How to Integrate Rational Functions

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The increasing availability of computer algebra systems has raised questions about how traditional topics in calculus are to be taught. In this note we look at integration of rational functions and propose a different approach, which has the following advantages: i) it is easily implemented on a computer or calculator algebra system, ii) it allows the students to use the computer algebra system in a meaningful way, and avoids routine calculations by hand, iii) it provides the students with some understanding of the general methods computer algebra systems actually use to integrate rational functions.

Rational function integration is important for itself and also because many integrals can be reduced to it by suitable substitutions, for example many trigonometric integrals and the so-called binomial integral [Subramaniam, Klambauer]. A rational function is traditionally integrated by expressing it in partial fractions form. This involves the following steps:

(1) Factor the denominator into linear and irreducible quadratic factors.

(2) Find the partial fraction decomposition. This involves solving a system of linear equations, with as many equations and unknowns as the degree of the polynomial in the denominator.

(3) Integrate each partial fraction. Those involving a quadratic factor require a trigonometric substitution or a reduction formula.

In the light of this recipe, consider the following integrals (of which the second and third are taken from our references):

$$\int \frac{8x^5 - 10x^4 + 5}{(2x^5 - 10x + 5)^2}\,dx$$

$$\int \frac{4x^5 - 1}{(x^5 + x + 1)^2}\,dx$$

$$\int \frac{4x^4 + 4x^3 + 16x^2 + 12x + 8}{x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1}\,dx$$

$$\int \frac{dx}{x^7 + 1}.$$

Each of these—as we shall see—has a simple antiderivative. However, in the first, the denominator is not solvable by radicals [Hungerford] and we cannot even get started, except by using numerical approximations to the roots. In the second and third the denominators factorize over the integers though this is not obvious. In the fourth the roots of the denominator are the seventh roots of unity. The partial fractions computation is quite involved. In these problems, even after factorization there is a great deal of algebra and integration left to do. Clearly this method can be quite tedious, if not impossible.

Recall that the integral of a rational function is the sum of a rational function together with a sum of logarithms and arctangents of polynomials. These are called respectively the rational and the transcendental parts of the integral. In this note we show how the rational part can be found without any integration, even when the factorization of the denominator is not known. All that is needed is the ability to calculate the g.c.d. of polynomials and to solve systems of linear equations. The algorithm is simple enough to work on any computer algebra system, even an HP-28S calculator. (In an appendix we briefly consider the HP-28S implementation.) We also consider to what extent the transcendental part can be determined. What follows is scattered between classical sources and the computer algebra literature and does not appear to be well known. We feel it is useful to write down an elementary and coherent account. Our bibliography should be consulted for a deeper study.

I. In what follows, P/Q is a rational function over the rationals; we assume that the leading coefficient of Q is one. We begin by proving the following proposition (the Hermite-Ostrogradski formula). Our proof is simpler than the ones we have been able to find (exemplified by [Davenport et al.] and [Klambauer]). We avoid the use of partial fractions.

Proposition 1. *Let P/Q be a rational function. Let $Q = \Pi_{i=1}^{n} h_i^{\alpha_i}$ be the factorization of Q into linear and irreducible quadratic factors, and let $Q_1 = \Pi_{i=1}^{n} h_i^{\alpha_i - 1}$ and $Q_2 = \Pi_{i=1}^{n} h_i$. Then there are polynomials P_1 and P_2 such that*

$$\int \frac{P(x)}{Q(x)}\,dx = \frac{P_1(x)}{Q_1(x)} + \int \frac{P_2(x)}{Q_2(x)}\,dx. \tag{1}$$

Note that the proposition says what is intuitively clear—in a partial fractions decomposition, the repeated factors of the denominator give us the rational part and the factors without repetition give us the transcendental part.

We prove this proposition by considering two cases. The first case is when Q has only one distinct irreducible factor: either $Q(x) = (x-c)^m$ or $Q(x) = (x^2 + ax + b)^m$ where the quadratic is irreducible and $m \geq 1$.

If $Q(x) = (x-c)^m$ then our integral is

$$\int \frac{P(x)}{(x-c)^m}\,dx,$$

where $P(x)$ is a polynomial. Write $P(x) = \Sigma_{k=0}^{n} a_k (x-c)^k$. Then

$$\int \frac{P(x)}{(x-c)^m}\,dx = \sum_{k=-m}^{n-m} a_{k+m} \int (x-c)^k\,dx.$$

If we integrate all the terms except the one for which $k = -1$, we get the desired equation (1), since $Q_2 = x - c$ and the integrated terms have the common denominator $Q_1 = (x-c)^{m-1}$.

If $Q(x) = (x^2 + ax + b)^m$, we can essentially do the same thing, but it becomes slightly more complicated. This is the price we pay for avoiding complex arithmetic. Divide $P(x)$ by $Q_2(x) = x^2 + ax + b$: $P(x) = R(x)Q_2(x) + S(x)$, where $S(x)$ is linear. It follows that

$$\int \frac{P(x)}{Q(x)}\,dx = \int \frac{R(x)}{Q_2(x)^{m-1}}\,dx + \int \frac{S(x)}{Q_2(x)^m}\,dx.$$

There is a standard reduction formula (easily obtained by integration by parts) of the form

$$\int \frac{Ax+B}{(x^2+ax+b)^m}\,dx = \frac{M(x)}{(x^2+ax+b)^{m-1}} + \int \frac{N}{(x^2+ax+b)^{m-1}}\,dx,$$

where $M(x)$ is a linear polynomial and N is constant. This formula, applied repeatedly, yields

$$\int \frac{Ax+B}{(x^2+ax+b)^m}\,dx = \frac{M(x)}{(x^2+ax+b)^{m-1}} + \int \frac{N}{x^2+ax+b}\,dx,$$

where now $M(x)$ is a polynomial and N is a constant. From this formula we obtain

$$\int \frac{P(x)}{Q(x)}\,dx = \int \frac{R(x)}{Q_2(x)^{m-1}}\,dx + \frac{M(x)}{Q_2(x)^{m-1}} + \int \frac{N}{Q_2(x)}\,dx.$$

The process can be repeated on $\int R(x)/Q_2(x)^{m-1}\,dx$; this will ultimately lead to the equation (1), since $Q_2(x) = (x^2+ax+b)$ and $Q_1(x) = (x^2+ax+b)^{m-1}$.

In the second case, when Q has at least two distinct irreducible factors, we proceed by induction on the number k of distinct irreducible factors. Accordingly, assume that (1) holds for $k < K$ $(K > 1)$. Let $Q(x)$ have K distinct irreducible factors and let $Q(x) = \prod_{i=1}^{K} h_i^{\alpha_i}$ be the irreducible factorization of Q. Since h_1 and $\prod_{i=2}^{K} h_i(x)^{\alpha_i} = g(x)$ are relatively prime, by the Euclidean algorithm for polynomials there are polynomials $a(x)$ and $b(x)$ for which

$$P(x) = a(x)h_1(x)^{\alpha_1} + b(x)g(x).$$

Then

$$\int \frac{P(x)}{Q(x)}\,dx = \int \frac{a(x)}{g(x)}\,dx + \int \frac{b(x)}{h_1(x)^{\alpha_1}}\,dx.$$

By the inductive hypothesis and the first case, each integral on the right can be expressed in the form (1). If we write them that way and collect terms, we have the formula (1) for $\int P/Q$. The proof is complete.

We remark that if degree $P <$ degree Q then P_1 and P_2 can be found with degree $P_1 <$ degree Q_1 and degree $P_2 <$ degree Q_2. Indeed, if degree $P_2 \geq$ degree Q_2, divide P_2 by Q_2, integrate the polynomial quotient and absorb it into P_1/Q_1. Now if degree $P_1 =$ degree Q_1, then P_1/Q_1 is a constant plus a proper rational function, and the constant may be dropped from the equation. Finally, if degree $P_1 >$ degree Q_1, then P_1/Q_1 is a polynomial of degree at least one plus a proper rational function. But this is impossible, for then the limit at infinity of the derivative of the right hand side of (1) would not be zero. In fact, P_1 and P_2 are unique. We do not prove this since we don't need this fact. Finally, note that the last integral in (1) is a sum of logarithms and arctangents.

II. The real utility of the Hermite-Ostrogradski formula comes from the fact that it is possible to calculate P_1, P_2, Q_1, and Q_2 without factorizing Q (see [Horowitz] or [Klambauer].) We now show how this can be done.

It is clear that $Q_1 = \text{g.c.d.}(Q, Q')$ and $Q_2 = Q/Q_1$. Also it is easy to see that Q_1 divides $Q_1'Q_2$ whence $S = Q_1'Q_2/Q_1$ is a polynomial. If we differentiate both sides of (1) we get

$$P/Q = \frac{Q_1P_1' - P_1Q_1'}{Q_1^2} + \frac{P_2}{Q_2} = \frac{P_1' - P_1Q_1'/Q_1}{Q_1} + \frac{P_2}{Q_2}.$$

Clearing the denominators we have $P = P_1'Q_2 - P_1S + P_2Q_1$.

In this equation, P, Q_1, Q_2, and S are known polynomials and we can solve for P_1 and P_2 by the method of undetermined coefficients. Here then is the algorithm:

Input: Polynomials P and Q, with degree $P <$ degree Q.

Output: P_1/Q_1, the rational part of $\int P/Q$, and P_2/Q_2, the integrand of the transcendental part of $\int P/Q$.

(1) $Q_1 :=$ g.c.d.(Q, Q'); $Q_2 := Q/Q_1$
(2) $S := Q_1'Q_2/Q_1$
(3) $q :=$ degree Q_1; $p :=$ degree Q_2.
(4) Write $P_1(x) := A_{q-1}x^{q-1} + A_{q-2}x^{q-2} + \cdots + A_0$ and $P_2(x) := B_{p-1}x^{p-1} + B_{p-2}x^{p-2} + \cdots + B_0$.
(5) Compute $T := P_1'Q_2 - P_1S + P_2Q_1$.
(6) Equate the coefficients of T with those of P.
(7) Solve this linear system of equations for the unknowns A_i and B_i.

If deg $Q = d$, then in step 7 we solve a system of d equations in d unknowns, which is the same amount of work as in the method of partial fractions, except that now there is no integration left to do for the rational part. The algorithm involves only polynomial arithmetic and solving systems of linear equations. We illustrate by an example (which was done on the HP-28S). It is example #3 of the introduction.

Example:

$$\int \frac{4x^4 + 4x^3 + 16x^2 + 12x + 8}{x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1}\,dx.$$

$$\begin{aligned}
Q_1 &= g.c.d.(x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1,\\
&\qquad 6x^5 + 10x^4 + 12x^3 + 12x^2 + 6x + 2)\\
&= x^3 + x^2 + x + 1\\
Q_2 &= Q/Q_1 = x^3 + x^2 + x + 1\\
P_1 &= Ax^2 + Bx + C, \qquad P_2 = Dx^2 + Ex + F\\
T &= P_1'Q_2 - P_1S + P_2Q_1\\
&= Dx^5 + (-A + D + E)x^4 + (-2B + D + E + F)x^3\\
&\quad + (A - B - 3C + D + E + F)x^2 + (2A - 2C + E + F)x\\
&\quad + (B - C + F).
\end{aligned}$$

Equating coefficients with $P = 4x^4 + 4x^3 + 16x^2 + 12x + 8$ and solving the resulting system of equations for A, B, C, D, E, and F, we get the result

$$-\frac{x^2 - x + 4}{x^3 + x^2 + x + 1} + \int \frac{3x + 3}{x^3 + x^2 + x + 1}\,dx.$$

The last term is

$$3\int\frac{dx}{x^2+1}=3\tan^{-1}x.$$

Examples #1 and #2 of the introduction can be worked the same way. One finds that

$$\int\frac{8x^5-10x^4+5}{(2x^5-10x+5)^2}\,dx=\frac{1-x}{2x^5-10x+5}$$

and

$$\int\frac{4x^5-1}{(x^5+x+1)^2}\,dx=-\frac{x}{x^5+x+1}.$$

In a hand computation or when using a computer or calculator algebra system without a built-in g.c.d. function, the g.c.d. can be calculated using the algorithm of [Kung]. However, g.c.d. calculations can lead to a large increase in the size of the intermediate results, this is "intermediate expression swell" (see [Knuth] or [Collins]). See the Appendix for another way to determine the system of equations.

III. We have seen that the rational part and the integrand of the transcendental part can be found using only polynomial arithmetic and linear algebra. Since most of the computational complexity of the method of partial fractions comes from the repeated factors, this is a considerable simplification in that the denominator of the integral still to be evaluated is now square free. One could, of course, now use the method of partial fractions to evaluate this integral. We, however, show now that if the roots of the denominator are known, there is a closed formula for the transcendental part. In fact,

$$\int P(x)/Q(x)\,dx=\sum\frac{P(a)}{Q'(a)}\operatorname{Log}(x-a), \tag{2}$$

where the sum ranges over all the roots a of $Q(x)$ (including the complex ones). In this formula, we use the complex logarithm.

To establish this formula, note that we are assuming that Q has no repeated roots and we may assume degree $P <$ degree Q. Let a be a root of $Q(x)$, and write $Q(x)=(x-a)Q_1(x)$, with $Q_1(a)\neq 0$. (Note that we are using Q_1 with a different meaning now.) We wish to write

$$P/Q=\frac{A}{x-a}+\frac{P_1(x)}{Q_1(x)}$$

for a constant A and polynomial P_1 (again, we use P_1 with a different meaning). This is possible, for if we choose $A=P(a)/Q_1(a)$ then

$$\begin{aligned}P_1(x)&=Q_1(x)\left\{\frac{P(x)}{Q(x)}-\frac{P(a)}{Q_1(a)}\frac{1}{x-a}\right\}\\&=\frac{1}{x-a}\left\{P(x)-\frac{P(a)}{Q_1(a)}Q_1(x)\right\}\end{aligned} \tag{3}$$

Since $P(x)-(P(a)/Q_1(a))Q_1(x)$ has a as a root, $P_1(x)$ is a polynomial. Also $Q'(a)=Q_1(a)$, and thus we have

$$P(x)/Q(x)=\frac{P(a)/Q'(a)}{x-a}+\frac{P_1(x)}{Q_1(x)}.$$

We now establish that

$$\frac{P_1(b)}{Q_1'(b)} = \frac{P(b)}{Q'(b)}$$

for every root b of Q_1. First note that $Q'(x) = (x - a)Q_1'(x) + Q_1(x)$, so $Q'(b) = (b - a)Q_1'(b)$. Also $P_1(b) = P(b)/(b - a)$ by (3). It follows that $P(b)/Q'(b) = P_1(b)/Q_1'(b)$. We may now repeat our process, expressing

$$P_1(x)/Q_1(x) = \frac{P_1(b)/Q_1'(b)}{x - b} + \frac{P_2(x)}{Q_2(x)}$$

where $Q_1(x) = (x - b)Q_2(x)$ and P_2 is a polynomial. We now have

$$P(x)/Q(x) = \frac{P(a)/Q'(a)}{x - a} + \frac{P(b)/Q'(b)}{x - b} + \frac{P_2(x)}{Q_2(x)}$$

with

$$\frac{P_2(c)}{Q_2'(c)} = \frac{P(c)}{Q'(c)}$$

for every root c of $Q_2(x)$. Since the degrees of the polynomials $P(x), P_1(x), P_2(x), \dots$ strictly decrease, we eventually arrive at the formula

$$P(x)/Q(x) = \sum_{a \mid Q(a)=0} \frac{P(a)/Q'(a)}{x - a}.$$

If we integrate each term we obtain the formula (2). However, if $P(x)$ and $Q(x)$ are real polynomials, the complex roots of $Q(x)$ come in conjugate pairs and a real formula for $\int P(x)/Q(x)\,dx$ can be obtained as follows.

Let $\bar{a}$ and a be a complex conjugate pair of roots of $Q(x)$.

If $P(a)/Q'(a) = c + id$, then $P(\bar{a})/Q'(\bar{a}) = c - id$, and

$$\begin{aligned}
&\frac{P(a)}{Q'(a)}\frac{1}{x - a} + \frac{P(\bar{a})}{Q'(\bar{a})}\frac{1}{x - \bar{a}} \\
&\quad = c\left(\frac{1}{x - a} + \frac{1}{x - \bar{a}}\right) + id\left(\frac{1}{x - a} - \frac{1}{x - \bar{a}}\right) \\
&\quad = c\frac{2x - 2\operatorname{Re}(a)}{x^2 - 2\operatorname{Re}(a) + |a|^2} - 2d\frac{\operatorname{Im}(a)}{x^2 - 2\operatorname{Re}(a) + |a|^2}.
\end{aligned}$$

Write $\alpha = \operatorname{Re}(a)$ and $\beta = \operatorname{Im}(a)$. We have

$$\begin{aligned}
\int \frac{P(a)}{Q'(a)}\frac{dx}{x - a} + \int \frac{P(\bar{a})}{Q'(\bar{a})}\frac{dx}{x - \bar{a}} &= c\log\left|(x - \alpha)^2 + \beta^2\right| - 2d\int \frac{\beta\,dx}{(x - \alpha)^2 + \beta^2} \\
&= c\log\left|(x - \alpha)^2 + \beta^2\right| - 2d\arctan\left(\frac{x - \alpha}{\beta}\right).
\end{aligned}$$

Thus

$$\int \frac{P(x)}{Q(x)}\,dx = \sum \frac{P(a)}{Q'(a)} \log|x-a|$$
$$+\sum\left\{\operatorname{Re}\left(\frac{P(a)}{Q'(a)}\right)\log\left|(x-\operatorname{Re}(a))^2+(\operatorname{Im}(a))^2\right|\right.$$
$$\left. -2\operatorname{Im}\left(\frac{P(a)}{Q'(a)}\right)\arctan\left(\frac{x-\operatorname{Re}(a)}{\operatorname{Im}(a)}\right)\right\},$$

where the first sum is over all real roots a, while the second sum is over all pairs of complex conjugate roots $a, \bar{a}$.

This formula is often superior to the method of partial fractions. For instance, if the roots are found numerically, finding the coefficients in partial fractions will compound round-off errors, unlike this formula. Even when the roots are known in a closed form this formula is preferable.

Example. $\int dx/(x^7+1)$ (this is example #4 of the introduction). The roots of the denominator are $w_n = e^{i(2n+1)\pi/7}$ for $0 \le n \le 6$, and

$$\frac{P(w_n)}{Q'(w_n)} = \frac{1}{7}e^{-i(2n+1)6\pi/7} \quad \text{for } 0 \le n \le 6.$$

Thus

$$\int \frac{dx}{x^7+1} = -\frac{1}{7}\ln|x+1|$$
$$+\frac{1}{7}\sum_{j=0}^{2}\left\{\cos\frac{(2j+1)6\pi}{7}\ln\left(x^2-2\cos\frac{(2j+1)\pi}{7}+1\right)\right.$$
$$\left. +2\sin\frac{(2j+1)6\pi}{7}\arctan\frac{x-\cos\frac{(2j+1)\pi}{7}}{\sin\frac{(2j+1)\pi}{7}}\right\}.$$

If the roots of the denominator are known in a closed form, the transcendental part of the integral can be written in a closed form.

IV. We now show that if the roots of the denominator cannot be expressed in a closed form, then in general the integral cannot be expressed in a closed form. First, we make precise what we mean by a closed form.

Definition: A field F is said to be a radical extension of $\mathcal{Q}$ if there is a chain of fields

$$\mathcal{Q} = F_0 \subseteq F_1 \cdots \subseteq F_n = F$$

such that for i with $1 \le i \le n$, $F_i = F_{i-1}(u_i)$ with some power of u_i in F_{i-1}.

Now in (2), collecting terms with the same coefficients we have

$$\int \frac{P(x)}{Q(x)}\,dx = \sum_i b_i \operatorname{Log} R_i(x) \tag{4}$$

where each $R_i(x)$ is a polynomial.

We say that $\int P/Q$ can be expressed in *closed form* if there is a radical extension F of $\mathcal{Q}$ with b_i in F and $R_i(x)$ in $F[x]$. Note that this simply means that b_i and the coefficients of $R_i(x)$ can be expressed by repeated use of arithmetical operations and root extractions on rational numbers.

We now have this proposition:

Proposition. *Suppose $\int P/Q$ can be expressed in closed form over F. If Q is irreducible over F then $P = CQ'$ for some C in F.*

Proof: We can assume that P and Q have no common factors, and can also assume that in (4) R_i and R_i' have no common factors, for otherwise R_i would have a repeated factor S^n and this would just give us another summand $b_i n \operatorname{Log} S$. Similarly, we may assume that R_i and R_j for $i \neq j$ have no common factors. Then differentiating (4) we have

$$PR_1 \cdots R_n = Q \sum_i b_i R_1 \cdots R_i' \cdots R_n.$$

Now R_j divides all the summands on the right except (by our assumption) $R_1 \cdots R_j' \cdots R_n$. Hence R_j, and more generally $R_1 \cdots R_n$, divides Q. Since P and Q have no common factors Q divides $R_1 \cdots R_n$ and $Q = C(R_1 \cdots R_n)$ for C in F. This contradicts our assumption that Q is irreducible over F (unless $n = 1$). Hence, say, $Q = CR_1$ and $P/Q = b_1 R_1'/R_1$.

As an example, since $x^2 - 2$ is irreducible over $\mathcal{Q}$, $\int dx/(x^2 - 2)$ cannot be written without irrationals. Risch [Risch] pointed out that this integral cannot be expressed without involving $\sqrt{2}$.

For another example, we have already noted that $2x^5 - 10x + 5$ cannot be solved by radicals. Hence this polynomial is irreducible over any radical extension F of $\mathcal{Q}$.

It follows that

$$\int \frac{P(x)}{2x^5 - 10x + 5}\,dx$$

cannot be expressed in closed form unless P is a multiple of $x^4 - 1$.

The close relationship between the problem of integrating rational functions in closed form and the solvability of polynomials in radicals is hardly surprising. As the recent book [Ebbinghaus, et al.] makes clear, the hard basic questions that led to the Fundamental Theorem of Algebra arose in part from the problem of integrating rational functions. As Hardy [Hardy] put it nearly a century ago, "The solution of the problem [of integration] in the case of rational functions may be said to be complete; for the difficulty with regard to the explicit solution of algebraic equations is not one of inadequate knowledge but of proved impossibility."

In particular cases we may be able to express the transcendental part of the integral without some or even any of the roots. Consider $\int x/(x^4 + 1)\,dx$. A calculus student would substitute $u = x^2$ and get the antiderivative $(\frac{1}{2})\arctan(x^2)$. It is instructive to work this example using our method (3) above. The roots of

x^4+1 are $\pm 1/\sqrt{2} \pm i/\sqrt{2}$. We get the solution

$$\int \frac{x\,dx}{x^4+1} = \frac{1}{2}\left\{\arctan(\sqrt{2}x-1) - \arctan(\sqrt{2}x+1)\right\}.$$

This incidentally is how Mathematica expresses the answer, which is of course expressed over $\mathcal{Q}(\sqrt{2})$. But by using the addition formula $\arctan A + \arctan B = \arctan((A+B)/(1-AB))$ we have

$$\int \frac{x\,dx}{x^4+1} = \frac{1}{2}\arctan\frac{-1}{x^2} = \frac{-1}{2}\left(\frac{\pi}{2} - \arctan(x^2)\right) = \frac{1}{2}\arctan(x^2) + C.$$

Thus the integral of a rational function may be expressible over a smaller field than the one that contains the roots of the denominator, i.e. its splitting field. Indeed the integration of the transcendental part turns on the solvability of a polynomial different from the denominator. (See [Trager] or [Lazard].)

Definition: If $b = P(a)/Q'(a)$ for some root a of Q, we say b is a residue of P/Q.

Observe that:

1) b is a residue if and only if $P(a) - bQ'(a) = 0$ for some a such that $Q(a) = 0$, and this holds if and only if $P(x) - bQ'(x)$ and $Q(x)$ have a common root.

2) If g.c.d.$(P(x) - bQ'(x), Q(x)) = R(x)$, then the roots of $R(x)$ are precisely the roots of $Q(x)$ which have b as their residue.

We collect together terms with the same coefficients in (2) to get

$$\int \frac{P(x)}{Q(x)}\,dx = \sum_i b_i \operatorname{Log} R_i(x),$$

where the b_i are the complex numbers b such that $P - bQ'$ and Q have a common root and $R_i(x) = $ g.c.d.$(P - b_iQ', Q)$. Thus if we can compute the residues b_i, a g.c.d. calculation (perhaps over an extension field) will give us the integral.

The problem of finding common roots of two polynomials is classical and is solved in terms of the resultant of the polynomials [Uspensky, Knuth, Griffiths, Davenport et al.]. We can avoid the resultant by realizing that if $P(x) - bQ'(x)$ and $Q(x)$ have a common factor then if we calculate their g.c.d. we will obtain as a remainder a polynomial in b which must be zero (the first remainder which is independent of x). This is a factor of the resultant. The calculation also yields the g.c.d. in terms of b.

We illustrate this by redoing our previous example $\int x/(x^4+1)\,dx$.

We need to find b such that $x - 4bx^3$ and x^4+1 have a common root. We compute their g.c.d. (the algorithm of [Kung] works nicely) and get the polynomial $1 + 16b^2 = 0$, with the g.c.d. being $1 - 4bx^2$. (The resultant is $(1+16b^2)^2$.) Thus $b = i/4$ or $-i/4$. We substitute these values into the g.c.d. $1 - 4bx^2$ to obtain

$$\int \frac{x}{x^4+1}\,dx = \sum_i b_i \operatorname{Log} R_i(x) = \frac{i}{4}\operatorname{Log}(1 - ix^2) - \frac{i}{4}\operatorname{Log}(1 + ix^2)$$

$$= -\frac{i}{4}\operatorname{Log}\left(\frac{x^2 - i}{x^2 + i}\right).$$

The answer is expressed over $\mathcal{Q}(i)$, the splitting field of $1 + 16b^2$. It is the further relation between log and arctan

$$\arctan x = \left(\frac{1}{2i}\right)\log((x - i)/(x + i)) - \frac{\pi}{2},$$

which gives us the answer $(\frac{1}{2})\arctan(x^2)$ over $\mathcal{Q}$. This example illustrates that if the resultant has multiple roots then the integral may be expressible over a smaller field than the splitting field of the denominator. If not, then the problem of finding the residues is no better than the problem of finding the roots of the denominator. In that sense Hardy's observation still holds. It is worth noting that even in the case where the denominator is a cubic with three real roots, in general $\mathcal{Q}(i)$ will be required to express the integral, since the roots in general cannot be expressed in closed form using real radicals only.

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APPENDIX. We briefly discuss the HP-28S implementation. The routines for polynomial arithmetic can be found in the booklet *Mathematical Applications* published by the Hewlett-Packard Company in 1988. These can even be made to work over rational arithmetic, using routines available from the first-named author. In these routines, a polynomial is stored as a list of coefficients. The denominator in example #3 of the introduction for instance is stored as $\{1,2,3,4,3,2,1\}$. At the end of step 5 we get the polynomial T represented by the list $\{D,\ -A + D + E,\ -2B + D + E + F,\ A - B - 3C + D + E + F,\ 2A - 2C + E + F,\ B - C + F\}$. Now if we set all the variables except A to be zero and A to be one (and successively for the other variables) we get a matrix whose transpose is the coefficient matrix for the system of equations in step 7. This is easily implemented on the HP-28S. The first named author will be happy to provide the codes on request.

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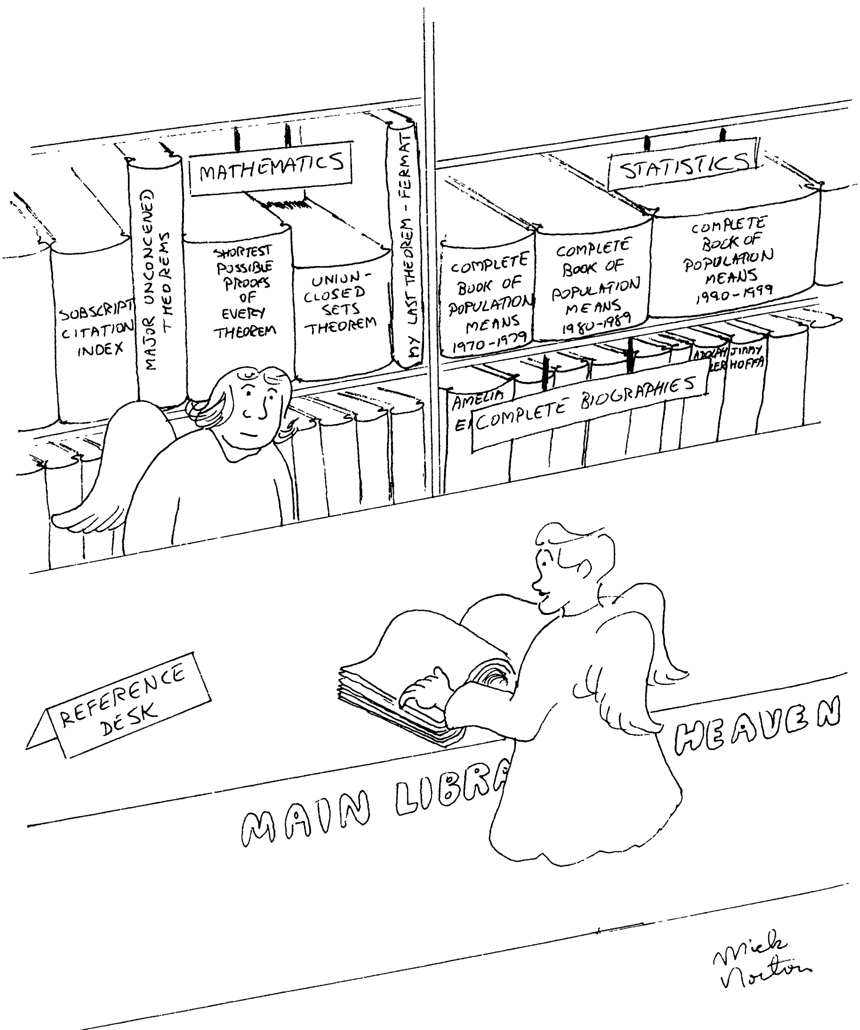

Ha! All that time wasted! It's true but not provable.