Equilibria of a noncooperative game for heterogeneous users of an ALOHA network

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Abstract—**A noncooperative group of users sharing a channel via ALOHA is considered. Depending on their quality of service requirements and willingness to pay, the users will select a desired throughput. The users then participate in a noncooperative game wherein they adjust their transmission-probability parameters in an attempt to attain their desired throughputs. The possible equilibrium points reached by such a community of users are studied.**

I. Introduction

In a multiservice internet equipped with a corresponding differential billing policy, users will compete for limited resources. Such user dynamics are typically studied in the setting of noncooperative games. In [10], the authors consider a Stackelberg game in which users choose routes in a wired network after the leader has chosen routes for its own traffic; in choosing, the leader controls user behavior to optimize some network utility or to achieve some other global goal. Their problem formulation admits a closed-form expression for the equilibrium which can then be steered by the leader to a preordained operating point (incentive compatibility). In [3], [4], the game's leader was itself a selfish user of network. In [6], [13], the authors formulate CDMA power control games of which the equilibrium point is studied. In the framework of Kelly [8], [9], TCP users in a wired Internet are studied in [12]. Rate-based flow control is studied in [1].

In this paper we study the behavior of competing users sharing a single channel using the ALOHA medium access protocol. Users desire more or less throughput depending on both their willingness to pay and their need [7], [8]. We study the existence of equilibrium points that could possibly be reached by the users for given user throughput demands. The users' convergence to equilibrium points is analyzed using a specified potential function that governs their dynamics.

II. A game for slotted ALOHA

In our incarnation of slotted ALOHA, users advertise their transmission-probabilities[1] q_n to other users but keep their desired throughput (demand) y_n private. Typically, y_n depends on utility function and given price [7], [8]. Let $\underline{q}^0$ be the initial N-vector of users' transmission-probabilities chosen so that $q_n^0 < 1$ for all n. For slotted ALOHA, the n^{th} user's throughput at this time is $\theta_n^0 \equiv q_n^0 x_n^0$ where $x_n^m \equiv \prod_{i \neq n}(1-q_i^m)$ for all m, n [11] (here we have assumed that every user's transmission queue is continuously backlogged). The users adjust their transmission-probabilities in an attempt to attain their desired throughputs y_n.

Here we choose the dimension packets per time-slot for parameters θ and y. Thus, these quantities are naturally bounded above by one since there is obviously at most one (fixed-length) packet per time-slot.

At the m^{th} iteration of the game, the n^{th} user will choose a new transmission-probability

$$q_n^{m+1} \quad = \quad \min\{y_n/x_n^m,\ 1\}, \qquad (1)$$

and then announce this choice to the other users. All other users will likewise update their choice of transmission-probability so that, given initial transmission-probabilities $\underline{q}^0 \in [0,1)^N$, we get, after the first step of the game, the new vector of transmission-probabilities $\underline{q}^1 \in [0,1]^N$. The users advertise their new transmission-probabilities $\underline{q}^1$ and the process repeats resulting in the sequence

$$\underline{q}^0,\ \underline{q}^1,\ \underline{q}^2,\ \ldots \qquad (2)$$

If this sequence converges, say to $\underline{q}^* \in [0,1]^N$, then $\underline{q}^*$ is a (Nash [7]) equilibrium point (or "fixed point") of

[1] The quantity $1 - q_n$ is called the backoff parameter of the n^{th} user.

the mapping (1):

$$q_n^* \quad = \quad \min\{y_n/x_n^*,\ 1\} \quad \text{for all } n, \tag{3}$$

where $x_n^* \equiv \prod_{i\neq n}(1-q_i^*)$. Clearly, at this point the average throughput for the n^{th} user is $\theta_n^* \equiv q_n^* x_n^*$ for all $n = 1, ..., N$.

There may exist an equilibrium point in which $q_n^* = 1$ for some n and $q_i^* = 0$ for all $i \neq n$ ($\Rightarrow \theta_n^* = 1$ and $\theta_i^* = 0$ for all $i \neq n$). Also, a completely deadlocked equilibrium point may be possible in which $q_n^* = 1$ and $q_k^* = 1$ for some $n \neq k$. For example, if $N = 3$, $y_1 = y_2 = 7/16$, $y_3 = 1/16$, and $q_n^0 = y_n$ for all n, then we will arrive at deadlock in two steps, i.e., $q_1^3 = q_2^3 = 1$. Clearly, both cases may not be desirable and can be avoided by restricting all transmission-probabilities q_n to all be less than some large $Q < 1$, say $Q = 0.9$, so that the equilibrium point satisfies (4) instead of (3).

$$q_n^* \quad = \quad \min\{y_n/x_n^*,\ Q\} \tag{4}$$

III. Equilibria of the ALOHA game

That an equilibrium point (3) or (4) exists is an immediate consequence of the Brouwer's fixed point theorem [2]. We will now study the equilibrium point for the ALOHA game described above.

First consider the unconstrained dynamics leading to the equilibrium point

$$q_i^* \quad = \quad y_i/\prod_{j\neq i}(1-q_j^*) \quad \text{for } 1 \leq i \leq N, \tag{5}$$

i.e., we neither restrict the transmission "probabilities" to be less than one (or Q) nor greater than zero. Let $a_{ij} \equiv y_i/y_j$ where we make the assumption that $y_j > 0$ for all j for the remainder of this section. In the following, the superscript "*" for the equilibrium point is dropped.

Lemma 1: If $\underline{q}$ solves (5) and if $0 < q_i < 1$ for some i, then $0 < q_j < 1$ for all $j \neq i$.

Proof:

For $i \neq j$, divide the i^{th} equation (5) (i.e., with $n = i$) by the j^{th} to get $q_i(1-q_j) = a_{ij}q_j(1-q_i)$. Thus,

$$q_j \quad = \quad \frac{a_{ji}q_i}{1-q_i+a_{ji}q_i}. \tag{6}$$

If we fix the index i, the statement of the lemma immediately follows from this expression for q_j in terms of q_i. $\square$

Fixing again the index i and substituting into (5) the expression for q_j in (6) for all $j \neq i$, we get the following polynomial equation in the component q_i of the equilibrium point:

$$G_i(q_i) \ \equiv \ q_i(1-q_i)^{N-1} - y_i\prod_{j\neq i}(1+(a_{ji}-1)q_i) = 0. \tag{7}$$

The following proposition is a corollary of the previous lemma.

Proposition 1: If, for some i, equation (7) admits a real solution in the interval (0,1), then (5) has a solution in $(0,1)^N$.

For example, two equilibria, $(4/5,\ 1/3)$ and $(2/3,\ 1/5)$, exist when $y_1 = 8/15$ and $y_2 = 1/15$. Therefore, the mapping $\underline{q}^m \rightarrow \underline{q}^{m+1}$ in (1) is *not* a contraction mapping.

It turns out that only the no-overbooking condition, $\sum_{n=1}^N y_n < 1$ (i.e., total desired throughput is less than the channel capacity) is *not* sufficient to guarantee a solution in the interval (0,1) for equation (7). For example, if $y_1 = 3/4$ and $y_2 = 1/5$, the solutions to (7) with $i = 1$ are complex and the only fixed point (4) is (Q, Q). This is not surprising in light of the well-known throughput limitations of slotted ALOHA.

Any solutions to (7) that may lie in $[0, 1]$ are readily found using Newton's method. One can easily show that, in general, any solution q_i of (7), $G_i(q_i) = 0$, satisfies $q_i > y_i$. Note that $G_i(y_i) < 0$ and $G_i(1) < 0$. Therefore, if $G_i(v) > 0$ for some $v \in (0, 1)$ then, by the intermediate value theorem, there is a solution q_i to $G_i(q_i) = 0$ in (y_i, v) and in $(v, 1)$.

IV. Local convergence to an equilibrium point using a modified game

Define the function $f_n(\underline{q}^*)$ as the right-hand-side of (4) and define the function $F : [0, Q]^N \rightarrow [0, Q]^N$ so that $F = (f_1,\ ...,\ f_N)^T$. So, the sequence (2) was generated by

$$\underline{q}^{m+1} \quad = \quad F(\underline{q}^m). \tag{8}$$

Consider the following modification of the game (8):

$$\underline{q}^{m+1} = \underline{q}^m + \varepsilon(F(\underline{q}^m) - \underline{q}^m) \tag{9}$$

for a fixed small $\varepsilon > 0$; this is simply the Jacobi update scheme, see equation (15) of [12]. For small ε, we can approximate (9) by the continuous-time game

$$\dot{q}(t) \quad = \quad F(\underline{q}(t)) - \underline{q}(t). \tag{10}$$

Now we will study the convergence property of the corresponding Jacobi update scheme. Note that if $\varepsilon = 1$, (9) becomes the original game. Define the following "potential" function on $[0, Q]^N$:

$$\Lambda(\underline{q}) = -\prod_{j=1}^{N} \frac{y_j}{1-q_j} + \sum_{j=1}^{N} \left(\frac{q_j}{1-q_j} + \log(1-q_j) \right) \prod_{i \neq j} y_i.$$

We assume that all $y_i > 0$ in the following. Since

$$\frac{\partial \Lambda(\underline{q})}{\partial q_j} = -\frac{1}{y_j(1-q_j)^2}(f_j(\underline{q}) - q_j) \prod_{i=1}^{N} y_i,$$

$\nabla \Lambda(\underline{q}) = 0$ in $[0, Q)^N$ if and only if $F(\underline{q}) = \underline{q}$, i.e., $\underline{q}$ is an equilibrium point of interest. Further note that, under the dynamics (10) and for every $\underline{q}(t) \in [0, Q)^N$ that is *not* an equilibrium point,

$$\begin{aligned} \frac{\partial \Lambda(\underline{q}(t))}{\partial t} &= \ < \nabla \Lambda(\underline{q}(t)),\ \dot{\underline{q}}(t) > \\ &= -\sum_{j=1}^{N} \frac{1}{y_j(1-q_j)^2} \left(\prod_{i=1}^{N} y_i \right) (f_j(\underline{q}) - q_j)^2 \ < \ 0. \end{aligned}$$

Note that the components of the Hessian H of Λ are

$$\frac{\partial^2 \Lambda(\underline{q})}{\partial q_j \partial q_n} = \begin{cases} -\frac{1}{(1-q_n)(1-q_j)} \prod_{i=1}^{N} \frac{y_i}{1-q_i} & \text{if } j \neq n \\ \frac{\prod_{i \neq j} y_i}{(1-q_j)^3}((1-q_j) - 2(f_j(\underline{q}) - q_j)) & \text{if } j = n \end{cases}$$

At a *stable* equilibrium point $\underline{q}$ of (10) (i.e., $\underline{q}$ is a local minimum of Λ), we require positive definiteness of Hessian $H(\underline{q})$. Diagonal dominance of H is a *sufficient* condition for positive definiteness (see Section 5.2 of [5]). The Hessian of Λ is diagonally dominant at $\underline{q}$ if, for all j, $1 \ > \ q_j \sum_{n \neq j} (1-q_n)^{-1}$. For the example of the previous section in which $y_1 = 8/15$ and $y_2 = 1/15$, the Hessian H of Λ at the equilibrium point $(2/3, 1/5)$ is diagonally dominant (and, therefore, positive definite) but is not diagonally dominant at $(4/5, 1/3)$.

Using Λ as a Lyapunov function locally, in this section we have shown the following proposition.

Proposition 2: If there is an equilibrium point $\underline{q}^* \in [0, Q)^N$ with $H(\underline{q}^*) \ > \ 0$ then there is a neighborhood $B \subset [0, Q)^N$ of $\underline{q}^*$ such that: for any initial transmission-probabilities $\underline{q}(0) \in B$, the function $\underline{q}(t)$ obeying the dynamics (10) will converge to $\underline{q}^* \in \overline{B}$ as $t \to \infty$.

For sufficiently small ε, this proposition can be adjusted to make a statement about the convergence of (9) about the equilibrium point $\underline{q}^*$. We postulate that equilibria that are stable for (10) are also stable when $\varepsilon = 1$, i.e., for the original iteration (1). Finally, note that the vector field $F(\underline{q}) - \underline{q}$ itself does not lead to a potential function in general.

V. Discussion

In [7], we studied a network pricing mechanism in this context; the user demands y_n were determined by the network's price per transmitted packet and user utility functions that accounted for user need and willingness to pay. In an alternative approach to throughput negotiation, the network sets a price and the users simply communicate to the network their throughput demands y_n; then the *network* simply computes an equilibrium point for the users and stipulates (broadcasts) a transmission-probability parameter for each user.

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