Optimal Allocation of a Divisible Good to Strategic Buyers

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Abstract— **We address the problem of allocating a divisible resource to buyers who value the quantity they receive, but strategize to maximize their net payoff (value minus payment). An allocation mechanism is used to allocate the resource based on bids declared by the buyers. The bids are equal to the payments, and the buyers are assumed to be in Nash equilibrium. For two buyers such an allocation mechanism is found that guarantees that the aggregate value is always greater than $\frac{7}{8}$ of the maximum possible, and it is shown that no other mechanism achieves a larger ratio. For a general finite number of buyers an allocation mechanism is given and an expression is given for its worst case efficiency. For three buyers the expression evaluates to 0.8737, for four buyers to 0.8735 and numerical computations suggest that the numerical value does not decrease when the number of buyers is increased beyond four. A potential application of this work is the allocation of communication bandwidth on a single link.**

I. Introduction

Players in non-cooperative games try to maximize their own payoff functions. If such a game has a designer with preferences on the outcomes, it may be possible for the designer to decide on strategy spaces and the corresponding outcomes (i.e. the *mechanism*) so that the players' strategic behavior will not lead to an outcome that is far from desirable.

Consider the game involving the allocation of a single unit of a divisible resource to competing buyers each of whom has to make a payment in compensation. Each buyer obtains a certain amount of value from allocations of the resource made to it, but strategizes to maximize his net payoff - the value minus payment. An allocation of the resource that maximizes the aggregate value (given the buyer value functions) is said to be efficient, other allocations with lower aggregate value are inefficient.

Nash equilibria are fundamental in the study of games of this nature. That inefficient allocations may occur at Nash equilibria is well known in the economics literature [9]. The question addressed in this paper is: how to design an allocation mechanism that results in the worst-case inefficiency of a Nash equilibrium being as low as possible ?

Recent research contains many examples of efforts to quantify the inefficiency of Nash equilibria in games related to resource allocation. Koutsoupias and Papadimitriou [5] and Roughgarden and Tardos [4] quantify worst-case inefficiencies of Nash equilibria in routing games. Johari and Tsitsiklis [1] quantify the inefficiency in allocating a divisible good with a uniform price. All of these papers use the ratio of the welfare at Nash equilibrium to the welfare at the social optimum, thus evaluating fractional efficiency.

Auctions of divisible goods have also received much attention. Besides [1] mentioned above, Maheswaran and Başar [7] and Gopalkrishnan and Hajek [2] also deal with allocation of a single divisible good. Kelly [6] shows that if the buyers are not strategic but price-taking, allocation of a divisible good can be made in a socially optimum way. In the economics literature this model appears in Back and Zender [8] who advocate using discriminatory pricing for the sale of treasury bonds.

For two buyers this paper finds the mechanism that has the lowest worst-case fractional inefficiency for allocating a divisible good when the bids that the buyers place are the payments they will make. The mechanism ensures that any Nash equilibrium (for any set of buyer preferences) will be at least $\frac{7}{8}^{th}$ as efficient as the optimal allocation. The extension to n buyers has not been completely evaluated, but numerical computations suggest that the worst possible case (with any number of buyers) has a fractional efficiency of 0.8735. Other characteristics like uniqueness of

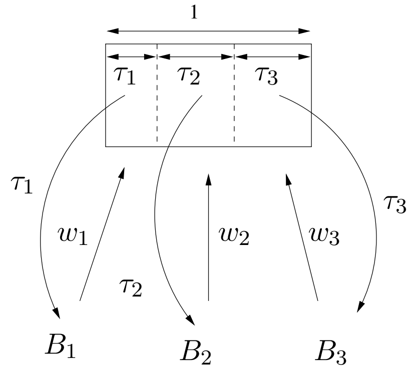

Fig. 1. The allocation rule τ: buyer B_i pays w_i and receives τ_i

equilibria are also investigated.

II. The Setup

Consider the situation where one unit of a divisible good is to be split up among n buyers. Let $\mathbf{w} = [w_1, \ldots, w_n]$ be the (non-negative) *payments* that the buyers make and let $\mathbf{x} = [x_1, \ldots, x_n]$ be the quantities they are allocated as a result. The allocation is made according to a pre-specified *allocation mechanism* τ, so that given the payments $\mathbf{w}$ the allocation to buyer i is given by $x_i = \tau_i(\mathbf{w})$. This procedure is depicted in Figure.

An allocation rule τ is said to be a *valid* allocation mechanism if it has the following properties:

- **A1** *It is an allocation:* $\tau_i \geq 0$ and $\sum_i \tau_i(\mathbf{w}) = 1$ for all values of $\mathbf{w}$ such that $\sum_i w_i > 0$. Also, $\tau_i(0, \mathbf{w}_{-i}) = 0$ for all $\mathbf{w}_{-i}$ i.e. a zero bid $w_i = 0$ will get zero allocation (even if all bids zero).
- **A2** *It is smooth:* $\tau_i(w_i, \mathbf{w}_{-i})$ is differentiable, increasing and concave in w_i for all $\mathbf{w}_{-i}$, except in the case when $\mathbf{w}_{-i} = 0$. Also for each i, $\tau_i(\mathbf{w})$ and $\frac{\partial \tau_i}{\partial w_i}(\mathbf{w})$ are continuous in the vector of payments $\mathbf{w}$ over the set $\mathbb{R}_+^n - \{0\}$.
- **A3** *It is symmetric in the buyer indices:* $\tau_i(\mathbf{w}) = \tau_{\sigma(i)}(\sigma(\mathbf{w}))$ for all permutations σ of the indices $i = 1, \ldots, n$
- **A4** *It is scale free*: for all real $\gamma > 0$ and $0 \leq i \leq n$, $\tau_i(\gamma \mathbf{w}) = \tau_i(\mathbf{w})$. [1]

One example of a mechanism in the class above is the proportional allocation (or uniform price mechanism) that divides the good so that each buyer gets a quantity proportional to the payment it made.

[1]Stated differently, the mechanism is independent of the units in which the payments are made.

The buyers each have a value function $U_i(x_i)$ for the amount of the good they are allocated. A value function is said to be *valid* if it is differentiable, concave and strictly increasing, with $U_i(0) = 0$. This paper deals only with valid value functions.

The aggregate value (or social welfare) of an allocation $\mathbf{x}$ is the sum of the individual values. For a set of value functions, an allocation $\mathbf{x}^*$ is *efficient* if it has the largest total value:

$$\sum_i U_i(x_i^*) \geq \sum_i U_i(x_i) \quad \forall \ \mathbf{x}$$

Whether an allocation is efficient depends only on the functions U_i and not on τ. The *efficiency* of any allocation $\mathbf{x}$ is defined to be the fraction

$$\frac{\sum_i U_i(x_i)}{\sum_i U_i(x_i^*)}$$

By definition the efficiency lies between 0 and 1.

Given the payments $\mathbf{w}$, the profit P_i of each buyer is the value derived from the allocation minus the payment made:

$$P_i(w_i, \mathbf{w}_{-i}) = U_i(\tau_i(\mathbf{w})) - w_i$$

Note that P_i is concave in w_i for every fixed value of the other amounts $\mathbf{w}_{-i}$.

The buyers play a non-zero-sum non-cooperative game with strategy variables w_i and payoff (or reward) functions P_i respectively. A vector of payments $\widetilde{\mathbf{w}}$ is said to be a Nash equilibrium if individual deviations (changes in payment made) cannot help a buyer:

$$P_i(\widetilde{w}_i, \widetilde{\mathbf{w}}_{-i}) \geq P_i(w_i, \widetilde{\mathbf{w}}_i) \quad \forall \, i, \ w_i \geq 0$$

For a vector $\widetilde{\mathbf{w}}$ to be an Nash equilibrium, i.e. for $\widetilde{\mathbf{w}} \in \mathcal{N}(\tau, \{U_i\})$ the necessary and sufficient conditions are that $\widetilde{w}_i > 0$ for at least two buyers i and

$$U_i'(\tau_i(\widetilde{\mathbf{w}})) \left(\frac{\partial \tau_i}{\partial w_i}(\widetilde{\mathbf{w}}) \right) - 1 \begin{cases} = 0 & \text{if } \widetilde{w}_i > 0 \\ \leq 0 & \text{if } \widetilde{w}_i = 0 \end{cases} \tag{2}$$

for all $i = 1, \ldots, n$.

It can be shown that Nash equilibria exist with the assumptions stated. The way to do this is to consider, for the given set of value functions U_i, the ϵ-game where each of the buyers are forced to bid at least ϵ. Also, the payments would never exceed the value obtained from having the entire unit of the resource, so we can assume that $w_i \leq U_i(1)$. Then, by Theorem

1 in Rosen [3] there exists an NEP $\widetilde{w}_\epsilon$ for the ϵ-game. A sequence $\epsilon_n \to 0$ will have a sequence of corresponding equilibria $\widetilde{w}_{\epsilon_n}$. Consider a convergent subsequence in the sequence of equilibria, and let $\widetilde{w}$ be its limit point. Then it can be shown that $\widetilde{w}$ will satisfy the necessary and sufficient conditions as stated above. [2]

Given a set of value functions $\{U_i\}$ and valid allocation mechanism τ, let $\mathcal{N}(\tau, \{U_i\})$ be the set of Nash equilibria and let $\mathbf{x}^*$ be an efficient allocation. Then, the worst case efficiency when the good is allocated according to τ is given by

$$\inf_{\{U_i\}} \inf_{\widetilde{\mathbf{w}} \in \mathcal{N}} \frac{\sum_i U_i(\tau_i(\widetilde{\mathbf{w}}))}{\sum_i U_i(x_i^*)}$$

The worst case is taken over a very broad set of buyer preferences and numbers, and indeed many allocation mechanisms are likely to have zero worst-case efficiency. Stated in these terms, [1] proves that the worst case efficiency of the proportional allocation mechanism is $\frac{3}{4}$.

Our objective is to find the allocation mechanism with the highest worst-case efficiency, in the class of valid mechanisms (i.e. those that satisfy assumptions **A1-4**). We also want to find the value of this best possible worst case efficiency:

$$\sup_{\tau} \inf_{\{U_i\}} \inf_{\widetilde{\mathbf{w}} \in \mathcal{N}} \frac{\sum_i U_i(\tau_i(\widetilde{\mathbf{w}}))}{\sum_i U_i(x_i^*)} \tag{3}$$

III. Reduction to Linear Value Functions

Following [1], which considered only the proportional allocation mechanism, it is shown in this section that for the purposes of evaluating the worst case performance of any valid allocation it is sufficient to assume that the buyers have linear value functions. This is crucial to the analysis that follows, for two reasons: it simplifies the space over which the infimum has to be taken, and the efficient allocation has the simple form of giving all to the buyer with the highest slope. Note that the results in this section do not use the scale-free assumption **A4** and have no restrictions on the number of buyers except that it be finite.

Lemma 1 is reproduced from [1], where it appears as Lemma 4. The reader is referred to the original paper for a proof.

[2]If assumption **A2** does not hold it is possible to construct examples where there are no Nash equilibria.

Lemma 1 (Johari and Tsitsiklis [1]): For any value functions $\{U_i\}$ if an efficient allocation is $\mathbf{x}^*$ and $\mathbf{x}$ is any other allocation, the following inequality holds:

$$\frac{\sum_i U_i(x_i)}{\sum_i U_i(x_i^*)} \geq \frac{\sum_i U_i'(x_i)x_i}{\max_i U_i'(x_i)}$$

As in [1], Lemma 1 implies that considering only linear value functions for the buyers is sufficient.

Proposition 1: For every τ satisfying **A1-3**, valid value functions $\{U_i\}$ and $\widetilde{\mathbf{w}} \in \mathcal{N}(\tau, \{U_i\})$ there exist linear value functions $\widehat{U}_i(x_i) = \alpha_i x_i$ such that $\widetilde{\mathbf{w}} \in \mathcal{N}(\tau, \{\widehat{U}_i\}) \triangleq \mathcal{N}(\tau, \alpha)$ and

$$\frac{\sum_i U_i(\tau_i(\widetilde{\mathbf{w}}))}{\sum_i U_i(x_i^*)} \geq \frac{\sum_i \alpha_i \tau_i(\widetilde{\mathbf{w}})}{\max_i \alpha_i} \tag{4}$$

where $\mathbf{x}^*$ is an efficient allocation for $\{U_i\}$.

Proof: Since $\widetilde{\mathbf{w}} \in \mathcal{N}(\tau, \{U_i\})$, the vector $\widetilde{\mathbf{w}}$ satisfies the conditions (2). Consider linear value functions defined by

$$\widehat{U}_i(x_i) = \alpha_i x_i = U_i'(\tau_i(\widetilde{\mathbf{w}}))x_i$$

It is easy to see that $\widetilde{\mathbf{w}} \in \mathcal{N}(\tau, \alpha)$ by checking that the conditions (2) hold for the new linear value functions $\alpha_i x_i$. Equation (4) is exactly the same as the statement of Lemma 1 with $x_i = \tau_i(\widetilde{\mathbf{w}})$ and writing $U_i'(x_i) = \alpha_i$ in the RHS. □

Note that the LHS of (4) is the efficiency of a Nash equilibrium for $\tau, \{U_i\}$ and the RHS is the efficiency of the same Nash equilibrium for τ, α (since for linear value functions an efficient allocation gives all of the good to the buyer with the biggest slope). Thus Proposition 1 says that the efficiency of a particular Nash equilibrium pair is lowest when the corresponding value functions are linear. Thus, for the infimum in the objective function (3) it is sufficient to consider linear value functions for both the users.

If $\alpha_i x_i$ are the value functions of the buyers and $\mathcal{N}(\tau, \alpha)$ is the set of Nash equilibria for these under allocation mechanism τ then the objective function (3) can be rewritten as

$$\sup_{\tau} \inf_{\alpha} \inf_{\widetilde{\mathbf{w}} \in \mathcal{N}(\tau,\alpha)} \frac{\sum_i \alpha_i \tau_i(\widetilde{\mathbf{w}})}{\max_i \alpha_i} \tag{5}$$

The rest of the analysis in this paper deals with the objective function given above.

IV. Two Buyers

Consider first the case of there being only two buyers. Since assumption **A3** asks for invariance with respect to permutation of indices, it is sufficient for the two-buyer case to specify a mechanism by the amount it allocates to the higher buyer (who makes the larger payment) and to the lower buyer.

Let w_l be the lower payment made and w_h be the higher payment, i.e. $w_l \leq w_h$ and let $\tau^*(w_l, w_h)$ be the allocation mechanism given by

$$\tau_l^* = \frac{w_l}{2w_h} \quad \text{and} \quad \tau_h^* = 1 - \frac{w_l}{2w_h} \tag{6}$$

where τ_l^* is the allocation to the lower buyer and τ_h^* is to the higher buyer. In this section we will prove that this is indeed the optimal valid mechanism: it has the highest worst-case efficiency.

The price (payment made for unit quantity received) each buyer pays is different, making this a discriminatory price mechanism with the prices determined by the willingness of each buyer to pay. The mechanism has a "volume discount": a higher buyer pays a lower price, and so has an incentive to bid high and - as a result - get a higher allocation. This discount partially offsets the effect that a high buyer tends to bid up the price and work against himself, causing a strategic high buyer to buy less quantity than is efficient.

The following lemma is easily verified, and we omit the proof.

Lemma 2: τ^* is a valid allocation mechanism.

The theorem that follows shows that this is the optimal valid two-buyer mechanism. First though we need a proposition:

Proposition 2: For any two-buyer valid allocation mechanism τ there exists a function $\phi : [0,1] \to [0,1]$ such that

$$\tau_l(w_l, w_h) = \phi\left(\frac{w_l}{w_h}\right) \text{ and } \tau_h(w_l, w_h) = 1 - \phi\left(\frac{w_l}{w_h}\right)$$

whenever $w_h > 0$. We say that the allocation τ is *based on* ϕ. The function ϕ further satisfies the following properties:

- **B1** $0 \leq \phi(v) \leq 1$ for all $v \in [0,1]$, with $\phi(0) = 0$ and $\phi(1) = \frac{1}{2}$
- **B2** $\phi(v)$ is differentiable, increasing and concave in v for all $v \in [0,1]$.

Proof: Given a valid allocation τ, define $\phi(v) \triangleq \tau_l(v, 1)$, i.e. the allocation to the lower buyer when the lower payment is v and the higher is 1. Then, it is easy to see that when $w_h > 0$,

$$\tau_l(w_l, w_h) = \tau_l\left(\frac{w_l}{w_h}, 1\right) = \phi\left(\frac{w_l}{w_h}\right)$$

where the first equality follows from the scale-free assumption **A4** with $\gamma = \frac{1}{w_h}$. This and **A1** immediately imply that $\tau_h(w_l, w_h) = 1 - \phi(\frac{w_l}{w_h})$ and that $0 \leq \phi(v) \leq 1$. Also by **A1**, when the lower payment is 0 the lower allocation is 0, and this gives us $\phi(0) = 0$. When both payments are equal and non-zero, say $w_l = w_h = w > 0$, the symmetry assumption **A3** implies that $\tau_l(w, w) = \frac{1}{2}$ and hence $\phi(1) = \frac{1}{2}$. Thus ϕ satisfies **B1**. The assumption **A2** that τ_l be increasing, concave and differentiable in w_l when $w_h > 0$ implies that **B2** holds for ϕ. Thus the proposition is proved. □

Notice that the converse does not hold: not all functions ϕ satisfying **B1, B2** correspond to valid allocation mechanisms, since **B1, B2** do not address the issue of concavity and differentiability in one of the payments w_1 at the point of the other payment w_2, i.e. in the region where $w_1 < w_2$ becomes $w_1 > w_2$. Maximizing the worst case efficiency over the space of functions satisfying **B1, B2** thus yields an upper bound on the efficiency of valid allocations:

$$\sup_{\tau} \inf_{\alpha} \inf_{\tilde{\mathbf{w}} \in \mathcal{N}(\tau,\alpha)} \frac{\sum_i \alpha_i \tau_i(\tilde{\mathbf{w}})}{\max_i \alpha_i}$$

$$\leq \sup_{\phi} \inf_{\alpha} \inf_{\tilde{\mathbf{w}} \in \mathcal{N}(\phi,\alpha)} \frac{\sum_i \alpha_i \phi_i(\tilde{\mathbf{w}})}{\max_i \alpha_i}$$

where ϕ_i is the allocation to buyer i from an allocation based on function ϕ. The following theorem evaluates the right hand side for two users.

Theorem 1: When two buyers are present, the worst case efficiency of τ^* is $\frac{7}{8}$, and no valid mechanism can achieve a higher ratio.

Proof: We will first prove that $\frac{7}{8}$ represents an upper bound, and then show achievability.

By the reasoning in Section III, for the purposes of worst case analysis it is sufficient to consider linear value functions with slopes α_l and α_h for the two users, both being strictly positive. In light of Proposition 2 above we will consider functions ϕ satisfying **B1-2**. If $(\tilde{w}_l, \tilde{w}_h) \in \mathcal{N}(\phi, \alpha)$, then the Nash equilibrium conditions of (2) give

$$\frac{1}{\widetilde{w}_h}\phi'\left(\frac{\widetilde{w}_l}{\widetilde{w}_h}\right) = \frac{1}{\alpha_l} \quad \text{and} \quad \frac{\widetilde{w}_l}{\widetilde{w}_h^2}\phi'\left(\frac{\widetilde{w}_l}{\widetilde{w}_h}\right) = \frac{1}{\alpha_h}$$

The above equations imply that for a given pair of linear value functions there will be unique Nash equilibrium $(\widetilde{w}_l, \widetilde{w}_h)$ and combining the two equations above gives $\frac{\alpha_l}{\alpha_h} = \frac{\widetilde{w}_l}{\widetilde{w}_h}$. The efficiency of the unique Nash equilibrium for the given slopes is thus

$$E_\alpha = \frac{\alpha_l \phi\left(\frac{\widetilde{w}_l}{\widetilde{w}_h}\right) + \alpha_h\left(1 - \phi\left(\frac{\widetilde{w}_l}{\widetilde{w}_h}\right)\right)}{\alpha_h}$$

Minimizing this over all α such that $\alpha_l \leq \alpha_h$ will give the worst case performance of ϕ, and maximizing this over all ϕ will give the upper bound. Denoting $v = \frac{\alpha_l}{\alpha_h}$, the upper bound is given by

$$\sup_{\phi} \min_{0 \leq v \leq 1} v\phi(v) + 1 - \phi(v)$$

The function $\phi^*(v) = \frac{v}{2}$ satisfies **B1-2** and has $\phi(v) \leq \phi^*(v)$ for any other ϕ that also satisfies **B1-2**. Thus it achieves the sup above, and the upper bound can be evaluated:

$$\min_{0 \leq v \leq 1} \frac{v^2}{2} + 1 - \frac{v}{2} = \frac{7}{8} \tag{7}$$

where the minimum is achieved at $v = \frac{1}{2}$. Thus $\frac{7}{8}$ represents an upper bound on the worst case performance of valid mechanisms.

For achievability, note that τ^* is the mechanism based on $\phi^*(v) = \frac{v}{2}$ (in the sense of Proposition 2), and Lemma 2 guarantees that it is valid. All that is needed now is a direct verification of the worst case performance.

As before, if α_l and α_h are the slopes of the value functions of the buyers, then there will be a unique Nash equilibrium $(\widetilde{w}_l, \widetilde{w}_h)$ and furthermore we have that $\frac{\alpha_l}{\alpha_h} = \frac{\widetilde{w}_l}{\widetilde{w}_h}$. Thus the worst-case efficiency of a Nash equilibrium for τ^* is given by

$$\min_{\alpha_l \leq \alpha_h} \frac{\alpha_l}{\alpha_h}\left(\frac{\alpha_l}{2\alpha_h}\right) + 1 - \frac{\alpha_l}{2\alpha_h}$$

Denoting $v = \frac{\alpha_l}{\alpha_h}$ makes this exactly the same as the LHS of equation (7), and hence the worst case efficiency is indeed $\frac{7}{8}$. $\square$

This worst case occurs when the slope of the value function of one buyer is half that of the other.

Unique Equilibria in τ^*

While the existence of Nash equilibria is guaranteed, uniqueness is not: it is possible that more than one Nash equilibria exist for a given set of value functions of the buyers and an allocation mechanism. There are many ways in which the notion of an equilibrium in a game can be refined to "choose" one of them. It would however still be of interest to see if given an allocation mechanism τ there is always a unique Nash equilibrium for any set of buyer value functions.

The proof of Theorem 1 indicates that allocating according to τ^* results in a unique Nash equilibrium when the buyers have linear value functions. The following theorem proves that this is the case for *any* pair of buyer value functions.

Theorem 2: For the case when there are only two buyers, the allocation mechanism τ^* results in a unique Nash equilibrium for any pair of valid buyer value functions U_1, U_2.

Proof: Let 1 and 2 be the two buyers, and $\mathcal{N}(\tau^*, U_1, U_2)$ be the set of Nash equilibria when allocation is done according to τ^*. Define two types of Nash equilibria:

Type I: $(\widetilde{w}_1, \widetilde{w}_2) \in \mathcal{N}(\tau^*, U_1, U_2)$ with $\widetilde{w}_1 \leq \widetilde{w}_2$
Type II: $(\widetilde{w}_1, \widetilde{w}_2) \in \mathcal{N}(\tau^*, U_1, U_2)$ with $\widetilde{w}_1 \geq \widetilde{w}_2$

Note that an equilibrium will be of both types iff $\widetilde{w}_1 = \widetilde{w}_2$. Any Nash equilibrium has to satisfy the condition given by (2). For τ^* allocation and Type I equilibria this can be rewritten as $\widetilde{w}_1 > 0$, $\widetilde{w}_2 > 0$ and

$$\frac{1}{2}U_1'\left(\frac{\widetilde{w}_1}{2\widetilde{w}_2}\right) = \frac{\widetilde{w}_1}{2\widetilde{w}_2}U_2'\left(1 - \frac{\widetilde{w}_1}{2\widetilde{w}_2}\right) = \widetilde{w}_2$$

From $U_2(x)$ define $\widehat{U}_2(x)$ as follows:

$$\widehat{U}_2(0) = 0 \quad \text{and} \quad \widehat{U}_2'(x) = U_2'(x)(1 - x)$$

Note that $\widehat{U}_2$ is strictly concave for all valid U_2. Also, if we define $\widetilde{x} = \frac{\widetilde{w}_1}{2\widetilde{w}_2}$ then we have that

$$\widetilde{x} = \arg\max_{0 \leq x \leq 1} \frac{U_1(x)}{2} + \widehat{U}_2(1 - x)$$

The above function is strictly concave in x, hence there exists a unique $\widetilde{x}$ maximizing it. For there to exist Type I equilibria the maximum above should be achieved with $\widetilde{x} \leq \frac{1}{2}$: if this is not the case there will be no Type I equilibrium. This and the fact that $\widetilde{w}_2 = \frac{1}{2}U_1'\left(\frac{\widetilde{w}_1}{2\widetilde{w}_2}\right)$ means that there can be at most one

Type I equilibrium. By similar reasoning there can be at most one Type II equilibrium. Also, at least one Nash equilibrium point exists. Thus all we need to prove is that if there is an equilibrium of each type, then these two equilibria are the same point.

To do this, define $\widehat{U}_1$ from U_1 in the same way that $\widehat{U}_2$ was defined using U_2, and let

$$\widetilde{y} \;=\; \arg\max_{0\le y\le 1} \widehat{U}_1(1-y) + \frac{U_2(y)}{2}$$

Suppose equilibria of both types exist. Then, from the definitions of $\widetilde{x}$ and $\widetilde{y}$ we have that

$$\frac{U_1'(\widetilde{x})}{2} \;=\; \widetilde{x}\,U_2'(1-\widetilde{x}) \tag{8}$$

$$\widetilde{y}\,U_1'(1-\widetilde{y}) \;=\; \frac{U_2'(\widetilde{y})}{2} \tag{9}$$

Also, we should have that $\widetilde{x} \le \frac{1}{2}$ and $\widetilde{y} \le \frac{1}{2}$. This gives us

$$\widetilde{y}\,U_1'(1-\widetilde{y}) \;\le\; \widetilde{y}\,U_1'\left(\frac{1}{2}\right) \;\le\; \frac{1}{2}\,U_1'\left(\frac{1}{2}\right) \;\le\; \frac{U_1'(\widetilde{x})}{2} \tag{10}$$

$$\widetilde{x}\,U_2'(1-\widetilde{x}) \;\le\; \widetilde{x}\,U_2'\left(\frac{1}{2}\right) \;\le\; \frac{1}{2}\,U_2'\left(\frac{1}{2}\right) \;\le\; \frac{U_2'(\widetilde{y})}{2} \tag{11}$$

Together (8)-(11) give us that $\widetilde{x} = \widetilde{y} = \frac{1}{2}$, which for given U_i means that there is only one NEP given by

$$\widetilde{w_1} = \widetilde{w_2} = \frac{1}{2}U_1'\left(\frac{1}{2}\right) = \frac{1}{2}U_2'\left(\frac{1}{2}\right)$$

Thus, there is a unique equilibrium for the two-player game when allocation is done according to τ^*. $\square$

The above technique shows the uniqueness of Nash equilibria by formulating an equivalent optimization problem. Similar methodologies have been used before in [2] and also in [1].

V. n Buyers

The optimal valid mechanism for two buyers has a natural extension to the case when there are n buyers. Specifically, we saw that the optimal two-buyer mechanism was the one that gave a higher buyer a lower price as compared to a lower buyer, and these prices were determined by the total amounts each buyer was willing to pay.

In this section we will omit the proofs, since they use essentially the same ideas as in the two-buyer case but are more technical.

Given a payment vector $\mathbf{w}$ let w_{max} denote the maximum payment, and consider the allocation mechanism τ^* that allocates to each buyer i according to

$$\tau_i^* \;=\; \frac{w_i}{w_{max}} \int_{s=0}^{1} \prod_{j\ne i} \left(1 - s\frac{w_j}{w_{max}}\right) ds$$

if $w_{max} > 0$ and $\tau_i^* = 0$ for all i if $w_{max} = 0$. The value of the empty product is taken to be 1.

Note that for $n = 2$ it simplifies to the two-buyer τ^* of the previous section. It is easy to see that this allocation is symmetric and scale-free, i.e. that it satisfies **A3** and **A4**. Verifying **A1** and **A2** is technical but straightforward, we do not include the proof here but state the result below.

Lemma 3: The mechanism τ^* as defined above is valid, i.e. it satisfies assumptions **A1-4**.

Following the method of Theorem 2 above, we can derive an expression for the worst-case efficiency of τ^*. Unfortunately we have not been able to evaluate it analytically for general n:

Theorem 3: The worst case efficiency E_n of τ^* is given by

$$\begin{aligned} E_n \;=\; \min\nolimits_{\mathbf{v}\in[0,1]^{n-1}} \quad & \left(1 + \sum_{i=1}^{n-1} v_i\right) \int_0^1 \prod_{i=1}^{n-1} (1 - s v_i)\, ds \\ & - \left(\sum_{i=1}^{n-1} v_i\right) \prod_{i=1}^{n-1} (1 - v_i) \end{aligned} \tag{13}$$

It can be shown analytically that for $n = 2$ the above gives a value of $\frac{7}{8}$, and for $n = 3$ it gives 0.8737. This compares favorably with the proportional allocation, which was shown in [1] to have a worst case efficiency of $2(\sqrt{2} - 1) \approx 0.8284$ when there are two buyers.

The minimum for $n \ge 4$ is still unsolved analytically, but numerical computations suggest (counter-intuitively) that the worst case over all n occurs when $n = 4$, i.e. there are only 4 users with non-zero payments. The value then is 0.8735. We state this as a conjecture:

Conjecture 1: With efficiency E_n as defined above, the worst case efficiency occurs when $n = 4$:

$$\inf_{n\ge 2} E_n \;=\; E_4 \;=\; 0.8735$$

If true, the above conjecture means that τ^* compares quite favorably to the proportional allocation: not only is the worst case ratio for τ^* higher than the $\frac{3}{4}$ for the proportional allocation, there is also no degradation as the number of buyers grows.

An interesting property of τ^* is that it is the unique "linear in the lower bids" mechanism:

Lemma 4: If τ is an allocation mechanism satisfying assumptions **A1-3** and also has the following property

L *Linearity:* $\tau_i(\mathbf{w})$ is linear in w_i in the region $0 \leq w_i \leq \max_{j \neq i} w_j$, when $\max_{j \neq i} w_j > 0$,

then $\tau = \tau^*$.

It is obvious from (12) that property **L** holds for τ^*. Property **L** has an interesting feature when the value functions of the lower buyers are linear: at Nash equilibrium the payment of each lower buyers equals the it obtains from the allocation.

VI. Conclusion and Extensions

This paper proposes a mechanism for allocating a good to buyers based only on the payments they make, so that the loss in value created due to strategizing is minimized. The mechanism gives the lowest price to the highest buyer, thus giving an incentive for buyers who value the good more to bid higher. This results in an increase in efficiency.

It is however not yet a complete piece of work. The most immediate task is of course to prove (or disprove) the optimality of τ^* for $n \geq 3$, and evaluate the worst case value given by (13). A second task is to determine whether Theorem 1 is still true if assumption **A4** is dropped. A third task is to see if the τ^* allocation rule with n buyers has a unique Nash equilibrium for general buyer value functions, possibly along the lines of the techniques used for the two-buyer case.

Notice that τ^* is a discriminatory price auction in which a buyer with a larger payment pays a strictly lower price than one with a strictly lower payment. This would suggest that the mechanism is immune to *strategic splitting*, wherein one buyer enters the game as multiple buyers who collaborate. The pricing suggests that it would be in its best interests to avoid splitting, as for the same amount of total payment made it would receive the highest quantity of resource if it made a single large payment. On the other hand though, this reasoning suggests that buyers will stand to gain by cooperating. These notions have to be formalized.

Another interesting (and obvious) feature of the general allocation τ^* with n buyers is its *hierarchical property*: if m of the n buyers submit a payment of 0 (or are "phantom"), the resulting allocation would be the same as it would have been if the $n-m$ "serious" buyers were allocated quantities according to the τ^* with $n-m$ buyers.

Extending the above sort of analysis to multiple inter-related goods (like e.g. capacities on links in a network) is also of interest. Specifically, it would be interesting to see if the worst case buyer value functions turn out to be linear in that case as well.

Besides this, another question in the single-good scenario is one of revenue maximization for the seller. A similar fractional formulation for the problem of either worst-case or best-case revenue loss due to strategizing can also be made, and allocation mechanisms for minimizing this can be found. Again, it would be of interest to see if the extremal value functions in this scenario are linear.

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