Nash Bargaining Solution and Alternating Offer Games

MIT 14.126-Game Theory

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Nash Bargaining Model

Formulation

Axioms & Implications

Elements

- The bargaining set: S
  - Utility pairs achievable by agreement
  - When? Immediate agreement?
- Disagreement point: $d \in R^2$
  - Result of infinitely delayed agreement?
  - Payoff during bargaining?
  - Outside option?
- Solution: $f(S,d) \in R^2$ is the predicted bargaining outcome

Impossibility of Ordinal Theory

- Fix (S,d) as follows

$$d = (0,0), S = \{x \geq 0 \mid x_1 + x_2 \leq 1\}$$

- Represent payoffs "equivalently" by (u_1,u_2) where

$$u_1 = x_1^{.5}, u_2 = 1-(1-x_2)^{.5}$$

- Then, the bargaining set is:

$$d' = (0,0), S' = \{u \geq 0 \mid u_1 + u_2 \leq 1\}$$

- Ordinal preferences over bargaining outcomes contain too little information to identify a unique solution.

Nash's Initial Assumptions

- Cardinalization by risk preference
  - Why?
  - What alternatives are there?
- Assume bargaining set S is convex
  - Why?

Nash's Axioms

- Independence of Irrelevant Alternatives (IIA)
  - If $f(S,d) \in T \subset S$, then $f(T,d)=f(S,d)$
- Independence of positive linear transformations (IPLT)
  - Let $h_i(x_i)=\alpha_i x_i+\beta_i$, where $\alpha_i>0$, for $i=1,2$.
  - Suppose $a=f(S,d)$. Let $S'=h(S)$ and $d'=h(d)$. Then, $f(S',d')=h(a)$.
- Efficiency
  - $f(S,d)$ is on the Pareto frontier of S
- Symmetry
  - Suppose $d'=(d_2,d_1)$ and $x \in S \Leftrightarrow (x_2,x_1) \in S'$. Then, $f_1(S,d)=f_2(S',d')$ and $f_2(S,d)=f_1(S',d')$.

Independence of Irrelevant Alternatives (IIA)

- Statement of the IIA condition
  - If $f(S,d) \in T \subset S$, then $f(T,d)=f(S,d)$
- Definitions.
  - $Vex(x,y,d)$ = convex hull of $\{x,y,d\}$.
  - xP_dy means $x=f(Vex(x,y,d),d)$.
  - xPy means $x=f(Vex(x,y,0),0)$
- By IIA, these are equivalent
  - $x=f(S,d)$
  - xP_dy for all y in S

Efficiency

- Statement of the efficiency condition
  - $f(S,d)$ is on the Pareto frontier of S
- Implications
  - The preference relations P_d are "increasing"

Positive Linear Transformations

- Statement of the IPLT condition
  - Let $h_i(x_i)=\alpha_i x_i+\beta_i$, where $\alpha_i>0$, for i=1,2.
  - Suppose a=f(S,d). Let S'=h(S) and d'=h(d). Then, f(S',d')=h(a).
- Implications
  - xP_dy if and only if (x-d)P(y-d)
  - Suppose d=0 and $x_1x_2=1$.
    - If $(x_1,x_2)P(1,1)$ then $(1,1)P(1/x_1,1/x_2)=(x_2,x_1)$

Symmetry

- Statement of the symmetry condition
  - Suppose $d'=(d_2,d_1)$ and $x\in S \Leftrightarrow (x_2,x_1)\in S'$. Then, $f_1(S,d)=f_2(S',d')$ and $f_2(S,d)=f_1(S',d')$.
- Implication
  - When d=(0,0), (x_1,x_2) is indifferent to (x_2,x_1).
- IPLT + Symmetry imply
  - $x_1x_2=1 \Rightarrow$ x is indifferent to (1,1).
  - $x_1x_2=y_1y_2 \Rightarrow$ x is indifferent to y.

A Nash Theorem

- Theorem. The unique bargaining solution satisfying the four axioms is given by:

$$f(S,d)\in \underset{x\in S}{\operatorname{argmax}}(x_1-d_1)(x_2-d_2)$$

- Question: Did we need convexity for this argument?

Alternating Offer Bargaining

- Two models
  - Both models have two bargainers, feasible set S
  - Multiple rounds: bargainer #1 makes offers at odd rounds, #2 at even rounds
  - An offer may be
    - Accepted, ending the game
    - Rejected, leading to another round
    - Possible outcomes
      - No agreement is ever reached
      - Agreement is reached at round t

Model #1: Risk of Breakdown

- After each round with a rejection, there is some probability p that the game ends and players receive payoff pair d.
  - Best equilibrium outcome for player one when it moves first is a pair (x_1,x_2) on the frontier of S.
  - Worst equilibrium outcome for player two when it moves first is a pair (y_1,y_2) on the frontier of S.
  - Relationships:

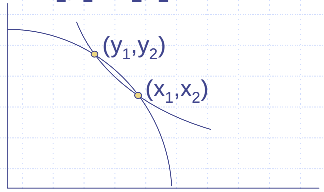
$$x_2 = (1-p)y_2 + pd_2$$
$$y_1 = (1-p)x_1 + pd_1$$

The Magical Nash Product

- Manipulating the equations:

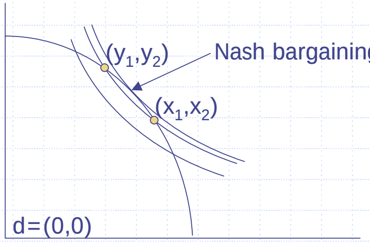
$$x_2 - d_2 = (1-p)(y_2 - d_2)$$
$$(1-p)(x_1 - d_1) = y_1 - d_1$$
$$(x_1 - d_1)(x_2 - d_2) = (y_1 - d_1)(y_2 - d_2)$$

- Taking d=(0,0), a solution is a 4-tuple (x_1,y_1, x_2,y_2) such that $x_1y_1=x_2y_2$, as follows:



Main Result

- Theorem. As p→0, (x_1,x_2) and (y_1,y_2) (functions of p) converge to f(S,d).
- Proof. Note: $y_1=(1-p)x_1$ and $x_2=(1-p)y_2$ and...



Commentary

- Facts and representations
  - Cardinal utility enters *because risk is present*
  - The risk is that the disagreement point *d* may be the outcome.
  - Comparative statics (risk aversion hurts a bargainer) is interpretable in these terms.

Outside Options

- Modify the model so that at any time t, either bargainer can quit and cause the outcome $z \in S$ to occur.
  - Is z a suitable threat point?
- Two cases:
  - If $z_1 \le y_1$ and $z_2 \le x_2$, then the subgame perfect equilibrium outcome is unchanged.
  - Otherwise, efficiency plus

$$x_2 = \max[z_2, (1-p)y_2]$$
$$y_1 = \max[z_1, (1-p)x_1]$$

Model #2: Time Preference

- An outcome consists of an agreement x and date t.
- Assumptions to model time preference
  - A time indifferent agreement n exists
  - Impatience: (x,0)P(n,0) and t<t' imply (x,t)P(x,t')
  - Stationarity: (x,t)P(x',t') implies (x,t+s)P(x',t'+s).
  - Time matters (+continuity): (x,0)P(y,0)P(n,0) implies there is some t such that (y,t)**I**(x,0).
- Theorem. For all $\delta \in (0,1)$, there is a function u such that (x,t)P(x',t') if and only if $u(x)\delta^t > u(x')\delta^{t'}$. In particular, u(n)=0.

Proof Exercise

- Insight: Same axioms imply that preferences can be written as:

$$v(x) - t \cdot \ln(\delta)$$

- Exercise: Interpret t as cash instead of time.
  - State similar axioms about preferences over (agreement, payment) pairs.
  - Use these to prove the quasi-linear representation that there exists a function v such that (x,0) is preferred to (y,t) if and only v(x)>v(y)+t.

Representing Time Preference

- Theorem. Suppose that u and v are positive functions with the property that $v(x)=[u(x)]^A$ for some A>0. Then $u(x)\delta^t$ and $v(x)\varepsilon^t$ represent the same preferences if and only if $\varepsilon = \delta^A$.

- Proof. Exercise.

Comparative Statics

- The following changes in preferences are equivalent
  - From $u(x)\delta^t$ to $u(x)\varepsilon^t$
  - From $u(x)\delta^t$ to $v(x)\delta^t$, where $v(x)=u(x)^A$ and $A=\ln(\delta)/\ln(\varepsilon)$.
- Hence, for fixed δ, greater impatience is associated with "greater concavity" of u.

Bargaining with Time Preference

- This model is identical in form to the risk preference model, but has a different interpretation.
- Fix $\delta\in(0,1)$ and corresponding utility functions u_1 and u_2 such that bargainer j's preferences over outcome (z,t) are represented by $x_j= \delta^t u_j(z)$.
  - Best equilibrium outcome for player one when it moves first is a pair (x_1,x_2) on the frontier of S.
  - Worst equilibrium outcome for player two when it moves first is a pair (y_1,y_2) on the frontier of S.
  - Relationships:

$$x_2 = \delta y_2, y_1 = \delta x_1$$
$$x_1 x_2 = y_1 y_2$$

General Conclusions

- Cardinalization principle
  - The proper way to cardinalize preferences depends on the source of bargaining losses that drives players to make a decision.
- Outside option principle
  - Outside options are not "disagreement points" and affect the outcome only if they are better for at least one party than the planned bargaining outcome.

End