Karatsuba's Integer Multiplication

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Abstract

We use several examples to analyze the computing time of the known (school) algorithm for integer multiplication and then present Karatsuba's ideas to speed it up[1].

Contents

1 Introduction

Early in school we learn to multiply two long integers a and b, say $a = 1234$ and $b = 5678$.

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> 1234*5678
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$$7006652$$

[1]Based on the paper "Multiplikation langer Zahlen ... schneller als in der Schule", by Arno Eigenwillig and Kurt Mehlhorn.

This multiplication is done as follows:

$$\begin{array}{rcl}
1234 & \Leftarrow & a \\
\underbrace{5678} & \Leftarrow & b \\
9872 & \Leftarrow & 1234 \times 8 \times 10^0 \\
86380 & \Leftarrow & 1234 \times 7 \times 10^1 \\
740400 & \Leftarrow & 1234 \times 6 \times 10^2 \\
\underbrace{6170000} & \Leftarrow & 1234 \times 5 \times 10^3 \\
7006652 & \Leftarrow & \text{product}
\end{array}$$

When the integers a and b are long, that is they consist of many digits, this school algorithm is "time consuming". In the sequel we will examine:

- what do we mean by "time consuming", and
- how can we multiply more efficiently.

1.1 What does "time consuming" mean?

When talking about a "time consuming" algorithm we do not refer to the number of seconds needed to perform it — because next year there will be faster computers. Instead we refer to the number of basic operations performed during the execution of this algorithm.

A basic operation is something that a computer (human or machine) can do in *one step*. In the multiplication algorithm, basic operations are computations with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 from which the integers a and b are made.

1.1.1 Multiplication of two digits

From the multiplication table (2 through 9) we know how to multiply two given digits s and t in *one step*. That is, we can easily compute u and v such that $s \times t = 10 \times u + v$. For example, for $s = 6$ and $t = 7$ we have $6 \times 7 = 42 = 10 \times 4 + 2$.

1.1.2 Addition of three digits

We want addition of three digits so that we can take care of the "carry". So, given three integers r, s and t we can compute in *one step* u and v such that $r + s + t = 10 \times u + v$. For example, for $r = 5$, $s = 6$ and $t = 2$ we have $5 + 6 + 2 = 13 = 10 \times 1 + 3$.

1.2 Addition of integers

Assume we want to add the integers a and b, both containing n digits (if one has less prepend the necessary number of zeros). What we do is to write them one under the other and — moving from right to left — to add them using the method described in (1.1.2). The result is of course

$10 \times u + v$, where v is written as the result and u is used as the "carry" and is written in the next column. For examle:

$$\begin{array}{rcl} 1234 & \Leftarrow & a \\ 5678 & \Leftarrow & b \\ \underbrace{0110} & \Leftarrow & \text{carries} \\ 6912 & \Leftarrow & \text{sum} \end{array}$$

Obviously, to add two integers we perform n basic operations — the digit additions of each column.

2 Basic operations of the school multiplication method

Before we can answer this question we need to talk about the number of basic operations needed for the following multiplication.

2.1 Multiplication of an integer times a digit

We will multiply the integer $b = 5678$ times 4, the last digit of the integer $a = 1234$ and count the number of basic operations in this multiplication. The result computed is one of the four partial products of the multiplication $a \times b$.

$$\begin{array}{rcl} 5678 & \Leftarrow & b \\ \underbrace{4} & \Leftarrow & \text{digit} \\ 32 & \Leftarrow & 8 \times 4 \times 10^0 \\ 280 & \Leftarrow & 7 \times 4 \times 10^1 \\ 2400 & \Leftarrow & 6 \times 4 \times 10^2 \\ 20000 & \Leftarrow & 5 \times 4 \times 10^3 \\ \underbrace{00100} & \Leftarrow & \text{carries} \\ 22712 & \Leftarrow & \text{product} \end{array}$$

Here we multiply each digit d of the integer $b = 5678$ times 4 — moving from right to left — and the product $10 \times u + v$ is written in a new row in such a way that v is in the same column as d and u in the next column to the left. We then add all these "2-digit" results to obtain one of— the four — partial products of $a \times b$.

In general, given an n-digit integer a how many basic operations do we performe if we multiply it times a digit d?

From the above example we see that each digit of a will be multiplied times d in one step, as discussed in (1.1.1); so, we have n basic operations from these 2-digit multiplications.

Next, we have at most $n + 1$ columns with subresults that need to be added as discussed in (1.1.2) — since there might be a "carry" involved. However, since in the rightmost column we have only one digit, no addition is performed and hence we are left with n 3-digit additions.

Therefore, the total number of basic operations to multiply an integer times a digit is $2n$.

2.2 Multiplication of two integers

We are now ready to analyze the school method for the multiplication of two integers, a, b, both having n digits; if one has fewer digits than n, then we prepend the necessary number of zeros.

Let us look again at the example we stated in the introduction — where now we have also prepended zeros to each partial product to make their length $2n = 8$ digits..

$$\begin{array}{rcl}
1234 & \Leftarrow & a \\
\underbrace{\underline{5678}} & \Leftarrow & b \\
00009872 & \Leftarrow & 1234 \times 8 \times 10^0 \\
00086380 & \Leftarrow & 1234 \times 7 \times 10^1 \\
00740400 & \Leftarrow & 1234 \times 6 \times 10^2 \\
\underbrace{\underline{06170000}} & \Leftarrow & 1234 \times 5 \times 10^3 \\
07006652 & \Leftarrow & \text{product}
\end{array}$$

As discussed in (2.1), each partial product is computed with $2n$ basic operations, and therefore, all n of them are computed with

$$2n^2$$

basic operations.

Subsequently, we perform $(n-1)$ additions to add these products; as discussed in (1.2), each such addition is computed with $2n$ basic operations — where $2n$ is the length of the integers. Hence, these additions are computed with

$$(n-1) \times (2n) = 2n^2 - 2n$$

basic operations.

In total, the school method for multiplication is performed with

$$4n^2 - 2n$$

basic operations — of which n^2 are multiplications.

So, for example, to multiply two integers of 1000 digits each we will perform

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> 4*1000^2-2*1000
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$$3998000$$

basic operations, of which

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> 1000^2
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$$1000000$$

are digit multiplications.

3 Analysis of Karatsuba's method

The Karatsuba method for multiplying two integers was the first multiplication algorithm asymptotically faster than the quadratic school method. An interesting account of its discovery can be found in http://en.wikipedia.org/wiki/Karatsuba_algorithm. Below we examine several cases related to the length n of the integers and compute the number of basic operations needed for the general case.

3.1 Multiplication of single-digit integers

This is the simplest case and to perform it we need 1 basic operation. For example $5 \times 5 = 25$.

3.2 Multiplication of two-digit integers

In this case, to compute $a \times b$, we write the integers as:

$$\begin{aligned} a &= a_1 \times 10 + a_0 \\ b &= b_1 \times 10 + b_0 \end{aligned}$$

For example, if $a = 35$ and $b = 67$ then $a_1 = 3$ and $a_0 = 5$ and $b_1 = 6$ and $b_0 = 7$.

In this way, the product $a \times b$ can be written as:

$$\begin{aligned} a \times b &= (a_1 \times 10 + a_0) \times (b_1 \times 10 + b_0) \\ &= (a_1 \times b_1) \times 100 + (a_1 \times b_0 + a_0 \times b_1) \times 10 + a_0 \times b_0 \end{aligned}$$

For our example we have:

$$35 \times 67 = (3 \times 6) \times 100 + (3 \times 7 + 5 \times 6) \times 10 + 5 \times 7 = 1800 + 510 + 35 = 2345$$

The way of multiplying the two integers requires *four* single digit multiplications — and addition of the intermediary results — exactly what the school method does.

Karatsuba was able to reduce the number of multiplications from 4 to 3 by computing the following quantities:

$$\begin{aligned} u &= a_1 \times b_1 \\ v &= (a_1 - a_0) \times (b_1 - b_0) \\ w &= a_0 \times b_0 \end{aligned} \tag{1}$$

Indeed, the number of multiplications was reduced to 3 but two single digit subtractions were added in the number of basic operations.

Why does (1) help improve on multiplication? The answer is that now the following equation holds:

$$u + w - v = a_1 \times b_1 + a_0 \times b_0 - (a_1 - a_0) \times (b_1 - b_0) = a_1 \times b_0 + a_0 \times b_1 \tag{2}$$

which leads to the following way of computing the product $a \times b$:

$$a \times b = u \times 10^2 + (u + w - v) \times 10 + w \tag{3}$$

For our example $a = 35$ and $b = 67$ we have:

$$\begin{aligned} 35 \times 67 &= 18 \times 100 + (18 + 35 - (3 - 5) \times (6 - 7)) \times 10 + 35 \\ &= 1800 + 510 + 35 \\ &= 2345 \end{aligned}$$

3.3 Multiplication of four-digit integers

To multiply two 4-digit integers a, b we proceed as before and write them as:

$$\begin{aligned} a &= a_1 \times 10^2 + a_0 \\ b &= b_1 \times 10^2 + b_0 \end{aligned}$$

We then compute the quantities u, v, w

$$\begin{aligned} u &= a_1 \times b_1 \\ v &= (a_1 - a_0) \times (b_1 - b_0) \\ w &= a_0 \times b_0 \end{aligned}$$

and write the product as:

$$a \times b = u \times 10^4 + (u + w - v) \times 10^2 + w$$

For example, our original example $a = 1234$ and $b = 5678$ is computed as follows with Karatsuba's method. First we computed the quantities u, v, w:

$$\begin{aligned} u &= 12 \times 56 = 672 \\ v &= (12 - 34) \times (56 - 78) = 484 \\ w &= 34 \times 78 = 2652 \end{aligned}$$

and then we have

$$\begin{aligned} 1234 \times 578 &= 672 \times 10^4 + (672 + 2652 - 484) \times 10^2 + 2652 \\ &= 6720000 + 284000 + 2652 \\ &= 7006652 \end{aligned}$$

3.4 Multiplication of n-digit integers

Finally we consider the case where both a and b have n digits, $n = 2^k$, for some positive integer k. In this case we split a and b as follows:

$$\begin{aligned} a &= a_1 \times 10^{\frac{n}{2}} + a_0 \\ b &= b_1 \times 10^{\frac{n}{2}} + b_0 \end{aligned}$$

and the product is computed — with the help of three multiplications of integers of length $\frac{n}{2} = 2^{k-1}$ — as follows:

$$a \times b = a_1 \times b_1 \times 10^n + (a_1 \times b_1 + a_0 \times b_0 - (a_1 - a_0) \times (b_1 - b_0)) \times 10^{\frac{n}{2}} + a_0 \times b_0$$

Consequently, we have the following table comparing the number of digit multiplications of the school method with Karatsuba's.

Length	School Method	Karatsuba's Method
$n = 2^k$	4^k	3^k
$1 = 2^0$	1	1
$2 = 2^1$	4	3
$4 = 2^2$	16	9
$8 = 2^3$	64	27
$\vdots$	$\vdots$	$\vdots$
$1024 = 2^{10}$	1.048.576	59.059
	$\vdots$	

If we consider that $k = \log_2(n)$, we see that the school method needs

$$4^k = 4^{\log_2(n)} = n^{\log_2(4)} = n^2$$

multiplications, whereas Karatsuba's method

$$3^k = 3^{\log_2(n)} = n^{\log_2(3)} = n^{1,58}$$

4 Conclusion

Two are the basic ideas of Karatsuba's method:

a) To multiply two integers of length n it uses several multiplications of integers of length $\frac{n}{2}$. This idea is inducively applied until the problem is reduced to the "two-digit" multiplication ("*divide and conquer*").

b) It uses the trick with three multiplications instead of four. This "small" detail yields — over the course of the recursion — tremendous savings and results in the advantages of Karatsuba's method over the classical school method.