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ANALYSIS OF TWO BARGAINING PROBLEMS WITH INCOMPLETE INFORMATION

by

Roger B. Myerson

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J. L. Kellogg Graduate School of Management
Northwestern University
Evanston, Illinois 60201

Abstract. This paper considers two bilateral trading problems with incomplete information: the symmetric uniform trading problem, and the lemon problem of Akerlof. For each example, we characterize the feasible and efficient trading mechanisms, and the neutral bargaining solutions that might actually be implemented by the traders in face-to-face negotiations. Experimentally testable predictions are developed.

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1. Introduction

To analyze a cooperative game with incomplete information, there are three kinds of solution concepts that should be considered. First, we should characterize the set of coordination mechanisms or decision rules that are feasible for the players when they cooperate, taking account of the incentive constraints that arise because the players cannot always trust each other. Second, we should characterize the mechanisms that are efficient within this feasible set. Efficiency criteria for games with incomplete information have been discussed in detail by Holmström and Myerson [1983]. Third, we should try to identify equitable mechanisms on the efficient frontier that are likely to actually be implemented by the players if they are sophisticated negotiators with equal bargaining ability. (We might also want to consider cases where one player has more bargaining ability than the others, as in principal-agent problems.) For this analysis, a concept of neutral bargaining solution has been axiomatically derived by Myerson [1983, 1984].

In this paper, we analyze two bilateral trading problems with incomplete information in terms of these three solution concepts. In Sections 2, 3, and 4, we consider the symmetric uniform trading problem, a simple problem in which the buyer and seller each have private information about how much the object being traded is worth to him. This problem was first studied by Chatterjee and Samuelson [1983], and was also considered by Myerson and Satterthwaite [1983]. In Sections 5 and 6, we consider the lemon problem, in which only the seller has private information, but the value of the object to the buyer may depend on this information. Akerlof [1970] first studied a version of the lemon problem, in a market context, and Samuelson [1981] characterized the seller's ex-ante optimal mechanisms.

Section 7 contains the more technical proofs relating to the neutral bargaining solutions. A reader who is not familiar with the earlier papers on this subject may prefer to omit this final section.

2. The Symmetric Uniform Trading Problem: Feasibility

In this section and the next two, we consider a bargaining problem in which there is only one seller (trader #1) and one potential buyer (trader #2) for a single indivisible object. Both buyer and seller have risk-neutral utility for money. We let $\tilde{V}_1$ denote the value of the object to the seller and $\tilde{V}_2$ denote the value to the buyer. We assume that $\tilde{V}_1$ and $\tilde{V}_2$ are independent random variables, and that each is uniformly distributed over the interval from 0 to 1 (in some monetary scale). Thus, the bargaining situation may be referred to as the symmetric uniform trading problem.

We assume that each trader i knows his own valuation $\tilde{V}_i$ at the time of bargaining, but he considers the other's valuation as a random variable. Furthermore, neither trader can directly observe the other's valuation. The traders can communicate with each other, but each would be free to lie about the value of the object to him, if he expected to get a better price by doing so.

A direct trading mechanism is one in which each trader simultaneously reports his valuation to a mediator or broker, who then determines whether the object is transferred from seller to buyer and how much the buyer must pay the seller. A direct mechanism is thus characterized by two outcome functions, denoted by $p(\cdot,\cdot)$ and $x(\cdot,\cdot)$, where $p(v_1,v_2)$ is the probability that the object is transferred to the buyer and $x(v_1,v_2)$ is the expected payment to the seller if v_1 and v_2 are the reported valuations of the seller and buyer. A direct mechanism is (Bayesian) incentive compatible if honest reporting forms a Bayesian Nash equilibrium. That is, in an incentive-compatible mechanism, each trader can maximize his expected utility by reporting his true valuation, given that the other trader is expected to report honestly.

We can, without loss of generality, restrict our attention to incentive-compatible direct mechanisms. This is because, for any Bayesian equilibrium of any bargaining game, there is an equivalent incentive-compatible direct mechanism that always yields the same outcomes (when the honest equilibrium is played). This result, which is well known and very general, is called the revelation principle. The essential idea is that, given any equilibrium of any bargaining game, we can construct an equivalent incentive-compatible direct mechanism as follows. First we ask the buyer and seller each to confidentially report his valuation; then we compute what each would have done in the given equilibrium strategies with these valuations; and then we implement the outcome (transfers of money and the object) as in the given game for this computed behavior. If either individual had any incentive to lie to us in this direct mechanism, then he would have had an incentive to lie to himself in the original game, which is a contradiction of the premise that he was in equilibrium in the original game. (For more on this revelation principle, see Myerson [1979].)

Given a direct mechanism with outcome functions (p,x), we define the following quantities

$$\bar{x}_1(v_1) = \int_0^1 x(v_1,t_2)\, dt_2\ , \qquad \bar{x}_2(v_2) = \int_0^1 x(t_1,v_2)\, dt_1,$$

$$\bar{p}_1(v_1) = \int_0^1 p(v_1,t_2)\,dt_2\ , \qquad \bar{p}_2(v_2) = \int_0^1 p(t_1,v_2)\,dt_1,$$

$$U_1(v_1,p,x) = \bar{x}_1(v_1) - v_1\,\bar{p}_1(v_1), \quad U_2(v_2,p,x) = v_2\,\bar{p}_2(v_2) - \bar{x}_2(v_2).$$

Thus, $U_1(v_1,p,x)$ is the expected profits or gains from trade for the seller if his valuation is v_1, since $\bar{x}_1(v_1)$ is his expected revenue and $\bar{p}_1(v_1)$ is his probability of losing the object given $\tilde{V}_1 = v_1$. Similarly, $U_2(v_2,p,x)$ is the expected gains from trade for the buyer, $\bar{x}_2(v_2)$ is the buyer's expected payment, and $\bar{p}_2(v_2)$ is the buyer's probability of getting the object, if his valuation is v_2.

In this formal notation, (p,x) is <u>incentive compatible</u> iff

$$U_1(v_1,p,x) \geq \bar{x}_1(t_1) - v_1\,\bar{p}_1(t_1)$$

and

$$U_2(v_2,p,x) \geq v_2\,\bar{p}_2(t_2) - \bar{x}_2(t_2)$$

for every v_1, v_2, t_1, and t_2 between 0 and 1. These two inequalities assert that neither trader should expect to gain in the mechanism by reporting valuation t_i when v_i is his true valuation.

We say that a mechanism (p,x) is <u>individually rational</u> iff each trader gets nonnegative expected gains from trade given any valuation, that is,

$$U_1(v_1,p,x) \geq 0 \quad \text{and} \quad U_2(v_2,p,x) \geq 0$$

for every v_1 and v_2 between 0 and 1. Since each individual already knows his valuation when he enters the bargaining process and neither individual can be forced to trade, a feasible mechanism should be individually rational in this sense, as well as incentive compatible. We say that a mechanism is <u>feasible</u> iff it is both individually rational and incentive compatible.

Many bargaining games satisfy a stronger individual-rationality condition: that neither individual ever consents to a trade that leaves him worse off ex post. Formally this condition is

$$x(v_1,v_2) - v_1\, p(v_1,v_2) \geq 0 \quad \text{and} \quad v_2\, p(v_1,v_2) - x(v_1,v_2) \geq 0$$

for every v_1 and v_2. If (p,x) satisfies this condition, then we must have

$$U_1(1,p,x) = 0 \quad \text{and} \quad U_2(0,p,x) = 0.$$

That is, the seller expects no gains from trade if $\tilde{V}_1 = 1$, since he knows that the buyer's valuation is lower; and similarly the buyer expects no gains from trade if $\tilde{V}_2 = 0$. We may say that a feasible mechanism (p,x) is <u>normal</u> iff $U_1(1,p,x) = 0 = U_2(0,p,x)$.

The following proposition completely characterizes the set of feasible mechanisms for the symmetric uniform trading problem.

<u>Proposition 1</u>. Given any function $p:[0,1]\times[0,1] \to [0,1]$, there exists some function $x(\cdot,\cdot)$ such that (p,x) is a feasible mechanism for the symmetric uniform trading problem if and only if $\bar{p}_1(\cdot)$ is a weakly decreasing function, $\bar{p}_2(\cdot)$ is a weakly increasing function, and

$$(1) \qquad 0 \leq \int_0^1 \int_0^1 (v_2 - v_1 - .5)\, p(v_1,v_2)\, dv_1\, dv_2.$$

Furthermore, x can be constructed so that (p,x) is normal if and only if (1) is satisfied with equality. In general, for any incentive-compatible mechanism (p,x)

$$(2) \qquad U_1(1,p,x) + U_2(0,p,x) = 2 \int_0^1 \int_0^1 (v_2 - v_1 - .5)\, p(v_1,v_2)\, dv_1\, dv_2$$

and, for every v_1 and v_2,

$$(3) \qquad U_1(v_1,p,x) = U_1(1,p,x) + \int_{v_1}^1 \bar{p}_1(s_1)\, ds_1$$

$$(4) \qquad U_2(v_2,p,x) = U_2(0,p,x) + \int_0^{v_2} \bar{p}_2(s_2)\, ds_2.$$

Proof. This proposition is a special case of Theorem 1 of Myerson and Satterthwaite [1983]. Q.E.D.

It is straightforward to check that

$$\int_0^1 \int_0^{v_2} (v_2 - v_1 - (1/3))\, dv_1\, dv_2 = 0.$$

Thus, conditional on the event that $\tilde{V}_2 \geq \tilde{V}_1$ (so that the individuals have something to gain from trading), the expected value of $\tilde{V}_2 - \tilde{V}_1$ equals 1/3. However, condition (1) asserts that, conditional on the event that a trade actually occurs, the expected value of $\tilde{V}_2 - \tilde{V}_1$ must be at least 1/2, for any feasible mechanism. Thus, it is not possible to construct a feasible mechanism in which trade occurs if and only if $\tilde{V}_2 \geq \tilde{V}_1$.

Condition (1) has experimentally testable implications. If we observe many instances of the symmetric uniform trading problem, with $\tilde{V}_1$ and $\tilde{V}_2$ chosen independently each time, and with each buyer and seller facing each other at most once (to avoid the complications of a repeated game), then the average difference $\tilde{V}_2 - \tilde{V}_1$ in those instances where trade occurs should be close to 1/2. This prediction holds true no matter what are the social conventions that regulate the negotiation process. We only need to assume that buyer and seller in each instance are playing some Bayesian Nash equilibrium of some bargaining game in which neither individual ever has to trade at a loss.

To interpret Proposition 1 geometrically, consider Figure 0. The dotted line represents the set of points where $v_2 = v_1 + 1/2$. If we draw any increasing curve in the unit square such that the center of gravity of the region above the curve lies on or above the dotted line, then there exists some feasible mechanism such that trade occurs if and only if $(\tilde{V}_1, \tilde{V}_2)$ is above the curve. For a normal mechanism, the center of gravity must be exactly on the dotted line.

[Insert Figure 0 here]

3. The Symmetric Uniform Trading Problem: Efficient Mechanisms

If two individuals can communicate effectively in a bargaining problem, then we may expect them to use a trading mechanism that is efficient, in the sense that there is no other incentive-compatible mechanism which they both would surely prefer. That is, we may say that an incentive-compatible mechanism (p,x) is *efficient* iff there does not exist any other incentive-compatible mechanism $(\hat{p},\hat{x})$ such that

$$U_1(v_1,\hat{p},\hat{x}) > U_1(v_1,p,x) \quad \text{and} \quad U_2(v_1,\hat{p},\hat{x}) > U_2(v_2,p,x)$$

for every v_1 and v_2 between 0 and 1. In the terminology of Holmström and Myerson [1983], this concept of efficiency corresponds to a weak form of *interim incentive efficiency*.

Using a standard separation argument, one can show that this definition is equivalent to the following more tractable characterization. A given incentive-compatible mechanism is efficient iff there exist two weakly increasing functions $L_1:[0,1] \to [0,1]$ and $L_2:[0,1] \to [0,1]$, with $L_1(0) = L_2(0) = 0$ and $L_1(1) = L_2(1) = 1$, such that the given mechanism maximizes

$$(5) \qquad \int_0^1 U_1(v_1,p,x)\, dL_1(v_1) + \int_0^1 U_2(v_2,p,x)\, dL_2(v_2)$$

over all incentive-compatible mechanisms (p,x). (It can be easily shown that $L_1(1) - L_1(0)$ must equal $L_2(1) - L_2(0)$, because otherwise a lump-sum transfer of money could make (5) arbitrarily large.) If L_1 and L_2 are differentiable, with $L_i' = \ell_i$, then the Riemann-Stieltjes integrals in (5) may be rewritten as

$$\int_0^1 U_1(v_1,p,x)\ \ell_1(v_1)\ dv_1 + \int_0^1 U_1(v_2,p,x)\ \ell_2(v_2)\ dv_2.$$

Proposition 2 below gives us a direct computational procedure for verifying efficiency of a mechanism.

<u>Proposition 2.</u> Suppose that (p,x) is an incentive-compatible mechanism for the symmetric uniform trading problem and $L_1(\cdot)$ and $L_2(\cdot)$ are weakly increasing functions such that $L_1(0) = L_2(0) = 0$ and $L_1(1) = L_2(1) = 1$. Suppose also that, for every v_1 and v_2 between 0 and 1,

$$p(v_1,v_2) = \begin{cases} 1 & \text{if} \quad 2v_1 - L_1(v_1) < 2v_2 - L_2(v_2), \\ 0 & \text{if} \quad 2v_1 - L_1(v_1) > 2v_2 - L_2(v_2). \end{cases}$$

Then (p,x) is efficient.

<u>Proof.</u> By Proposition 1, if (p,x) is incentive compatible, then

$$\int_0^1 U_1(v_1,p,x)\ dL_1(v_1) + \int_0^1 U_1(v_2,p,x)\ dL_2(v_2) =$$

$$= U_1(1,p,x) + \int_0^1 \int_{v_1}^1 \bar{p}_1(s_1)ds_1 dL_1(v_1) + U_2(0,p,x) + \int_0^1 \int_0^{v_2} \bar{p}_2(s_2)ds_2 dL_2(v_2)$$

$$= U_1(1,p,x) + U_2(0,p,x) + \int_0^1 L_1(s_1)\ \bar{p}_1(s_1)\ ds_1 + \int_0^1 (1 - L_2(s_2))\ \bar{p}_2(s_2)\ ds_2$$

$$= \int_0^1\int_0^1 (2v_2 - 2v_1 - 1)p(v_1,v_2)dv_1 dv_2 + \int_0^1\int_0^1 (L_1(s_1) + 1 - L_2(s_2))p(s_1,s_2)ds_1 ds_2$$

$$= \int_0^1 \int_0^1 \big((2v_2 - L_2(v_2)) - (2v_1 - L_1(v_1))\big)\ p(v_1,v_2)\ dv_1\ dv_2.$$

The conditions in Proposition 2 imply that p maximizes this double integral over all functions from $[0,1]\times[0,1]$ to $[0,1]$. Q.E.D.

Let us now consider three specific mechanisms which were studied by Chatterjee and Samuelson [1983].

The first mechanism corresponds to a game in which the seller has the authority to demand any price for his object, and then the buyer can either take it or leave it. The seller's optimal price in this game is $q_1 = (1 + \tilde{V}_1)/2$, which maximizes his expected profit $(1 - q_1)(q_1 - \tilde{V}_1)$. Thus, this mechanism is represented by (p^1,x^1) where

$$p^1(v_1,v_2) = \begin{cases} 1 & \text{if} \quad v_2 \geqslant (1 + v_1)/2, \\ 0 & \text{if} \quad v_2 < (1 + v_1)/2, \end{cases}$$

$$x^1(v_1,v_2) = \begin{cases} (1 + v_1)/2 & \text{if} \quad v_2 \geqslant (1 + v_1)/2, \\ 0 & \text{if} \quad v_2 < (1 + v_1)/2. \end{cases}$$

It is straightforward to verify that (p^1,x^1) is efficient, using Proposition 2 with

$$L_1(v_1) = v_1, \quad L_2(v_2) = \begin{cases} 0 & \text{if} \quad v_2 = 0, \\ 1 & \text{if} \quad v_2 > 0. \end{cases}$$

Figure 1 shows the trading region for this mechanism (p^1,x^1).

[Insert Figure 1 here]

The second mechanism corresponds to a game in which the buyer can commit himself to any offer price for the object, and then the seller can only accept it or reject it. The buyer's optimal price in this game is $q_2 = \tilde{V}_2/2$, which maximizes his expected profit $q_2(\tilde{V}_2 - q_2)$. Thus, this mechanism is represented by (p^2,x^2), where

$$p^2(v_1,v_2) = \begin{cases} 1 & \text{if} \quad v_2/2 \geq v_1, \\ 0 & \text{if} \quad v_2/2 < v_1, \end{cases}$$

$$x^2(v_1,v_2) = \begin{cases} v_2/2 & \text{if} \quad v_2/2 \geq v_1, \\ 0 & \text{if} \quad v_2/2 < v_1. \end{cases}$$

To verify that (p^2,x^2) is efficient, use Proposition 2 with

$$L_2(v_2) = v_2, \qquad L_1(v_1) = \begin{cases} 0 & \text{if} \quad v_1 < 1, \\ 1 & \text{if} \quad v_1 = 1. \end{cases}$$

Figure 2 shows the trading region for (p^2,x^2).

[Insert Figure 2 here]

The third mechanism corresponds to a game in which the seller and buyer each simultaneously announce a bid price. If the seller's bid is lower than the buyer's bid, then the buyer gets the object for the average of the two bids. On the other hand, if the seller's bid is higher than the buyer's bid, then there is no trade. Chatterjee and Samuelson [1983] have shown that the equilibrium bids for this game are $q_1 = \frac{2}{3}\tilde{V}_1 + \frac{1}{4}$ and $q_2 = \frac{2}{3}\tilde{V}_2 + \frac{1}{12}$. Notice $q_1 \geq q_2$ if and only if $\tilde{V}_2 \geq \tilde{V}_1 + \frac{1}{4}$. Thus this mechanism is represented by (p^3,x^3) where

$$p^3(v_1,v_2) = \begin{cases} 1 & \text{if} \quad v_2 \geqslant v_1 + \frac{1}{4}, \\ 0 & \text{if} \quad v_2 < v_1 + \frac{1}{4}, \end{cases}$$

$$x^3(v_1,v_2) = \begin{cases} (v_1 + v_2 + .5)/3 & \text{if} \quad v_2 \geqslant v_1 + \frac{1}{4}, \\ 0 & \text{if} \quad v_2 < v_1 + \frac{1}{4}. \end{cases}$$

To verify that (p^3,x^3) is efficient, use Proposition 2 with

$$L_1(v_1) = \begin{cases} \frac{2}{3} v_1 & \text{if} \quad v_1 < 1, \\ 1 & \text{if} \quad v_1 = 1, \end{cases}$$

$$L_2(v_2) = \begin{cases} 0 & \text{if} \quad v_2 = 0, \\ \frac{2}{3} v_2 + \frac{1}{3} & \text{if} \quad v_2 > 0. \end{cases}$$

Figure 3 shows the trading region for (p^3,x^3).

[Insert Figure 3 here]

Myerson and Satterthwaite [1983] showed that (p^3,x^3) maximizes the expected sum of the two traders' profits over all feasible mechanisms. To verify this, let L_1 and L_2 be as in the preceeding paragraph, and observe that

$$\int_0^1 U_1(v_1,p,x)\, dL_1(v_1) + \int_0^1 U_2(v_2,p,x)\, dL_2(v_2)$$

$$= \frac{2}{3}[\int_0^1 U_1(v_1,p,x)dv_1 + \int_0^1 U_2(v_2,p,x)dv_2 + \frac{1}{2} U_1(1,p,x) + \frac{1}{2} U_2(0,p,x)].$$

The expression in brackets may be interpreted as the Lagrangian function for the problem of maximizing the expected sum of the traders' profits, when we give a shadow price of 1/2 to each of the individual-rationality constraints $U_1(1,p,x) \geq 0$ and $U_2(0,p,x) \geq 0$. Since (p^3,x^3) maximizes this expression over all incentive-compatible mechanisms (by the proof of Proposition 2) and satisfies these two individual-rationality constraints with equality, it maximizes the expected sum of profits over all feasible mechanisms.

4. The Symmetric Uniform Trading Problem: Neutral Solutions

Let us suppose now that the seller and buyer in the symmetric uniform trading problem can negotiate face to face (perhaps with some time limit) to try to determine a mutually acceptable price for the object. In a realistic setting, such negotiations would be much more complicated than the three simple games that we discussed in the preceding section. In real negotiations, each trader's strategy is a plan for making a sequence of demands, offers, and arguments, which may be chosen from the infinite richness of human language. Obviously, we have no simple mathematical model of the traders' strategy sets in such face-to-face negotiations. However, if one could construct a realistic model of face-to-face negotiations as a noncooperative game in strategic form, any equilibrium of the model would still correspond to some feasible mechanism, by the revelation principle. Thus, instead of trying to model the negotiation process as a game in strategic form, we may try to model the negotiation process as a direct mechanism. That is, by analyzing the various incentive-compatible mechanisms, we may hope to find one that is a realistic description of face-to-face negotiations.

A concept of neutral bargaining solutions has been defined by Myerson [1984] for general bargaining problems with incomplete information. This solution concept generalizes Nash's [1950] bargaining solution, and is based on axioms of equity, efficiency, and independence of irrelevant alternatives. For the symmetric uniform trading problem, this solution concept identifies a new efficient mechanism, different from the three mechanisms that were discussed in the preceding section. However, before we consider this mechanism and argue why it may be a good model of face-to-face negotiations for this symmetric uniform trading problem, let us reconsider the mechanism (p^3,x^3) discussed in the preceding section.

At first, (p^3,x^3) seems to have many good properties to recommend it to us as a bargaining solution for symmetric uniform trading. As we have seen, (p^3,x^3) is efficient. It treats the two traders symmetrically. It is also ex-ante efficient, in the sense that, among all feasible mechanisms for the symmetric uniform problem, (p^3,x^3) maximizes the sum of the two traders' ex-ante expected gains from trade. Thus, if the traders could commit themselves to a mechanism before either learns his own valuation $\tilde{V}_i$, then the best symmetric mechanism for both would be (p^3,x^3).

However, each trader already knows his actual valuation $\tilde{V}_i$ when he negotiates, and this is not assumed to be a repeated game. Thus, each trader cares only about his conditionally expected gains given his actual valuation. Ex-ante expected gains are not relevant to the actual traders during negotiations, so ex-ante efficiency should be irrelevant to our theory of negotiations. In fact, if the seller's valuation is higher than .75, then the mechanism (p^3,x^3) is among the seller's least-preferred mechanisms, since $U_1(v_1,p^3,x^3) = 0$ for all $v_1 \geq .75$.

Suppose, for example, that the seller's valuation is $\tilde{V}_1 = .8$, and he is negotiating with a buyer who wants to play the simultaneous-bid split-the-difference game with the equilibrium that is equivalent to (p^3,x^3). The seller knows that he has nothing to gain by playing this game, as the buyer will never bid above .75 in it. Thus, the seller has nothing to lose by refusing to play by its rules, and instead trying to make a nonnegotiable first-and-final offer to sell at price 0.9. The buyer may be antagonized by such an arrogant "Boulware" strategy, but if $\tilde{V}_2 \geq .9$, there should be at least some positive probability that the buyer would accept. Thus, the seller would be strictly better off than in the mechanism (p^3,x^3).

Similarly, $U_2(v_2,p^3,x^3) = 0$ if $v_2 \leq .25$, so the buyer would have nothing to lose by refusing to participate in the (p^3,x^3) mechanism and instead trying to make a nonnegotiable first-and-final offer to buy at some low price. Thus, the mechanism that accurately describes the real negotiation process should have more trade occuring when $\tilde{V}_1 \geq .75$ or $\tilde{V}_2 \leq .25$ than in the (p^3,x^3) mechanism. To satisfy the "center of gravity" condition (1) of Proposition 1, the mechanism must also have less trade than (p^3,x^3) under some other circumstances, when $\tilde{V}_1$ and $\tilde{V}_2$ are in the middle of their range.

The following mechanism (p^4,x^4) satisfies the conditions for a neutral bargaining solution from Myerson [1984], and it differs qualitatively from (p^3,x^3) exactly as described above.

$$p^4(v_1,v_2) = \begin{cases} 1, & \text{if} \quad v_2 \geq 3\,v_1 \quad \text{or} \quad 3\,v_2 - 2 \geq v_1 \,, \\ 0, & \text{if} \quad v_2 < 3\,v_1 \quad \text{and} \quad 3\,v_2 - 2 < v_1 \,, \end{cases}$$

$$x^4(v_1,v_2) = \begin{cases} p^4(v_1,v_2)\, v_2/2 & \text{if} \quad v_2 \leq 1 - v_1 \,, \\ p^4(v_1,v_2)\,(1 + v_1)/2 & \text{if} \quad v_2 > 1 - v_1 \,. \end{cases}$$

Figure 4 shows the trading region for (p^4,x^4). (The kink in the boundary of the trading region is at (.25, .75).)

[Insert Figure 4 here]

It is straightforward to check that this neutral mechanism (p^4,x^4) is incentive compatible and individually rational. To check that (p^4,x^4) is efficient, use Proposition 2 with

$$(6) \qquad L_1(v_1) = \begin{cases} 0 & \text{if } v_1 < \frac{1}{4}, \\ \frac{4}{3}v_1 - \frac{1}{3} & \text{if } v_1 \geq \frac{1}{4}, \end{cases}$$

$$L_2(v_2) = \begin{cases} \frac{4}{3}v_2 & \text{if } v_2 \leq \frac{3}{4}, \\ 1 & \text{if } v_2 > \frac{3}{4}. \end{cases}$$

We say that the seller is in a strong bargaining position if $\tilde{V}_1$ is close to 1, since he has very little to lose by not trading. Similarly, we say that the buyer is in a strong bargaining position if $\tilde{V}_2$ is close to 0. The formula for x^4 can then be interpreted as follows. If $\tilde{V}_2 < 1 - \tilde{V}_1$, then the buyer is in a stronger bargaining position than the seller (since $\tilde{V}_2$ is closer to 0 than $\tilde{V}_1$ is to 1). In this case, if trade occurs then it is at price $\tilde{V}_2/2$, which is the buyer's optimal first-and-final offer, as in (p^2,x^2). If $\tilde{V}_2 > 1 - \tilde{V}_1$ then the seller is in a stronger bargaining position than the buyer, and any trade is at the seller's optimal first-and-final offer $(1 + \tilde{V}_1)/2$, as in (p^1,x^1). Thus, if the seller is stronger than the buyer, then the neutral bargaining solution (p^4,x^4) resembles the mechanism (p^1,x^1), in which the seller controls the price, except that the trading region is slightly smaller (compare the upper wedge in Figure 4 with Figure 1). Similarly, if the buyer is stronger than the seller, then the neutral bargaining solution (p^4,x^4) resembles the mechanism (p^2,x^2) in which the buyer controls the price, except again the trading region is slightly smaller.

(Compare the lower wedge in Figure 4 with Figure 2.)

The neutral bargaining solution concept of Myerson [1984] is meant to be applied to two-person bargaining problems with incomplete information in which the two players have equal bargaining ability. Here "bargaining ability" means the ability to argue articulately and persuasively in the negotiation process. Myerson [1983] defined a theory of solutions for cooperative games with incomplete information in which one individual has all of the bargaining ability. In the terminology of that paper, if the seller had all of the bargaining ability than (p^1,x^1) would be the seller's neutral optimum (because it is undominated for the seller and is safe, in the sense that it would be incentive compatible and individually rational even if the buyer knew the seller's valuation). Similarly, (p^2,x^2) would be the buyer's neutral optimum if he had all of the bargaining ability.

Thus, the neutral bargaining solution (p^4,x^4) is a first illustration of the following important property, which we may call arrogance of strength. If two individuals of symmetric bargaining ability negotiate with each other, but one individual has a surprisingly strong bargaining position (that is, the range of agreements that would be better for him than the disagreement outcome is smaller than the other individual expects), then the outcome of the neutral bargaining solution tends to be similar to what would have been the outcome if the strong individual had had all of the bargaining ability, except that the probability of disagreement (no trade) is higher.

The proof that (p^4,x^4) is a neutral bargaining solution for the symmetric uniform trading problem is given in Section 7. However, it may be helpful to discuss here the essential properties of (p^4,x^4) that identify it as a bargaining solution. The neutral bargaining solutions were defined by Myerson [1984] using axioms that generalize the axioms of Nash's [1950] bargaining solution. Then, in a theorem, it was shown that these neutral bargaining solutions can also be characterized by two properties: efficiency and virtual equity. The efficiency property has already been discussed above, but the virtual equity property needs more explanation.

Given any L_1 and L_2 as in Proposition 2, we define functions W_1 and W_2 by

$$W_1(v_1) = 2\,v_1 - L_1(v_1), \qquad W_2(v_2) = 2\,v_2 - L_2(v_2).$$

We call $W_i(v_i)$ the virtual valuation of the object to trader i if v_i is his true valuation. For L_1 and L_2 as in (6) the virtual valuations are

$$\text{(7)} \qquad W_1(v_1) = \begin{cases} 2\,v_1 & \text{if } v_1 < \frac{1}{4}, \\ (2\,v_1 + 1)/3 & \text{if } v_1 \geqslant \frac{1}{4}, \end{cases}$$

$$W_2(v_2) = \begin{cases} \frac{2}{3}\,v_2 & \text{if } v_2 \leqslant \frac{3}{4}, \\ 2\,v_2 - 1 & \text{if } v_2 > \frac{3}{4}. \end{cases}$$

By Proposition 1, for any feasible mechanism there must be a positive probability of negotiations ending without a trade when the object is worth more to the buyer than to the seller. Such a conclusion may seem paradoxical if the traders have the option to continue negotiating. Why should they stop negotiating when they know that there is still a possibility of mutual gains from trade? One possible explanation is that each trader i deliberately distorts his preferences in bargaining, in response to the other's distrust, and acts as if the object was worth the virtual valuation $W_i(\tilde{V}_i)$ to him, instead of the actual valuation $\tilde{V}_i$. (In (7), $W_1(v_1) \geqslant v_1$ and $W_2(v_2) \leqslant v_2$, so the seller is overstating and the buyer is understating the object's value.) The mechanism (p^4,x^4) has trade occuring if and only if $W_2(\tilde{V}_2) \geqslant W_1(\tilde{V}_1)$, so there is no possibility of further virtual gains from trade after (p^4,x^4).

Of course, any efficient mechanism that satisfies Proposition 2 would satisfy a similar property (which we may call virtual ex-post efficiency) in terms of some other virtual valuation function. But (p^4,x^4) is also virtually equitable, in terms of the same virtual valuations (7) that make it virtually ex post efficient. To see this, consider any $v_1 \geqslant 1/4$. If the seller's true valuation is v_1, then his conditionally expected virtual gains in (p^4,x^4) are

$$\int_0^1 \left(x^4(v_1,v_2) - W_1(v_1)\, p^4(v_1,v_2)\right) dv_2$$

$$= \int_{(v_1+2)/3}^{1} \left((1 + v_1)/2 - (2v_1 + 1)/3\right) dv_2$$

$$= (1 - v_1)^2/18,$$

which is equal to his conditional expectation of the buyer's virtual gains in (p^4,x^4):

$$\int_0^1 \left(W_2(v_2)\, p^4(v_1,v_2) - x^4(v_1,v_2)\right) dv_2$$

$$= \int_{(v_1+2)/3}^{1} \left((2v_2 - 1) - (1 + v_1)/2\right) dv_2$$

$$= (1 - v_1)^2/18,$$

Similarly, if $\tilde{V}_2 = v_2 \leqslant 3/4$, then the buyer's conditional expectation of his own virtual gains in (p^4,x^4)

$$\int_0^{v_2/3} (2v_2/3 - v_2/2)\, dv_1 = (v_2)^2/18$$

is equal to his conditional expectation of the seller's virtual gains

$$\int_0^{v_2/3} (v_2/2 - 2v_1)\, dv_1 = (v_2)^2/18.$$

For $v_1 \leqslant 1/4$ or $v_2 \geqslant 3/4$, these equalities do not hold, but L_1 and L_2 from (6) are constant over these intervals, so that the corresponding objective function (5) puts no weight on these valuations. Thus, with respect to the virtual valuations in (7), (p^4,x^4) is both virtually ex post efficient and virtually equitable, except for some weak types that get no weight in the corresponding objective function. These are necessary conditions for a neutral bargaining solution derived in Myerson [1984]. But more importantly, they demonstrate that (p^4,x^4) can be justified as both efficient and equitable, in a newly recognized sense.

5. The Lemon Problem: Feasibility and Efficiency

Let us now consider some trading problems in which the seller has private information related to the quality of the object being sold, so that the value of the object to the buyer is a function of the seller's valuation. To keep the problem tractable, let us assume that the seller knows this function and the buyer has no private information. We may call this the lemon problem, after Akerlof's [1970] seminal paper, "The Market for Lemons," which studied a special case of this problem, in a market context. (In colloquial American, a bad used car is a "lemon.")

So again let trader #1 be the only seller and trader #2 be the only potential buyer of a single indivisible object. Both have risk-neutral utility for money. The quality of the object, which is known only to the seller, is measured by the random variable $\tilde{V}_1$, which is the value of the object to the seller. The buyer has a probability distribution for $\tilde{V}_1$ with cumulative distribution $F(v_1) = \Pr(\tilde{V}_1 \leqslant v_1)$, and with a continuous density

$f(v_1) = F'(v_1)$ that is positive over a bounded interval $0 \leq v_1 \leq M$. The value of the object to the buyer is $g(\tilde{V}_1)$, where $g:[0,M] \to \mathbb{R}$ is a continuous function.

A <u>direct trading mechanism</u> for the lemon problem is characterized by two outcome functions $p:[0,M] \to [0,1]$ and $x:[0,M] \to \mathbb{R}$, where $p(v_1)$ is the probability of trade occuring and $x(v_1)$ is the expected revenue to the seller, if the seller's valuation equals v_1. The expected gain to the buyer from (p,x) is

$$U_2(p,x) = \int_0^M (g(v_1)\ p(v_1) - x(v_1))\ dF(v_1).$$

The expected gain to the seller from (p,x) if his valution equals v_1 is

$$U_1(v_1,p,x) = x(v_1) - v_1\ p(v_1).$$

In this context, mechanism (p,x) is <u>incentive compatible</u> iff, for every v_1 and t_1 in $[0,M]$

$$U_1(v_1,p,x) \geq x(t_1) - v_1\ p(t_1).$$

Mechanism (p,x) is <u>individually rational</u> iff $U_2(p,x) \geq 0$ and, for every v_1 in $[0,M]$, $U_1(v_1,p,x) \geq 0$. As before, a mechanism is <u>feasible</u> iff it is incentive compatible and individually rational.

(In this formulation, we are assuming that the terms of trade cannot be made conditional on the actual quality of the object, only on the seller's report of it. Presumably the buyer will eventually learn the quality of the object if he buys it, but too late to renegotiate the price.)

The following proposition characterizes the set of feasible mechanisms.

Proposition 3. Given any function $p:[0,M] \to [0,1]$, there exists some function $x(\cdot)$ such that (p,x) is a feasible mechanism for the lemon problem if and only if $p(\cdot)$ is a weakly decreasing function and

$$\int_0^M \left(g(v_1) - v_1 - F(v_1)/f(v_1)\right) p(v_1)\, f(v_1)\, dv_1 \geq 0.$$

In general, for any incentive-compatible mechanism (p,x), $p(\cdot)$ is weakly decreasing,

$$U_1(M,p,x) + U_2(p,x) = \int_0^M \left(g(v_1) - v_1 - F(v_1)/f(v_1)\right) p(v_1)\, f(v_1)\, dv_1$$

and, for every v_1 in $[0,M]$,

$$U_1(v_1,p,x) = U_1(M,p,x) + \int_{v_1}^M p(s)\, ds.$$

Proof. The proof of the equation for $U_1(v_1,p,x)$ and of p decreasing is exactly as in the proof of Theorem 1 of Myerson and Satterthwaite [1983]. The equation for $U_1(M,p,x) + U_2(p,x)$ is derived from the following chain of equalities:

$$\int_0^M (g(v_1) - v_1)\, p(v_1)\, f(v_1)\, dv_1$$

$$= \int_0^M U_1(v_1,p,x)\, f(v_1)\, dv_1 + U_2(p,x)$$

$$= \int_0^M \int_{v_1}^M p(s)\, ds\, f(v_1)\, dv_1 + U_1(M,p,x) + U_2(p,x)$$

$$= \int_0^M F(v_1)\, p(v_1)\, dv_1 + U_1(M,p,x) + U_2(p,x).$$

Finally, if p is weakly decreasing and satisfies the inequality in Proposition 3, then one can construct a feasible mechanism by using

$$x(v_1) = v_1\ p(v_1) + \int_{v_1}^{M} p(s)\ ds,$$

as is straightforward to check. Q.E.D.

As in the symmetric uniform example, our next task is to characterize the efficient mechanisms for the lemon problem. As before, we use the term "efficient" in the sense of weak interim incentive-efficiency: (p,x) is <u>efficient</u> iff there is no other incentive-compatible mechanism $(\hat{p},\hat{x})$ such that $U_2(\hat{p},\hat{x}) > U_2(p,x)$ and, for every v_1, $U_1(v_1,\hat{p},\hat{x}) > U_1(v_1,p,x)$.

For any number s between 0 and M, let $(p^{(s)},x^{(s)})$ denote the mechanism

$$p^{(s)}(v_1) = \begin{cases} 1 & \text{if } v_1 \leqslant s, \\ 0 & \text{if } v_1 > s, \end{cases}$$

$$x^{(s)}(v_1) = \begin{cases} s & \text{if } v_1 \leqslant s, \\ 0 & \text{if } v_1 > s. \end{cases}$$

We may refer to any such mechanism $(x^{(s)},p^{(s)})$ as a <u>simple</u> mechanism, since there are only two possible outcomes: either the object is sold for s dollars (if $\tilde{V}_1 \leqslant s$) or it is not sold at all. These simple mechanisms are important because they are the extreme points of the set of incentive-compatible mechanisms for the lemon problem, up to addition of a lump-sum transfer between buyer and seller. To see why, notice that any incentive-compatible mechanism differs by a lump-sum transfer (a constant added to $x(\cdot)$) from an incentive-compatible mechanism with $U_1(M,p,x) = 0$. By Proposition 3, any such mechanism is then completely characterized by the weakly decreasing function p; and each $U_1(v_1)$ and U_2 are linear functions of p. But any weakly decreasing function from [0,M] into [0,1] can be approximated arbitrarily closely (except possibly on a countable set) by a convex combination of the step-functions $\{p^{(s)}\}$. Since we are assuming that F is a continuous distribution, changing p on a countable set would not change any of the expected payoffs in Proposition 3. (Without this continuity assumption, we would have to distinguish $(p^{(s)},x^{(s)})$ from the mechanism in which the object is sold for s dollars if and only if $\tilde{V}_1 < s$, and we would add such mechanisms also to the list of extreme points.)

A mechanism is efficient for the lemon problem if and only if it maximizes some linear functional of the form

$$(8) \qquad \int_0^M U_1(v_1,p,x)\, dL_1(v_1) + U_2(p,x)$$

over the set of all incentive-compatible mechanisms, where $L_1(\cdot)$ is weakly increasing, $L_1(0) = 0$, and $L_1(M) = 1$. But the maximum of any such linear functional must be attained at some simple mechanism $(p^{(s)},x^{(s)})$, because these are the extreme points. Samuelson's [1981] result that the ex ante optimum for the seller is always a simple mechanism can be derived from this fact.

To characterize the set of efficient mechanisms for the lemon problem, we need some further definitions. Let Y(s) denote the expected gain to the buyer from mechanism $(p^{(s)},x^{(s)})$, that is

$$Y(s) = U_2(p^{(s)},x^{(s)}) = \int_0^s (g(v_1) - s)\, f(v_1)dv_1.$$

Let $\bar{Y}:[0,M] \to \mathbb{R}$ be the lowest concave function that is greater than or equal to $Y(\cdot)$ and has a slope between 0 and -1 everywhere. That is, $\bar{Y}$ differs from the concave hull of Y only in that $\bar{Y}$ is constant over the interval where the concave hull is increasing, and $\bar{Y}$ has a slope -1 over any interval where the concave hull is decreasing at a steeper slope than -1. Finally, let $L_1^*:[0,M] \to [0,1]$ be defined so that $L_1^*(0) = 0$, $L_1^*(M) = 1$, and

$$L_1^*(v_1) = -\bar{Y}'(v_1)$$

at every v_1 in $(0,M)$ where the derivative $\bar{Y}'$ is defined. (Define L_1^* by left continuity when $\bar{Y}'$ jumps.) Notice that L_1^* is an increasing function, since $\bar{Y}$ is concave. Notice also that $\bar{Y}(0) = \underset{s\in[0,M]}{\text{maximum}}\ Y(s)$.

[Insert Figure 5 here]

The set of efficient mechanisms for the lemon problem has a remarkably simple structure: it is a flat set contained in a hyperplane. That is, given any two efficient mechanisms, their convex combination is also efficient. The function L_1^* gives us the normal to this flat efficient set, as shown in the following proposition.

<u>Proposition 4</u>. Let (p,x) be any incentive-compatible mechanism for the lemon problem. Then (p,x) is efficient if and only if

$$\int_0^M U_1(v_1,p,x)\, dL_1^*(v_1) + U_2(p,x) = \bar{Y}(0). \tag{9}$$

Equivalently, (p,x) is efficient if and only if p satisfies the following three conditions: $p(0) = 1$ if $\bar{Y}(0) > 0$; $p(M) = 0$ if $\bar{Y}(M) > Y(M)$; and

$$\int_0^M (\bar{Y}(v_1) - Y(v_1))\, dp(v_1) = 0,$$

so that p must be constant over any interval in which $\bar{Y} > Y$.

<u>Proof</u>. Notice first that, from the definition of Y,

$$Y(0) = 0, \quad \text{and} \quad Y'(v_1) = (g(v_1) - v_1)\, f(v_1) - F(v_1).$$

Now, using Proposition 3, for any incentive-compatible mechanism (p,x):

$$\int_0^M U_1(v_1,p,x)\, dL_1^*(v_1) + U_2(p,x)$$

$$= \int_0^M \int_{v_1}^M p(s)\, ds\, dL_1^*(v_1) + U_1(M,p,x) + U_2(p,x)$$

$$= \int_0^M L_1^*(v_1)\, p(v_1)\, dv_1 + \int_0^M \big((g(v_1) - v_1)\, f(v_1) - F(v_1)\big)\, p(v_1)\, dv_1$$

$$= \int_0^M (Y'(v_1) - \bar{Y}'(v_1))\, p(v_1)\, dv_1$$

$$= \bar{Y}(0)\, p(0) - (\bar{Y}(M) - Y(M))\, p(1) + \int_0^M (\bar{Y}(v_1) - Y(v_1))\, dp(v_1).$$

Since $\bar{Y}(v_1) \geq Y(v_1)$ for all v_1, the decreasing function p that maximizes the last expression must have $p(0) = 1$ if $\bar{Y}(0) > 0$, $p(M) = 0$ if $\bar{Y}(M) > Y(M)$, and must be constant over any interval in which $\bar{Y} > Y$. (Notice that the integral is nonpositive, because p is decreasing.) Such a function p does exist and gives the maximum value $\bar{Y}(0)$. Thus, p is efficient if it satisfies (9).

Let r_1 be the lowest number in $[0,M]$ such that $\bar{Y}(r_1) = Y(r_1)$, and let r_2 be the highest such number. (See Figure 5.) Now consider any simple mechanism $(p^{(s)},x^{(s)})$ that does not satisfy (9). Then

$$\bar{Y}(0) > \int_0^M U_1(v_1,p^{(s)},x^{(s)})\, dL_1^*(v_1) + U_2(p^{(s)},x^{(s)})$$

$$= \bar{Y}(0) - (\bar{Y}(s) - Y(s)),$$

and so $\bar{Y}(s) > Y(s)$. We shall show that $(p^{(s)},x^{(s)})$ is not efficient. There are three cases to consider: $s < r_1$; $s > r_2$; and $r_1 < s < r_2$.

If $s < r_1$, then $Y(s) < Y(r_1) = \bar{Y}(0)$. So the buyer would strictly prefer $(p^{(r_1)},x^{(r_1)})$ over $(p^{(s)},x^{(s)})$. The seller also prefers $(p^{(r_1)},x^{(r_1)})$ over $(p^{(s)},x^{(s)})$, since

$$U_1(v_1,p^{(s)},x^{(s)}) = \max\{0,\ s - v_1\}$$

is increasing in s. So $(p^{(s)},x^{(s)})$ is not efficient.

If $s > r_2$ then $Y(s) < Y(r_2) + (r_2 - s)$, since the slope of $\bar{Y}$ is -1 for all $v_1 > r_2$. So the buyer would strictly prefer to pay $s - r_2$ as a lump-sum transfer and then implement $(p^{(r_2)},x^{(r_2)})$. It is easy to see that the seller would also prefer this change, so $(p^{(s)},x^{(s)})$ is not efficient.

If $r_1 < s < r_2$, then there exist numbers s_1, s_2, and λ such that $s = \lambda s_1 + (1 - \lambda)s_2$, $0 \leq \lambda \leq 1$, and $Y(s) < \lambda Y(s_1) + (1 - \lambda)Y(s_2)$. So the buyer would strictly prefer to randomize between $(p^{(s_1)},x^{(s_1)})$ with probability λ and $(p^{(s_2)},x^{(s_2)})$ with probability $1 - \lambda$, rather than use $(p^{(s)},x^{(s)})$. Since $U_1(v_1,p^{(s)},x^{(s)})$ is a convex function of s, the seller would prefer this randomization also. Thus $(p^{(s)},x^{(s)})$ is not efficient if it violates (9).

Any efficient mechanism must be equal to some convex combination of efficient simple mechanisms plus a lump-sum transfer. Thus, any efficient mechanism must satisfy condition (9) in Proposition 4. Q.E.D.

To illustrate these results, consider first the example studied by Akerlof [1970], in which $M = 2$, $F(v_1) = .5\ v_1$, and $g(v_1) = 1.5\ v_1$. That is, the seller's valuation is uniformly distributed over [0,2], and the object would always be worth 50% more to the buyer, if he knew the seller's valuation. For this example

$$g(v_1) - v_1 - F(v_1)/f(v_1) = -.5\, v_1 < 0.$$

So by Proposition 3, there does not exist any feasible mechanism with a positive probability of trade for Akerlof's example.

For a second example, let $M = 1$, $F(v_1) = v_1$ and $g(v_1) = v_1 + \alpha$, where $0 < \alpha < 1$. We may call this the uniform additive lemon problem. For this example, there are many feasible mechanisms (for example, $(p^{(s)},x^{(s)})$ for every $s \leqslant 2\alpha$). To apply Proposition 4,

$$Y(s) = \int_0^s (v_1 + \alpha - s)\, dv_1 = \alpha\, s - .5\, s^2,$$

and so

$$\bar{Y}(s) = \begin{cases} \alpha\, s - .5\, s^2 = Y(s), & \text{if} \quad \alpha \leqslant s \leqslant 1, \\ .5\, \alpha^2 > Y(s), & \text{if} \quad s < \alpha. \end{cases}$$

Thus, an incentive-compatible mechanism (p,x) is efficient if and only if $p(v_1) = 1$ for every v_1 such that $0 \leqslant v_1 < \alpha$.

6. The Uniform Additive Lemon Problem: Neutral Solutions

As in Section 4, let us now try to make some prediction as to which efficient mechanism may actually be implemented by the seller and buyer in the lemon problem if they negotiate face to face. To simplify the analysis, we will consider only one specific case: the uniform additive case with $\alpha = 0.4$. That is, the seller knows his valuation $\tilde{V}_1$, which is a uniform random variable on $[0,1]$, and if the buyer gets the object then it will be ultimately worth $\tilde{V}_1 + .4$ to him. The seller is free to make statements to the buyer about $\tilde{V}_1$, but there is no way for the buyer to verify whether these claims are true or false until after the negotiations end and the terms of trade are fixed.

For simplicity, let us begin with the assumption that the buyer has all of the bargaining ability, perhaps because he is much more articulate and persuasive in negotiations. The best feasible mechanism for the buyer is the simple mechanism $(p^{(.4)},x^{(.4)})$. That is, if the buyer can control the negotiations, he wants to make a nonnegotiable first-and-final offer to buy the object for a price of 0.4. To see that this mechanism is optimal, notice that

$$Y(s) = \int_0^s (v_1 + .4 - s)\,dv_1 = .4\,s - .5\,s^2$$

which is maximized at $s = 0.4$. The buyer's expected gain from his optimal mechanism is $Y(.4) = 0.08$.

Now, let us assume that the seller has all of the bargaining ability. The problem of determining which mechanism he should implement is a problem of mechanism design by an informed principal, as studied in Myerson [1983].

Among the simple mechanisms, $U_1(v_1,p^{(s)},x^{(s)})$ is increasing in s, and $U_2(p^{(s)},x^{(s)}) \geq 0$ if and only if $s \leq 0.8$. That is, for any price s that is higher than 0.8, the expected value of the object to the buyer conditional on $\tilde{V}_1 \leq s$ is $.5s + .4$, which is less than s, so the buyer expects to lose. Thus, if the seller were to implement a simple mechanism, his best one would be $(p^{.8)},x^{(.8)})$. (Even though the object is always worth more to the buyer than to the seller, there is no feasible mechanism in which the buyer always gets the object, because the inequality in Proposition 3 would fail if $p(v_1) = 1$ for all v_1.)

The mechanism $(p^{(.8)},x^{(.8)})$ maximizes both the probability of trade and the seller's ex-ante expected gains $(\int_0^1 U_1(v_1,p,x)\,dv_1)$ over all feasible mechanisms for this example. Thus, if the seller could have selected any feasible mechanism before he learned $\tilde{V}_1$, he would certainly have selected

$(p^{(.8)},x^{(.8)})$. However, this argument is not necessarily relevant to our analysis of negotiations, because we are assuming that the seller already knows $\tilde{V}_1$ when the negotiations begin, and this is not a repeated game.

There exist other mechanisms that the seller would prefer to $(p^{(.8)},x^{(.8)})$ if $\tilde{V}_1$ is relatively high. (Notice that $U_1(v_1,p^{(.8)},x^{(.8)}) = 0$ if $v_1 \geq 0.8$.) For example, consider $(\hat{p},\hat{x})$ defined by

$$\hat{p}(v_1) = e^{-v_1/.4}, \quad \hat{x}(v_1) = (v_1 + .4)\ \hat{p}(v_1).$$

That is, the seller demands that the buyer should pay the full value $q = \tilde{V}_1 + .4$, and the buyer accepts with probability $e^{-(q-.4)/.4}$. It is straightforward to check that $(\hat{p},\hat{x})$ is individually rational and incentive compatible. If the seller demanded a higher price, the decrease in probability of acceptance would be just enough to prevent him from gaining more. Among all mechanisms in which the buyer never loses ex post (<u>safe</u> mechanisms, in the terminology of Myerson [1983]), $(\hat{p},\hat{x})$ is the best for the seller. If $\tilde{V}_1 \geq .74$ then the seller would prefer $(\hat{p},\hat{x})$ over $(p^{(.8)},x^{(.8)})$ $(.4e^{-.74/.4} > .8 - .74)$.

One theory of negotiations which <u>cannot</u> be valid is to suggest that the seller would implement $(p^{(.8)},x^{(.8)})$ if $\tilde{V}_1 < .74$ and would implement $(\hat{p},\hat{x})$ if $\tilde{V}_1 \geq .74$. The buyer would refuse to buy the object for .8 if he believed that the seller would only make this demand when $\tilde{V}_1 < .74$, because the conditionally expected value of the object to him would be only $.74/2 + .4 = .77$. On the other hand, the buyer would never expect losses in $(\hat{p},\hat{x})$, even if he inferred that $\tilde{V}_1 \geq .74$. So $(p^{(.8)},x^{(.8)})$ is blocked for the seller by $(\hat{p},\hat{x})$, since the buyer knows that $(p^{(.8)},x^{(.8)})$ would be implemented by the seller only if $\tilde{V}_1$ were in $[0,.74]$, where the buyer expects to lose on average.

However, $(\hat{p},\hat{x})$ is not an efficient mechanism, because any efficient mechanism must have $p(v_1) = 1$ for all v_1 in the interval $[0,.4)$ (where $\bar{Y}(v_1) > Y(v_1)$), as was shown at the end of Section 5. For example, $(\hat{p},\hat{x})$ is dominated by the mechanism (p^*,x^*) defined by

$$p^*(v_1) = \begin{cases} 1 & \text{if } v_1 < .4 , \\ .5\, e^{-(v_1-.4)/.4} & \text{if } v_1 \geq .4 , \end{cases}$$

$$x^*(v_1) = \begin{cases} .6 & \text{if } v_1 < .4 , \\ (v_1 + .4)\, p(v_1) & \text{if } v_1 \geq .4 . \end{cases}$$

It is straightforward to check that (p^*,x^*) is incentive compatible, $U_2(p^*,x^*) = U_2(\hat{p},\hat{x}) = 0$, and $U_1(v_1,p^*,x^*) > U_1(v_1,\hat{p},\hat{x})$ for all v_1. Also, (p^*,x^*) is efficient because a sale will occur for sure if $0 \leq \tilde{V}_1 < .4$. If $\tilde{V}_1 \geq .4$ then the seller insists on getting the buyer's reservation price $\tilde{V}_1 + .4$, and the buyer's probability of acceptance decreases in the price in such a way as to keep the seller honest. It can be shown (see Section 7) that this mechanism (p^*,x^*) is a neutral optimum for the seller, in the sense of Myerson [1983].

Thus, we predict that outcome of negotiations would be as in $(p^{(.4)},x^{(.4)})$ if the buyer had all of the bargaining ability, and would be as in (p^*,x^*) if the seller had all of the bargaining ability.

Let us now assume that the buyer and seller have equal bargaining ability. In this case, the solution theory of Myerson [1984] identifies the average of these mechanisms $(p^o,x^o) = .5(p^{(.4)},x^{(.4)}) + .5(p^*,x^*)$ as a neutral bargaining solution. That is, the neutral bargaining solution is

$$p^o(v_1) = \begin{cases} 1 & \text{if} \quad v_1 < .4 , \\ .25\, e^{-(v_1-.4)/.4} & \text{if} \quad v_1 \geq .4 , \end{cases}$$

$$x^o(v_1) = \begin{cases} .5 & \text{if} \quad v_1 < .4 , \\ (v_1 + .4)\ p^o(v_1) & \text{if} \quad v_1 \geq .4 . \end{cases}$$

Notice that if $\tilde{V}_1 \geq .4$, the seller fully exploits the buyer in (p^o,x^o) by charging him $\tilde{V}_1 + .4$ when trade occurs, just as in (p^*,x^*). However, the probability of trade occurring when $\tilde{V}_1 \geq .4$ in (p^o,x^o) is half of what it is in (p^*,x^*). Thus, the neutral bargaining solution (p^o,x^o) has the property called <u>arrogance of strength</u> in Section 4. That is, if the traders have equal bargaining ability but the seller is in a surprisingly strong bargaining position, then the outcome is like when the seller has all of the bargaining ability, except that the probability of disagreement is higher.

The mechanism (p^o,x^o) may seem more equitable when we look at virtual-utility payoffs. For this example, the function L_1^* which supports all efficient mechanisms (as stated in Proposition 4) is

$$L_1^*(v_1) = -\bar{Y}'(v_1) = \begin{cases} 0 & \text{if} \quad v_1 < .4 \\ v_1 - .4 & \text{if} \quad .4 \leq v_1 < 1 \\ 1 & \text{if} \quad v_1 = 1 \end{cases}$$

Because the seller's valuation is uniformly distributed over [0,1], his virtual valuation is $2\tilde{V}_1 - L_1^*(\tilde{V}_1)$ (as in the symmetric uniform trading problem), which equals $\tilde{V}_1 + .4$ if $\tilde{V}_1 \geq .4$ (except at the endpoint $\tilde{V}_1 = 1$, which has zero probability). Thus, when $\tilde{V}_1 \geq .4$, the seller's virtual valuation equals the buyer's valuation, and so $\tilde{V}_1 + .4$ is the only virtually equitable price. (Since the buyer has no private information in this example, his virtual and real valuations are equal.) When $\tilde{V}_1$ is the interval [0,.4), the seller's average virtual valuation $(2\,\tilde{V}_1)$ is .4, and the buyer's average valuation $(\tilde{V}_1 + .4)$ is .6, so the price .5 in (p^o, x^o) is virtually equitable on average.

7. Derivation of the Neutral Solutions

To show how the solution concepts of Myerson [1983] and [1984] are applied to the examples of this paper, let us first consider a discrete approximation to the lemon problem. That is, let δ be a small number, and let $T_1 = \{0, \delta, 2\delta, 3\delta, \ldots, M\}$ be the set of possible seller's valuations for the object. Let $f(v_1)\delta = F(v_1) - F(v_1 - \delta)$ be the probability that $\tilde{V}_1 = v_1$ for any v_1 that is a multiple of δ. Given an increasing function L_1 as in (8), let $\ell(v_1) = (L_1(v_1) - L_1(v_1 - \delta))/\delta$. Thus, the discrete analogue of (8) is

$$(10) \qquad \sum_{v_1 \in T_1} \Big(\big(x(v_1) - v_1\, p(v_1)\big)\, \ell(v_1)\, \delta + \big(g(v_1)\, p(v_1) - x(v_1)\big)\, f(v_1)\, \delta \Big).$$

It can be shown that, for the discrete lemon problem, local incentive-compatibility implies global incentive-compatibility. (That is, if for every v_1, the seller with valuation v_1 could not gain by reporting $v_1 + \delta$ or $v_1 - \delta$, then the seller cannot gain by any lie.) Furthermore, in most cases, the binding incentive constraint for the seller is the one in the upward direction: that the seller should not gain by reporting $v_1 + \delta$ when his valuation is v_1. So let $A(v_1)$ denote the shadow price of this incentive constraint in the problem of maximizing (10) among all incentive-compatible mechanisms. Then the Lagrangian function for this problem can be written

(11) $$\sum_{v_1 \varepsilon T_1} \Big(\big(x(v_1) - v_1\ p(v_1)\big)\ \ell(v_1)\ \delta + \big(g(v_1)\ p(v_1) - x(v_1)\big)\ f(v_1)\ \delta$$

$$+ A(v_1)\ \Big(\big(x(v_1) - v_1\ p(v_1)\big) - \big(x(v_1+\delta) - v_1\ p(v_1+\delta)\big)\Big)\Big)$$

$$= \sum_{v_1 \varepsilon T_1} \Big(\Big[\big(\ell(v_1)\ \delta + A(v_1)\big)\big(x(v_1) - v_1\ p(v_1)\big) - A(v_1-\delta)\big(x(v_1) - (v_1-\delta)p(v_1)\big)\Big]$$

$$+ \big(g(v_1)\ p(v_1) - x(v_1)\big)\ f(v_1)\ \delta\Big).$$

The coefficient of $x(v_1)$ in this Lagrangian must be zero, since $x(v_1)$ is an unconstrained variable. Thus, we must have, for all v_1,

$$A(v_1) - A(v_1-\delta) = f(v_1)\delta - \ell(v_1)\delta,$$

and so

$$A(v_1) = F(v_1) - L_1(v_1).$$

The seller's <u>virtual</u> <u>utility</u> is defined in Myerson [1984] as the bracketed expression in the above Lagrangian formula divided by the probability $f(v_1)\delta$. That is, if the seller's valuation is v_1 and his expected revenue is $y = x(v_1)$ and his probability of sale is $q = p(v_1)$, then his <u>virtual-utility</u> payoff $z_1(v_1)$ is defined to be

$$z_1(v_1) = [(\ell(v_1)\ \delta + A(v_1))(y - v_1\ q) - A(v_1-\delta)(y - (v_1-\delta)\ q)]/(f(v_1)\ \delta)$$

$$= y + \Big(v_1 + \frac{F(v_1-\delta) - L_1(v_1-\delta)}{f(v_1)}\Big)\ q.$$

Equivalently, if we let $u_1(v_1) = y - v_1 q$ denote the seller's actual utility payoff when his valuation is v_1 (and let $u(v_1-\delta) = y - (v_1-\delta)q$), then the seller's virtual-utility payoff can be rewritten

$$z_1(v_1) = u_1(v_1) + \left(\frac{F(v_1-\delta) - L_1(v_1-\delta)}{f(v_1)}\right)\left(\frac{u_1(v_1) - u_1(v_1-\delta)}{\delta}\right).$$

At the maximum of (10), the product of incentive constraint times shadow price is always zero, by complementary slackness, so (11) implies that

$$\sum_{v_1 \varepsilon T_1} (x(v_1) - v_1\ p(v_1))\ \ell(v_1)\ \delta = \sum_{v_1 \varepsilon T_1} z_1(v_1)\ f(v_1)\ \delta.$$

Now, letting δ go to zero, let us return to the continuous version of the lemon problem. The above three equations become

$$(12) \qquad z_1(v_1) = x(v_1) + \left(v_1 + \left(\frac{F(v_1) - L_1(v_1^-)}{f(v_1)}\right)\right)\ p(v_1),$$

$$(13) \qquad z_1(v_1) = u_1(v_1) + \left(\frac{F(v_1) - L_1(v_1^-)}{f(v_1)}\right)\ u_1'(v_1),$$

$$(14) \qquad \int_0^M u_1(v_1)\ dL_1(v_1) = \int_0^M z_1(v_1)\ f(v_1)\ dv_1,$$

where $\quad L_1(v_1^-) = \lim_{\delta \downarrow 0} L_1(v_1 - \delta).$

The seller's <u>virtual</u> <u>valuation</u> for the object is, from (12),

$$(15) \qquad W_1(v_1) = v_1 + (F(v_1) - L_1(v_1^-))/f(v_1).$$

In the uniform case with $F(v_1) = v_1$ on $[0,1]$, when L_1 is continuous this virtual valuation is just $W_1(v_1) = 2\ v_1 - L_1(v_1)$, as in Section 4.

Since the buyer has no private information in the lemon problem and gets a weight of 1 in the objective functions (8) and (10), the buyer's virtual utility is the same as his real utility.

We are now ready to verify that (p^o,x^o) is the neutral bargaining solution for the uniform-additive lemon problem with $g(v_1) = v_1 + .4$. To prove this, we must apply Theorem 4 of Myerson [1984], which gives necessary and sufficient conditions for a neutral bargaining solution. This theorem requires us to consider a sequence of virtual-utility scales, each of which is generated by an objective function that puts positive weight on all types of all players. (For the lemon problem, this means that L_1 must be strictly increasing over the whole range of possible valuations.) For each virtual-utility scale, we must first compute the virtually-equitable allocations, in which the traders plan to divide the available virtual gains equally among themselves in every state; then we must solve equations (13) and (14) to find what allocations of real utility would correspond to the equitable allocations of virtual utility. The corresponding allocations of real utility are called the warranted claims of the seller and the buyer. If the limit of these warranted claims (over the sequence of virtual-utility scales) does not exceed the actual expected utility generated by our mechanism for any type, then that mechanism is a neutral bargaining solution.

The sequence of objectives that supports (p^o,x^o) is

$$L_1^\varepsilon(v_1) = \begin{cases} \varepsilon\, v_1 & \text{if } \ 0 \leq v_1 < .4/(1-\varepsilon), \\ v_1 - .4 & \text{if } \ .4/(1-\varepsilon) \leq v_1 < 1, \\ 1 & \text{if } \ v_1 = 1, \end{cases}$$

where the index ε is positive and converging to zero. Notice that each L_1^ε is strictly increasing over $[0,1]$, and converges to L_1^* of Section 6 as ε goes to zero.

With respect to $L_1 = L_1^\varepsilon$, if the seller's actual valuation is v_1 then his virtual valuation (from (15)) is

$$W_1(v_1) = \begin{cases} (2-\varepsilon)\ v_1 & \text{if } 0 \leq v_1 < .4/(1-\varepsilon), \\ v_1 + .4 & \text{if } .4/(1-\varepsilon) \leq v_1 \leq 1; \end{cases}$$

and so the total available virtual gains from trade are

$$g(v_1) - W_1(v_1) = \begin{cases} .4 - (1-\varepsilon)\ v_1 & \text{if } 0 \leq v_1 < .4/(1-\varepsilon), \\ 0 & \text{if } .4/(1-\varepsilon) \leq v_1 \leq 1. \end{cases}$$

The virtually-equitable allocation with respect to L_1^ε would give the seller half of these virtual gains; that is, he would get

$$z_1(v_1) = \begin{cases} .2 - .5\ (1-\varepsilon)\ v_1 & \text{if } 0 \leq v_1 < .4/(1-\varepsilon) \\ 0 & \text{if } .4/(1-\varepsilon) \leq v_1 \leq 1 \end{cases}$$

in virtual utility when his valuation is v_1. The seller's warranted claims with respect to L_1^ε are the values of $u_1(v_1)$ that satisfy (13) and (14) for this z_1 function. That is, u_1 must satisfy

$$.2 - .5\ (1-\varepsilon)\ v_1 = u_1(v_1) + (1-\varepsilon)\ v_1\ u_1'(v_1) \quad \text{if } 0 \leq v_1 < .4/(1-\varepsilon),$$

$$0 = u_1(v_1) + .4\ u_1'(v_1) \quad \text{if } .4/(1-\varepsilon) \leq v_1 \leq 1,$$

$$\int_0^1 \left(.2 - .5\ (1-\varepsilon)\ v_1\right) dv_1 =$$

$$= \int_0^{.4/(1-\varepsilon)} u_1(v_1)\ \varepsilon\ dv_1 + \int_{.4/(1-\varepsilon)}^1 u_1(v_1)\ dv_1 + .4\ u_1(1).$$

(The term $.4\ u_1(v_1)$ comes from the jump in L_1^ε at $v_1 = 1$.) The unique solution to these equations is

$$u_1(v_1) = \begin{cases} .2 - .5\ \left(\frac{1-\varepsilon}{2-\varepsilon}\right)\ v_1 & \text{if } 0 \leq v_1 < .4/(1-\varepsilon), \\ .2\ \left(\frac{1-\varepsilon}{2-\varepsilon}\right)\ e^{(1/(1-\varepsilon) - v_1/.4)} & \text{if } .4/(1-\varepsilon) \leq v_1 \leq 1. \end{cases}$$

As ε goes to zero, these warranted claims converge to

$$u_1(v_1) = \begin{cases} .2 - v_1/4 & \text{if } 0 \leq v_1 < .4, \\ .1\ e^{-(v_1-.4)/.4} & \text{if } .4 \leq v_1 \leq 1. \end{cases}$$

The seller's actual payoff from the mechanism (p^o,x^o) is

$$U_1(v_1,p^o,x^o) = \begin{cases} .5 - v_1 & \text{if } 0 \leq v_1 < .4, \\ .1\ e^{-(v_1-.4)/.4} & \text{if } .4 \leq v_1 \leq 1, \end{cases}$$

so $U_1(v_1,p^o,x^o) \geq u_1(v_1)$ for all v_1 in $[0,1]$.

Since the buyer has only one possible type in this problem, his warranted claim with respect to L_1^ε is simply half of the expected virtual gains from trade:

$$u_2 = \int_0^{.4} (.2 - .5\ (1-\varepsilon)\ v_1)\ dv_1 = .04\ (1+\varepsilon).$$

As ε goes to zero, this converges to $.04 = U_2(p^o,x^o)$.

So the mechanism (p^o,x^o) fulfills all of the limiting warranted claims, and therefore is a neutral bargaining solution, by Theorem 4 of Myerson [1984].

If the seller had all of the bargaining ability then the seller's warranted claims for each type would be computed in the same way, except that he would get all of the virtual gains from trade, instead of half. This would simply double the values of z_1 and u_1 throughout the above derivation. Since $U_1(v_1,p^*,x^*) = 2\ U_1(v_1,p^o,x^o)$ for all v_1, (p^*,x^*) satisfies the conditions for a seller's neutral optimum, given in Theorem 7 of Myerson [1983].

Let us now consider the symmetric uniform trading problem and show that (p^4,x^4) is a neutral bargaining solution, as was claimed in Section 5. The formulas for the seller's virtual utility (12)-(15) can be derived for the symmetric uniform trading problem exactly as in the lemon problem. (Now $M = 1$, $F(v_1) = v_1$, and $f(v_1) = 1$.) Analogous formulas define virtual utility for the buyer, who now also has private information.

For any small $\varepsilon > 0$, we let

$$L_1^\varepsilon(v_1) = \begin{cases} \varepsilon\, v_1 & \text{if } 0 \leq v_1 < 1/(4-2\varepsilon), \\ \left(\frac{4-3\varepsilon}{3-2\varepsilon}\right) v_1 - \left(\frac{1-\varepsilon}{3-2\varepsilon}\right) & \text{if } 1/(4-2\varepsilon) \leq v_1 \leq 1, \end{cases}$$

$$L_2^\varepsilon(v_2) = \begin{cases} \left(\frac{4-3\varepsilon}{3-2\varepsilon}\right) v_2 & \text{if } 0 \leq v_2 \leq \left(\frac{3-2\varepsilon}{4-2\varepsilon}\right), \\ \varepsilon\, v_2 + (1-\varepsilon) & \text{if } \left(\frac{3-2\varepsilon}{4-2\varepsilon}\right) < v_2 \leq 1. \end{cases}$$

Notice that each L_i^ε is strictly increasing over $[0,1]$, and converges to L_i of equation (6) as ε goes to zero.

The seller's warranted claims with respect to L_1^ε and L_2^ε are determined by the following equations:

$$u_1(v_1) + (v_1 - L_1^\varepsilon(v_1))\, u_1'(v_1) = z_1(v_1),$$

$$\int_0^1 u_1(v_1)\, dL_1^\varepsilon(v_1) = \int_0^1 z_1(v_1)\, dv_1,$$

where

$$z_1(v_1) = .5 \int_0^1 \max\{0,\ W_2(v_2) - W_1(v_1)\}\, dv_2,$$

$$W_1(v_1) = 2\, v_1 - L_1^\varepsilon(v_1),$$

$$W_2(v_2) = 2\, v_2 - L_2^\varepsilon(v_2).$$

It can be shown (somewhat tediously) that the unique solution to these equations is

$$u_1(v_1) = \begin{cases} \left(\frac{3-\varepsilon}{16-8\varepsilon}\right) - .5\, v_1 + \left(\frac{2-\varepsilon}{4}\right)(v_1)^2 & \text{if } 0 \leq v_1 < 1/(4-2\varepsilon), \\ \left(\frac{2-\varepsilon}{12-8\varepsilon}\right)(1 - v_1)^2 & \text{if } 1/(4-2\varepsilon) \leq v_1 \leq 1. \end{cases}$$

As ε converges to zero, these warranted claims converge to

$$u_1(v_1) = \begin{cases} (3 - 8\,v_1 + 8(v_1)^2)/16 & \text{if } 0 \leq v_1 < .25, \\ (1 - v_1)^2/6 & \text{if } .25 \leq v_1 \leq 1. \end{cases}$$

If the seller's valuation is v_1, then his actual expected utility from the mechanism (p^4,x^4) is

$$U_1(v_1,p^4,x^4) = \begin{cases} (6 - 15\,v_1 + 12(v_1)^2)/32 & \text{if } 0 \leq v_1 < .25, \\ (1 - v_1)^2/6 & \text{if } .25 \leq v_1 \leq 1. \end{cases}$$

It is straightforward to check that $U_1(v_1,p^4,x^4) \geq u_1(v_1)$ for every v_1, so (p^4,x^4) fulfills all of the seller's limiting warranted claims. A symmetric argument shows that (p^4,x^4) satisfies the buyer's limiting warranted claims for every v_2 as well. Thus (p^4,x^4) satisfies the conditions for a neutral bargaining solution.

A final remark about the uniqueness of these solutions is in order. The general conditions for a neutral bargaining solution are well-determined, in the sense of giving us as many equations as unknowns (see Theorem 5 of Myerson [1984]), but there is no general uniqueness theorem. For the symmetric uniform trading problem, some inessential nonuniqueness is known. There exist other functions x such that (p^4,x) is a feasible mechanism, and all of these mechanisms are neutral bargaining solutions giving the same expected utility allocations as (p^4,x^4). But other than this nonuniqueness in x, it is this author's unproven belief that (p^4,x^4) is probably the unique neutral bargaining solution for the symmetric uniform trading problem, and that (p^o,x^o) is the unique neutral bargaining solution for the uniform-additive lemon problem considered in Section 6.

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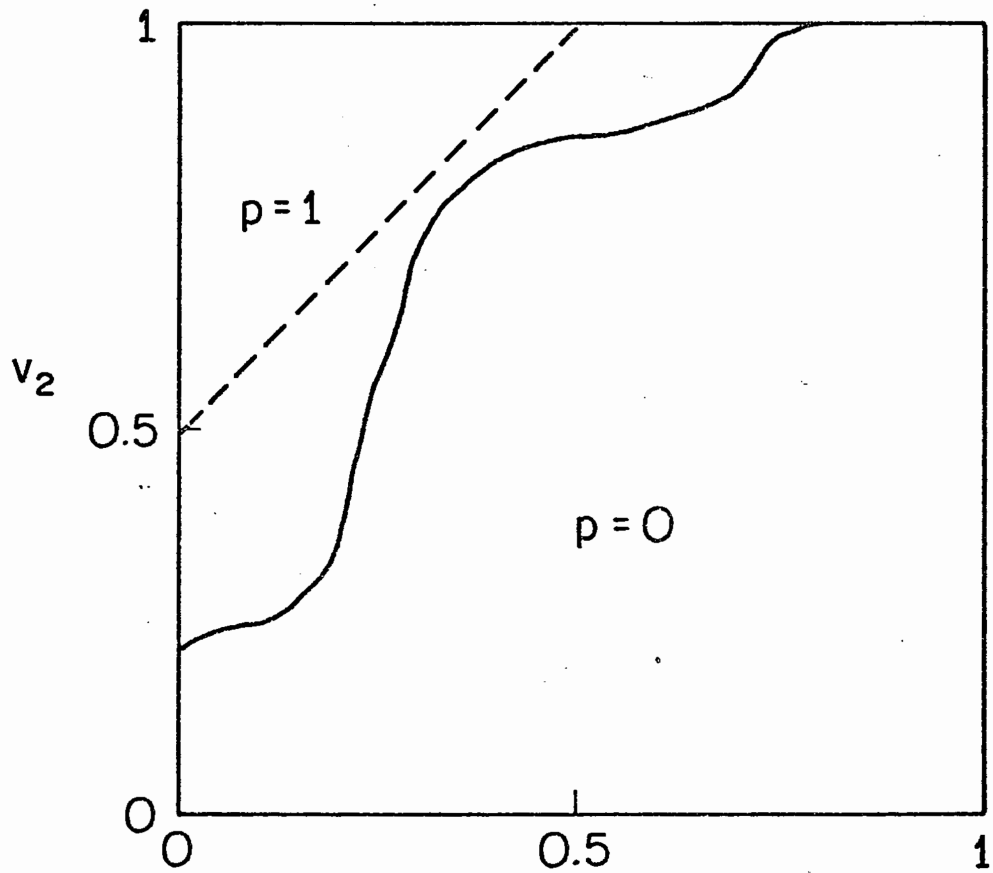

Figure O

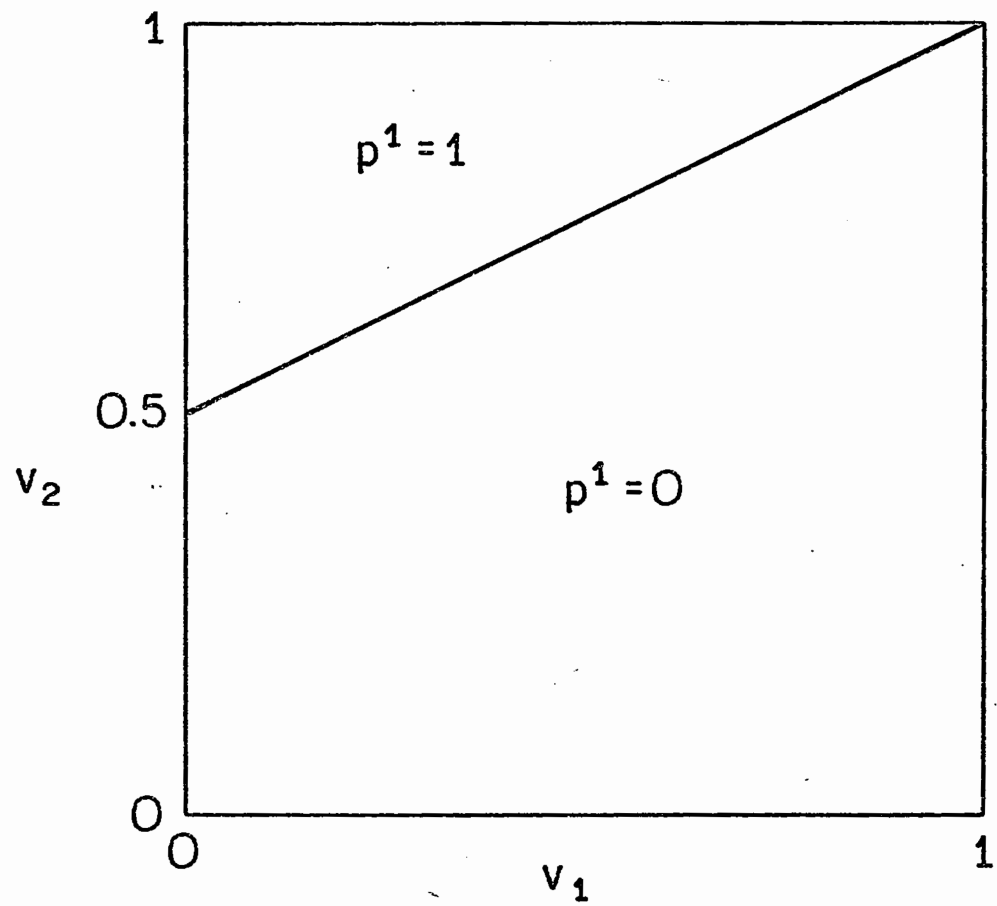

Figure 1

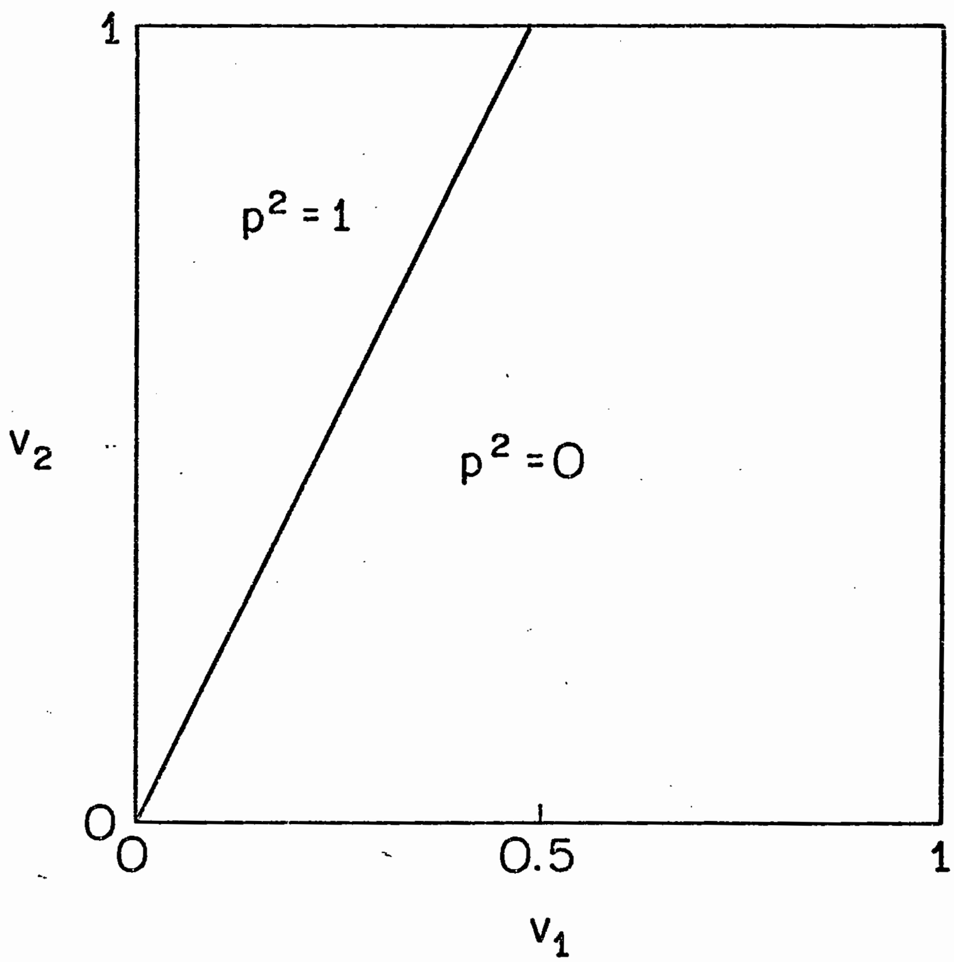

Figure 2

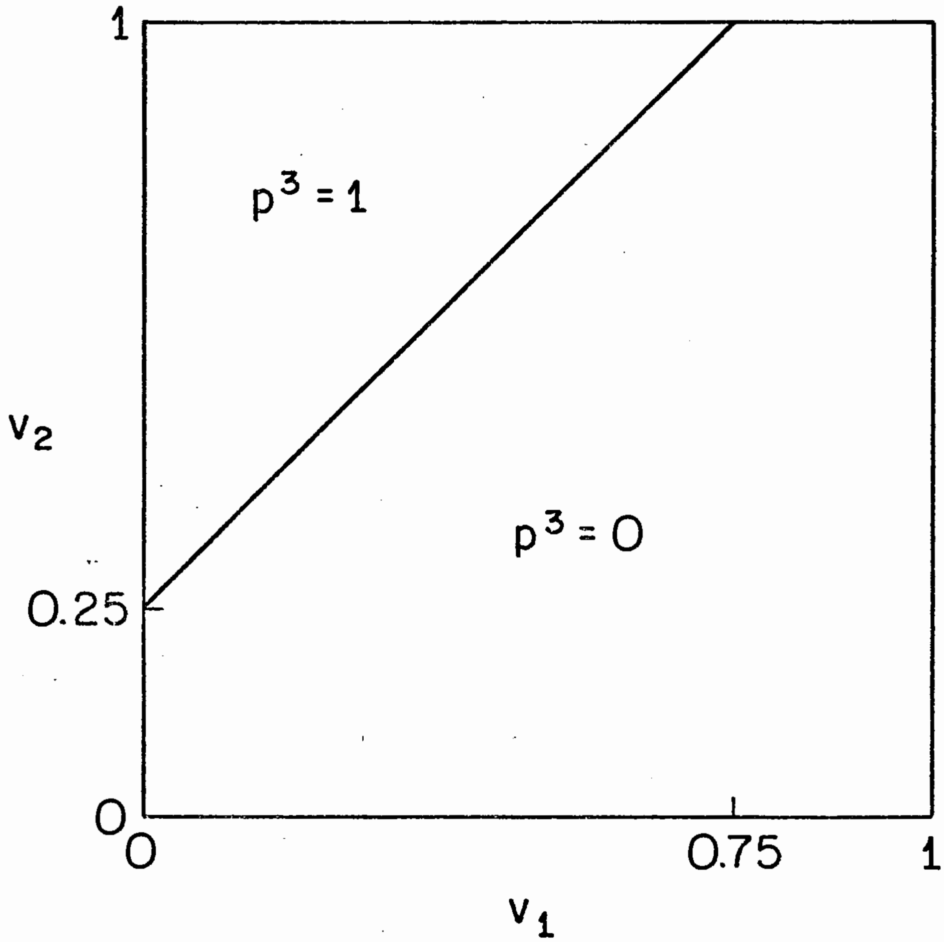

Figure 3

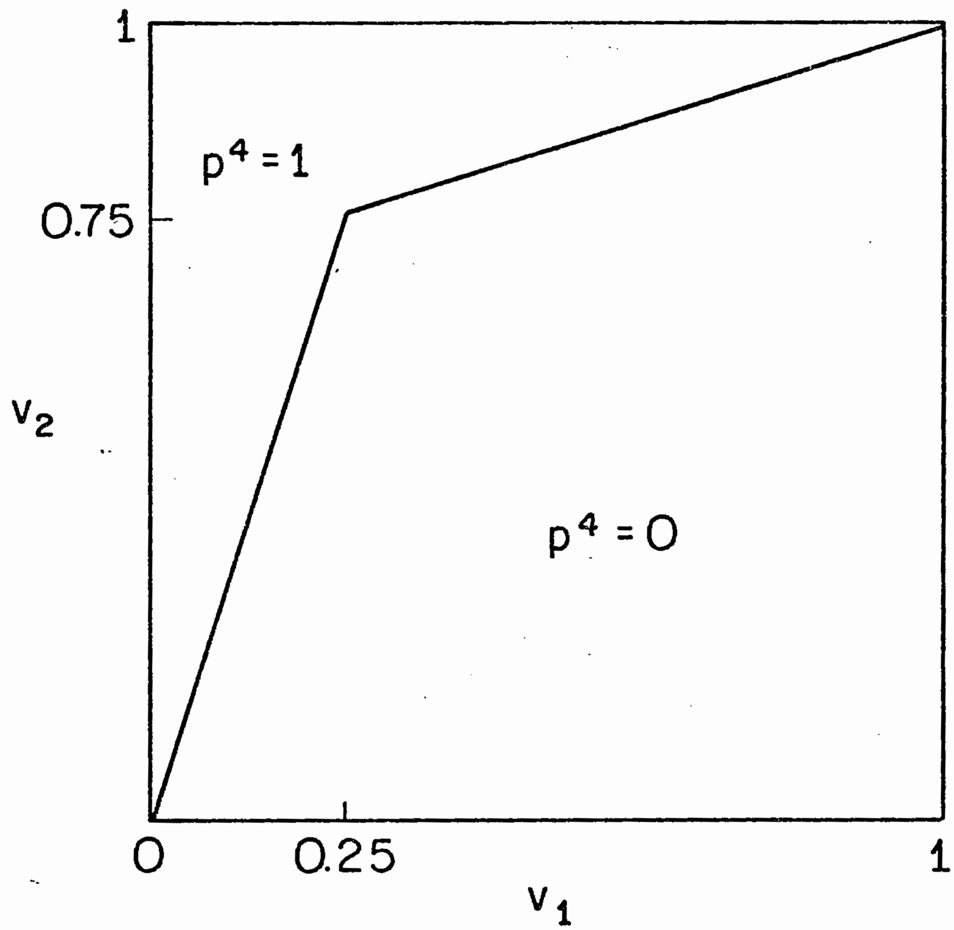

Figure 4

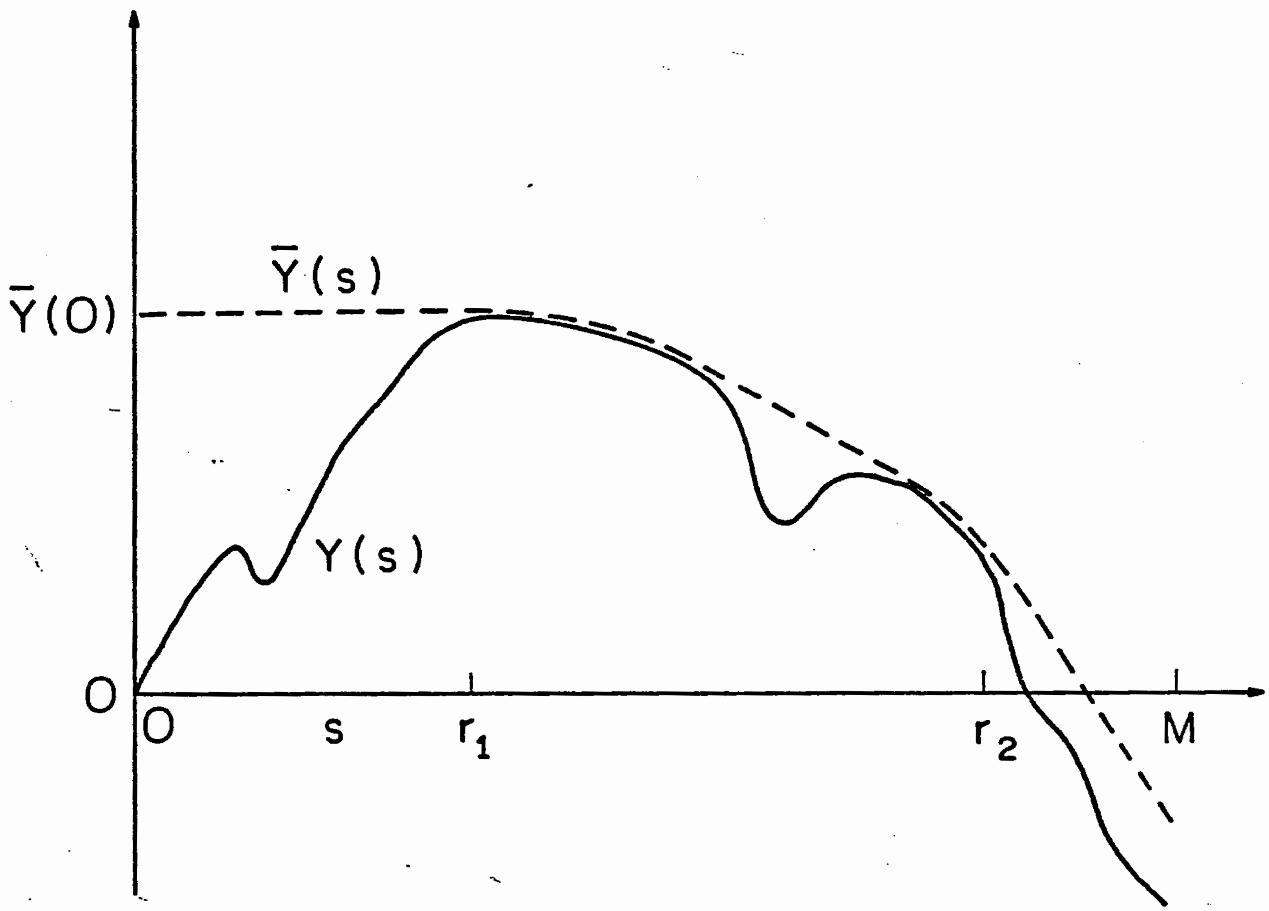

Figure 5