Cores of Convex Games [1])

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Abstract: The core of an n-person game is the set of feasible outcomes that cannot be improved upon by any coalition of players. A convex game is defined as one that is based on a convex set function. In this paper it is shown that the core of a convex game is not empty and that it has an especially regular structure. It is further shown that certain other cooperative solution concepts are related in a simple way to the core: The value of a convex game is the center of gravity of the extreme points of the core, and the von Neumann-Morgenstern stable set solution of a convex game is unique and coincides with the core.

1. Introduction

The core of an n-person game is the set of feasible outcomes that cannot be improved upon by any coalition of players [3]). A convex game is one that is based on a convex set function (see below); intuitively this means that the incentives for joining a coalition increase as the coalition grows, so that one might expect a "snowballing" or "band-wagon" effect when the game is played cooperatively.

In this paper we show that the core of a convex game is not empty – in fact, it is quite large – and that it has an especially regular structure. We further show that certain other cooperative solution concepts are related in a simple way to the core. Specifically (1) the value of a convex game is the center of gravity of the extreme points of the core, and (2) the VON NEUMANN-MORGENSTERN stable set solution of a convex game is unique and coincides with the core. In a subsequent paper [MASCHLER, PELEG, and SHAPLEY] rather similar results will be presented for two other cooperative solutions: the kernel and the bargaining set.

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[1]) Presented at the Fifth Informal Conference on Game Theory, held at Princeton University in April 1965. This paper is based on a Rand Corporation research memorandum [SHAPLEY, 1965], written under the sponsorship of Air Force Project Rand. The author wishes to acknowledge the stimulus of a query from Jack Edmonds concerning the properties of convex set functions.

[2]) The Rand Corporation, Santa Monica, California.

[3]) The core is sometimes incorrectly described as the set of outcomes "that cannot be blocked by any coalition". This unfortunately misleading description has arisen as a result of the counterintuitive use of the word "block" in the technical terminology of several early papers. (The present author must bear part of the blame!) In fact, the core is concerned with what coalitions can *do*, not what they can *prevent*. The distinction is especially striking in economic game models, where the bargaining power that certain groups may acquire through their ability to *obstruct* trade or production is completely ignored in the essentially *constructive* conditions that define the core.

1.1. Notation

We shall systematically use the letters $n, s, t, \ldots$ to denote the number of elements in the finite sets $N, S, T, \ldots$ The letter "O" will denote the empty set, and "$\subset$", "$\supset$" will denote strict inclusion. "Payoff vectors" are elements of the n-dimensional linear space E^N with coordinates indexed by the elements of N. If $a \in E^N$ and $S \subseteq N$ we shall often write $a(S)$ for $\Sigma_S a_i$, treating a as an additive function on the subsets of N. The hyperplane in E^N defined by the equation $a(S) = v(S)$, $0 \subset S \subseteq N$, will be denoted by H_S.

2. Convex Games

In this paper, a *game* is a function v from a lower-case ring $\mathcal{N}$ to the reals, satisfying

$$v(0) = O\,. \tag{1}$$

It is *superadditive* if

$$v(S) + v(T) \leqq v(S \cup T)\,, \quad \text{all} \quad S,T \in \mathcal{N} \text{ with } \quad S \cap T = O\,. \tag{2}$$

It is *convex* [1]) if

$$v(S) + v(T) \leqq v(S \cup T) + v(S \cap T)\,, \quad \text{all} \quad S,T \in \mathcal{N}. \tag{3}$$

It is *strictly convex* if inequality holds in (3) whenever neither $S \subseteq T$ nor $T \subseteq S$, i.e., whenever $S, T, S \cup T$, and $S \cap T$ are all different. Note that (1) and (3) together imply (2).

To appreciate the term "convex", define for each $R \in \eta$ a differencing operator Δ_R by:

$$[\Delta_R v](S) = v(S \cup R) - v(S - R)\,, \quad \text{all} \quad S \in \mathcal{N}\,,$$

and let $\Delta_{QR} v$ denote $\Delta_Q(\Delta_R v)$. Then (3) is equivalent to the assertion that these "second differences" are everywhere nonnegative, i.e., that

$$[\Delta_{QR} v](S) \geqq 0\,, \quad \text{all} \quad Q,R,S \in \mathcal{N}\,. \tag{4}$$

This is analogous to the nonnegative second derivatives associated with convex functions in real analysis.

Two games are termed *equivalent* if their difference is an additive game, i.e., obeys the equality in (2) (or (3) or (4)). It is easily seen that any game equivalent to a convex game is convex; that any positive scalar multiple of a convex game is convex; and that the sum of two or more convex games is convex. It follows that the ensemble of convex games (for fixed $\mathcal{N}$) forms a convex cone in a suitable linear space (say $E^{\mathcal{N}-\{0\}}$), and that this cone contains the subspace of additive games.

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[1]) In potential theory, functions satisfying the reverse inequality to (3) (i.e., negatives of convex functions) are called *strongly subadditive* or, when certain other properties are adduced, *capacities* [CHOQUET; MEYER]. In lattice theory and combinatorial mathematics, such functions, often restricted to be monotonic and integer valued, as in the rank function of a matroid, are called *submodular* or *lower semi-modular*, making our convex functions supermodular or upper semi-modular [WHITNEY; EDMONDS, ROTA; CRAPO, ROTA; EDMONDS].

2.1. Interpretation

Throughout this paper $\mathcal{N}$ will be the lower-case ring of subsets of a *finite* set N – thus, $\mathcal{N} = 2^N$ [1]). In the standard application in game theory the elements of N are "players", the elements of $\mathcal{N}$ are "coalitions", and $v(S)$, called the "characteristic function", gives for each coalition the best payoff it can achieve without help from other players.

Superadditivity (2) arises naturally in this interpretation, but convexity (3) is another matter. For example, in a voting situation S and T, but not $S \cap T$, might be winning coalitions, causing (3) to fail. To see what convexity does entail, regard the function m:

$$m(S,T) = v(S \cup T) - v(S) - v(T) \tag{5}$$

as defining the "incentive to merge" between disjoint coalitions S and T. Then it is a simple exercise to verify that (3) is equivalent to the assertion that $m(S,T)$ is nondecreasing in each variable – whence the "snowballing" or "bandwagon" effect mentioned in the introduction.

Another condition that is equivalent to (3) (provided $\mathcal{N}$ is finite) is to require that [2])

$$v(S \cup \{i\}) - v(S) \leqq v(T \cup \{i\}) - v(T), \tag{6}$$

for all individuals $i \in N$ and all $S \subseteq T \subseteq N - \{i\}$. This expresses a sort of increasing marginal utility for coalition membership, and is analogous to the "increasing returns to scale" associated with convex production functions in economics.

2.2. Convex Measure Games

Let v be given by

$$v(S) = f(\mu(S)), \quad \text{all} \quad S \in \eta, \tag{7}$$

where μ is a nonnegative additive function on η, i.e., a measure, and f is a real function with $f(0) = 0$ [3]). Then it is easy to verify that v is convex if f is convex, and strictly convex if f is strictly convex. A game that is a convex function of a measure is called a *convex measure game* [4]).

Not all convex games are convex measure games. A simple counterexample is the two-person convex game with $v(\{1\}) = -v(\{2\}) = 1$, $v(\{1,2\}) = v(O) = 0$. But this game, being additive, is equivalent to the convex measure game $v(S) \equiv 0$, so we might ask whether every convex game is at least equivalent to

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[1]) For the infinite case see ROSENMÜLLER.

[2]) Cf. WHITNEY.

[3]) Actually, any game can be put into this form; we need merely choose μ so that all the numbers $\mu(S)$ are different and define f accordingly. Thus, the general concept of "measure game", with f and μ not further restricted, is of no interest in the finite case. [Cf. AUMANN and SHAPLEY].

In economic applications, μ may represent the initial distribution of some resource, while f may represent a production function.

[4]) Curiously, if f is a function of several variables and μ a vector of measures, then convexity of f does not imply convexity of v in (7), or even superadditivity. Indeed, in the case of a homogeneous function $f(\lambda x) \equiv \lambda f(x)$, it is concavity of f, not convexity, that implies superadditivity of v.

Table 1. Extremal convex games on four players

$\emptyset$	1	2	3	4	12	13	14	23	24	34	123	124	134	234	1234
0	0	0	0	0	0	1	1	1	1	1	2	2	2	2	4
0	0	0	0	0	1	0	1	1	1	1	2	2	2	2	4
0	0	0	0	0	1	1	0	1	1	1	2	2	2	2	4
0	0	0	0	0	1	1	1	0	1	1	2	2	2	2	4
0	0	0	0	0	1	1	1	1	0	1	2	2	2	2	4
0	0	0	0	0	1	1	1	1	1	0	2	2	2	2	4
0	0	0	0	0	1	0	0	0	0	0	1	1	1	1	2
0	0	0	0	0	0	1	0	0	0	0	1	1	1	1	2
0	0	0	0	0	0	0	1	0	0	0	1	1	1	1	2
0	0	0	0	0	0	0	0	1	0	0	1	1	1	1	2
0	0	0	0	0	0	0	0	0	1	0	1	1	1	1	2
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	2
0	0	0	0	0	0	0	1	0	1	1	1	2	2	2	3
0	0	0	0	0	0	1	0	1	0	1	2	1	2	2	3
0	0	0	0	0	1	0	0	1	1	0	2	2	1	2	3
0	0	0	0	0	1	1	1	0	0	0	2	2	2	1	3
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	2
0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	2
0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	2
0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	2
0	0	0	0	0	1	1	1	1	1	1	2	2	2	2	3
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
0	0	0	0	0	1	1	0	1	0	0	2	1	1	1	2
0	0	0	0	0	1	0	1	0	1	0	1	2	1	1	2
0	0	0	0	0	0	1	1	0	0	1	1	1	2	1	2
0	0	0	0	0	0	0	0	1	1	1	1	1	1	2	2
0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	1
0	0	0	0	0	0	1	0	0	0	0	1	0	1	0	1
0	0	0	0	0	0	0	1	0	0	0	0	1	1	0	1
0	0	0	0	0	0	0	0	1	0	0	1	0	0	1	1
0	0	0	0	0	0	0	0	0	1	0	0	1	0	1	1
0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	1

a convex measure game. Again the answer is no. Indeed, if this were true, then every convex game would be the sum of three convex measure games, since every additive game is the sum of two [1]). But convex games exist that are not the sum of

[1]) Simply express the game as the difference of two measures, $\mu_1 - \mu_2$, and let $f_1(x) \equiv -f_2(x) \equiv x$.

any number of convex measure games; indeed, the first twelve games of Table 1 have this property.

The situation can be viewed geometrically. As already remarked, the convex games form a cone in $E^{\mathscr{N}}$ which is invariant under equivalence. The subcone of convex measure games, however, is neither convex nor invariant under equivalence (unless n is trivially small). The convex hull of that subcone, which is what we obtain when we take all sums of convex measure games, is invariant under equivalence but still does not include all convex games if $n > 3$.

The thirty-seven extremals of the cone of four-person convex games were determined in 1965 by S. A. COOK [1]) (Table 1). The games in the first seven symmetry classes listed are indecomposable (see below); the remainder are obtained from smaller games by adding dummy players. For larger n, little is known about the set of all extremals [2]).

2.3. Decomposable Games [3])

Let $P = \{N_1, \ldots, N_p\}$ be a partition of N into $p \geqq 2$ nonempty subsets. The game v on $\mathscr{N}$ is said to be *decomposable* (with respect to P) if v is additive across the partition, i.e., for each $S \in \mathscr{N}$,

$$v(S) = v(S \cap N_1) + v(S \cap N_2) + \cdots + v(S \cap N_p). \tag{8}$$

Note that v is completely determined by its values on the subsets of the N_i. The p smaller games, obtained by restricting v to the subsets of each N_i in turn, are the *components* of the decomposition. A decomposable game has a unique finest decomposition, that is, a decomposition in which none of the components are themselves decomposable [4]).

Theorem 1.
(a) A decomposable game is convex if and only if each component is convex.
(b) A convex game v is decomposable if and only if

$$v(N) = v(N_1) + v(N_2) + \cdots + v(N_p) \tag{9}$$

holds for some partition $\{N_1, \ldots, N_p\}$ of N into $p \geqq 2$ nonempty subsets.

Proof [5]).
The first statement is immediate. For the second, suppose (9) holds and let $S \in \mathscr{N}$. By convexity,

[1]) Private communication.
[2]) For the infinite case, see ROSENMÜLLER.
[3]) This section is based on MASCHLER, PELEG and SHAPLEY [1967].
[4]) The notion of decomposition was introduced by VON NEUMANN and MORGENSTERN; see also GILLIES [1953, 1959] and PELEG [1965a, b].
[5]) Several proofs can be given; this one is due to Y. KANNAI.

$$\begin{aligned} v(S) + v(N_1) &\leqq v(S \cap N_1) + v(S \cup N_1) \\ v(S \cup N_1) + v(N_2) &\leqq v(S \cap N_2) + v(S \cup N_1 \cup N_2) \\ \cdots \qquad & \qquad \cdots \\ v(S \cup N_1 \cup \cdots \cup N_{p-1}) + v(N_p) &\leqq v(S \cap N_p) + v(S \cup N_1 \cup \cdots \cup N_p). \end{aligned}$$

Adding these inequalities to each other and to (9) yields

$$v(S) \leqq v(S \cap N_1) + v(S \cap N_2) + \cdots + v(S \cap N_p). \tag{10}$$

But equality must hold here, by superadditivity (2). This completes the proof.

A coalition is said to be "inessential" if its characteristic-function value is no greater than required for superadditivity (2). Theorem 1 (b) expresses the fact that in a convex game, decomposability is equivalent to inessentiality of the all-player coalition.

Corollary.

A strictly convex game is indecomposable.

3. Core Geometry

A payoff vector $a \in E^N$ is said to be *feasible* (for v) if $a(N) \leqq v(N)$[1]. The core of v is defined as the set C of all feasible $a \in E^N$ such that

$$a(S) \geqq v(S), \quad \text{all} \quad S \subseteq N. \tag{11}$$

The core is obviously a subset of the hyperplane H_N (see § 1.1), and since the inequalities $a_i \geqq v(\{i\})$ are included in (11), the core is bounded. Thus, C is a compact convex polyhedron, possibly empty, of dimension at most $n - 1$. By a "full-dimensional" core we shall mean one of dimension exactly $n - 1$.

In order to discuss the facial structure of C, we define $C_S = C \cap H_S$ for $O \subset S \subseteq N$; note that $C_N = C$. It is convenient also to define $C_O = C$. If none of the C_S are empty, we say that the configuration $\{C_S\}$ is *complete*; if they all have the highest possible dimension we say that $\{C_S\}$ is *strictly complete.* In the latter case, the core is full-dimensional and has exactly $2^n - 2$ polyhedral faces of dimension $n - 2$.

Figure 1 shows a core configuration for $n = 3$ that is complete but not strictly complete, since $C_{\{2\}}$ is only a point. Figure 2 depicts a strictly complete core[2] for $n = 4$, there being fourteen 2-dimensional faces. (The extra lines and points, included for perspective, indicate the intersections of the H_S for $s = 3$ with the

[1]) In application, there may be also a lower bound β, such that only payoff vectors with $\beta \leqq a(N) \leqq v(N)$ are truly "feasible". Only the upper bound, however, will be of any significance in the present work.

[2]) Note that if we are given only the point set C, we cannot in general identify all of the sets C_S, or even tell whether any are empty. In fact, it is not difficult to show that C uniquely determines v, and hence $\{C_S\}$, only in the strictly complete case. Thus, it makes sense to speak of a "strictly complete core", but not a "complete core".

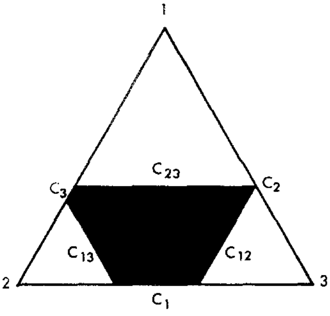

Fig. 1: Core configuration of a 3-person convex game

simplex A in H_N defined by the H_S for $s = 1$.) We see from the key that parallel faces on opposite sides of the core correspond to complementary coalitions[1]).

Note that completeness implies a "large" core – one which touches all of the $(n - 2)$-dimensional faces of A.

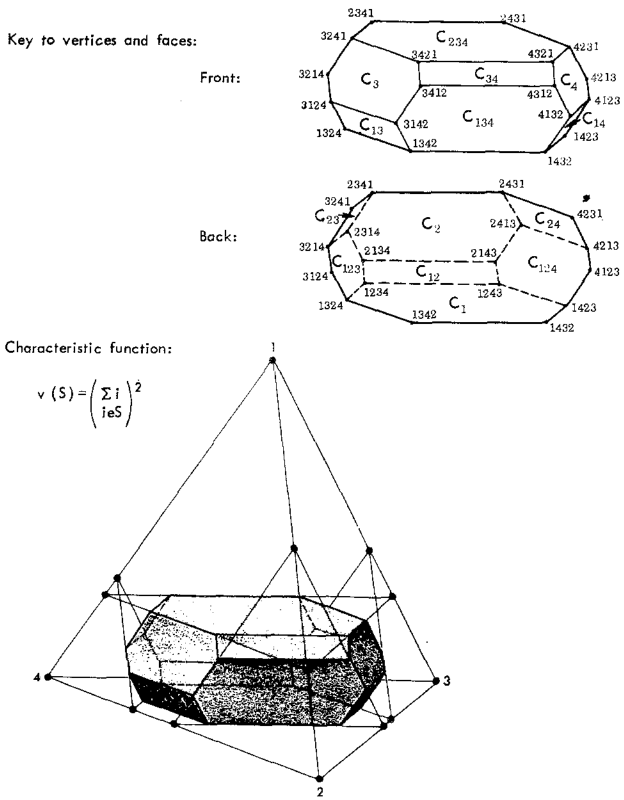

Fig. 2: Core of a four-person convex game

1) In SCHMEIDLER a game with a complete core configuration is called an *exact* game.

3.1. Regular Core Configurations

Although the hyperplanes H_S have the same slopes in every games v, their positions in E^N relative to one another are not predetermined, and even in the case of a strictly complete core there are many ways (if $n > 3$) in which the faces C_S can fit together. But one "natural" arrangement stands out from the rest; it occurs, for example, when all faces C_S, $S \neq O, N$, are equidistant from a common center – i.e., tangent to the inscribed $(n - 2)$-sphere[1]). In this arrangement, it can be shown that two faces touch if and only if their corresponding coalitions are comparable, i.e.,

$$C_S \cap C_T \neq O \quad \text{if and only if} \quad S \subseteq T \quad \text{or} \quad T \subseteq S. \tag{12}$$

We shall call property [MEYER] *strict regularity*. It obviously implies completeness (take $S = T$); we shall see presently that it implies strict completeness. The configuration of Fig. 2 is strictly regular, but not that of Fig. 1, since C_{12} and C_{23} meet.

We shall work primarily with a weaker form of (12). A core configuration $\{C_S\}$ will be called *regular* if $C_N \neq O$ and

$$C_S \cap C_T \subseteq C_{S \cup T} \cap C_{S \cap T}, \quad \text{all} \quad S, T \subseteq N. \tag{13}$$

We see at once that strict regularity implies regularity, as the terminology would suggest. Condition (13) is equivalent to the statement that for each $a \in E^N$, the family of sets

$$\mathscr{S}_a = \{S : a \in C_S\} \tag{14}$$

is closed under union and intersection.

The formal similarity between (13) and (3) is not accidental; in due course we shall show that convexity of a game is equivalent to regularity of its core configuration. First we shall develop some geometrical consequences of (13).

3.2. Faces

Theorem 2.

In a regular core configuration $\{C_S\}$ we have

$$C_{S_1} \cap C_{S_2} \cap \cdots \cap C_{S_m} \neq O \tag{15}$$

for any increasing sequence $S_1 \subset S_2 \subset \cdots \subset S_m$. In particular (take $m = 1$), a regular core configuration is complete.

The proof proceeds with the aid of two lemmas concerning regular cores. Let us write $S \subset \subset T$ to mean that $S \subset T$ and $t - s \geqq 2$.

Lemma 1.

If $S \subset \subset T$ and $a \in C_S \cap C_T$, then there exists Q and b such that $S \subset Q \subset T$ and $b \in C_S \cap C_Q \cap C_T$. Moreover, we can require that $b_i = a_i$, all $i \in S$, and that $j \in Q$, $k \notin Q$ for any two preassigned elements j,k of $T - S$.

[1]) Such a game is $v(S) \equiv s - \sqrt{s(n - s)/(n - 1)}$.

Proof.

Fix $j,k \in T - S$. Define $b \in C$ by $b_j = a_j - \rho$, $b_k = a_k + \rho$, and $b_i = a_i$ for $i \neq j,k$, giving ρ the largest possible value compatible with $b \in C$; this can be done because C is compact and contains a. Clearly, $b(S) = a(S)$, $b(T) = a(T)$; hence $b \in C_S \cap C_T$. Since a larger ρ would have taken b out of the core entirely, there must be a set R, containing j and *not* containing k, such that $b \in C_R$. Let $Q = (S \cup R) \cap T$. Since $\mathscr{S}_b$ (see (14)) is closed under "$\cup$" and "$\cap$", we have $b \in C_Q$. But Q has all the required properties: $S \subset Q \subset T$, $j \in Q$, and $k \notin Q$; so the lemma is proved.

Lemma 2.

Let $S \subset \subset N$ and $a \in C_S$. Then, for any $j \in N - S$ there exists $b \in C_S \cap C_{S \cup \{j\}}$ such that $b_i = a_i$, all $i \in S$.

Proof.

Set $T = N$ and use Lemma 1 to find Q, b such that $S \cup \{j\} \subseteq Q \subset N$ and $b \in C_S \cap C_Q$, the latter agreeing with a on S. If $Q = S \cup \{j\}$ we are through; if not, set $T = Q$ and repeat the argument.

Proof of Theorem 2.

We may assume that the sequence $\{S_k\}$ is of maximum length, i.e., that $m = n + 1$. Then we have $C_{S_1} = C_O$, which is nonempty by definition. Take $a_{(1)} \in C_{S_1}$, and for $k = 1, \ldots, n-1$, apply Lemma 2 to find a point $a_{(k+1)} \in C_{S_k} \cap C_{S_{k+1}}$, agreeing with $a_{(k)}$ on S_k. The point $a_{(n)}$ will be in all of the C_{S_k}, $1 \leqq k \leqq n$, and it will also be in $C_{S_{n+1}} = C_N = C$. Hence (15) follows.

3.3. Vertices

Let ω represent a simple ordering of the players. Specifically, let ω be one of the $n!$ functions that map N onto $\{1,2,\ldots,n\}$. Define

$$S_{\omega,k} = \{i \in N : \omega(i) \leqq k\}, \quad k = 0,1,\ldots,n\,; \tag{16}$$

these are the "initial segments" of the ordering. Thus, $S_{\omega,0} = O$ and $S_{\omega,n} = N$. Consider the equations $a(S_{\omega,k}) = v(S_{\omega,k})$, $k = 1,2,\ldots,n$. Linear independence is assured, and it is easy to solve the set of equations to find the coordinates of the intersection of the hyperplanes $H_{S_{\omega,k}}$, namely

$$a_i^{\omega} = v(S_{\omega,\omega(i)}) - v(S_{\omega,\omega(i)-1}), \quad \text{all} \quad i \in N\,. \tag{17}$$

In this way, each ordering ω defines a payoff vector a^{ω}; there is, of course, no assurance that the $n!$ points a^{ω} are all distinct.

Theorem 3 [1]).

The vertices of the core in a regular configuration are precisely the points a^{ω}.

[1]) Theorem 22 of EDMONDS gives a similar result for "polymatroids", which are similar in many respects to cores of convex games. Thus, Theorem 8 of EDMONDS may be compared with our (11), Theorem 42 of EDMONDS with our (13), etc.

Proof.

We have, for any ω,

$$C_{S_{\omega,1}} \cap \cdots \cap C_{S_{\omega,n}} \subseteq H_{S_{\omega,1}} \cap \cdots \cap H_{S_{\omega,n}}.$$

The left-hand side is not empty, by Theorem 2; it therefore consists of the single point a^ω. Hence a^ω is in the core. If it were not a vertex (i.e., an extreme point), then for some nonzero vector d we would have both $a^\omega \pm d \in C$. But this is impossible, since at least one of the hyperplanes $H_{S_{\omega,k}}$ that meet at and determine a^ω must pass strictly between the two points $a^\omega \pm d$, excluding one of them from possible membership the core. Hence a^ω is a vertex of C, for each ω.

It remains to show that C has no other vertices. Let a be any vertex of C and let $O = S_1 \subset S_2 \subset \cdots \subset S_m = N$ be a "longest" increasing sequence of members of $\mathscr{S}_a$ (see (14)), i.e., one that maximizes m. If there are no gaps in this sequence, i.e., if $m = n + 1$, then a is of the form a^ω, and we are through. Suppose therefore that a gap exists, so that $S_k \subset\subset S_{k+1}$ for some k, and let $i, j \in S_{k+1} - S_k$, $i \neq j$. Since a is a vertex, it is not only a solution of all the equations $a(S) = v(S)$, $S \in \mathscr{S}_a$, it is their unique solution. But in order to determine the coordinates a_i and a_j, and not just their sum $a_i + a_j$, there must be an equation that separates them, i.e., an $R \in \mathscr{S}_a$ that contains precisely one of i,j. Since $\mathscr{S}_a$ is closed under "$\cup$" and "$\cap$", we have $Q \in \mathscr{S}_a$, where $Q = (S_k \cup R) \cap S_{k+1}$. But $S_k \subset Q \subset S_{k+1}$, contradicting the maximality of m. This completes the proof.

3.4. Strictly Regular Cores

This section is mainly descriptive, and formal proofs are omitted. Strict regularity was defined by (12) above, but the one-sided condition

$$C_S \cap C_T \neq O \quad \text{implies} \quad S \subseteq T \quad \text{or} \quad T \subseteq S, \tag{18}$$

together with $C \neq O$, would have sufficed. To see this, note that (18) with $C \neq O$ implies regularity, which in turn implies the converse of (18) by way of Theorem 2 with $m = 2$.

Strict regularity implies strict completeness, and hence is a property of the point set C. A strictly regular core has $n!$ distinct extreme points a^ω. The core in a regular configuration that is not strictly regular, on the other hand, always has fewer than $n!$ extreme points, although it may still be strictly complete, i.e., have the maximum number of $(n-2)$-dimensional faces.

The facial arrangement of a strictly regular core is completely determined in all dimensions. For example, an $(n-2)$-dimensional face C_S is bounded by exactly $2^s + 2^{n-s} - 4$ $(n-3)$-dimensional faces of the form $C_S \cap C_T$ with $0 \subset T \subset S$ or $S \subset T \subset N$. The automorphisms of this highly symmetric combinatorial structure are generated by the permutations of N, together with complementation, the latter automorphism mapping each vertex a^ω into its antipode $a^{\omega'}$, where ω' denotes the reversal of ω. This is illustrated for $n = 4$ in Fig. 2.

We can now distinguish several classes of games, according to the kinds of cores they posses:

Table 2

nonempty cores	full-dimensional cores
complete core configurations	strictly complete cores
regular core configurations	strictly regular cores

Each class includes those listed below it and to the right. For $n > 3$, the inclusions are all strict. If a game is regarded as a point in $E^{\mathcal{N}-\{O\}}$ (see § 2.2), then each of the six classes describes a convex cone; and those on the left [right] are the closures [interiors] of those on the right [left]. In particular, every regular core is a limit of strictly regular cores – indeed it can be approximated by both increasing and decreasing nested sequences of strictly regular cores.

4. Solutions of Convex Games

Many different "solution concepts" have been proposed for n-person games in characteristic-function form. We shall be concerned here with three of them: the core, already defined; the (SHAPLEY) value (§ 4.2); and the (VON NEUMANN-MORGENSTERN) solutions, which we call stable sets (§ 4.3). We shall see that the three concepts are intimately related when the game is convex.

4.1. The Core

The core of a game, already defined at (11), may be interpreted as the set of "socially stable" outcomes, in that no coalition can improve upon any of them. In a game with an empty core, at least one set of players must fail to realize its full potential, no matter how the winnings are divided.

First we require

Theorem 4.

The core of a convex game is not empty.

Proof.

It suffices to show that a^{ω} is in the core, for some ω (see § 3.3). Let $T \subset N$ and let j be the "ω-earliest" element of $N - T$, so that all the ω-predecessors of j are in T, but not j itself. Then we have at once

$$T \cup S_{\omega,\omega(j)} = T \cup \{j\},$$
$$T \cap S_{\omega,\omega(j)} = S_{\omega,\omega(j)-1},$$

using the notation of (16). Hence, by convexity of v,

$$v(T) + v(S_{\omega,\omega(j)}) \leqq v(T \cup \{j\}) + v(S_{\omega,\omega(j)-1})$$

or, by (17),

$$a_j^\omega \leqq v(T \cup \{j\}) - v(T).$$

Hence,

$$a^\omega(T) - v(T) \geqq a^\omega(T \cup \{j\}) - v(T \cup \{j\}).$$

Repeating the argument $n - t - 1$ times, we obtain

$$a^\omega(T) - v(T) \geqq a^\omega(N) - v(N) = 0.$$

Since T was arbitrary, a^ω is in the core.

Theorem 5.

A game is convex if and only if its core configuration is regular.

Proof.

Suppose we have regularity (13). Let $S, T \subseteq N$. Since $S \cup T \supseteq S \cap T$, we can find $a \in C_{S \cup T} \cap C_{S \cap T}$, by Theorem 2. Then we have

$$v(S \cup T) + v(S \cap T) = a(S \cup T) + a(S \cap T) = a(S) + a(T) \geqq v(S) + v(T).$$

Hence v is convex.

Conversely, suppose that v is convex. Then $C \neq O$ by Theorem 4 and it remains only to establish (13). Suppose a is in $C_S \cap C_T$. Then we have

$$v(S \cup T) + v(S \cap T) \geqq v(S) + v(T) = a(S) + a(T) = a(S \cup T) + a(S \cap T).$$

But we also have $a(S \cup T) \geqq v(S \cup T)$ and $a(S \cap T) \geqq v(S \cap T)$, because a is in the core. Hence equality prevails everywhere, and $a \in C_{S \cup T} \cap C_{S \cap T}$, as required.

Corollary.

A game is strictly convex if and only if its core is strictly regular.

We omit the simple proof.

Theorem 6 [1]).

(a) The core of an indecomposable convex game is full-dimensional.

(b) The core of a decomposable convex game is the cartesian product of the cores of the components of any decomposition. Its dimension is $n - p$, where p is the number of components in the finest decomposition.

Proof.

(*a*) If C is less than full-dimensional, then $C = C_S$ for some $O \subset S \subset N$. We have $C_{N-S} \neq 0$, by completeness. Let $a \in C_{N-S} \subseteq C = C_S$. Then

$$v(S) + v(N - S) = a(S) + a(N - S) = a(N) = v(N),$$

and decomposability follows from (b) of Theorem 1.

[1]) Based on MASCHLER, PELEG, and SHAPLEY [1967]. Note that the "cartesian product" property does not depend on convexity, but is valid for any decomposable game.

(b) Let v be additive across the partition $\{N_i\}$, and first let a be in the cartesian product of the component cores. Then for all i and S we have

$$a(N_i \cap S) \geqq v(N_i \cap S) \quad \text{and} \quad a(N_i) = v(N_i).$$

Applying (8) we obtain $a(S) \geqq v(S)$ and $a(N) = v(N)$, showing that a is in the core. Next let a be a payoff vector that is not in the cartesian product of the component cores. Then for some i,

$$\text{either } a(S) < v(S) \quad \text{for some} \quad S \subseteq N_i\,, \quad \text{or} \quad a(N_i) > v(N_i).$$

In the first case, $a \notin C$. In the second case, we have

$$a(N) - a(N - N_i) = a(N_i) > v(N_i) = v(N) - v(N - N_i),$$

using (9). Hence either $a(N) > v(N)$ or $a(N - N_i) < v(N - N_i)$, so again $a \notin C$.

Finally, to determine the dimension of C let $\{N_1, \ldots, N_p\}$ yield the finest decomposition. Then the components are all indecomposable and have full-dimensional cores, by part (a), so that their cartesian product has dimension

$$\sum_1^p (n_i - 1) = n - p.$$

This completes the proof.

4.2. The Value

The *value* of a game v is the payoff vector $\varphi \in E^N$ defined by

$$\varphi_i = \sum_{\substack{S \subseteq N \\ S \ni i}} \frac{(s-1)!(n-s)!}{n!} [v(S) - v(S - \{i\})], \quad \text{all} \quad i \in N. \tag{19}$$

We have $\varphi(N) = v(N)$ and, for superadditive games, $\varphi_i \geqq v(\{i\})$ for all $i \in N$ [1]). The quantity φ_i may be interpreted as the "equity value" associated with the position of the i-th player in the game.

Theorem 7.

The value of a convex game is an element of the core.

Proof.

It is well known (see SHAPLEY [1953b]) that

$$\varphi = \frac{1}{n!} \sum_{\omega \in \Omega} a^\omega, \tag{20}$$

where Ω is the set of all orderings of N. Hence, by Theorems 3 and 5, the value belongs to the core – indeed, it is in the relative interior since it is a center of gravity of the vertices.

[1]) See SHAPLEY [1953b]. It is also shown in SHAPLEY [1953b] that the value of a decomposable game is the cartesian product of the values of its components.

For completeness, we sketch a proof of (20). Fix i. Given $S \ni i$, we ask: for how many orderings ω do we have $S_{\omega,\omega(i)} = S$? It is clearly both necessary and sufficient that ω be such that $\omega(j) < \omega(i)$ for all $j \in S - \{i\}$ and $\omega(j) > \omega(i)$ for all $j \in N - S$. Thus the number of orderings is $(s-1)!(n-s)!$. The result now follows easily from (19) and (17).

4.3. Stable Sets

A payoff vector b is said to be *dominated* by a payoff vector a if there is a non-empty $S \in \mathcal{N}$ such that

$$a(S) \leqq v(S), \quad \text{and} \quad a_i > b_i \quad \text{all} \quad i \in S. \tag{21}$$

A set V of feasible[1]) payoff vectors is said to be *stable* if every feasible[1]) payoff vector is either a member of V or dominated by a member of V, but not both. A stable set may be interpreted as a "standard of behavior", i.e., a set of "conventional" outcomes which are always given a chance to dominate any proposal that might be put forward during pre-play negotiations.

It is easily verified that every stable set contains the core. Since no stable set can properly include another, it follows that if the core is stable then it is the only stable set[2]).

Theorem 8.

The core of a convex game is stable (i.e., is the unique VON NEUMANN-MORGENSTERN solution).

Proof.

Let $\{C_S\}$ be regular, and take any feasible b not in C. We shall show that b is dominated by an element of C. Define $g(O) = 0$ and

$$g(S) = \frac{v(S) - b(S)}{s}, \quad \text{all} \quad O \subset S \subseteq N,$$

and let g attain its maximum, g^*, at $S = S^*$. Since $b \notin C$ but is feasible we see that $g^* > 0$ and hence $S^* \neq O$. By completeness, $C_{S^*} \neq O$. Let $c \in C_{S^*}$, and define $a \in E^N$ by:

$$a_i = \begin{cases} b_i + g^*, & i \in S^* \\ c_i, & i \in N - S^*. \end{cases}$$

[1]) The classical definition [VON NEUMANN and MORGENSTERN] replaces "feasible" in these two places by "feasible and individually rational", where $a \in E^N$ is *individually rational* if and only if $a_i \geqq v(\{i\})$, all $i \in N$. This distinction (discussed at length in SHAPLEY [1951] – see also LUCE and RAIFFA [p. 215ff.]) is not important here, since it can be shown that the core is stable in the one sense if and only if it is stable in the other. (See e.g., Theorems 12 and 13 of GILLIES [1959].)

[2]) The early conjectures [VON NEUMANN and MORGENSTERN; SHAPLEY, 1953] that stable sets exist for all superadditive games and that the core is the intersection of all stable sets have been disproved; see LUCAS [1967, 1968].

Since $a(S^*) = b(S^*) + s^*g^* = v(S^*)$, condition (21) is satisfied, and we see that a dominates b. Moreover, since $a(N) = v(S^*) + c(N - S^*) = c(N) = v(N)$, we see that a is feasible. It remains to show that a satisfies the core inequalities (11).

Let $T \subseteq N$ be arbitrary, and break up T into $Q = T \cap S^*$ and $R = T - S^*$. Then we have, in easy steps,

$$\begin{aligned} a(T) &= a(Q) + a(R) \\ &= b(Q) + qg^* + c(R) \\ &\geqq b(Q) + qg(Q) + c(T \cup S^*) - c(S^*) \\ &\geqq v(Q) + v(T \cup S^*) - v(S^*) \\ &\geqq v(T)\,. \end{aligned}$$

Hence a is in the core.

4.4. Remarks

1. An example of a stable core that does not come from a regular, or even complete, configuration is shown in Fig. 3. The core is a perfect cube, with faces

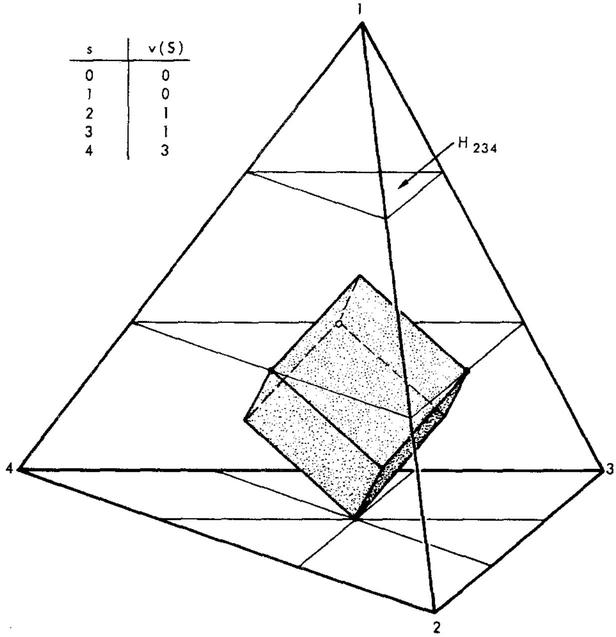

Fig. 3: Example of a stable, nonregular core

$\{C_S : s = 2\}$ and vertices at $(1,1,1,0)$, $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2})$, etc. The sets C_S, $s = 3$, are empty.

2. We have obtained a complete theory of core stability in *symmetric games*, i.e., games of the form

$$v(S) \equiv f(s)\,;$$

this will be presented elsewhere.

3. Theorem 8 implies that a "positive fraction" of all games are solvable – more precisely, that the set of solvable games on $\mathcal{N}$ includes a full-dimensional (i.e., $(2^n - 1)$-dimensional) cone in $E^{\mathcal{N}-\{O\}}$. This was first pointed out by GILLIES[1]), using the cone generated by the games in 0,1-normalization that satisfy $v(S) < 1/n$ for $2 \leqq s < n$. This cone includes some but not all convex games, as well as some that are not convex.

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[1]) GILLIES [1953, 1959]; see also BONDAREVA.

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