6

The bi-matrix is read as follows: if, for example, Harold confesses but William does not, then Harold's payoff is **0** , representing immediate release, and William's payoff is **-9** representing 9 months in jail.

… representation as a game …

By definition, representation as a *normal form game* specifies:

- The *players* in the game
- The action alternatives, called *strategies* for each player
- The payoffs received by the players for each possible combination of strategies.

In prisoners' dilemma game, the players are Harold and William. Each player has two strategies available: confess, denoted by *c*, and not confess, *nc*. The payoffs to the players depending on the chosen pair of strategies are given in the corresponding cell of bi-matrix.

... Nash equilibrium to the game ...

We can easily deduce the *solution* to the prisoners' dilemma game. After having learnt the consequences of their crime, William thinks in his cell what is his best choice, or *best* *response strategy*, if Harold chooses to play *c* or *nc*. His best response in both cases is *c*. Similarly, Harold thinks in his cell to play *c* because it is always his best response to whatever William might choose. Hence both players want to confess and the strategy pair (*c*,*c*) is a self evident solution to the prisoners' dilemma game and both players will be 6 months in jail. This solution is called the *Nash equilibrium solution* of the game.

By definition, a strategy pair is *Nash equilibrium solution* if each player's strategy is the best response to the other player's strategy.

In Nash equilibrium neither of the players wants to deviate from his strategy alone.

… Pareto optimal outcomes …

Note, however, that there is an outcome in this game, see the bi-matrix, which would give both players a better payoff, namely -1,-1, meaning one month in jail for both of them. Because the self-evident Nash equilibrium giving worse payoffs is chosen, there is a dilemma hidden in this game. The outcome -1, -1 is called a *Pareto optimal* outcome.

By definition, an outcome is *Pareto optimal* if any other outcome gives a worse outcome for at least one of the players.

Hence, the outcomes 0, -9 and -9, 0 are Pareto optimal as well.

Math.: **... the game in mathematical form ...**

Let s_i denote an arbitrary strategy for player *i*; *i* refers either to player 1 or 2. The set of all strategies available to player ***i*** is denoted by S_i and called *i*'s *strategy set*. We denote $s_i \in S_i$

to indicate that s_i belongs to the strategy set S_i. Let u_i denote player *i*'s *payoff function*. It specifies all possible payoffs for player *i* for each combination (s_1, s_2) that might be chosen by the players.

In prisoners' dilemma game we let player 1 be Harold and player 2 William. Hence, if $s_1 \in S_1$, it is either *c* or *nc* and similarly for $s_2 \in S_2$. We may also write $S_1 = S_2 = \{c, nc\}$. From the bi-matrix we can read the values u_i for all combinations (s_1, s_2), e.g., $u_1(nc,nc)=-1$, $u_1(c,nc)=0$, $u_2(nc,c)=-9$, $u_2(c,c)=-6$.

Math.: **... the solution to the game ...**

The *solution* to the game above is a strategy pair (s_1^*, s_2^*) with the following property: The strategy s_1^* is the best strategy for player 1, provided that player 2 chooses to play s_2^* and vice versa. Mathematically the solution satisfies

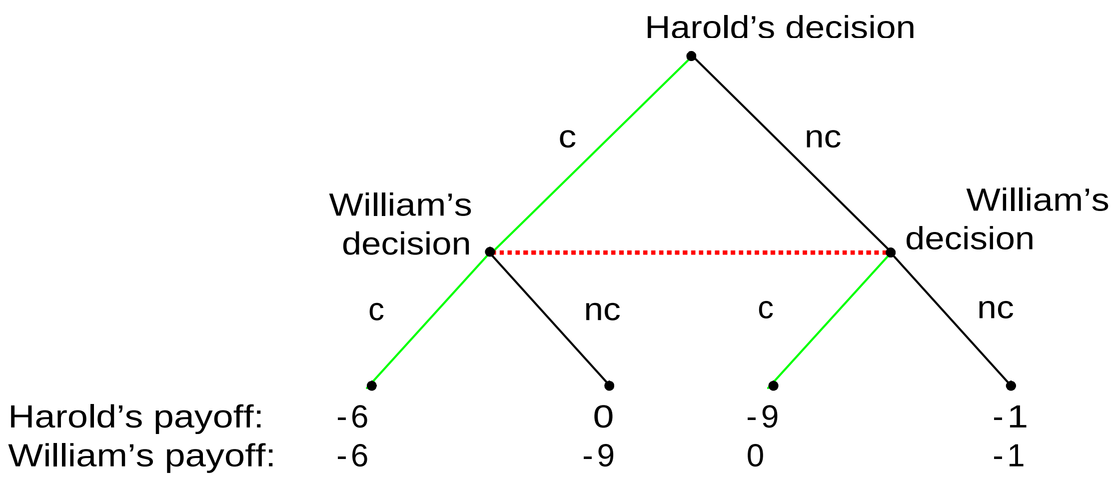
$$u_1(s_1^*, s_2^*) \geq u_1(s_1, s_2^*) \text{ for all } s_1 \in S_1,$$
$$u_2(s_1^*, s_2^*) \geq u_2(s_1^*, s_2) \text{ for all } s_2 \in S_2.$$

Hence, each player is willing to choose the strategy indicated by the solution provided the other player also does so.

This solution to the game is called Nash equilibrium.

... tree representation of the game ...

The game can be represented as a tree form, too. It is called an *extensive form game*:



By definition, in the *game tree*, there are:

- *nodes* at which the players make their decisions
- *arcs* connecting them.

On the top of the game tree, there is first Harold's *decision node*, called *initial node*, but since here the timing of the decisions is irrelevant it could be William's decision node, as well.

… how to read the tree? …

For example, let Harold play *c*. Then William's decision node on the left is reached and William makes his decision. William's decision is then followed by a *terminal node* and the game ends. The payoffs for the players are shown below the terminal nodes.

Note that when William is making his decision, he does not know Harold's decision, i.e., in which of his decision nodes he actually is in the game tree. This ignorance is denoted by connecting the corresponding nodes with the red dotted line.

The problem of the commons

... the friends establish farming business...

It is spring. William and Harold are now released from jail and they both are going to graze goats in the summer on a common green. In the autumn, they are going to sell their goats. Their problem now is to decide how many goats they should graze. This problem is known since Hume (1739) and it is called the problem of the commons.

... what is the value of a goat for a farmer? ...

The more a goat has grass the better it survives. If there are only a few goats on the green, adding one more does not harm the ones already grazing. But, if there are many goats, adding one more is harmful for all the goats and the value of a goat decreases remarkably.

… representation as a game …

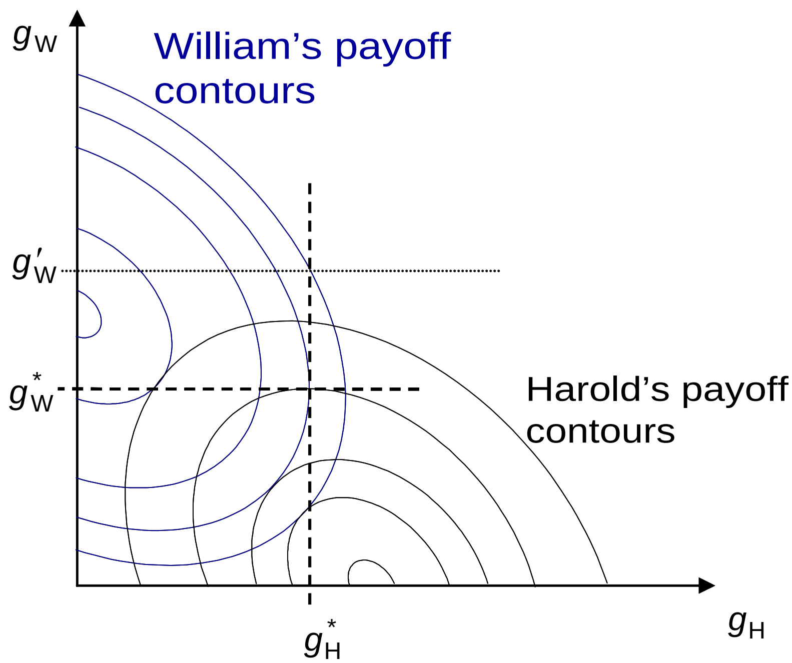
- Players: Harold and William
- Strategy for each player: The number of the goats he owns.
- The payoff for a player: His value of owning the goats; it is the total selling price minus the total cost of the goats.

… the solution of the game …

Denote the number of goats for Harold by g_H and for William by g_W. Assume that the goats are continuously divisible, i.e., g_H and g_W are real numbers instead of being integer values only.

Recall that the solution to the prisoners' dilemma game was defined by using the players' best response strategies.

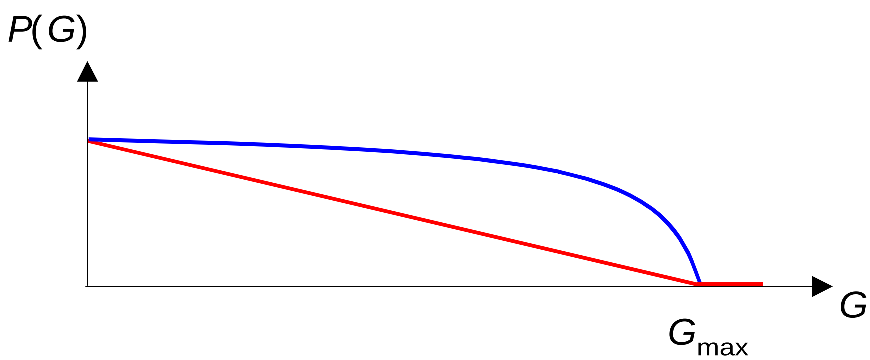
Similarly we can define the Nash equilibrium (g_H^*, g_W^*) here. Let us look at the following figure:



In the figure, there are some contours of the players' payoff functions. Along a contour a player's payoff is constant. It is assumed that the value of the inner contour is always greater than that of the outer contour. Let us first check if the point (g_H^*, g_W^*) in the figure is the Nash equilibrium solution of the game. If Harold decides to play g_H^*, then William chooses his best response strategy and maximises his payoff on the black slashed vertical line. He chooses g_W^*, where the vertical line touches one of his contours. This is because for any other choice, like g'_W, Harold's payoff is smaller. Likewise, g_H^* maximises Harold's payoff on the black slashed horizontal line corresponding to William's choice g_W^*. By definition the pair (g_H^*, g_W^*) is the Nash equilibrium of the game. In the following it is shown that the Nash equilibrium can be found by computing the intersection of the players' best response curves.

Math: **… the solution by using mathematics …**

The payoff function for Harold is $u_H(g_H, g_W) = [P(g_H + g_W) - c]\, g_H$, where c is the cost of buying and caring for and $P(g_H + g_W)$ is the selling price of a goat per goat. Adding one more goat to the green harms the rest more if there are many goats than if there were only a few goats on the green. This means a bigger drop for the selling price per goat in the former case. Therefore, the shape of the function *P* as a function of the total number of goats G, is as shown by the blue curve in the figure below. Here G_{max} is the carrying capacity of the green.

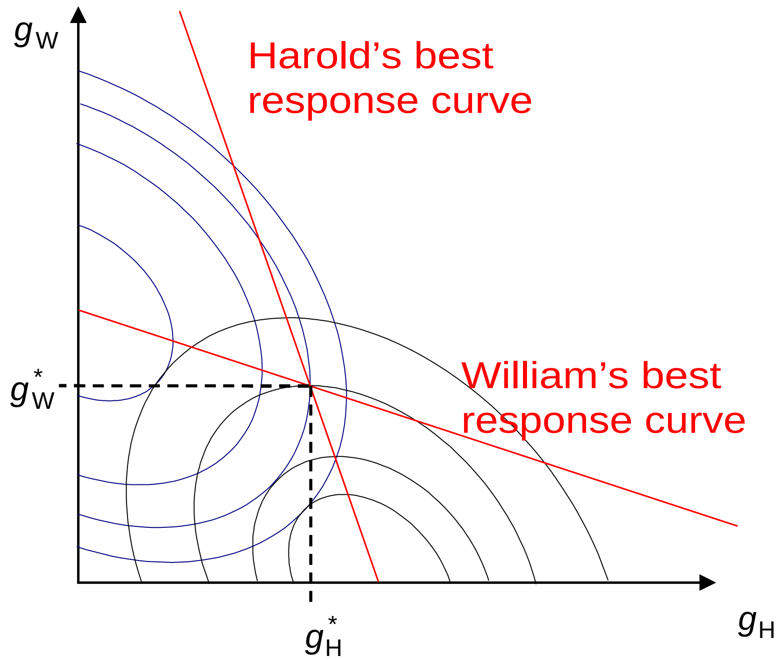


William's payoff is similar to that of Harold but with g_H and g_W interchanged. For simplicity, approximate the blue curve in the figure by the red line, i.e., assume that $P(g_H+g_W)=(G_{max}-g_H-g_W)$ if $g_H+g_W=G_{max}$ and $P=0$ otherwise. Then $u_H(g_H,g_W)=(G_{max}-g_H-g_W)\ g_H$ and $u_W(g_H,g_W)=(G_{max}-g_H-g_W)\ g_W$.

The Nash equilibrium can now be computed using the *best response curves*. For Harold it is defined as in the prisoners' dilemma game so that it gives him the best response for any choice g_W of William. This best response function is denoted by $R_H(g_W)$, and it satisfies

$$u_H(R_H(g_W), g_W) \geq u_H(g_H, g_W) \text{ for all } g_H.$$

The best response curves $g_H = R_H(g_W)$ and $g_W = R_W(g_H)$, that appear to be linear in this example, are shown by the red lines in the figure below.



Since g_H^* is Harold's best response to William's Nash equilibrium strategy g_W^*, and vice versa, the Nash equilibrium satisfies

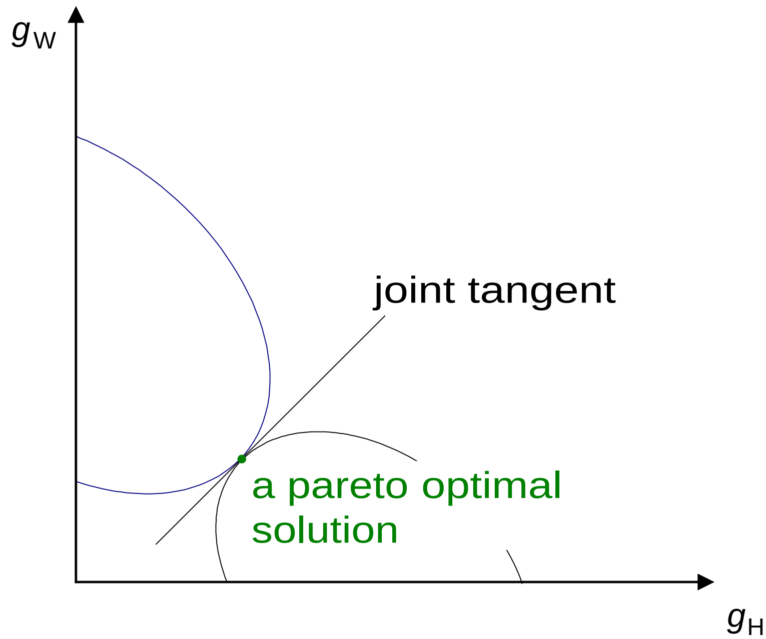
$$g_H^* = R_H(g_W^*),$$
$$g_W^* = R_W(g_H^*),$$

i.e., it is defined by the intersection of the two best response functions.

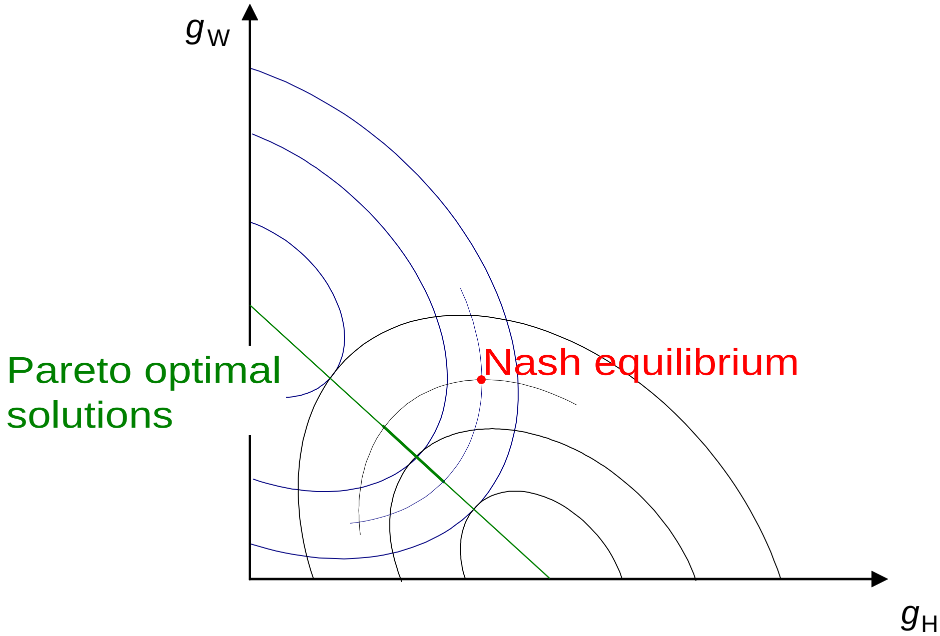
Pareto optimal solutions are defined in terms of Pareto optimal outcomes.

By definition, we say that (g_H^P, g_W^P) is a *Pareto optimal solution* if any other strategy pair gives a worse outcome for at least one of the players.

The definition implies that the Pareto optimal solutions are defined by the points of tangency of the players' payoff contours. One such point is illustrated in the figure below. This result can be verified easily by considering any other point, which is worse for either one of the players or both of them.



In the Nash equilibrium, Harold and William overutilize the common green because they compete about it. Therefore, the Nash equilibrium is sometimes called *competitive equilibrium*. If you look at the figure you can see that there are plenty of Pareto optimal points indicated by the green line segment. The bolded part of the line segment presents those Pareto optimal points which give better payoffs for the players than the Nash equilibrium because the inner contours give better payoff for the players. The phenomenon was called prisoners' dilemma in our previous example. Now it is called the tragedy of commons.


g_W
Pareto optimal solutions
Nash equilibrium
g_H

Further reading

- Nash, J. (1951). "Non Cooperative Games". *Annals of Mathematics*, 54, 286-95. Nash defines the Nash equilibrium.
- Luce, R., and H. Raiffa (1957). *Games and Decisions*. John Wiley & Sons. A book on different types of games including co-operative game theory.
- Gibbons, R. (1992). *A Primer in Game Theory*. Prentice Hall. An easy introduction to game theory.
- Hume, D. (1739). *Treatise of Human Nature*. Reprint. London: J.M. Dent (1952). An early book on social philosophy presenting the tragedy of the commons.
- Hardin, G. (1968). "Tragedy of Commons". *Science*, 162, 1243-1248. Hardin describes the problem of the commons in the society from the quantitative perspective.

Axiomatic bargaining

The principles of axiomatic bargaining are illustrated by studying further the problem of the commons faced by Harold and William in axiomatic bargaining.

It is commonly approved that a solution should desirably satisfy Pareto optimality in any bargaining or negotiation situation. This is because it guarantees that there exists no other outcome unanimously preferred by each player. Pareto optimality alone does not, however, offer a unique solution but a set of efficient solutions that are more or less preferable from a player's viewpoint. The paradigm of choosing a fair Pareto optimal solution for a game is addressed by *axiomatic bargaining theory* that is a field of game theory. It consists of formulating axioms on how a solution for a set of games should be selected and checking if the implied solutions seem appealing, see, e.g., Thomson and Lensberg (1989; especially Chapter 2).

Originally Nash (1950) introduced this axiomatic approach by presenting the Nash bargaining solution, which is an outcome maximizing the product of utilities perceived by the players, and the related axioms. Some of Nash’s axioms were criticized in the literature and consequently several modifications were presented. The best known variation of Nash bargaining solution, the Kalai-Smorodinsky solution, was contributed by Kalai and Smorodinsky (1975). As game theory, axiomatic bargaining is descriptive in its nature too and it does not either offer practical aid on how to reach the outcomes implied by the axioms in practice.

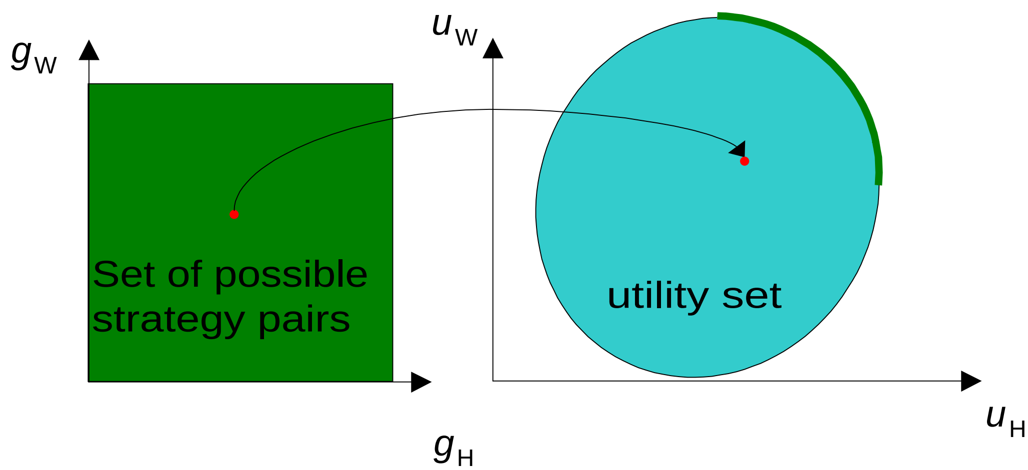
The bargaining problem

... establishing co-operation ...

Harold and William ended up to the competitive Nash equilibrium and they are worried about overutilizing the green. Hence they sit in a negotiation table and work co-operatively to find a *fair*, jointly satisfying and Pareto optimal solution, or *agreement*, for their problem.

... bargaining problem in the utility set ...

Axiomatic bargaining describes the bargaining problem in the *utility set*, which is the set of possible payoff pairs the players can receive. The players can construct the utility set by plotting the payoff pairs for each possible strategy pair in (u_H, u_W)-plane, see figure below.



The green bolded frontier in the utility set indicates the Pareto optimal payoff pairs.

By definition, the problem of selecting a particular point in the utility set is called the *bargaining problem*.

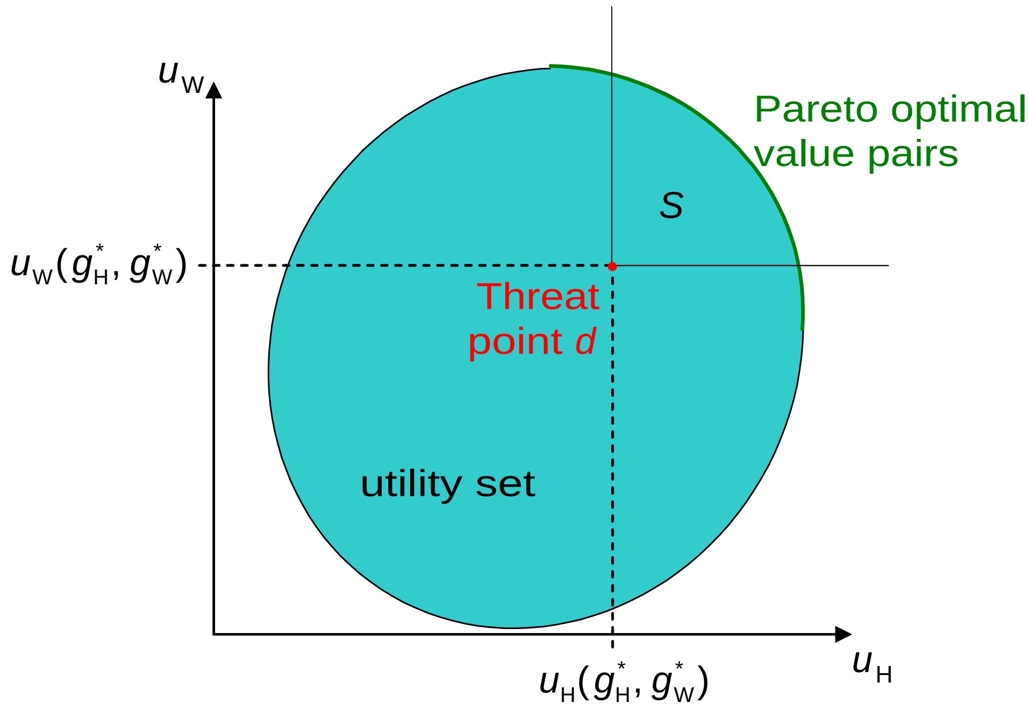
... what if their co-operation fails? ...

If Harold and William can find a jointly accepted point in the utility set, then they get it. However, if they fail to agree, then they end up to the *disagreement point* $d=(d_H,d_W)$. In this game, the disagreement point is the payoff pair implied by the non-cooperative solution, the Nash equilibrium strategy pair, and hence

$$d=(u_H(g_H^*,g_W^*),u_W(g_H^*,g_W^*)).$$

... the role of the disagreement point? ...

Harold and William plot now the disagreement point d in the utility set.

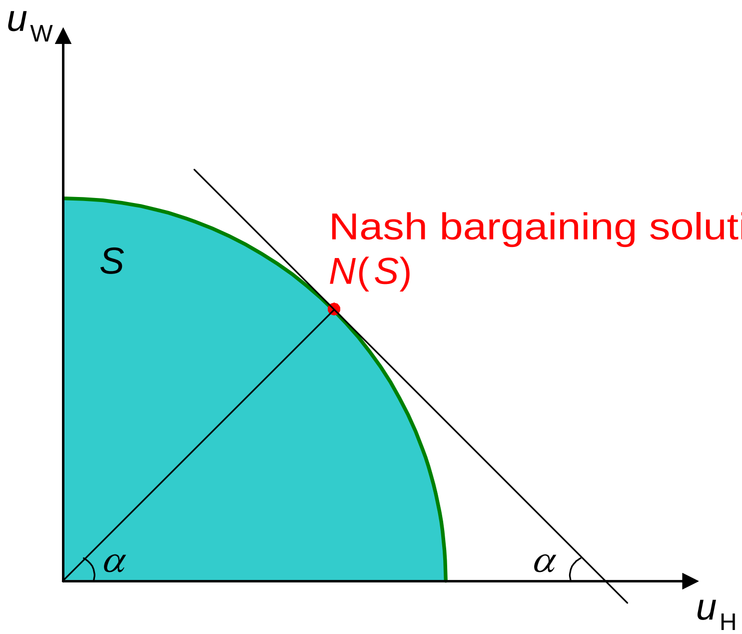


The players are not willing to choose any point in the utility set, because some of them yield worse payoff for them than the threat agreement. Therefore, they search for a solution only in the subset *S* that consists of the payoff pairs that are jointly preferred to *d*.

Obviously, the players should now select a Pareto optimal point in the utility set, but which one of them is fair?

Nash solution to the co-operative game

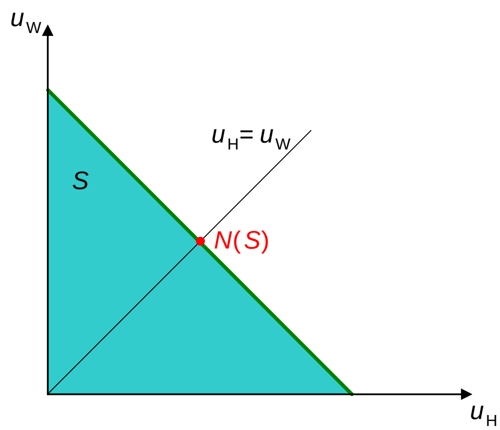
The players first find the Nash bargaining solution, which is denoted by $N(S)$, simply by maximising Nash's product $(u_H-d_H)(u_W-d_W)$ in S:



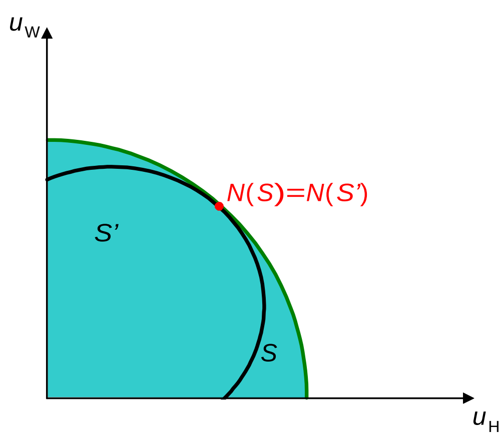
Geometrically it is found, for instance, by drawing a triangle such that its one side tangents the set S and one vertex is at the threat point (origin).

Nash's solution $N(S)$ is implied by the following *axioms*:

1. $N(S)$ is Pareto optimal and jointly preferred to the disagreement point.
2. $N(S)$ is independent of the units of the decision-makers: the solution does not vary if the utility of a decision-maker is multiplied by a positive constant.
3. Symmetry: the utilities of the decision-makers at the solution are equal if the set S is symmetric. By definition, S is symmetric if the set S is not affected even if the roles of the players are changed, see figure below as an example for symmetrical set.



4. Independence of the irrelevant alternatives: if there is a negotiation set *S* and *S* is narrowed to produce a new smaller negotiation set *S'* such that the *N*(*S*) is in *S'*, then *N*(*S'*) = *N*(*S*), see figure below for illustration.

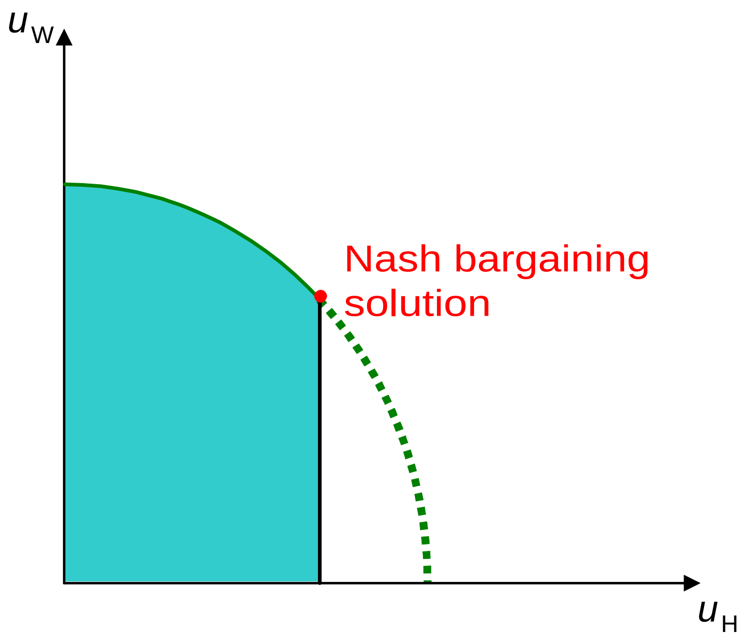


… so what? …

Should Harold and William accept $N(S)$? Is it fair solution? If they consider the axioms fair so perhaps they should make an agreement.

… some criticism …

Nash's fourth axiom, independence of irrelevant alternatives, has been the source of considerable contention. Imagine that the game is modified such that its representation in the utility set is as in figure below.



Due to the axiom of irrelevant alternatives the original and the modified game have the Nash bargaining solution in common. The dilemma is, that Harold obtains maximal payoff and William achieves only about 80% of the maximal payoff in the modified case.

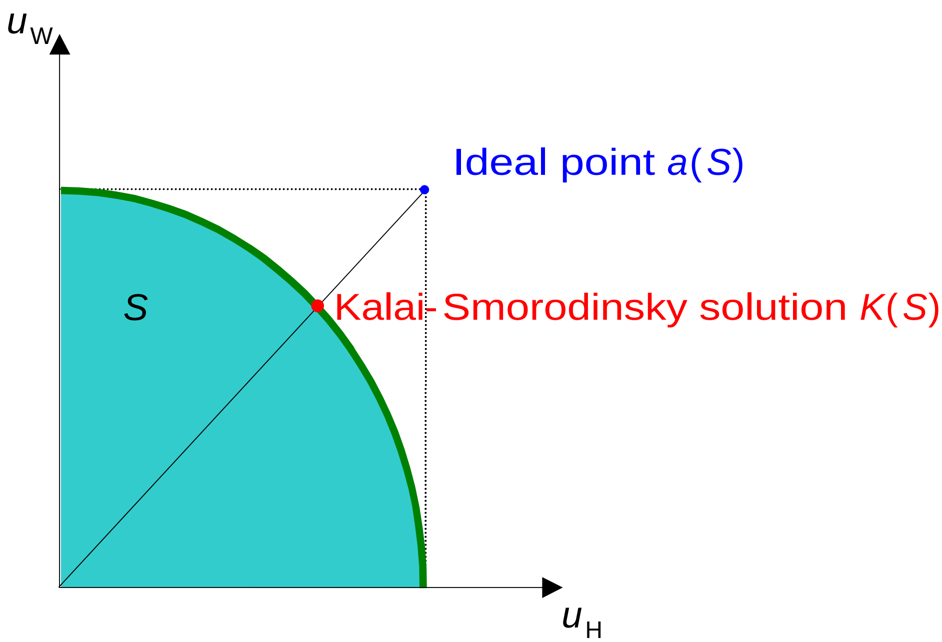
Kalai-Smodrodinsky solution

Kalai and Smorodinsky (1975) modified the axioms presented by Nash. They replaced his arguing fourth axiom by monotonicity axiom:

4’. If the negotiation set *S* is enlargened such that the maximum utilities of the players remain unchanged, then neither of the players must not suffer from it.

… its graphical representation …

This solution concept can be described graphically, as well. Define an *ideal point* $a(S)$ gives the maximum payoff for Harold and William separately, see the figure below.



Now, connect the origin and that ideal point by a line segment. The Kalai-Smorodinsky solution $K(S)$ is maximal point in S on that line segment.

Further reading

- Nash, J. (1950). "The Bargaining Problem". *Econometrica*, 18, 155-62. Nash presents his axioms and the implied solution.
- Nash, J. (1953). "Two-Person Cooperative Games". *Econometrica*, 21, 128-40. Presents an interesting equilibrium property of the Nash's bargaining solution.
- Kalai, E. and M. Smorodinsky (1975). "Other Solutions to the Nash's Bargaining Problem". *Econometrica* 43, 513-18.
- Thomson, W. and T. Lensberg (1989). *Axiomatic Theory of Bargaining with a Variable number of Agents*. Cambridge University Press. Contains a short review on different axiomatic solutions and their generalisations.

Negotiation analysis

In the game theory section we showed that a natural outcome for Harold and William is the Nash equilibrium. Yet there are other outcomes, although not sharing an equilibrium property, giving a better result in terms of utility payoffs. For example, think about the jointly improving Pareto optimal outcomes. In axiomatic bargaining theory the Pareto optimal outcomes were analysed in the utility set but no attention was paid to how to reach them. In negotiation analysis, such questions are addressed and methods to generate Pareto optimal outcomes are analysed.

The term negotiation analysis was first introduced by Raiffa (1982) who gave a comprehensive description about the field and the possible methods by integrating elements from game theory and multiple criteria decision analysis. In this section basic concepts in negotiation analysis are introduced.

... what are the parties? ...

In Harold's and William's problem, it is very easy to identify the negotiating *parties*. Nevertheless, it is not always easy to recognize different parties.

For instance, consider a water system regulation policy making. Usually there are many stakeholders involved in such a policy making. Suppose that it is possible to identify some interest groups such as power producers, farmers, environmentalists, recreational users and fishermen and every stakeholder belongs to some group. Nevertheless, it is not at all clear if these groups can act as parties in negotiation since it is well possible that the members within a group may not agree among themselves in such a situation. Thus there emerges pressure to splinter the groups to smaller ones.

... what are the issues? ...

In the case of Harold and William the *issues* are easily identified: the number of goats they choose. Nevertheless, they may include other issues, like money, to their negotiation or link completely separate negotiations together establishing the possibility to achieve better outcomes. In principle, the issues are chosen so that they describe sufficiently well the underlying negotiation situation. Note that the issue selection can be considered as a negotiation problem of its own.

... the mediator ...

Often the negotiating parties ask a third party to intervene the negotiation. His or her role is to suggest agreements or to offer facilities, like the communication forms, for the parties. One type of intervenor is a *mediator* that is a neutral party gathering some confidential information from the parties, making suggestions for them and assisting them to find a jointly accepted agreement. Usually a mediator is a person using a software to support the negotiation, but also a software alone can take the role of a mediator. In the latter case the parties use the supporting software by themselves.

Totally another type of intervenor is an *arbitrator* that analyses the problem and, unlike the mediator, dictates the solution for the parties.

... how to negotiate? ...

Here we present the main types of negotiation procedures as classified by Raiffa (1982):

1. Whether the value functions of the parties are elicited or not.

2. Whether the parties

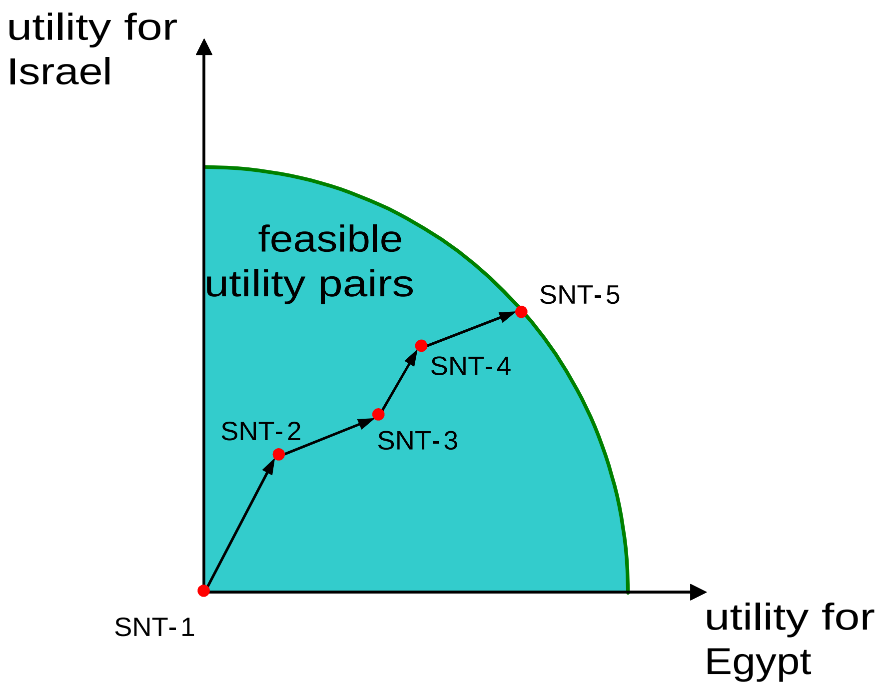
- *make concessions* or
- seek for joint improvements from a reference outcome.

... joint improvements seeking methods ...

The methods that step by step seek for joint improvements are called *single negotiation text* (*SNT*) methods. This type of negotiation was developed by Fisher and Ury (1981) and it was first applied in the Middle East peace negotiations between Egypt and Israel at Camp David in 1978; Raiffa (1982). There were seven issues to be decided upon and a U.S. team worked as an assisting mediator, who presented an initial tentative agreement, called SNT-1, by putting initial suggestive values for the issues and asked the parties to evaluate it. Based on the evaluations the U.S. team remodified the tentative agreement iteratively and this way went through several SNT's until no joint improvements were possible. As a result, after five tentative agreements, the parties concluded the peace. The U.S. president Jimmy

Carter worked as the head of the U.S. team and his mediation was qualified for the Nobel Peace Prize in 2002.

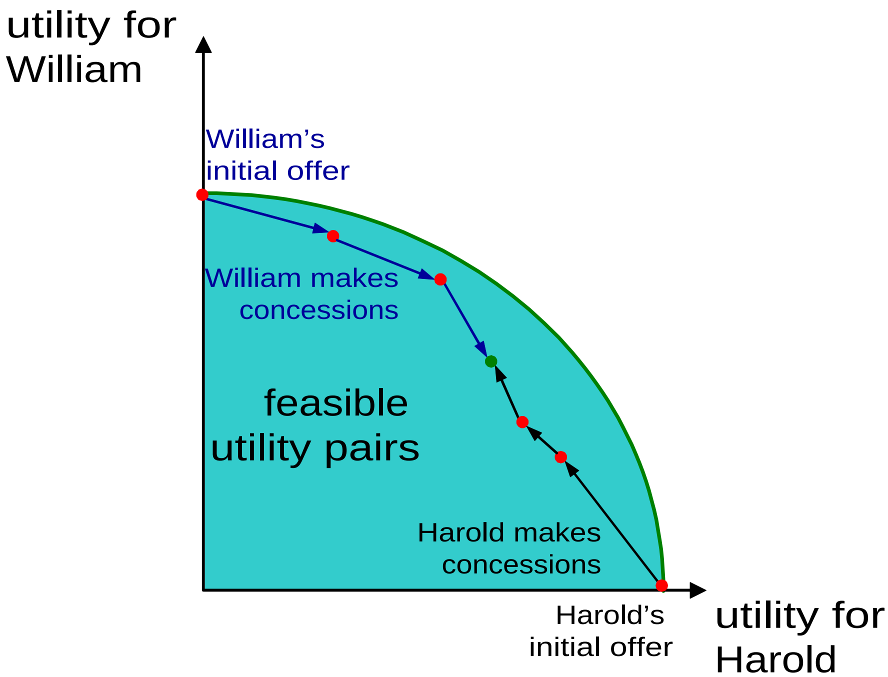
The figure below presents the Camp David negotiation process in the utility set.



... concession based methods ...

In concession based methods, the parties start the negotiation from separate positions and they are supposed to make *concessions* from their current positions until they finally reach each other.

For instance, suppose that Harold and William both insist, that the other is not allowed to graze his goats in the green at all. So, they are in the different initial positions and they do not have a tentative agreement as in the SNT-procedure. They make concessions by giving in more and more to the other party until they finally shake hands. The figure below shows an example about how they might proceed in the utility set.



As a result of concessions, they reach an agreement which might not be Pareto optimal. Hence, there still exists room for joint improvements and they could apply SNT-method.

... value function based methods ...

In *value function based methods*, the parties' utility, or value functions as they are called in the negotiation analysis context, functions are elicited. This makes it possible to construct all Pareto optimal outcomes and to let the parties finally choose among these. One possibility then is to select a particular outcome by using the axiomatic bargaining theory.

... interactive methods ...

Since the elicitation of the parties' value functions may be a formidable task a good alternative is to use *interactive methods*. They are used to prevent the elicitation of the value functions as a whole. In interactive methods, only local preference information from the parties is required. In practise, a mediator could ask the parties only a few relatively simple questions. For instance, the mediator might ask them to compare some outcomes.

Further reading

- Raiffa, H., J. Richardson and D. Metcalfe (2002). *Negotiation Analysis: The Science and Art of Collaborative Decision Making*. Belknap Press of Harvard University.
- Raiffa, H. (1982). *Art and Science of Negotiation*. Harvard University Press.
- Fisher, R., and W. Ury (1981). *Getting to Yes*. Arrow.

Jointly improving direction method

The jointly improving direction method is developed by Ehtamo, Verkama and Hämäläinen (1999) and Ehtamo, Kettunen and Hämäläinen (2001) and it is a mathematical formalisation of the SNT procedure.

The method is an interactive method; nevertheless it can be used to generate Pareto solutions also in the case where the value functions are explicitly known. Here we solve the negotiation problem associated with the problem of the commons in both ways.

... the method ...

The method consists of three different phases that are repeated until no joint improvements can be achieved:

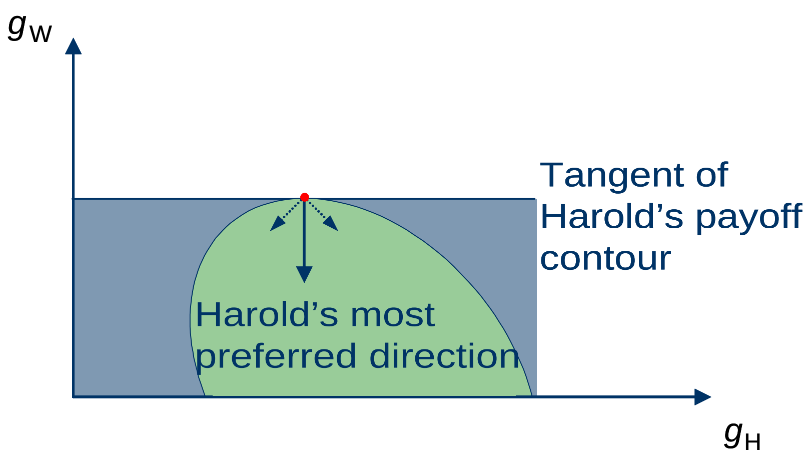
1. The mediator helps the parties to *criticise* the tentative agreement
2. The mediator generates a compromise direction
3. The mediator helps the parties to find a jointly preferred outcome in the compromise direction

... criticizing the tentative agreement ...

Recall that Harold and William ended up to the Nash equilibrium N which is their result if the negotiation does not end to a jointly accepted agreement. Therefore, it is their reference agreement and they select it as the initial tentative agreement which they try to jointly improve. First Harold criticises it.

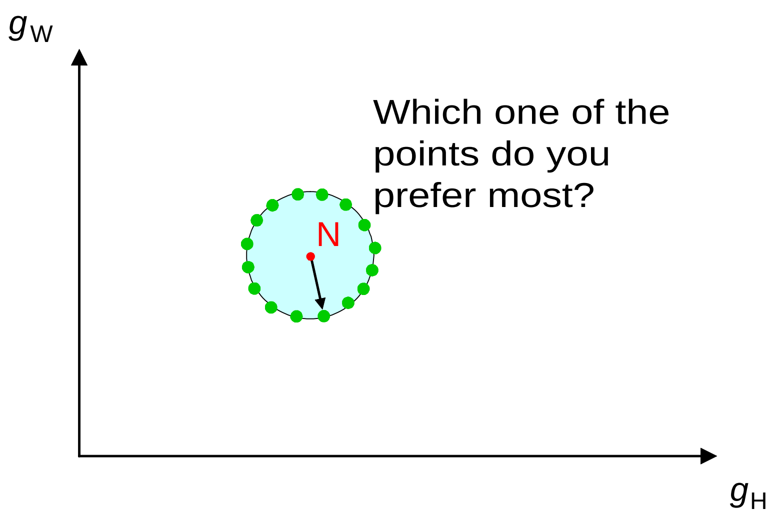
By definition, neither of the parties wants to deviate the Nash equilibrium outcome alone. Thus one party will gain only if the other or they both will accept a move from Nash equilibrium.

Recall that the points strictly inside of a party's contour give a better payoff for him/her. Thus a move along any arrow which starts from the Nash equilibrium outcome (N) and ends to a point inside the contour going through N gives a positive payoff gain for Harold. We call such arrows Harold's *improving directions* some of which are included in the figure below. The arrow that is perpendicular to the tangent line of the contour at N is called Harold's *criticism*. It is the *most improving direction* or the steepest ascent direction, as it is called in the mathematics literature, of Harold's utility payoff at N.



... finding the most preferred direction interactively ...

The mediator draws a small circle, or ellipse, centered at the initial tentative agreement and takes some points on it. The mediator asks Harold to choose the point he prefers most. After Harold has made his choice, the mediator is able to approximate Harold's most preferred direction by drawing a line segment from the tentative agreement to the point Harold chose, see figure below.



Note that the approximation of the most preferred direction need not be exactly perpendicular to the contour but the smaller the circle is the better is the approximation. On the other hand the circle must not be too small because otherwise it would be too difficult to find the best point in it.

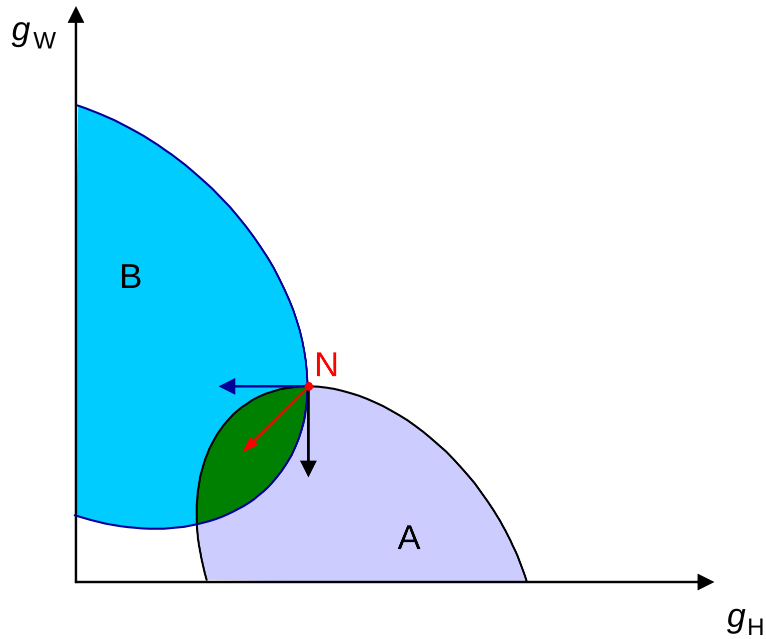
... finding compromise direction ...

Similarly, William can find his most preferred direction and his improving directions as well. The set of *jointly improving directions* can now be found by taking the directions that are improving for both of them.

By definition, a direction is *jointly improving* if by taking any sufficiently small step to the direction an outcome that is better for each party is reached.

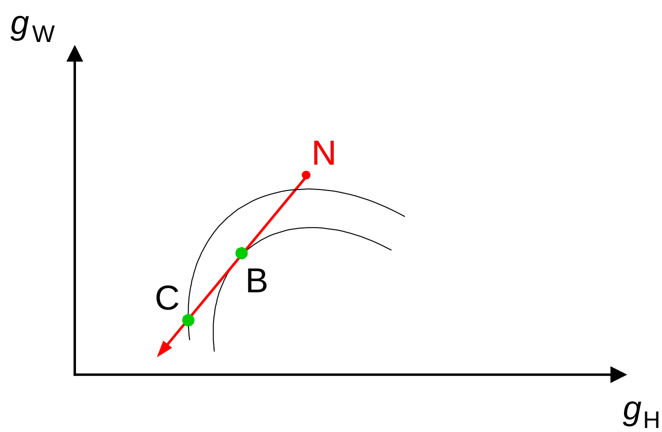
The mediator selects one such direction as a *compromise direction*. Here we define the compromise direction to be the direction bisecting the angle between their most preferred directions. The mediator also takes a suitable step to the compromise direction to propose a jointly preferred outcome to the parties.

In the figure below the intersection of A and B define the jointly improving directions. Namely an arrow from N to a point in the intersection is a jointly improving direction. The compromise direction is the bisecting direction of the parties most preferred directions.



... choosing the most preferred agreement ...

A suitable step length can be found, e.g., by asking Harold and William to choose their most preferred outcomes along the compromise direction.

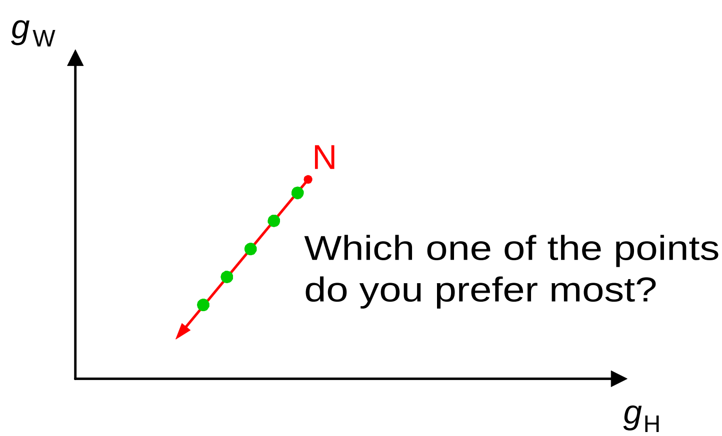


Let us check out that Harold’s most preferred outcome on the compromise direction is the point B at which one of his payoff contours touches that direction. Consider any other outcome in the direction, e.g., outcome C, and draw a contour through it. Because B lies on the inner contour it gives better payoff for Harold than C.

If Harold and William choose different outcomes in the compromise direction, then the mediator chooses the one that is closer to the tentative agreement and proposes it as a new tentative agreement. This guarantees that the step is not too long and hence the proposal is jointly preferred.

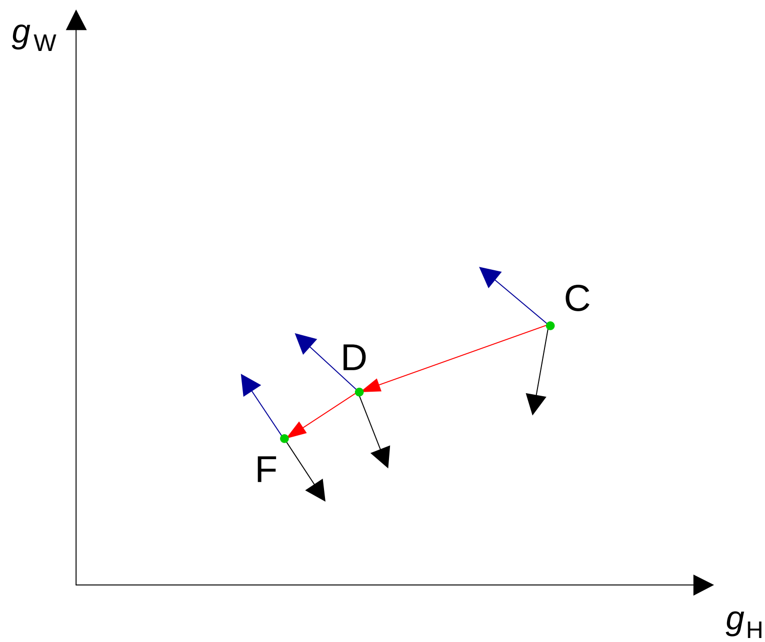
... finding tentative agreement interactively ...

The mediator may help Harold and William to find their most preferred outcomes on the compromise direction if they do not know their payoff functions explicitly. For instance, the mediator can take some points on the direction and ask Harold and William to choose the one they prefer most.



... seeking for more joint improvements ...

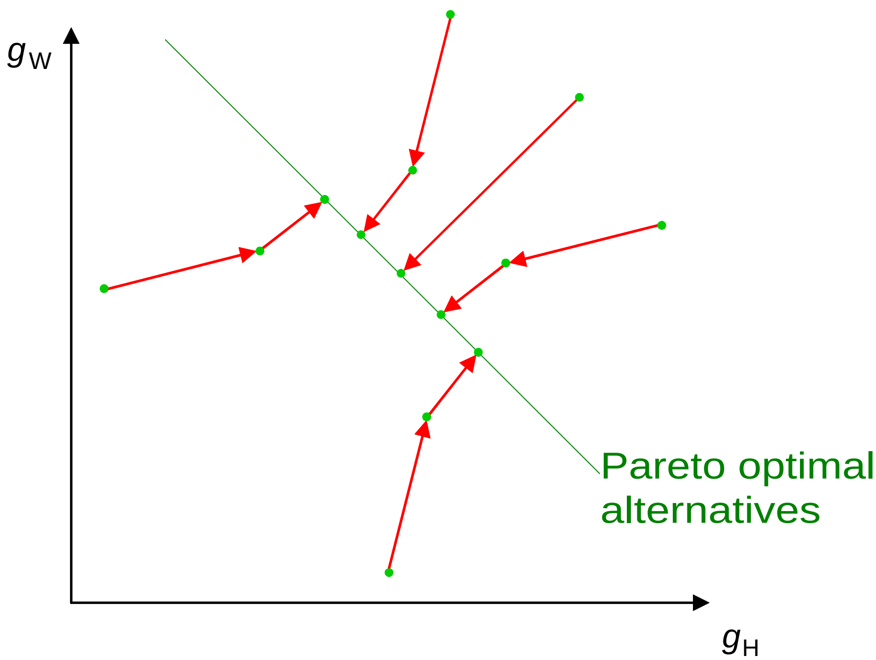
The parties should seek joint improvements starting from the new tentative agreement until a Pareto optimal outcome is reached.



The figure above shows how joint improvements are produced. When the parties' most preferred directions are opposite directions there will be no jointly improving directions and the tentative agreement is Pareto optimal.

... searching for more Pareto optimal outcomes ...

The jointly improving directions method allows the parties to find also other Pareto optimal outcomes. They only need to negotiate starting from different initial tentative agreements and produce as many Pareto optimal outcomes as they wish to.



Further reading

- Ehtamo, H., M. Verkama and R. Hämäläinen (1999). “How to Select Fair Improving Directions in a Negotiation Model over Continuous Issues”. *IEEE Transactions on Systems Man and Cybernetics – Part C: Applications and Reviews*, 29, 26-33. The method of improving directions is presented mathematically for two parties and two issues.
- Ehtamo, H. and R. Hämäläinen (2001). ”Interactive Multiple Criteria Methods for Reaching Pareto Optimal Agreements in Negotiations”. *Group Decision and Negotiation*, 10, 475-491. A non-mathematical presentation of the method of improving directions.
- Ehtamo, H., E. Kettunen and R. Hämäläinen (2001). “Theory and Methodology Searching for Joint Gains in Multi-Party Negotiations”. *European Journal of Operational Research*, 130, 54-69. A mathematical description and analysis of the method of improving directions.
- Hämäläinen, R., E. Kettunen, H. Ehtamo and M. Marttunen (2001). ”Evaluating a Framework for Multi-

Stakeholder Decision Support in Water Resources Management". *Group Decision and Negotiation*, 10, 331-353. Presents a multi-party decision making problem and some solution methods including the method of improving directions.

Joint Gains

Joint Gains is a Java applet that implements the method of improving directions. Joint Gains offers facilities for the parties, that can be spatially distributed all over the Internet, to negotiate through the web if they have Java enabled web browser installed.

The implementation of Joint Gains is interactive and it

1. helps the parties to evaluate the tentative agreement.
2. calculates a jointly improving direction.
3. helps the participants to find their most preferred alternatives along the compromising direction.
4. calculates a proposal
5. asks the parties to accept the proposal, that is to state if they prefer the proposal to the tentative agreement.

If all parties prefer the proposal to the tentative agreement, the proposal becomes the tentative agreement and the process continues from step 1 otherwise the procedure stops.

We illustrate Joint Gains is by a case: problem of buyer and seller, which shows how the Joint Gains helps the parties to evaluate the tentative agreement.

In Joint Gains, the following terms are used:

Term in Joint Gains	Concept
Facilitator	Mediator
decision variable	Issue
Participant	Party
resolution parameter	radius of the circle used in the evaluation elicitation

Glossary

- action, Game theory
- agreement, Axiomatic bargaining
- alternative, MCDA
- arbitrator, Negotiation analysis
- arc, Game theory
- axiom, Axiomatic bargaining
- best response curve, Game theory
- best response strategy, Game theory
- bi-matrix, Game theory
- bisection method, MCDA
- cardinal value function, MCDA
- competitive equilibrium, Game theory
- compromise direction, Method of improving directions
- concave, MCDA
- concessions, Negotiation analysis

- consequence, MCDA
- constraint, MCDA
- contour, MCDA
- criteria, MCDA
- decision node, Game theory
- decision variable, MCDA
- decreasing marginal utility, MCDA
- disagreement point, Axiomatic bargaining
- efficient, MCDA
- evaluation, MCDA
- extensive form game, Game theory
- game tree, Game theory
- gradient, MCDA
- ideal point, Axiomatic bargaining
- improving direction, MCDA
- indifference curve, MCDA
- initial node, Game theory
- interactive methods, Negotiation analysis