Evolutionary dynamics and potential games in non-cooperative routing

Eitan Altman[1] and Yezekael Hayel[2] and Hisao Kameda[3]

[1] INRIA Sophia Antipolis, eitan.altman@sophia.inria.fr
[2] LIA/University of Avignon, yezekael.hayel@univ-avignon.fr
[3] University of Tsukuba, kameda@cs.tsukuba.ac.jp

Abstract. We consider a routing problem in a network with a general topology. Considering a link cost which is linear in the link flow, we obtain a unique Nash equilibrium and show that the non-cooperative game can be expressed as a potential game. We establish various convergence and stability properties of of the equilibrium related to the routing problem being a potential game. We then consider the routing problem in the framework of a population game and study the evolution of the size of the populations when the replicator dynamics is used.

Key words: potential game, replicator dynamics, global optimization.

1 Introduction

Non-cooperative routing games have long been studied in the context road traffic in the framework of infinite number of players (drivers) where the solution concept is the Wardrop equilibrium [18]. In that context they can be modeled as potential games which allows one to obtain a unique equilibrium (in terms of link flows) as a solution of an equivalent (single player) optimization problem [4].

In the context of computer networks, non-cooperative routing have been introduced and studied in [10] in a context of finitely many users, each of which having to decide how to split flows between various links between sources and destinations. A user may correspond to a service provider that controls the routs taken by the traffic of its subscribers. This type of formulation, already studied in the context of road traffic ([6]), turns out to be much more difficult and does not enjoy in general from the structure of a potential game. In particular, counter examples are given in [10] for the non-uniqueness of the equilibrium. It is therefore of interest to identify conditions on the cost structure that allow to obtain a potential game in the setting of [10]. In this paper we show that the case of linear link costs provides such conditions. We then exploit the potential game structure to obtain convergence to equilibrium of schemes based on best responses. This extends to general topology some of the convergence results obtained in [2] for that restricted to parallel links. We further exploit the potential game structure to establish the convergence of the replicator dynamics which has an evolutionary interpretation.

Related work. Potential games in the case of finite number of players have been defined in [9]. It has been recently used to study networking problems like energy control in wireless networks [15, 7, 16] or to study interference avoidance in [8]. It has also been used for studying evolutionary dynamics of congestion in transportation network model in [13] in the context of Wardrop equilibrium (infinite population). The use of potential in finite networking games go back to Rosenthal [11, 12]. In his framework, however, a user can send one or zero units on each link; the flow is not splittable as in our setting. By allowing a user to send more than one unit of flow over a link in the framework of Rosenthal, the routing problem is no more a potential game.

The structure of the paper is as follows. We begin by introducing in the next section the Model. The potential game structure is then established in Section 3. Sections 4 and 5 then study the convergence of best response and of the replicator dynamics, respectively.

2 Model

We study non-cooperative routing problems in a general network topology. The network is modeled as a directed graph $\{V, L, f\}$ where V is the set of nodes, L is the set of directed arcs and $f = (f_l, l \in L)$ where $f_l : \mathcal{R} \to \mathcal{R}$ is the cost function of link l, which gives the cost per unit of traffic on the link l. We consider the following linear cost:

$$f_l(\lambda_l) = a_l \lambda_l + b_l,$$

where λ_l is the total load on that link, a_l and b_l are positive parameters. We consider N users numbered $1, 2, \ldots, N$ sharing the network. Each user i sends a fixed amount of traffic Λ_i from a source $s(i)$ to a destination $d(i)$. For $i = 1, \ldots, N$ and $l = 1, \ldots, M$, denote by λ_l^i the user's i rate on link l. We define, for $i = 1, 2, \ldots, N$ and $l = 1, 2, \ldots, M$,

$$C_l^i(\lambda_l) = \lambda_l^i \left(a_l \lambda_l + b_l\right). \tag{1}$$

This is the total link l cost payed by user i. The cost perceived by each user is the summation of the cost perceived on each link carrying his traffic. Each user's i cost is thus defined by

$$C^i(\lambda) = \sum_{l=1}^{M} \lambda_l^i \left(a_l \lambda_l + b_l\right) = \sum_{l=1}^{M} C_l^i(\lambda). \tag{2}$$

Each user has to determine the way his traffic is split in order to minimize his cost. We have a non-cooperative game with finite number of player and infinite space strategy.

The flows of each user i has to satisfy some feasibility conditions: positivity and flow conservation constraints:

$$\forall l \in L,\ \lambda_l^i \geq 0 \text{ and } \forall v \in V, r^i(v) + \sum_{l \in In(v)} \lambda_l^i = \sum_{l \in Out(v)} \lambda_l^i, \tag{3}$$

where

$$r^i(v) = \begin{cases} \Lambda^i & \text{if} \quad v = s(i) \\ -\Lambda^i & \text{if} \quad v = d(i) \\ 0 & \text{otherwise,} \end{cases}$$

and $In(v)$ (resp. $Out(v)$) is the set of links which are input (resp. output) to node v.

The multi-strategy (or vector strategy) λ is written as $\lambda = (\lambda^i, \lambda^{-i})$, where λ^{-i} is the vector of all the other rates of other users on each link.

We study in this paper Nash equilibria, i.e. multi-strategies λ satisfying

$$C^i(\lambda) \leq C^i(\gamma^i, \lambda^{-i})$$

for all players i and all strategies γ^i for user i.

3 Establishing the potential game structure

Introduce the following function:

$$P(\lambda) = \sum_{l=1}^{L} \frac{a_l}{2} \left\{ \sum_{i=1}^{N} (\lambda_l^i)^2 + (\sum_{i=1}^{N} \lambda_l^i)^2 \right\} + \sum_{l=1}^{L} b_l \lambda_l. \tag{4}$$

It is a potential of a game [9] if it satisfies for each player i, each multistrategy γ and each strategy λ_i for player i:

$$P(\lambda^i, \lambda^{-i}) - P(\gamma^i, \lambda^{-i}) = C^i(\lambda^i, \lambda^{-i}) - C^i(\gamma^i, \lambda^{-i}). \tag{5}$$

Considering the expression of the cost function (2), we have for all $l \in \{1, \ldots, L\}$

Proposition 1. *The function P defined by equation (4) is a* potential *function for the finite player game.*

Proof We define the following function for $l = 1, 2, \ldots, M$.

$$P_l(\lambda_l) = \frac{a_l}{2} \left\{ \sum_{i=1}^{N} (\lambda_l^i)^2 + (\sum_{i=1}^{N} \lambda_l^i)^2 \right\} + b_l \lambda_l. \tag{6}$$

Then we have

$$P(\lambda) = \sum_{l=1}^{M} P_l(\lambda_l) \text{ and } C^i(\lambda) = \sum_{l=1}^{M} C_l^i(\lambda_l). \tag{7}$$

Considering the function P defined in Equation (4), we have :

$$P_l(\lambda_l^i,\lambda_l^{-i}) - P_l(\gamma_l^i,\lambda_l^{-i}) = \frac{a_l}{2}\left\{\sum_{j\neq i}(\lambda_l^j)^2 + (\lambda_l^i)^2 + (\sum_{j\neq i}\lambda_l^j + \lambda_l^i)^2\right\},$$

$$+b_l\sum_{j\neq i}\lambda_l^j + b_l\lambda_l^i - \frac{a_l}{2}\left\{\sum_{j\neq i}(\lambda_l^j)^2 + (\gamma_l^i)^2 + (\sum_{j\neq i}\lambda_l^j + \gamma_l^i)^2\right\}$$

$$-b_l\sum_{j\neq i}\lambda_l^j - b_l\gamma_l^i,$$

$$= \frac{a_l}{2}\left\{(\lambda_l^i)^2 - (\gamma_l^i)^2 + (\sum_{j\neq i}\lambda_l^j + \lambda_l^i)^2 - (\sum_{j\neq i}\lambda_l^j + \gamma_l^i)^2\right\} + b_l(\lambda_l^i - \gamma_l^i),$$

$$= a_l((\lambda_l^i)^2 - (\gamma_l^i)^2) + a_l\sum_{j\neq i}\lambda_l^j(\lambda_l^i - \gamma_l^i) + b_l(\lambda_l^i - \gamma_l^i),$$

$$= C_l^i(\lambda_l^i,\lambda_l^{-i}) - C_l^i(\gamma_l^i,\lambda_l^{-i}).$$

Then, by summing up the left-hand side and the right-hand side of the above equation with respect to $l = 1, 2, \ldots, M$ and by noting the equation (7), we can prove the equation (5), that is, P is a potential considering the difference of cost link by link from strategy λ^i to γ^i. Then P is a potential function. ■

The function P is strictly convex in λ over the compact set defined by constraints (3).

Proposition 2. *The Nash Equilibrium of the game with N finite players is the minimum of the following constrained optimization problem* $\min_\lambda P(\lambda)$ *subject to constraints (3).*

Proof We construct the following Lagrangian function

$$L(\lambda,\alpha) = P(\lambda) - \sum_{i=1}^{N}\{\sum_{v\in V}\alpha_{iv}(r^i(v) + \sum_{l\in In(v)}\lambda_l^i - \sum_{l\in Out(v)}\lambda_l^i)\}.$$

The equilibrium of the non-cooperative game converges to the minimum solution of the Lagrangian. We call this solution as λ^* and α^*. As the potential function P is strictly convex, the minimum is unique and is λ^*. ■

Once the equilibrium exists, it follows that it is unique because it corresponds to the minimum of the potential function. Closed form expressions for this equilibrium are known however for very special topologies such as the parallel links [1].

4 Best response dynamics

Having shown that the game is a potential game, we proceed to obtain convergence.

Definition 1. Asynchronous Best-Response Update (ABBU): *Consider some strictly increasing time sequence T_n. An ABBU algorithm is an update rule in which (i) at each T_n, one user updates its strategy to the best response against the current strategy of the others, and (ii) the set of times at which user i updates its strategy is infinite.*

The next result follows from [9] by exploiting the fact that the the game has a potential.

Theorem 1 *The ABBU dynamics converges to the unique NE.*

5 Replicator dynamics

In this section we adopt a perspective of a population game in which the replicator dynamics is used to describe the evolution of strategies. We assume that the flow Λ^i generated by population i is constant over time. A user i with rate Λ_i can be considered as a population i with a global mass Λ_i of infinitesimal users. The proportion of population i who use strategy m (i.e. server or link m) is $x_m^i = \frac{\lambda_m^i}{\Lambda^i}$.

The replicator dynamics was first introduced in [17] in the context of discrete strategy space. In [5], the author presents this evolutionary dynamics for continuous strategy spaces. This dynamics comes from the basic tenet of the Darwinism related to a model of evolution through imitation, where the percentage growth rate of each strategy is given by the difference between that strategy's fitness and the average fitness of the population. Considering this dynamics, strategies with higher fitness will survive. In our model, we have to modify the mathematical expression of the derivative equations as we consider cost function and not fitness. **Hence, in our formulation of this dynamics, the strategy with the lowest cost will survive. Moreover, the fitness is by definition the payoff obtained by unit mass which is expressed as the marginal cost here following the definition given in [14].**

The cost C_l^i perceived by users in population i who use link l is rewritten by:

$$C_l^i(x_l) = (x_l^i \Lambda^i)\Big(a_l \sum_{i=1}^{N} x_l^i \Lambda^i + b_l\Big),$$

where $x_l = (x_l^1, x_l^2, \ldots, x_l^{N-1}, x_l^N)$. For all $i = \{1, \ldots, N\}$ and $m = \{1, \ldots, M\}$, the replicator dynamics is given by:

$$\dot{x}_l^i = x_l^i [F^i(x) - f_l^i(x_l)] := \varphi_l^i(x), \tag{8}$$

with f_l^i the marginal cost given by:

$$f_l^i(x_l) = \frac{\partial C^i}{\partial \lambda_l^i} = a_l \sum_{i=1}^{N} x_l^i \Lambda^i + a_l x_l^i \Lambda^i + b_l$$

and

$$F^i(x) = \sum_{l=1}^{M} x_l^i f_l^i(x)$$

the mean population cost.

When the constant cost b_l are equal for all link l,i.e. $b_l = b$, we prove that the NE x^* is a stationary point of the vector field as for all i and l $\varphi_l^i(x^*) = 0$. The NE obtained in [2] for a general network topology is given by

$$\forall i, l \quad (x_l^i)^* = \frac{1/a_l}{\sum_{l=1}^{M} 1/a_l}. \tag{9}$$

Proposition 3. *The NE is a stationary point for the replicator dynamics (8).*

Proof We have for all i and l:

$$\begin{aligned}
F^i(x^*) - f_l^i(x_l^*) &= \sum_{m=1}^{M} x_m^i f_m^i(x^*) - a_l \sum_{i=1}^{N} x_l^i \Lambda^i - a_l x_l^i \Lambda^i - b, \\
&= \sum_{m=1}^{M} x_m^i \left(a_m \sum_{i=1}^{N} x_m^i \Lambda^i + a_m x_m^i \Lambda^i + b \right) - a_l \sum_{i=1}^{N} x_l^i \Lambda^i - a_l x_l^i \Lambda^i - b, \\
&= \sum_{m=1}^{M} \left(\frac{1}{\sum_{l=1}^{M} 1/a_l} \sum_{i=1}^{N} \frac{\Lambda^i/a_m}{\sum_{l=1}^{M} 1/a_l} + \frac{x_m^i \Lambda^i}{\sum_{l=1}^{M} 1/a_l} + x_m^i b \right) \\
&\quad -a_m \sum_{i=1}^{N} \frac{1\Lambda^i/a_m}{\sum_{l=1}^{M} 1/a_l} - a_m \frac{\Lambda^i/a_m}{\sum_{l=1}^{M} 1/a_l} - b, \\
&= \frac{1}{\sum_{l=1}^{M} 1/a_l} \sum_{l=1}^{M} \frac{\Lambda/a_l}{\sum_{l=1}^{M} 1/a_l} + \frac{\Lambda^i \sum_{l=1}^{M} x_l^i}{\sum_{l=1}^{M} 1/a_l} + \sum_{l=1}^{M} x_l^i b \\
&\quad -a_l \frac{\Lambda/a_l}{\sum_{l=1}^{M} 1/a_l} - \frac{\Lambda^i}{\sum_{l=1}^{M} 1/a_l} - b \\
&= \frac{\Lambda}{\sum_{l=1}^{M} 1/a_l} + \frac{\Lambda^i}{\sum_{l=1}^{M} 1/a_l} + b - \frac{\Lambda}{\sum_{l=1}^{M} 1/a_l} - \frac{\Lambda^i}{\sum_{l=1}^{M} 1/a_l} - b \\
&= 0.
\end{aligned}$$

which implies the Proposition. ■

Remark If the constant costs b_l are different and if at equilibrium, each user sends a positive amount of traffic on each link, then the unique NE obtained in [2] is not a stationary point of the dynamic.

Considering the stability of the dynamic system, one can define two kinds of stability. The local stability around the equilibrium known as the *Lyapunov stability*, and the global stability known as the *asymptotical stability.*

The difficulty for the local stability, is to find a Lyapunov function but, for this perspective, we can use our potential function defined by equation (4).

Definition 2. *A stationnary point x^* is Lyapunov stable if there exists a function f, called a* Lyapunov function*, such that:*

1. *f is C^1,*
2. *for all $x \in \Omega \setminus x^*$, $f(x) > 0$*
3. *for all t and $x \in \Omega \setminus x^*$, $\frac{d}{dt} f(x(t)) \leq 0$,*

We prove now that our potential function obtained in the first part is a global Lyapunov function.

Proposition 4. *The function f defined by:*

$$f(x) = P(x) - P(x^*) \tag{10}$$

is a global Lyapunov function under the replicator dynamics.

Proof Taking the expression of the potential (4), the function P can be expressed by :

$$P(x) = \sum_{l=1}^{M} \frac{a_l}{2} \left(\sum_{i=1}^{N} (x_l^i \Lambda^i)^2 + (\sum_{i=1}^{N} x_l^i \Lambda^i)^2 \right) + \sum_{l=1}^{M} b_l \sum_{i=1}^{N} x_l^i \Lambda^i.$$

The function f is clearly C^1 and $f(x^*) = 0$. Moreover, for all $x \in \Omega \setminus x^*$ we have $f(x) > 0$ as x^* is the unique minimum of P. Finally, for all t and $x \in \Omega \setminus x^*$, we have

$$\frac{df(x(t))}{dt} = \nabla P(x(t)).\dot{x}(t) - \nabla P(x^*(t)).\dot{x^*}(t) = \nabla P(x(t)).\dot{x}(t)$$

Then,

$$\begin{aligned}
\nabla P(x(t)).\dot{x}(t) &= \sum_{i=1}^{N} \sum_{l=1}^{M} \frac{\partial P}{\partial x_l^i}(x(t)) \dot{x}_l^i(t), \\
&= \sum_{i=1}^{N} \sum_{l=1}^{M} \frac{\partial P}{\partial x_l^i}(x(t)) \varphi_l^i(x(t)), \\
&= \sum_{i=1}^{N} \sum_{l=1}^{M} \frac{\partial P}{\partial x_l^i}(x(t)) x_l^i \left(F^i(x) - f_l^i(x_l)\right), \\
&= \sum_{i=1}^{N} \sum_{l=1}^{M} (x(t)) \frac{x_l^i}{\Lambda^i} \left(F^i(x) - f_l^i(x_l)\right).
\end{aligned}$$

But we have for all i, l, $\frac{\partial P}{\partial \lambda_l^i} = \frac{\partial C^i}{\partial \lambda_l^i} = f_l^i$ because P is a potential. Then,

$$\nabla P(x(t)).\dot{x}(t) = \sum_{i=1}^{N} \left(\frac{1}{\Lambda^i} (\sum_{l=1}^{M} x_l^i f_l^i)^2 - \frac{1}{\Lambda^i} \sum_{l=1}^{M} x_l^i (f_l^i)^2 \right),$$
$$= \sum_{i=1}^{N} \frac{1}{\Lambda^i} \left((\sum_{l=1}^{M} x_l^i f_l^i)^2 - \sum_{l=1}^{M} x_l^i (f_l^i)^2 \right).$$

By Jensen's inequality, we have that the summation is non-positive [16]. Then we have proved that the function f is a Lyapunov function for the system. ■

We have proved that the NE point is Lyapunov stable. A stronger stability property is the asymptotical stability which implies Lyapunov stability and requires that in addition, the population returns to equilibrium after any small perturbation. We cannot use directly the result from [13] as the author assumes that the dynamic satisfies the non-extinction property[4], which is not the case for the replicator dynamics as $\varphi(0) = 0$ and it is not a NE. But, in our case, we can use the convexity property of the Lyapunov function and the uniqueness of the NE for the interior space.

Theorem 2 *The NE is asymptotically stable under the dynamics defined in (8) and any solution trajectory of this dynamics starting from an interior initial condition converges to the NE.*

Proof As the dynamics (8) admits a global Lyapunov function, every solution trajectory converges to a connected set of rest points. Using the strict convexity of the Lyapunov function, we obtain by contradiction, as in [14], the convergence of any interior initial condition to the NE x^*.

As x^* is strictly positive (all components non-negative), there exist a real strictly positive ϵ which defines a neighborhood A around the NE with strictly positive components. Then A is a local minimizer set of the potential function and all Nash equilibria in A are only the NE x^*. Also, for all $x \in A \setminus \{x^*\}$, we have $\frac{d}{dt} f(x(t)) > 0$ as it is equal to 0 if and only if $\varphi(x(t)) = 0$, which is equivalent to $x(t) = 0$ or $x(t) = x^*$. Thus, we can conclude that the function f is a strict local Lyapunov function for x^* and then x^* is asymptotically stable. ■

6 Conclusion and perspectives

We have investigated in this paper non-cooperative routing for link cost density (i.e. cost per packet) that are linear in the link flow, thus extending existing results for parallel links [2] to general topology. By identifying a potential structure

[4] This property requires that any extinct strategy can rebirth and is expressed by the following implication that $\varphi(x) = 0$ implies x is a NE.

for this game we have obtained convergence of both best response dynamics as well as of the replicator dynamics. As future work we plan to extend the results to the multiclass setting of [3] and to study the stability of other dynamics.

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